Volume 28, Number 1 January 2020
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE
SCOPE OF THE JOURNAL
An international publication of Eudoxus Press, LLC
(six times annually)
Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, St.Martin Univ.,Olympia,WA,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Editorial Board

Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Fractional: Differential Equations-Operators-Fourier Transforms, Special functions, Approximations, and Applications

Ram Verma
International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis, Numerical Analysis, Variational Inequalities, Optimization Theory, Computational Mathematics, Operator Theory

Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Xiao-Jun Yang
State Key Laboratory for Geomechanics and Deep Underground Engineering,
China University of Mining and Technology,
Xuzhou 221116, China
*Local Fractional Calculus and Applications, Fractional Calculus and Applications, General Fractional Calculus and Applications, Variable-order Calculus and Applications, Viscoelasticity and Computational methods for Mathematical Physics.*dyangxiaojun@163.com

Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

Ding-Xuan Zhou
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

Xin-long Zhou
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Jessada Tariboon
Department o□Mathemati□s□
King Mongkut□s □niversity o□
Te□hnology □. □angkok
□□□□ Pra□harat □ Rd.□□ ongsa□ang□
□angsue□□angkok□Thailand □□□□□
jessada.t@sci.kmutnb.ac.th, Time scales,
Di□erential□Di□eren□e □□uations□
□ra□tional Di□erential □□uations

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

<u>Journal Article</u>

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

<u>Book</u>

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

<u>Contribution to a Book</u>

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus <u>homepage</u>.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

GEHRING INEQUALITIES ON TIME SCALES

MARTIN BOHNER AND SAMIR H. SAKER

ABSTRACT. In this paper, we first prove a new dynamic inequality based on an application of the time scales version of a Hardy-type inequality. Second, by employing the obtained inequality, we prove several Gehring-type inequalities on time scales. As an application of our Gehring-type inequalities, we present some interpolation and higher integrability theorems on time scales. The results as special cases, when the time scale is equal to the set of all real numbers, contain some known results, and when the time scale is equal to the set of all integers, the results are essentially new.

1. INTRODUCTION

Let I be a fixed cube with sides parallel to the coordinate axes and let f and g be nonnegative measurable functions defined on I. The classical integral Hölder inequality

$$\int_I |f(x)g(x)|\mathrm{d}x \le \left[\int_I |f(x)|^p \mathrm{d}x\right]^{\frac{1}{p}} \left[\int_I |g(x)|^q \mathrm{d}x\right]^{\frac{1}{q}},$$

where $1/p + 1/q = 1$, shows that there is a natural scale of inclusion for the $\mathrm{L}^p(I)$-spaces, when the underlying space I has a finite measure $|I|$.

In 1972, Muckenhoupt [14] proved the first simplest reverse integral (mean) inequality, which can be considered as a reverse inclusion, of the form

$$\frac{1}{|I|}\int_I w(x)\mathrm{d}x \le \kappa \operatorname{essinf}_{x\in I} w(x), \tag{1.1}$$

where w is a nonnegative measurable function defined on I. A function verifying (1.1) is called an A_1-weight Muckenhoupt function. In [14] (see also [13]), it is proved that any A_1-weight Muckenhoupt function belongs to $\mathrm{L}^r(I)$, for $1 \le r < s$ and s depending on κ and the dimension of the space.

In 1973, Gehring [8] extended the result of Muckenhoupt for reverse mean inequalities. We say that w satisfies a Gehring condition (or a reverse Hölder inequality) if there exists $p > 1$ and a constant $\kappa > 0$ such that for every cube I with sides parallel to the coordinate axes, we have

$$\left(\frac{1}{|I|}\int_I w^p(x)\mathrm{d}x\right)^{1/p} \le \frac{\kappa}{|I|}\int_I w(x)\mathrm{d}x.$$

In this case we write $w \in \mathrm{RH}_p$. A well known result obtained by Gehring [8] states that if $w \in \mathrm{RH}_p$, then w satisfies a higher integrability condition, namely

1991 *Mathematics Subject Classification.* 26D07, 42B25, 42C10, 34N05.

Key words and phrases. Gehring's inequality, reversed Hölder inequality, Hardy-type inequality, interpolation, higher integrability, time scales.

1

for sufficiently small $\varepsilon > 0$, $q = p + \varepsilon$, we have for any cube I,

$$\left(\frac{1}{|I|}\int_I w^q(x)\mathrm{d}x\right)^{1/q} \leq \left(\frac{\kappa}{|I|}\int_I w^p(x)\mathrm{d}x\right)^{1/p}.$$

In other words, Gehring's result states that $w \in \mathrm{RH}_p$ implies that there exists $\varepsilon > 0$ such that $w \in \mathrm{RH}_{p+\varepsilon}$. The proof of Gehring's inequality is based on the use of the Calderón–Zygmund decomposition and the scale structure of L^p-spaces. In [12], the author extended Gehring's inequality by means of connecting it to the real method of interpolation by considering maximal operators, and via rearrangements reinterpreted the underlying estimates through the use of K-functionals. This technique allowed to quantify in a precise way, via reiteration, how Calderón–Zygmund decompositions have to be reparameterized in order to characterize different L^p-spaces.

Reverse integral inequalities (cf. [8, 9]) and its many variants and extensions are important in qualitative analysis of nonlinear PDEs, in the study of weighted norm inequalities for classical operators of harmonic analysis, as well as in functional analysis. These inequalities also appear in different fields of analysis such as quasiconformal mappings, weighted Sobolev embedding theorems, and regularity theory of variational problems (see [11]).

In recent years, the study of dynamic inequalities on time scales has received a lot of attention. For details, we refer to the books [2, 3, 5, 6] and the recent paper [1] and the references cited therein. The general idea in studying dynamic inequalities on time scales is to prove a result for an inequality, where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. This idea goes back to its founder Stefan Hilger [10]. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$, and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ with $q > 1$. The study of dynamic inequalities on time scales helps avoid proving results twice – once for differential inequalities and once again for difference inequalities.

Following this trend and to develop the study of dynamic inequalities on time scales, we aim in this paper to prove Gehring-type inequalities on time scales, which contain the classical integral inequalities of Gehring's type and their discrete versions as special cases. We believe that the reverse dynamic inequalities on time scales will be, just like in the classical case, similarly important for the analysis of dynamic equations on time scales.

The rest of the paper is organized as follows: In Section 2, we recall some definitions and notations related to time scales which will be used throughout the paper. Section 3 features some auxiliary results, in particular, a time scales version of Hardy's inequality. In Section 4, we present the proofs of our Gehring-type inequalities on time scales and give some interpolation results as well as some higher integrability theorems for monotone nonincreasing functions on time scales, see Section 5. As special cases, we offer discrete versions of the Gehring inequalities. To the best of the authors' knowledge, nothing is known regarding the discrete analogues of Gehring inequalities or even their extensions, and thus the presented discrete inequalities are essentially new.

2. Time Scales Preliminaries

We assume that the reader is familiar with time scales as presented in the monographs [5, 6]. For concepts concerning general measure and integration on time scales, see [6, Chapter 5] and [4, 7]. Here, we only state four facts that are essentially used in the proofs of our results. For a function $f : \mathbb{T} \to \mathbb{R}$, where $\mathbb{T}$ is a time scale, we denote the delta derivative by f^Δ and the forward shift by $f^\sigma = f \circ \sigma$, where σ is the time scales jump operator. The time scales *product rule* says that for two differentiable functions f and g, the product fg is differentiable with

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta. \tag{2.1}$$

On the other hand, the time scales *integration by parts rule* says that for two integrable functions $f, g : \mathbb{T} \to \mathbb{R}$ and $a, b \in \mathbb{T}$, we have

$$\int_a^b f^\Delta(t) g(\sigma(t))\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f(t)g^\Delta(t)\Delta t. \tag{2.2}$$

We also need the time scales *chain rule* which says that if $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable, then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable with

$$(f \circ g)^\Delta = g^\Delta \int_0^1 f'(hg^\sigma + (1-h)g)\mathrm{d}h. \tag{2.3}$$

Finally, we need the time scales *Hölder inequality* which says that for two nonnegative integrable functions $f, g : \mathbb{T} \to \mathbb{R}$ and $a, b \in \mathbb{T}$ and $p, q > 1$ with $1/p + 1/q = 1$, we have

$$\int_a^b f(t)g(t)\Delta t \le \left[\int_a^b f^p(t)\Delta t\right]^{1/p} \left[\int_a^b f^q(t)\Delta t\right]^{1/q}, \tag{2.4}$$

and p, q are called the corresponding *exponents.*

Throughout this paper, we assume that the functions in the statements of the theorems are nonnegative and rd-continuous functions, delta differentiable, locally delta integrable, and the integrals considered are assumed to exist (finite, i.e., convergent).

3. Auxiliary Results

In this section, we give some auxiliary results that are used in the proofs of our main results.

Definition 3.1. Throughout this paper, we suppose that $\mathbb{T}$ is a time scale with $0 \in \mathbb{T}$, and we let $T > 0$ with $T \in \mathbb{T}$. For any function $f : (0, T] \to \mathbb{R}$ which is Δ-integrable, nonnegative, and nonincreasing, we define the *average* function $\mathcal{A}f : (0, T] \to \mathbb{R}$ by

$$\mathcal{A}f(t) := \frac{1}{t}\int_0^t f(s)\Delta s \quad \text{for all} \quad t \in (0, T]. \tag{3.1}$$

Some simple facts about $\mathcal{A}f$ are given next.

Lemma 3.2. *If $f : (0, T] \to \mathbb{R}$ is Δ-integrable, nonnegative, and nonincreasing, then*

$$\mathcal{A}f \ge f. \tag{3.2}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Proof. Due to

$$\mathcal{A}f(t) = \frac{1}{t}\int_0^t f(s)\Delta s \geq \frac{1}{t}\int_0^t f(t)\Delta s = f(t),$$

(3.2) follows immediately. □

Lemma 3.3. *If $f : (0, T] \to \mathbb{R}$ is Δ-integrable, nonnegative, and nonincreasing, then so is $\mathcal{A}f$.*

Proof. In this proof, we write $F = \mathcal{A}f$ for brevity. We show that F inherits the nonincreasing nature of f. Let $t_1 < t_2$. Then

$$\begin{aligned} F(t_1) - F(t_2) &= \frac{1}{t_1}\int_0^{t_1} f(s)\Delta s - \frac{1}{t_2}\left[\int_0^{t_1} f(s)\Delta s + \int_{t_1}^{t_2} f(s)\Delta s\right] \\ &= \frac{t_2 - t_1}{t_2}\left[\frac{1}{t_1}\int_0^{t_1} f(s)\Delta s - \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(s)\Delta s\right] \\ &\geq \frac{t_2 - t_1}{t_2}\left[\frac{1}{t_1}\int_0^{t_1} f(t_1)\Delta s - \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(t_1)\Delta s\right] = 0, \end{aligned}$$

completing the proof. □

Now we present a Hardy inequality (see also [3, Corollary 1.5.1]) which, for completeness, we prove in our special setting.

Theorem 3.4. *If $q > 1$ and $f : (0, T] \to \mathbb{R}$ is Δ-integrable, nonnegative, and nonincreasing, then*

$$\mathcal{A}\left[(\mathcal{A}f)^\sigma\right]^q \leq \left(\frac{q}{q-1}\right)^q \mathcal{A}f^q. \tag{3.3}$$

Proof. In this proof, we write $F = \mathcal{A}f$ for brevity. Using Lemma 3.3, the chain rule shows that

$$\begin{aligned} (F^q)^\Delta &\overset{(2.3)}{=} qF^\Delta \int_0^1 (hF^\sigma + (1-h)F)^{q-1}\,\mathrm{d}h \\ &\leq qF^\Delta \int_0^1 (hF^\sigma + (1-h)F^\sigma)^{q-1}\,\mathrm{d}h = qF^\Delta (F^\sigma)^{q-1}. \end{aligned} \tag{3.4}$$

Moreover, since

$$tF(t) = \int_0^t f(s)\Delta s,$$

the product rule yields

$$f(t) \overset{(2.1)}{=} F(\sigma(t)) + tF^\Delta(t). \tag{3.5}$$

Now, putting $u(t) = t$ and $v(t) = F^q(t)$, we use integration by parts to find

$$\begin{aligned} \int_0^t (F(\sigma(s)))^q \Delta s &= \int_0^t u^\Delta(s) v(\sigma(s))\Delta s \\ &\overset{(2.2)}{=} u(t)v(t) - \lim_{s\to 0^+} u(s)v(s) - \int_0^t u(s)v^\Delta(s)\Delta s \\ &= tF^q(t) - \int_0^t s v^\Delta(s)\Delta s \end{aligned}$$

$$\begin{aligned}
&\geq -\int_0^t s v^\Delta(s)\Delta s \\
&\overset{(3.4)}{\geq} -q\int_0^t sF^\Delta(s)F^{q-1}(\sigma(s))\Delta s \\
&\overset{(3.5)}{=} -q\int_0^t [f(s)-F(\sigma(s))]\, F^{q-1}(\sigma(s))\Delta s \\
&= -q\int_0^t f(s)F^{q-1}(\sigma(s))\Delta s + q\int_0^t (F(\sigma(s)))^q\Delta s
\end{aligned}$$

so that, by using Hölder's inequality with exponents q and $q/(q-1)$,

$$\begin{aligned}
(q-1)\int_0^t (F(\sigma(s)))^q\Delta s &\leq q\int_0^t f(s)(F(\sigma(s)))^{q-1}\Delta s \\
&\overset{(2.4)}{\leq} q\left[\int_0^t (f(s))^q\Delta s\right]^{1/q}\left[\int_0^t (F(\sigma(s)))^q\Delta s\right]^{(q-1)/q},
\end{aligned}$$

resulting in (3.3). $\square$

In the main results of this paper, we assume that there exists a constant $\lambda \geq 1$ such that

$$\sigma(t) \leq \lambda t \quad \text{for all} \quad t \in \mathbb{T}. \tag{3.6}$$

We now apply the time scales chain rule to obtain some estimates that will be used later.

Lemma 3.5. *Let $x(t) = t$. If $0 < \gamma < 1$, then*

$$\left(x^{1-\gamma}\right)^\Delta \geq \frac{1-\gamma}{\sigma^\gamma}, \tag{3.7}$$

and if $\gamma > 1$ and (3.6) holds, then

$$\left(x^{1-\gamma}\right)^\Delta \geq \frac{(1-\gamma)\lambda^\gamma}{\sigma^\gamma}. \tag{3.8}$$

Proof. By the chain rule, we obtain

$$\begin{aligned}
\left(x^{1-\gamma}\right)^\Delta(t) &\overset{(2.3)}{=} (1-\gamma)x^\Delta(t)\int_0^1 \frac{\mathrm{d}h}{(hx(\sigma(t))+(1-h)x(t))^\gamma} \\
&= (1-\gamma)\int_0^1 \frac{\mathrm{d}h}{(h\sigma(t)+(1-h)t)^\gamma}.
\end{aligned}$$

Thus, if $0 < \gamma < 1$, then

$$\left(x^{1-\gamma}\right)^\Delta(t) \geq (1-\gamma)\int_0^1 \frac{\mathrm{d}h}{(h\sigma(t)+(1-h)\sigma(t))^\gamma} = \frac{1-\gamma}{(\sigma(t))^\gamma},$$

which is (3.7), and if $\gamma > 1$ and (3.6) holds, then

$$\left(x^{1-\gamma}\right)^\Delta(t) \geq (1-\gamma)\int_0^1 \frac{\mathrm{d}h}{(ht+(1-h)t)^\gamma} = \frac{1-\gamma}{t^\gamma} \overset{(3.6)}{\geq} \frac{(1-\gamma)\lambda^\gamma}{(\sigma(t))^\gamma},$$

which is (3.8). $\square$

Lemma 3.6. *If F is nonnegative and nondecreasing and $\gamma > 1$, then*

$$(F^\gamma)^\Delta \geq \gamma F^\Delta F^{\gamma-1}. \tag{3.9}$$

Proof. Again we apply the chain rule to see that

$$\begin{aligned}(F^\gamma)^\Delta &\overset{(2.3)}{=} \gamma F^\Delta \int_0^1 (hF^\sigma + (1-h)F)^{\gamma-1}\mathrm{d}h \\ &\geq \gamma F^\Delta \int_0^1 (hF + (1-h)F)^{\gamma-1}\mathrm{d}h \\ &= \gamma F^\Delta F^{\gamma-1},\end{aligned}$$

which shows (3.9). □

4. Main Results

We say that $f : (0,T] \to \mathbb{R}$ belongs to $\mathrm{L}^p_\Delta(0,T]$ provided $\int_0^T |f(t)|^p \Delta t < \infty$. The first theorem will be used later in the proof of the Gehring inequality.

Theorem 4.1. *If $f \in \mathrm{L}^p_\Delta(0,T]$ for $p > 1$ is nonnegative and nonincreasing, then, for any $q \in (0,p)$, we have*

$$\mathcal{A}f^p \leq \frac{q}{p}\left[\mathcal{A}f^q\right]^{p/q} + \frac{(p-q)\lambda^{p/q}}{p}\mathcal{A}\left[(\mathcal{A}f^q)^\sigma\right]^{p/q}. \tag{4.1}$$

Proof. From the Hardy inequality, see (3.3), we see that the second integral on the right-hand side of (4.1) is finite. Now, we consider this integral. Then, for $0 < q < p$, we put

$$\gamma = \frac{p}{q} > 1 \quad \text{and} \quad F(t) = \int_0^t f^q(s)\Delta s.$$

Using the notation from Lemma 3.5, we have

$$\begin{aligned}&\frac{(p-q)\lambda^{p/q}}{pt}\int_0^t \left[\frac{1}{\sigma(s)}\int_0^{\sigma(s)} f^q(\tau)\Delta\tau\right]^{p/q}\Delta s \\ &= \frac{(\gamma-1)\lambda^\gamma}{\gamma t}\int_0^t \left[\frac{1}{\sigma(s)}\int_0^{\sigma(s)} f^q(\tau)\Delta\tau\right]^{\gamma}\Delta s \\ &= \frac{(\gamma-1)\lambda^\gamma}{\gamma t}\int_0^t \left[\frac{F(\sigma(s))}{\sigma(s)}\right]^\gamma \Delta s \\ &= -\frac{1}{\gamma t}\int_0^t F^\gamma(\sigma(s))\frac{(1-\gamma)\lambda^\gamma}{(\sigma(s))^\gamma}\Delta s \\ &\overset{(3.8)}{\geq} -\frac{1}{\gamma t}\int_0^t F^\gamma(\sigma(s))\left(x^{1-\gamma}\right)^\Delta(s)\Delta s \\ &\overset{(2.2)}{=} \lim_{s\to 0+}\frac{F^\gamma(s)x^{1-\gamma}(s)}{\gamma t} - \frac{F^\gamma(t)x^{1-\gamma}(t)}{\gamma t} + \frac{1}{\gamma t}\int_0^t \left(F^\gamma\right)^\Delta(s)x^{1-\gamma}(s)\Delta s \\ &= \frac{1}{\gamma t}\int_0^t s^{1-\gamma}\left(F^\gamma\right)^\Delta(s)\Delta s + \frac{1}{\gamma t}\lim_{s\to 0+}\left[s\left(\frac{F(s)}{s}\right)^\gamma\right] - \frac{1}{\gamma}\left(\frac{F(t)}{t}\right)^\gamma \\ &\overset{(3.9)}{\geq} \frac{1}{\gamma t}\int_0^t \frac{\gamma F^\Delta(s)F^{\gamma-1}(s)}{s^{\gamma-1}}\Delta s - \frac{1}{\gamma}\left(\frac{F(t)}{t}\right)^\gamma \\ &= \frac{1}{t}\int_0^t f^q(s)\left[\mathcal{A}f^q(s)\right]^{\gamma-1}\Delta s - \frac{1}{\gamma}\left[\mathcal{A}f^q(t)\right]^\gamma\end{aligned}$$

$$\begin{aligned}
&\overset{(3.2)}{\geq} \frac{1}{t}\int_0^t f^q(s)\,[f^q(s)]^{\gamma-1}\,\Delta s - \frac{1}{\gamma}\,[\mathcal{A}f^q(t)]^\gamma \\
&= \frac{1}{t}\int_0^t [f^q(s)]^\gamma\,\Delta s - \frac{1}{\gamma}\,[\mathcal{A}f^q(t)]^\gamma \\
&= \mathcal{A}f^p(t) - \frac{q}{p}\,[\mathcal{A}f^q(t)]^{p/q}
\end{aligned}$$

from which (4.1) follows. □

Now, we are ready to state and prove our first time scales version of Gehring's mean inequality for monotone functions.

Theorem 4.2 (Gehring Inequality I)**.** *Assume* (3.6)*. If* $f \in \mathrm{L}^q_\Delta(0,T]$ *for* $q > 1$ *is nonnegative and nonincreasing such that*

$$\mathcal{A}f^q \leq \kappa\,[\mathcal{A}f]^q \quad \textit{for some} \quad \kappa > 0, \tag{4.2}$$

then $f \in \mathrm{L}^p_\Delta(0,T]$ *for any* $p > q$ *satisfying*

$$\tilde{\kappa} := \frac{q\kappa^{p/q}}{p - (p-q)(\lambda\kappa)^{p/q}\left(\frac{p}{p-1}\right)^p} > 0, \tag{4.3}$$

and in this case,

$$\mathcal{A}f^p \leq \tilde{\kappa}\,[\mathcal{A}f]^p. \tag{4.4}$$

Proof. Assuming (4.2), we find

$$\begin{aligned}
\frac{1}{t}\int_0^t f^p(s)\Delta s &\overset{(4.1)}{\leq} \frac{q}{p}\left[\frac{1}{t}\int_0^t f^q(s)\Delta s\right]^{p/q} \\
&\quad + \frac{(p-q)\lambda^{p/q}}{pt}\int_0^t \left[\frac{1}{\sigma(s)}\int_0^{\sigma(s)} f^q(\tau)\Delta\tau\right]^{p/q}\Delta s \\
&\overset{(4.2)}{\leq} \frac{q}{p}\kappa^{p/q}\left[\frac{1}{t}\int_0^t f(s)\Delta s\right]^p + \frac{(p-q)\lambda^{p/q}}{pt}\int_0^t \kappa^{p/q}\left[\frac{1}{\sigma(s)}\int_0^{\sigma(s)} f(\tau)\Delta\tau\right]^p\Delta s \\
&\overset{(3.3)}{\leq} \frac{q}{p}\kappa^{p/q}\left[\frac{1}{t}\int_0^t f(s)\Delta s\right]^p + \frac{(p-q)(\lambda\kappa)^{p/q}}{pt}\left(\frac{p}{p-1}\right)^p\int_0^t f^p(s)\Delta s
\end{aligned}$$

so that, due to (4.3),

$$\frac{1}{t}\int_0^t f^p(s)\Delta s \leq \tilde{\kappa}\left[\frac{1}{t}\int_0^t f(s)\Delta s\right]^p,$$

from which (4.4) follows. □

As a special case of Theorem 4.2 when $\mathbb{T} = \mathbb{R}$, we get the classical Gehring inequality (see Section 1) with $\lambda = 1$. In the case when $\mathbb{T} = \mathbb{N}$, we have the following result with $\lambda = 2$.

Corollary 4.3 (Discrete Gehring Inequality I)**.** *Let* $q > 1$ *and* $\{a_n\}_{n\in\mathbb{N}_0}$ *be a nonnegative and nonincreasing sequence such that*

$$\frac{1}{n}\sum_{i=0}^{n-1} a_i^q \leq \kappa\left(\frac{1}{n}\sum_{i=0}^{n-1} a_i\right)^q.$$

Then, for $p > q$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} a_i^p \leq \tilde{\kappa}\left(\frac{1}{n}\sum_{i=0}^{n-1} a_i\right)^p,$$

provided

$$\tilde{\kappa} := \frac{q\kappa^{p/q}}{p-(p-q)(2\kappa)^{p/q}\left(\frac{p}{p-1}\right)^p} > 0.$$

It is natural to ask what happens if in (4.4) we fix $p > 1$ and consider the improvement to this inequality that would result from lowering the exponent on the right-hand side. The following result gives an answer.

Theorem 4.4. *Suppose that the assumptions of Theorem 4.2 hold and define $\tilde{\kappa}$ as in* (4.3)*. Then, for all $0 < r < 1$, we have*

$$\mathcal{A}f^p \leq \overline{\kappa}\,[\mathcal{A}f^r]^{p/r}, \quad \text{where} \quad \overline{\kappa} := \tilde{\kappa}^{1/\theta} \quad \text{with} \quad \theta := \frac{1-\frac{1}{p}}{\frac{1}{r}-\frac{1}{p}}. \tag{4.5}$$

Proof. Note first that $\theta \in (0,1)$ and

$$\frac{1-\theta}{p} + \frac{\theta}{r} = 1.$$

Then, by the Hölder inequality with exponents $p/(1-\theta)$ and r/θ, we have

$$\begin{aligned}
\left[\frac{1}{t}\int_0^t f^p(s)\Delta s\right]^{1/p} &\overset{(4.4)}{\leq} \frac{\tilde{\kappa}^{1/p}}{t}\int_0^t f(s)\Delta s \\
&= \frac{\tilde{\kappa}^{1/p}}{t}\int_0^t f^{1-\theta}(s)f^{\theta}(s)\Delta s \\
&\overset{(2.4)}{\leq} \frac{\tilde{\kappa}^{1/p}}{t}\left[\int_0^t f^p(s)\Delta s\right]^{(1-\theta)/p}\left[\int_0^t f^r(s)\Delta s\right)^{\theta/r} \\
&= \tilde{\kappa}^{1/p}\left[\frac{1}{t}\int_0^t f^p(s)\Delta s\right]^{(1-\theta)/p}\left[\frac{1}{t}\int_0^t f^r(s)\Delta s\right]^{\theta/r}
\end{aligned}$$

so that, by dividing, we find

$$\left[\frac{1}{t}\int_0^t f^p(s)\Delta s\right]^{\theta/p} \leq \tilde{\kappa}^{1/p}\left[\frac{1}{t}\int_0^t f^r(s)\Delta s\right]^{\theta/r},$$

i.e., (4.5). ☐

By Theorem 4.4, under the assumptions of Theorem 4.2, if $f \in \mathrm{L}^r_\Delta(0,T]$ for $0 < r < 1$, then $f \in \mathrm{L}^p_\Delta(0,T]$ for $p > 1$. But in the general case when $p \neq r$, $\mathrm{L}^p_\Delta(0,T]$ neither includes nor is included in $\mathrm{L}^r_\Delta(0,T]$. The following theorem gives some results for $\mathrm{L}^p_\Delta(0,T]$-interpolation.

Theorem 4.5. *Suppose that $0 < p_0 < p_1 < \infty$ and that $0 < \theta < 1$.*

(i) *If $p = (1-\theta)p_0 + \theta p_1$ and $f \in \mathrm{L}^{p_0}_\Delta(0,T] \cap \mathrm{L}^{p_1}_\Delta(0,T]$, then $f \in \mathrm{L}^p_\Delta(0,T]$ and*

$$\mathcal{A}f^p \leq [\mathcal{A}f^{p_0}]^{1-\theta}[\mathcal{A}f^{p_1}]^{\theta}.$$

(ii) *If* $p = \frac{1}{\frac{1-\theta}{p_0}+\frac{\theta}{p_1}}$ *and* $f \in \mathrm{L}^{p_0}_{\Delta}(0,T] \cap \mathrm{L}^{p_1}_{\Delta}(0,T]$, *then* $f \in \mathrm{L}^{p}_{\Delta}(0,T]$ *and*

$$\mathcal{A}f^p \le [\mathcal{A}f^{p_0}]^{(1-\theta)p/p_0} [\mathcal{A}f^{p_1}]^{\theta p/p_1}.$$

Proof. For (i), we apply the Hölder inequality with exponents $1/(1-\theta)$ and $1/\theta$ to see that

$$\begin{aligned}\frac{1}{t}\int_0^t f^p(s)\Delta s &= \frac{1}{t}\int_0^t f^{(1-\theta)p_0}(s) f^{\theta p_1}(s)\Delta s\\ &\overset{(2.4)}{\le} \left[\frac{1}{t}\int_0^t f^{p_0}(s)\Delta s\right]^{1-\theta}\left[\frac{1}{t}\int_0^t f^{p_1}(s)\Delta s\right]^{\theta},\end{aligned}$$

which shows (i). For (ii), we apply the Hölder inequality with exponents $1/(1-\gamma)$ and $1/\gamma$, where

$$\gamma := \frac{\theta p}{p_1} \quad \text{so that} \quad 1-\gamma = \frac{(1-\theta)p}{p_0},$$

to see that

$$\begin{aligned}\frac{1}{t}\int_0^t f^p(s)\Delta s &= \frac{1}{t}\int_0^t f^{(1-\theta)p}(s) f^{\theta p}(s)\Delta s\\ &\overset{(2.4)}{\le} \left[\frac{1}{t}\int_0^t f^{(1-\theta)p/(1-\gamma)}(s)\Delta s\right]^{1-\gamma}\left[\frac{1}{t}\int_0^t f^{\theta p/\gamma}(s)\Delta s\right]^{\gamma}\\ &= \left[\frac{1}{t}\int_0^t f^{p_0}(s)\Delta s\right]^{(1-\theta)p/p_0}\left[\frac{1}{t}\int_0^t f^{p_1}(s)\Delta s\right]^{\theta p/p_1},\end{aligned}$$

which shows (ii). □

In the following, we give a new proof of Gehring's mean inequality on time scales. The inequality will be proved by using a condition similar to the condition (1.1) due to Muckenhoupt. In fact, we do not assume that the reverse Hölder inequality holds.

Theorem 4.6 (Gehring Inequality II)**.** *Assume* (3.6). *If* $f : (0,T] \to \mathbb{T}$ *is nonnegative and nonincreasing such that*

$$\mathcal{A}f^\sigma \le \nu f \quad \textit{for some} \quad \nu > 1, \tag{4.6}$$

then $f \in \mathrm{L}^p_\Delta(0,T]$ *for* $p \in [1, \alpha/(\alpha-1))$, *where* $\alpha = \lambda\nu$, *and we have*

$$\mathcal{A}(f^p)^\sigma \le \tilde{\nu}\,[\mathcal{A}f^\sigma]^p, \quad \textit{where} \quad \tilde{\nu} := \frac{\alpha}{\alpha - p(\alpha-1)} > 0. \tag{4.7}$$

Proof. For this proof, we put

$$F(t) := \int_0^t f^\sigma(s)\Delta s, \quad l(t) = \log(t), \quad L(t) = \log(F(t)).$$

By the chain rule, we get

$$\begin{aligned}\frac{1}{\alpha} l^\Delta(t) &\overset{(2.3)}{=} \frac{1}{\lambda\nu}\int_0^1 \frac{\mathrm{d}h}{h\sigma(t) + (1-h)t}\\ &\overset{(3.6)}{\le} \frac{1}{\lambda\nu}\int_0^1 \frac{\mathrm{d}h}{h\sigma(t) + (1-h)\frac{\sigma(t)}{\lambda}}\\ &\le \frac{1}{\lambda\nu}\cdot\frac{\lambda}{\sigma(t)} = \frac{1}{\nu\sigma(t)}\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

$$
\begin{aligned}
&\overset{(4.6)}{\leq} \frac{f(\sigma(t))}{F(\sigma(t))} = \frac{F^{\Delta}(t)}{F(\sigma(t))} \\
&= F^{\Delta}(t) \int_0^1 \frac{\mathrm{d}h}{hF(\sigma(t)) + (1-h)F(\sigma(t))} \\
&\leq F^{\Delta}(t) \int_0^1 \frac{\mathrm{d}h}{hF(\sigma(t)) + (1-h)F(t)} \\
&\overset{(2.3)}{=} L^{\Delta}(t),
\end{aligned}
$$

and hence, by integrating,

$$\log\left(\frac{t}{\sigma(s)}\right)^{1/\alpha} = \frac{1}{\alpha}l(t) - \frac{1}{\alpha}l(\sigma(s)) \leq L(t) - L(\sigma(s)) = \log\left(\frac{F(t)}{F(\sigma(s))}\right)$$

so that

$$f(\sigma(s)) \leq \frac{1}{\sigma(s)} \int_0^{\sigma(s)} f(\sigma(\tau))\Delta\tau = \frac{F(\sigma(s)}{\sigma(s)} \leq \left(\frac{\sigma(s)}{t}\right)^{1/\alpha} \frac{F(t)}{\sigma(s)},$$

and by integrating again, putting $\gamma := p(1-1/\alpha) \in (0,1)$, and using the notation from Lemma 3.5, we obtain

$$
\begin{aligned}
\frac{1}{t}\int_0^t f^p(\sigma(s))\Delta s &\leq \frac{F^p(t)}{t^{1+p/\alpha}} \int_0^t \frac{\Delta s}{(\sigma(s))^{p(1-1/\alpha)}} \\
&\overset{(3.7)}{\leq} \frac{F^p(t)}{(1-\gamma)t^{1+p/\alpha}} \int_0^t \left(x^{1-\gamma}\right)^{\Delta}(s)\Delta s \\
&= \frac{t^{1-\gamma}F^p(t)}{(1-\gamma)t^{1+p/\alpha}} = \frac{1}{1-\gamma}\left(\frac{F(t)}{t}\right)^p,
\end{aligned}
$$

proving (4.7). □

As a special case of Theorem 4.6 when $\mathbb{T} = \mathbb{N}$, we have the following result.

Corollary 4.7 (Discrete Gehring Inequality II). *Let $\{a_n\}_{n\in\mathbb{N}_0}$ be a nonnegative and nonincreasing sequence. If there exists a constant $\nu > 1$ such that*

$$\frac{1}{n}\sum_{i=1}^{n} a_i \leq \nu a_n,$$

then, for $p \in [1, \alpha/(\alpha-1)]$, where $\alpha = 2\nu$, we have

$$\frac{1}{n}\sum_{i=1}^{n} a_i^p \leq \tilde{\nu}\left[\frac{1}{n}\sum_{i=1}^{n} a_i\right]^p, \quad \text{where} \quad \tilde{\nu} := \frac{\alpha}{\alpha - p(\alpha-1)}.$$

5. Higher Integrability

In the following, as an application of Gehring's inequality (4.7), we prove a higher integrability theorem for monotone nonincreasing functions. First notice that for all nonnegative and nonincreasing functions $f \in \mathrm{L}^q_{\Delta}(0,T]$ with $q > 1$, we always have

$$\mathcal{A}f^q(t) = \frac{1}{t}\int_0^t f^q(s)\Delta s = \frac{1}{t}\int_0^t f^{q-1}(s)f(s)\Delta s \geq \frac{f^{q-1}(t)}{t}\int_0^t f(s)\Delta s. \tag{5.1}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Let us now consider the class of nonnegative and nonincreasing functions $f \in \mathrm{L}^q_\Delta(0,T]$ that satisfy the reverse of (5.1), namely

$$\mathcal{A}f^q \le \eta f^{q-1}\mathcal{A}f \quad \text{for some} \quad \eta > 1. \tag{5.2}$$

Theorem 5.1. *Assume* (3.6). *If $f \in \mathrm{L}^q_\Delta(0,T]$ for $q > 1$ is nonnegative and nonincreasing such that* (5.2) *holds, then $f \in \mathrm{L}^p_\Delta(0,T]$ for $p \in [q, q+c]$, $c \in (q,\eta)$, and we have*

$$\mathcal{A}(f^p)^\sigma \le \tilde{\eta}\,[\mathcal{A}f^q]^{p/q}, \quad \textit{where} \quad \tilde{\eta} := \frac{\lambda{\eta_q}^{1+p/q}}{\lambda\eta_q - \frac{p}{q}(\lambda\eta_q - 1)} \quad \textit{with} \quad \eta_q = \frac{\eta q}{q-1}. \tag{5.3}$$

Proof. In this proof, we write $F = \mathcal{A}f^q$ for brevity. By using the Hölder inequality with exponents $q/(q-1)$ and q, we obtain

$$\begin{aligned}
\frac{1}{t}\int_0^t F(\sigma(s))\Delta s &\overset{(5.2)}{\le} \frac{\eta}{t}\int_0^t (f(\sigma(s)))^{q-1}\cdot\frac{1}{\sigma(s)}\int_0^{\sigma(s)} f(\tau)\Delta\tau\Delta s \\
&\overset{(2.4)}{\le} \frac{\eta}{t}\left[\int_0^t (f(\sigma(s)))^q\Delta s\right]^{(q-1)/q}\left[\int_0^t\left(\frac{1}{\sigma(s)}\int_0^{\sigma(s)} f(\tau)\Delta\tau\right)^q\Delta s\right]^{1/q} \\
&\overset{(3.3)}{\le} \frac{\eta q}{(q-1)t}\left[\int_0^t (f(s))^q\Delta s\right]^{(q-1)/q}\left[\int_0^t (f(s))^q\Delta s\right]^{1/q} \\
&= \frac{\eta_q}{t}\int_0^t f^q(s)\Delta s = \eta_q F(t),
\end{aligned}$$

i.e.,

$$\mathcal{A}F^\sigma \le \eta_q F. \tag{5.4}$$

Since F is also nonnegative and nonincreasing (see Lemma 3.3), it satisfies the assumptions of Theorem 4.6, and thus

$$\frac{1}{t}\int_0^t [F(\sigma(s))]^r\,\Delta s \le \tilde{\eta}_q\left[\frac{1}{t}\int_0^t F(\sigma(s))\Delta s\right]^r \tag{5.5}$$

with

$$\tilde{\eta}_q = \frac{\alpha_q}{\alpha_q - r(\alpha_q - 1)} \quad \text{and} \quad \alpha_q = \lambda\eta_q \quad \text{for} \quad r = \frac{p}{q} \in \left[1, \frac{\alpha_q}{\alpha_q - 1}\right).$$

Noting that

$$F(t) = \frac{1}{t}\int_0^t f^q(s)\Delta s \ge f^q(t), \tag{5.6}$$

we obtain

$$\begin{aligned}
\frac{1}{t}\int_0^t (f(\sigma(s)))^p\Delta s &= \frac{1}{t}\int_0^t (f^q(\sigma(s)))^r\Delta s \\
&\overset{(5.6)}{\le} \frac{1}{t}\int_0^t (F(\sigma(s)))^r\Delta s \\
&\overset{(5.5)}{\le} \tilde{\eta}_q\left(\frac{1}{t}\int_0^t F^\sigma(s)\Delta s\right)^r \\
&\overset{(5.4)}{\le} \tilde{\eta}_q{\eta_q}^r\,[F(t)]^r = \tilde{\eta}\,[F(t)]^r
\end{aligned}$$

$$= \tilde{\eta}\left[\frac{1}{t}\int_0^t f^q(s)\Delta s\right]^{p/q},$$

proving (5.3). □

In Theorem 5.1, if $\mathbb{T}=\mathbb{R}$, then we have that $\sigma(t)=t$, $\alpha_q=\eta_q$, and we get the following result.

Corollary 5.2. *Let $\eta>1$ and $q>1$. Then every nonnegative nonincreasing function f satisfying*

$$\int_0^t f^q(x)\mathrm{d}x \le \eta f^{q-1}(t)\int_0^t f(x)\mathrm{d}x$$

belongs to $\mathrm{L}^p_\Delta(0,T](0,T]$ for $p\in[q,q+c]$ and $c\in(q,\eta)$, and we have

$$\frac{1}{t}\int_0^t f^p(x)\mathrm{d}x \le \tilde{\eta}\left(\frac{1}{t}\int_0^t f^q(x)\mathrm{d}x\right)^{p/q},$$

where

$$\tilde{\eta} := \frac{\left(\frac{\eta q}{q-1}\right)^{\frac{p}{q}+1}}{\frac{\eta q}{q-1}-\frac{p}{q}\left(\frac{\eta q}{q-1}-1\right)}.$$

In Theorem 5.1, if $\mathbb{T}=\mathbb{N}$, then we have that $\sigma(t)=t+1$, and by choosing $\lambda=2$, we get the following result.

Corollary 5.3. *Let $\eta>1$ and $q>1$. Suppose $\{a_n\}_{n\in\mathbb{N}_0}$ is a nonnegative and nonincreasing sequence satisfying*

$$\sum_{i=0}^{n-1} a_i^q \le \eta a_n^{q-1}\sum_{i=0}^{n-1} a_i.$$

Then, for $p\in[q,q+c]$, $c\in(q,\eta)$, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} a_i^p \le \tilde{\eta}\left(\frac{1}{n}\sum_{i=0}^{n-1} a_i^q\right)^{p/q},$$

where

$$\tilde{\eta} := \frac{2\left(\frac{\eta q}{q-1}\right)^{\frac{p}{q}+1}}{2\frac{\eta q}{q-1}-\frac{p}{q}\left(2\frac{\eta q}{q-1}-1\right)}.$$

References

[1] Ravi P. Agarwal, Martin Bohner, and Samir Saker. Dynamic Littlewood-type inequalities. *Proc. Amer. Math. Soc.*, 143(2):667–677, 2015.

[2] Ravi P. Agarwal, Donal O'Regan, and Samir Saker. *Dynamic inequalities on time scales.* Springer, Cham, 2014.

[3] Ravi P. Agarwal, Donal O'Regan, and Samir H. Saker. *Hardy type inequalities on time scales.* Springer, Cham, 2016.

[4] Ravi P. Agarwal, Victoria Otero-Espinar, Kanishka Perera, and Dolores R. Vivero. Basic properties of Sobolev's spaces on time scales. *Adv. Difference Equ.*, 14 pages, Art. ID 38121, 2006.

[5] Martin Bohner and Allan Peterson. *Dynamic equations on time scales.* Birkhäuser Boston, Inc., Boston, MA, 2001. An introduction with applications.

[6] Martin Bohner and Allan Peterson. *Advances in dynamic equations on time scales.* Birkhäuser Boston, Inc., Boston, MA, 2003.

[7] Alberto Cabada and Dolores R. Vivero. Expression of the Lebesgue Δ-integral on time scales as a usual Lebesgue integral: application to the calculus of Δ-antiderivatives. *Math. Comput. Modelling*, 43(1-2):194–207, 2006.

[8] Frederick W. Gehring. The L^p-integrability of the partial derivatives of a quasiconformal mapping. *Acta Math.*, 130:265–277, 1973.

[9] Frederick W. Gehring. The L^p-integrability of the partial derivatives of quasiconformal mapping. *Bull. Amer. Math. Soc.*, 79:465–466, 1973.

[10] Stefan Hilger. Analysis on measure chains—a unified approach to continuous and discrete calculus. *Results Math.*, 18(1-2):18–56, 1990.

[11] Tadeusz Iwaniec. On L^p-integrability in PDEs and quasiregular mappings for large exponents. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 7(2):301–322, 1982.

[12] Mario Milman. A note on Gehring's lemma. *Ann. Acad. Sci. Fenn. Math.*, 21(2):389–398, 1996.

[13] Benjamin Muckenhoupt. Hardy's inequality with weights. *Studia Math.*, 44:31–38, 1972. Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, I.

[14] Benjamin Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.

Missouri University of Science and Technology, Department of Mathematics and Statistics, Rolla, Missouri 65409-0020, USA

E-mail address: bohner@mst.edu

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt

E-mail address: shsaker@mans.edu.eg

Basin of attraction of the fixed point and period-two solutions of a certain anti-competitive map

Emin Bešo, Senada Kalabušić and Esmir Pilav
Department of Mathematics,
University of Sarajevo,
71000 Sarajevo,
Bosnia and Herzegovina
emin.beso@hotmail.com; kalabusicsenada06@gmail.com; esmirpilav@gmail.com

Naida Mujić
Faculty of Electrical Engineering,
University of Sarajevo,
71000 Sarajevo,
Bosnia and Herzegovina
mujicnaida@gmail.com;

February 26, 2019

Abstract

We investigate the periodic nature, the boundedness character, and the global asymptotic stability of solutions of the difference equation

$$x_{n+1} = \frac{\gamma x_{n-1}^2}{Cx_{n-1}^2 + x_n}$$

where the parameters γ, C are positive numbers and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers such that $x_{-1} + x_0 > 0$. We determine the basin of attraction of fixed point and period-two solutions. The associated map is not defined at the $(0,0)$. However, we show that there exist period two solutions on the axis that are locally asymptotically stable and two continuous invariant curves passing through the point $(0,0)$, which are boundaries of the basins of attractions of these period two solutions, such that every solution starting on these two curves or in the region between these two curves is attracted to the point $(0,0)$.

Key Words: Basin of attraction; difference equation; global attractivity; global stable manifold; monotonicity;
MSC(2010): Primary: 39A10,39A23, 39A30; Secondary: 37E05

1 Introduction and Preliminaries

In this paper we consider the following quadratic rational difference equation of second order

$$x_{n+1} = \frac{\gamma x_{n-1}^2}{C x_{n-1}^2 + x_n} \tag{1}$$

We assume that $\gamma, C > 0$ and initial conditions x_{-1}, x_0 are positive real numbers, such that $x_0 + x_{-1} > 0$. Notice that the map associated to this equation is not defined at the point $(0,0)$. The second iterate of the map associated to Equation (1) is *competitive map*. We call such map *anti-competitive*. See [7, 8]. Theory of competitive systems and maps in the plane have been extensively developed and main results are given in [2, 6, 11, 13, 14]. Equation (1) is a special case of the difference equation

$$x_{n+1} = \frac{\beta x_n x_{n-1} + \gamma x_{n-1}^2 + \delta x_n}{B x_n x_{n-1} + C x_{n-1}^2 + D x_n}, \quad n = 0, 1, 2, \ldots \tag{2}$$

where the parameters $\beta, \gamma, \delta, B, C, D$ are nonnegative numbers which satisfy $B + C + D > 0$ and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers such that $B x_n x_{n-1} + C x_{n-1}^2 + D x_n > 0$ for all $n \geq 0$. Locally stability of the equilibrium points of (2) has been studied in [10]. In this paper we describe global behavior of solutions of Equation (1).

Equation (1) is related to the difference equation

$$x_{n+1} = \frac{\gamma x_{n-1}}{B x_n + C x_{n-1}}, \; n = 0, 1, \ldots \tag{3}$$

where the parameters γ, B and C are positive real numbers and the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers such that $x_{-1} + x_0 > 0$, see [1, 9].

As we will see in this paper Equation (1) has very different behaviour than Equation (3) showing that introduction of quadratic terms can significantly change behaviour of the equation. We prove that parametric space splits into four regions given by $0 < \gamma < 1$, $1 < \gamma < 3, \gamma = 3$ and $\gamma > 3$. By using results from [3, 13] we obtain global result in each of these four regions, different than global results for Equation (3). For example in Section 3 we show that there exist two increasing continues invariant curves passing through the point $(0,0)$ which are the boundaries of basins of attractions of the period-two solutions such that every solution that starts on these two curves or in the region between these two curves is attracted to the point $(0,0)$.

We now present some basic notation about competitive map in the plane.
Consider a first order system of difference equations of the form

$$\left\{ \begin{array}{lcl} x_{n+1} & = & f(x_n, y_n) \\ y_{n+1} & = & g(x_n, y_n) \end{array} \right. , \quad n = 0, 1, 2, \ldots, \quad (x_{-1}, x_0) \in \mathcal{I} \times \mathcal{I} \tag{4}$$

where $f, g : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ are continuous functions on an interval $\mathcal{I} \subset \mathbb{R}$, $f(x,y)$ is non-decreasing in x and non-increasing in y, and $g(x,y)$ is non-increasing in x and non-decreasing in y. Such system is called *competitive*. One may associate a competitive map T to a competitive system (4) by setting $T = (f, g)$ and considering T on $\mathcal{B} = \mathcal{I} \times \mathcal{I}$.

A point $\mathbf{x} \in \mathcal{B}$ is a *fixed point* of T if $T(\mathbf{x}) = \mathbf{x}$, and a *minimal period-two point* if $T^2(\mathbf{x}) = \mathbf{x}$ and $T(\mathbf{x}) \neq \mathbf{x}$. A *period-two point* is either a fixed point or a minimal period-two point. In a similar fashion one can define a minimal period p point. The *orbit of* $\mathbf{x} \in \mathcal{B}$ is the sequence $\{T^\ell(\mathbf{x})\}_{\ell=0}^\infty$. A *minimal period-two orbit* is an orbit $\{\mathbf{x}_\ell\}_{\ell=0}^\infty$ for which $\mathbf{x}_0 \neq \mathbf{x}_1$ and $\mathbf{x}_0 = \mathbf{x}_2$. The *basin of attraction* of a fixed point $\mathbf{x}$ is the set of all $\mathbf{y}$ such that $T^n(\mathbf{y}) \to \mathbf{x}$. A fixed point $\mathbf{x}$ is a *global attractor* on a set $\mathcal{A}$ if $\mathcal{A}$ is a subset of the basin of attraction of $\mathbf{x}$. A fixed point $\mathbf{x}$ is a *saddle point* if T is differentiable at $\mathbf{x}$, and the eigenvalues of the Jacobian matrix of T at $\mathbf{x}$ are such that one of them lies in the interior of the unit circle in $\mathbb{R}^2$, while the other eigenvalue lies in the exterior of the unit circle. If $T = (T_1, T_2)$ is a map on $\mathcal{R} \subset \mathbb{R}^2$, define the sets $\mathcal{R}_T(-,+) := \{(x,y) \in \mathcal{R} : T_1(x,y) \leq x, \; T_2(x,y) \geq y \}$ and $\mathcal{R}_T(+,-) := \{(x,y) \in \mathcal{R} : T_1(x,y) \geq x, \; T_2(x,y) \leq y \}$.

If $\mathbf{v} = (u,v) \in \mathbb{R}^2$, we denote with $\mathcal{Q}_\ell(\mathbf{v})$, $\ell \in \{1,2,3,4\}$, the four quadrants in $\mathbb{R}^2$ relative to $\mathbf{v}$, i.e., $\mathcal{Q}_1(\mathbf{v}) = \{(x,y) \in \mathbb{R}^2 \; : x \geq u, \; y \geq v\}$, $\mathcal{Q}_2(\mathbf{v}) = \{(x,y) \in \mathbb{R}^2 \; : x \leq u, \; y \geq v\}$, and so on. Define the *South-East* partial order $\preceq_{se}$ on $\mathbb{R}^2$ by $(x,y) \preceq_{se} (s,t)$ if and only if $x \leq s$ and $y \geq t$. Similarly, we define the *North-East* partial order $\preceq_{ne}$ on $\mathbb{R}^2$ by $(x,y) \preceq_{ne} (s,t)$ if and only if $x \leq s$ and $y \leq t$. A stronger inequality

may be defined as $\mathbf{v} = (v_1, v_2) \ll \mathbf{w} = (w_1, w_2)$ if $\mathbf{v} \preceq \mathbf{w}$ with $v_1 \neq w_1$ and $v_2 \neq w_2$. For $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^2$, the *order interval* $[\![\mathbf{u}, \mathbf{v}]\!]$ is the set of all $\mathbf{x} \in \mathbb{R}^2$ such that $\mathbf{u} \preceq \mathbf{x} \preceq \mathbf{v}$.

A map T is *competitive* if $T(\mathbf{x}) \preceq_{se} T(\mathbf{y})$ whenever $\mathbf{x} \preceq_{se} \mathbf{y}$, and T is *strongly competitive* if $\mathbf{x} \preceq_{se} \mathbf{y}$ implies $T(\mathbf{x}) - T(\mathbf{y}) \in \{(u, v) : u > 0,\ v < 0\}$. If T is differentiable, a sufficient condition for T to be strongly competitive is that the Jacobian matrix of T at any $\mathbf{x} \in \mathcal{B}$ has the sign configuration

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix}.$$

For additional definitions and results (e.g., repeller, hyperbolic fixed points, stability, asymptotic stability, stable and unstable manifolds) see [6, 14] for competitive maps, and [9, 11] for difference equations.

This paper is structured as follows. In Section 2 we prove linearized stability results. Depending on parameter γ we determine the nature of equilibrium point and period-two solutions and then we prove convergence result for period-two solution. In Section 3 we describe completely global behaviour of Equation (1).

2 Linearized stability analysis and convergence result

In this section we prove linearized stability and convergence results for Equation(1).

Theorem 1 *If $\gamma > 1$ then Equation* (1) *has the unique equilibrium point $\bar{x}$ which is given by*

$$\bar{x} = \frac{\gamma - 1}{C}$$

and $\bar{x}$ is

a) locally asymptotically stable if $\gamma > 3$.

b) a non-hyperbolic point if $\gamma = 3$;

c) a saddle point if $1 < \gamma < 3$;

Proof. The proof follows from the well known linearized stability theorem, see [10].

□

Theorem 2 *For the Equation* (1) *the following holds:*

(a) For all values of parameters Equation (1) *has prime period-two solution*

$$\left\{0, \frac{\gamma}{C}\right\}$$

which is locally asymptotically stable.

(b) If $\gamma > 3$ then Equation (1) *has prime period-two solution*

$$\left\{\frac{\gamma - \sqrt{(\gamma-3)(\gamma+1)} + 1}{2C}, \frac{\gamma + \sqrt{(\gamma-3)(\gamma+1)} + 1}{2C}\right\}$$

which is a saddle point.

Proof.

(a) It is easy to check that $\{0, \frac{\gamma}{C}\}$ is period two solution for all values of parameters. This period two solution always exists.

(b) Assume that $\gamma > 3$. If $\ldots a, b, a, a, b, \ldots$ is a period two solution, then this solution satisfies the following system of algebraic equations

$$\begin{aligned} b &= \frac{\gamma b^2}{Cb^2 + a} \\ a &= \frac{\gamma a^2}{Ca^2 + b}. \end{aligned}$$

Straightforward calculations shows that under the condition $\gamma > 3$ the unique solution of this system is given by

$$a = \frac{\gamma - \sqrt{(\gamma-3)(\gamma+1)} + 1}{2C}, \quad b = \frac{\gamma + \sqrt{(\gamma-3)(\gamma+1)} + 1}{2C}.$$

By using linearized stability theorem, it is easy to see that this period two solution is a saddle point, see [10].

Note that it is not possible to obtain period two solution $\{0, \frac{\gamma}{C}\}$ by solving the previous system of algebraic equations.

□

Now, we show that every solution of Equation (1) converges to a period-two solution (not necessarily minimal).
Let

$$F(u, v) = \frac{\gamma v^2}{Cv^2 + u}.$$

It is easy to see that

$$F'_x = -\frac{\gamma v^2}{(Cv^2 + u)^2} \text{ and } F'_y = \frac{2\gamma uv}{(Cv^2 + u)^2}$$

Set

$$u_n = x_{n-1} \text{ and } v_n = x_n \text{ for } n = 0, 1, \ldots \tag{5}$$

We can rewrite Equation (1) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= \frac{\gamma u_n^2}{Cu_n^2 + v_n} \end{aligned} \tag{6}$$

for $n = 0, 1, \ldots$.
Let T be the map associated to Equation (1):

$$T(u, v) = (v, F(v, u)) = \left(v, \frac{\gamma u^2}{Cu^2 + v}\right). \tag{7}$$

then

$$(u_{n+1}, v_{n+1}) = T(u_n, v_n) \tag{8}$$

It is easy to see that

$$T^2(u, v) = T(T(u, v)) = (T_{21}(u, v), T_{22}(u, v)) \left(\frac{\gamma u^2}{Cu^2 + v}, \frac{\gamma v^2\left(Cu^2 + v\right)}{C^2u^2v^2 + Cv^3 + \gamma u^2}\right)$$

from which it follows that

$$(u_{2n+2}, v_{2n+2}) = T^2(u_{2n}, v_{2n}) \tag{9}$$

which is equivalent to

$$(x_{2n+1}, x_{2n+2}) = T^2(x_{2n-1}, x_{2n}).$$

The Jacobian matrix of the map T has the form:

$$J_T(u,v) = \begin{pmatrix} 0 & 1 \\ \frac{2uv\gamma}{(cu^2+v)^2} & -\frac{u^2\gamma}{(cu^2+v)^2} \end{pmatrix} \tag{10}$$

The determinant of (10) is given by

$$detJ_T(u,v) = -\frac{2\gamma uv}{(cu^2+v)^2} \tag{11}$$

The Jacobian matrix of the map T^2 has the form:

$$J_{T^2}(u,v) = \begin{pmatrix} \frac{2uv\gamma}{(Cu^2+v)^2} & -\frac{u^2\gamma}{(Cu^2+v)^2} \\ -\frac{2uv^3\gamma^2}{(\gamma u^2+Cv^2(Cu^2+v))^2} & \frac{u^2v(2Cu^2+3v)\gamma^2}{(\gamma u^2+Vv^2(Cu^2+v))^2} \end{pmatrix} \tag{12}$$

The determinant of (12) is given by

$$detJ_{T^2}(u,v) = \frac{4\gamma^3u^3v^2}{(Cu^2+v)\left(Cv^2\left(Cu^2+v\right)+\gamma u^2\right)^2} \tag{13}$$

The equilibrium curves of the map T^2 are given by

$$\mathcal{C}_1 := \left\{(x,y) \in [0,\infty)^2 : T_{21}(x,y) = x\right\} = \left\{(x,y) \in [0,\infty)^2 : y = \gamma x - Cx^2\right\}$$

and

$$\mathcal{C}_2 := \left\{(x,y) \in [0,\infty)^2 : T_{22}(x,y) = y\right\} = \left\{(x,y) \in [0,\infty)^2 : x = \frac{y\sqrt{\gamma - Cy}}{\sqrt{Cy(Cy-\gamma)+\gamma}}\right\}$$

By direct inspection of Equation (13) we obtain the following result:

Lemma 1 *The map T^2 is competitive on $[0,\infty)^2 \setminus \{(0,0)\}$ and strongly competitive on $(0,\infty)^2$.*

It is easy to see that the following holds.

Lemma 2 *For all $x_{-1}, x_0 \in [0,\infty)$, such that $x_{-1}+x_0 > 0$, the following holds $x_n \leq \frac{\gamma}{C}$ for $n \geq 1$.*

By using very powerful Theorem 1.5 from [4] and Lemma 2, we obtain the following convergence result.

Theorem 3 *Every solution of Equation (1) converges to a period-two solution or to zeros.*

3 Global behavior

In this section we consider the following four parametric regions $\gamma > 3$, $1 < \gamma < 3, \gamma = 3$ and $0 < \gamma < 1$. We completely describe the global behaviour of Equation (1) in these regions.

The following theorem details the case $\gamma > 3$.

Theorem 4 *Assume that*

$$\gamma > 3.$$

Then system (8) has a unique equilibrium point $E(\bar{u}, \bar{u})$ which is locally asymptotically stable and there exist two prime period-two solutions: $\{P_1(\bar{u}_1, \bar{v}_1), P_2(\bar{v}_1, \bar{u}_1)\}$ which is locally asymptotically stable and $\{P_3(\bar{u}_2, \bar{v}_2), P_4(\bar{v}_2, \bar{u}_2)\}$ which is a saddle point, where

$$\bar{u}_1 = 0, \quad \bar{v}_1 = \frac{\gamma}{C} \text{ and } \bar{u} = \frac{\gamma - 1}{C}$$

and

$$\bar{u}_2 = \frac{\gamma + 1 - \sqrt{(\gamma-3)(\gamma+1)}}{2C} \text{ and } \bar{v}_2 = \frac{\gamma + 1 + \sqrt{(\gamma-3)(\gamma+1)}}{2C}$$

Furthermore, global stable manifold of the periodic solution $\{P_3, P_4\}$ is given by $\mathcal{W}^s(\{P_3, P_4\}) = \mathcal{W}^s(P_3) \cup \mathcal{W}^s(P_4)$ where $\mathcal{W}^s(P_3)$ and $\mathcal{W}^s(P_4)$ are continuous increasing curves, invariant under the map T^2 and $T(\mathcal{W}^s(P_3)) = \mathcal{W}^s(P_4)$, and divide the first quadrant into two connected components, namely

$$\mathcal{W}_1^- := \{x \in \mathcal{R} \setminus \mathcal{W}^s(P_3) : \exists y \in \mathcal{W}^s(P_3) \text{ with } y \preceq_{se} x\}$$
$$\mathcal{W}_1^+ := \{x \in \mathcal{R} \setminus \mathcal{W}^s(P_3) : \exists y \in \mathcal{W}^s(P_3) \text{ with } x \preceq_{se} y\}$$

and

$$\mathcal{W}_2^- := \{x \in \mathcal{R} \setminus \mathcal{W}^s(P_4) : \exists y \in \mathcal{W}^s(P_4) \text{ with } y \preceq_{se} x\}$$
$$\mathcal{W}_2^+ := \{x \in \mathcal{R} \setminus \mathcal{W}^s(P_4) : \exists y \in \mathcal{W}^s(P_4) \text{ with } x \preceq_{se} y\}$$

respectively. In addition, $\mathcal{W}^s(P_3)$ is passing through the point P_3 and $\mathcal{W}^s(P_4)$ is passing through the point P_4 and the following holds:

i) If $(u_0, v_0) \in \mathcal{W}^s(P_3)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_3, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_4.

ii) If $(u_0, v_0) \in \mathcal{W}^s(P_4)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_4, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_3.

iii) If $(u_0, v_0) \in \mathcal{W}_1^+$ (the region above $\mathcal{W}^s(P_3)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_1, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_2.

iv) If $(u_0, v_0) \in \mathcal{W}_2^-$ (the region below $\mathcal{W}^s(P_4)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_2, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_1.

v) If $(u_0, v_0) \in \mathcal{W}_1^- \cap \mathcal{W}_2^+$ (the region between $\mathcal{W}_1^-$ and $\mathcal{W}_2^+$) then the sequence $\{(u_n, v_n)\}$ is attracted to E.

See Figure 1.

Proof. Theorem 1 implies that there exists a unique equilibrium point $E(\bar{x}, \bar{x})$ which is locally asymptotically stable. Theorem 2 implies that the periodic solution $\{P_1, P_2\}$ is locally asymptotically stable and $\{P_3, P_4\}$ is a saddle point. In view of (12) the map $T^2(u, v) = T(T(u, v))$ is competitive on $\mathcal{R} = \mathbb{R}_+^2 \setminus \{(0, 0)\}$ and strongly competitive on $int(\mathcal{R})$. It is easy to see that at each point, the Jacobian matrix of T^2 has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively.

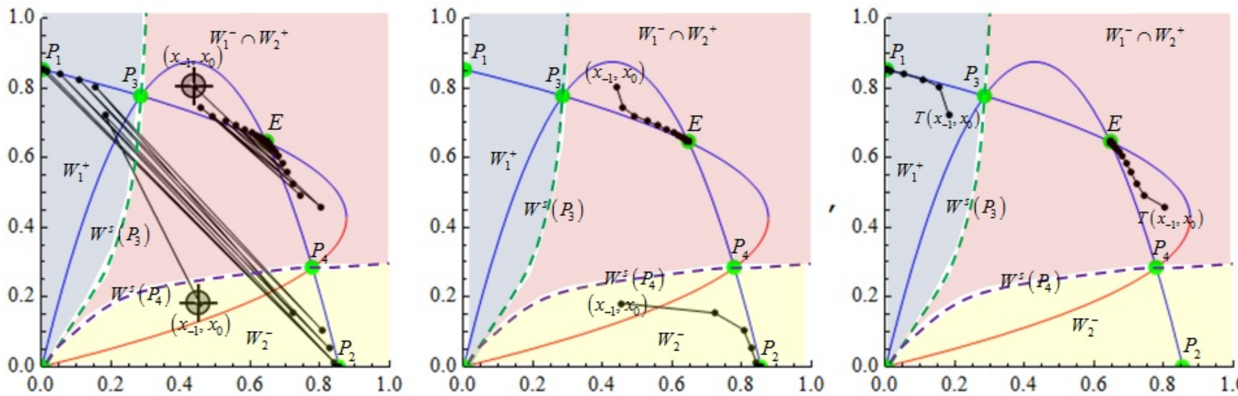


Figure 1: Visual illustration of Theorem 4.

In view of Theorem 3 we have that all solutions converge to period-two solution. Hence, all conditions of Theorem 4 in [13] are satisfied, which yields the existence of the global stable manifolds $\mathcal{W}^s(P_3)$ and $\mathcal{W}^s(P_4)$ which are the graphs of the strictly decreasing functions of the first coordinate on an interval.

By Theorem 4 in [13], we have that if $(u_0, v_0) \in \mathcal{W}^s(P_3)$ then $(u_{2n}, v_{2n}) = T^{2n}(u_0, v_0) \to P_3$ as $n \to \infty$ which implies that $(u_{2n+1}, v_{2n+1}) = T(T^{2n}(u_0, v_0)) \to T(P_3) = P_4$ as $n \to \infty$ from which it follows the statement i). The proof of the statement ii) is similar to the proof of the statement i).

Take $(u_0, v_0) \in \mathcal{W}_1^+ \cap \mathcal{R}$. By Theorem 4 in [13], we have that there exists $n_0 > 0$ such that, $T^{2n}(u_0, v_0) \in int(Q_2(P_3) \cap \mathcal{R}), \quad n > n_0$. In view of Theorem 1 in [11], since P_3 is a saddle point, we obtain that for all $(u_0, v_0) \in int(Q_2(P_3) \cap \mathcal{R})$, there exists $r_0 > 0$ such that $(u_0, v_0) \preceq_{se} P_3 - r_0\mathbf{v}_1$ and $T^2(P_3 - r_0\mathbf{v}_1) \preceq_{se} P_3 - r_0\mathbf{v}_1$. By monotonicity $T^{2n+2}(P_3 - r_0\mathbf{v}_1) \preceq_{se} T^{2n}(P_3 - r_0\mathbf{v}_1) \ll P_3$. In view of Lemma 2 we have that

$$T\left([0,\infty)^2 \setminus \{(0,0)\}\right) \subset \left[0, \frac{\gamma}{C}\right)^2 \setminus \{(0,0)\}.$$

From this and the fact that $P_1 \ll P_3 \ll E \ll P_4 \ll P_2$ we have that $T^{2n}(P_3 - r_0\mathbf{v}_1) \to P_1$ as $n \to \infty$. By monotonicity we have that $P_1 \preceq_{se} T^{2n}(u_0, v_0) \preceq_{se} T^{2n}(P_3 - r_0\mathbf{v}_1) \ll P_3$ which implies that $T^{2n}(u_0, v_0) \to P_1$ and $T^{2n+1}(u_0, v_0) = T(T^{2n}(u_0, v_0)) \to T(P_1) = P_2$ as $n \to \infty$ which proves the statement iii).

Take $(u_0, v_0) \in \mathcal{W}_2^- \cap \mathcal{R}$. By Theorem 4 in [13], we have that there exists $n_1 > 0$ such that, $T^{2n}(u_0, v_0) \in int(Q_4(P_4) \cap \mathcal{R}), \quad n > n_1$. In view of Theorem 1 in [11], since P_4 is a saddle point, we obtain that for all $(u_0, v_0) \in int(Q_4(P_4) \cap \mathcal{R})$, there exists $r_1 > 0$ such that $P_4 + r_1\mathbf{v}_1 \preceq_{se} (u_0, v_0)$ and $P_4 + r_1\mathbf{v}_1 \preceq_{se} T^2(P_4 + r_1\mathbf{v}_1)$. The rest of the proof of the statement iv) is similar to the proof of the statement iii) and we skip it here.

Now, we show that each orbit starting in the region $\mathcal{W}_1^- \cap \mathcal{W}_2^+$ converges to E. Take $(u_0, v_0) \in \mathcal{W}_1^- \cap \mathcal{W}_2^+$. By Theorem 4 in [13],we have that there exists $n_2 > 0$ such that, $T^{2n}(u_0, v_0) \in int(Q_4(P_3) \cap Q_2(P_4) \cap \mathcal{R}) = [[P_3, P_4]]$, for $n > n_2$. Since P_3 and P_4 are the saddle points and E is locally asymptotically stable, in view of Corollary 2 [12] we have that $T^{2n}(u', v') \to E$ and $T^{2n+1}(u', v') = T(T^{2n}(u', v')) \to T(E) = E$ as $n \to \infty$ for all $(u', v') \in [[P_3, E]]$ and that $T^{2n}(u'', v'') \to E$ and $T^{2n+1}(u'', v'') = T(T^{2n}(u'', v'')) \to T(E) = E$ as $n \to \infty$ for all $(u'', v'') \in [[E, P_4]]$. Then there exist the points $(u_0', v_0') \in [[P_3, E]]$ and $(u_0'', v_0'') \in [[E, P_4]]$ such that $(u_0', v_0') \preceq_{se} T^{2n_2+2}(u_0, v_0) \preceq_{se} (u_0'', v_0'')$. By monotonicity of the map T^2 we have that $T^{2n}(u_0, v_0) \to E$ and $T^{2n+1}(u_0, v_0) = T(T^{2n}(u_0, v_0)) \to T(E) = E$ as $n \to \infty$ for all $(u_0, v_0) \in \mathcal{W}_1^- \cap \mathcal{W}_2^+$. This completes the proof of statement v) of the Theorem.

□

The following theorem considers the case $1 < \gamma < 3$.

Theorem 5 *Assume that*

$$1 < \gamma < 3.$$

Then system (8) *has a unique equilibrium point $E(\bar{u}, \bar{u})$ which is a saddle point and prime period-two solution $\{P_1(\bar{u}_1, \bar{v}_1), P_2(\bar{v}_1, \bar{u}_1)\}$ which is locally asymptotically stable, where*

$$\bar{u}_1 = 0, \quad \bar{v}_1 = \frac{\gamma}{C} \text{ and } \bar{u} = \frac{\gamma - 1}{C}.$$

Global stable manifold $\mathcal{W}^s(E)$, which is continuous increasing curve, divides the first quadrant into two connected components

$$\mathcal{W}^-(E) := \{x \in \mathcal{R} \setminus \mathcal{W}^s(E) : \exists y \in \mathcal{W}^s(E) \text{ with } y \preceq_{se} x\}$$
$$\mathcal{W}^+(E) := \{x \in \mathcal{R} \setminus \mathcal{W}^s(E) : \exists y \in \mathcal{W}^s(E) \text{ with } x \preceq_{se} y\}$$

such that

$$\mathbb{R}_+^2 = \mathcal{W}^-(E) \cup \mathcal{W}^+(E) \cup \mathcal{W}^s(E).$$

In addition, $\mathcal{W}^s(E)$ passing through the point E and the following holds:

i) Every initial point (u_0, v_0) in $\mathcal{W}^s(E)$ is attracted to E.

ii) If $(u_0, v_0) \in \mathcal{W}^+(E)$ (the region below $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_2, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_1.

iii) If $(u_0, v_0) \in \mathcal{W}^-(E)$ (the region above $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(x_{2n}, v_{2n})\}$ is attracted to P_1, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_2.

See Figure 1.

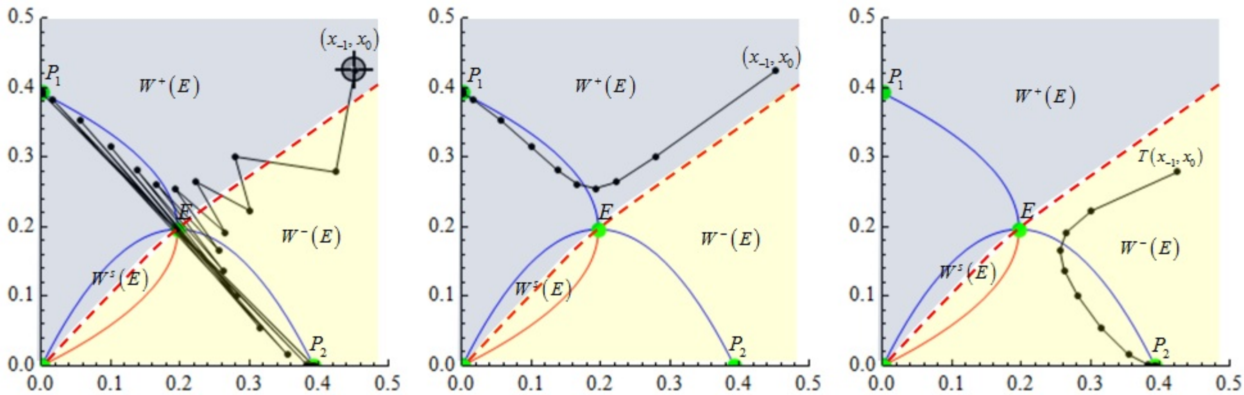


Figure 2: Visual illustration of Theorem 5 .

Proof. Theorem 1 implies that there exists a unique equilibrium point $E(\bar{x}, \bar{x})$ which is a saddle point. Theorem 2 implies that the period-two solution $\{P_1, P_2\}$ is locally asymptotically stable. Similar as in the proof of Theorem 4 all conditions of Theorem 4 in [13] are satisfied, which yields the existence of the global stable manifold $\mathcal{W}^s(E)$ which is the graph of the strictly increasing function.

Take $(u_0, v_0) \in \mathcal{W}^+ \cap \mathcal{R}$. By Theorem 4 in [13], we have that there exists $n_0 > 0$ such that, $T^{2n}(u_0, v_0) \in int(Q_2(E) \cap \mathcal{R}), \quad n > n_0$. In view of Theorem 1 in [11], since E is a saddle point, we obtain that for all $(u_0, v_0) \in int(Q_2(E) \cap \mathcal{R})$, there exists $r_0 > 0$ such that $(u_0, v_0) \preceq_{se} E - r_0\mathbf{v}_1 \preceq_{se} E$ and $T^2(E - r_0\mathbf{v}_1) \preceq_{se} E - r_0\mathbf{v}_1$. By monotonicity $T^{2n+2}(E - r_0\mathbf{v}_1) \preceq_{se} T^{2n}(E - r_0\mathbf{v}_1) \ll E$. In view of Lemma 2 we have that $T^n(u, v) \in [0, \gamma/C)^2 \setminus \{(0, 0)\}$. From this and the fact that $P_1 \ll E \ll P_2$ we have that $T^{2n}(E - r_0\mathbf{v}_1) \to P_1$ as $n \to \infty$. By monotonicity, $P_1 \preceq_{se} T^{2n}(u_0, v_0) \preceq_{se} T^{2n}(E - r_0\mathbf{v}_1) \ll E$ which implies that $T^{2n}(u_0, v_0) \to E$ and $T^{2n+1}(u_0, v_0) = T(T^{2n}(u_0, v_0)) \to T(E) = E$ as $n \to \infty$ which proves the statement ii).

The proof of the statement iii) is similar and we skip it here.

□

Now, we assume that $\gamma = 3$. The following theorem holds.

Theorem 6 *Assume that*

$$\gamma = 3.$$

Then System (8) has a unique equilibrium point $E(\bar{u}, \bar{u})$ which is a non-hyperbolic and prime period-two solution $\{P_1(\bar{u}_1, \bar{v}_1), P_2(\bar{v}_1, \bar{u}_1)\}$ which is locally asymptotically stable, where

$$\bar{u}_1 = 0, \quad \bar{v}_1 = \frac{3}{C} \text{ and } \bar{u} = \frac{2}{C}.$$

There exists a continuous increasing curve $\mathcal{C}_E$ which is a subset of the basin of attraction of E and it divides the first quadrant into two connected invariant components

$$\mathcal{W}^-(E) := \{x \in \mathcal{R} \setminus \mathcal{C}_E : \exists y \in \mathcal{C}_E \text{ with } y \preceq_{se} x\}$$
$$\mathcal{W}^+(E) := \{x \in \mathcal{R} \setminus \mathcal{C}_E : \exists y \in \mathcal{C}_E \text{ with } x \preceq_{se} y\}$$

such that the following holds:

i) Every initial point (u_0, v_0) in $\mathcal{C}_E$ is attracted to E.

ii) If $(u_0, v_0) \in \mathcal{W}^+(E)$ (the region above $\mathcal{C}_E$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_1, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_2.

ii) If $(u_0, v_0) \in \mathcal{W}^-(E)$ (the region below $\mathcal{C}_E$) then the subsequence of even-indexed terms $\{(x_{2n}, v_{2n})\}$ is attracted to P_2, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_1.

See Figure 3.

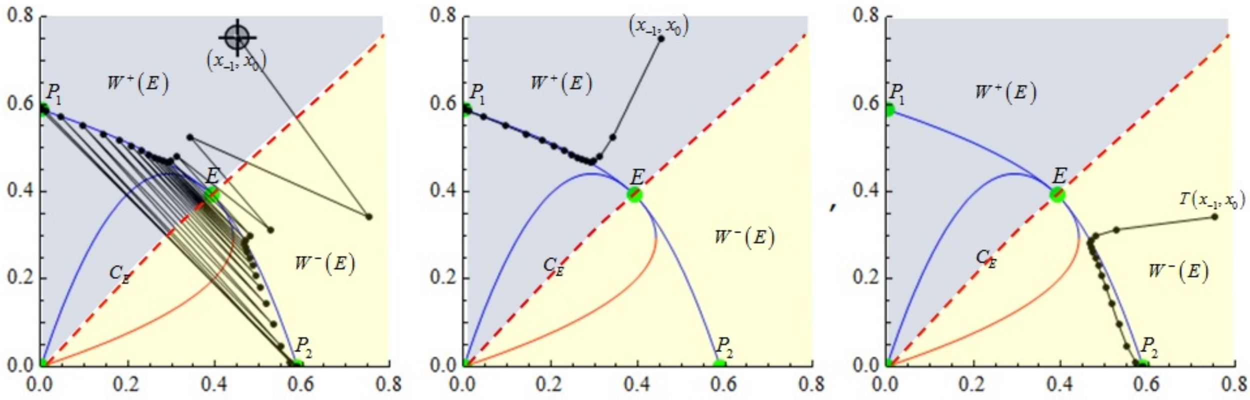


Figure 3: Visual illustration of Theorem 6 .

Proof. Theorem 1 implies that there exists a unique equilibrium point $E(\bar{x}, \bar{x})$ which is non-hyperbolic. Theorem 1(c) implies that the periodic solution $\{P_1, P_2\}$ is locally asymptotically stable. Similar as in the proof of Theorem 4 all conditions of Theorem 4 in [13] are satisfied, which yields the existence a continuous increasing curve $\mathcal{C}_E$ which is a subset of the basin of attraction of E and for every x $\in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(\mathrm{x}) \in \mathrm{int}\, \mathcal{Q}_2(E)$ for $n \geq n_0$ and for every x $\in \mathcal{W}_-$ there exists $n_1 \in \mathbb{N}$ such that $T^n(\mathrm{x}) \in \mathrm{int}\, \mathcal{Q}_4(E)$ for $n \geq n_1$.

Set $U(t) = t(3 - Ct)$. It is easy to see that $(t, U(t)) \preceq_{se} E$ if $t \in [\frac{3}{2C}, \bar{u}]$ and $E \preceq_{se} (t, U(t))$ if $t \in [\bar{u}, \frac{3}{C}]$ and $U(\bar{u}) = \bar{u}$. In view of Lemma 2 we have that

$$T\left([0, \infty)^2 \setminus \{(0,0)\}\right) \subset \left[0, \frac{\gamma}{C}\right)^2 \setminus \{(0,0)\}.$$

One can show that

$$T^2(t, U(t)) - (t, U(t)) = \left(0, \frac{t(Ct-3)(Ct-2)^3}{Ct(Ct-3)^2+1}\right)$$

which implies that $T^2(t, U(t)) \preceq_{se} (t, U(t))$ if $t < \bar{u}$ and $(t, U(t)) \preceq_{se} T^2(t, U(t))$ if $t > \bar{u}$. By monotonicity if $t < \bar{u}$ then we obtain that $T^{2n}(t, U(t)) \to P_1$ as $n \to \infty$ and if $t > \bar{u}$ then we have that $T^{2n}(t, U(t)) \to P_2$ as $n \to \infty$.

If $(u', v') \in int\mathcal{Q}_2(E)$ then there exists t_1 such that $P_1 \preceq_{se} (u', v') \preceq_{se} (t_1, U(t_1)) \ll_{se} E$. By monotonicity of the map T^2 we obtain that $P_1 \preceq_{se} T^{2n}(u', v') \preceq_{se} T^{2n}(t_1, U(t_1)) \ll_{se} E$ which implies that $T^{2n}(u', v') \to P_1$ and $T^{2n+1}(u', v') \to T(P_1) = P_2$ as $n \to \infty$ which proves the statement ii).

If $(u'', v'') \in int\mathcal{Q}_4(E)$ then there exists t_1 such that $E \ll (t_2, U(t_2)) \preceq_{se} (u'', v'') \preceq_{se} P_2$. By monotonicity of the map T^2 we obtain that $E \preceq_{se} T^{2n}(t_2, U(t_2)) \preceq_{se} T^{2n}(u'', v'') \ll_{se} P_2$ which implies that $T^{2n}(u'', v'') \to P_2$ and $T^{2n+1}(u'', v'') \to T(P_2) = P_1$ as $n \to \infty$ which proves the statement iii), and completes the proof of the Theorem.

□

First we notice the following. Theorem 3 and Lemma 2 imply that $T^{2n}(x_0, y_0)$ is asymptotic to either $P_1 = (0, \frac{\gamma}{C})$ or $P_2 = (\frac{\gamma}{C}, 0)$ or $(0,0)$, for all $(x_0, y_0) \in \mathcal{R} \setminus \{(0,0)\}$. Let $\mathcal{B}(P_1)$ be the basin of attraction of P_1 and $\mathcal{B}(P_2)$ be the basin of attraction of P_2 with respect to the map T^2. Let $\mathcal{C}^+$ denote the boundary of $\mathcal{B}(P_1)$ considered as a subset of *int* $Q_1(0,0)$ (the first quadrant relative to $(0,0)$) and $\mathcal{C}^-$ denote the boundary of $\mathcal{B}(P_2)$ considered as a subset of *int* $Q_1(0,0)$. It is easy to see that $(0,0) \in \mathcal{C}^+$ and $(0,0) \in \mathcal{C}^-$.

Now, similarly to the proof of the of Claim 1 and Claim 2 in [5], one can prove that the following lemma holds.

Lemma 3 *Let $\mathcal{C}^+$ and $\mathcal{C}^-$ be the sets defined above. Then the sets $\mathcal{C}^+$ and $\mathcal{C}^-$ are invariant under the map T^2 and they are the graphs of continuous strictly increasing functions. Further, $\mathcal{C}^+ \cup \mathcal{C}^- \subset \mathcal{B}(0,0)$.*

The following theorem details the existence two invariant strictly increasing curves passing through the point $(0,0)$, such that every solution that stars on these two curves or in the region between these two curves is attracted to the point $(0,0)$.

Theorem 7 *Assume that*

$$0 < \gamma < 1.$$

Then there exists prime period-two solution $\{P_1(\bar{u}_1, \bar{v}_1), P_2(\bar{v}_1, \bar{u}_1)\}$ which is locally asymptotically stable, where

$$\bar{u}_1 = 0, \quad \bar{v}_1 = \frac{\gamma}{C}$$

Furthermore, there exist sets $\mathcal{C}^+$ and $\mathcal{C}^-$ which are continuous increasing curves, invariant under the map T^2 and $T(\mathcal{C}^+) = \mathcal{C}^-$, and divide the first quadrant into two connected components, namely

$$\mathcal{W}_1^- := \{x \in \mathcal{R} \setminus \mathcal{C}^+ : \exists y \in \mathcal{C}^+ \text{ with } y \preceq_{se} x\} \quad \text{and} \quad \mathcal{W}_1^+ := \{x \in \mathcal{R} \setminus \mathcal{C}^+ : \exists y \in \mathcal{C}^+ \text{ with } x \preceq_{se} y\}$$

and

$$\mathcal{W}_2^- := \{x \in \mathcal{R} \setminus \mathcal{C}^- : \exists y \in \mathcal{C}^- \text{ with } y \preceq_{se} x\} \quad \text{and} \quad \mathcal{W}_2^+ := \{x \in \mathcal{R} \setminus \mathcal{C}^- : \exists y \in \mathcal{C}^- \text{ with } x \preceq_{se} y\}$$

respectively. In addition, $\mathcal{C}^+$ and $\mathcal{C}^-$ passing through the point $(0,0)$ and the following holds:

i) If $(u_0, v_0) \in \mathcal{W}_1^+$ (the region above $\mathcal{C}^+$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_1, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_2.

ii) If $(u_0, v_0) \in \mathcal{W}_2^-$ (the region below $\mathcal{C}^-$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_2, and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_1.

iii) If $(u_0, v_0) \in (\mathcal{C}^+ \cup \mathcal{C}^+) \cup (\mathcal{W}_1^- \cap \mathcal{W}_2^+)$ (the region between $\mathcal{C}^+$ and $\mathcal{C}^-$) then the sequence $\{(u_n, v_n)\}$ is attracted to $(0,0)$.

Proof. The proof follows from Lemma 3, and it is similar to the proof of Theorem 4, so we skip it. □

Based on a series of numerical simulations we pose the following hypothesis.

Conjecture 1 *Suppose that all assumptions of the Theorem 7 are satisfied, then the following holds: $\mathcal{C}^+ = \mathcal{C}^-$.*

References

[1] A.M. Amleh, D.A.Georgiou, E.A.Grove, and G.Ladas, On the recursive sequence $x_{n+1} = \alpha + \frac{x_{n+1}}{x_n}$. *J.Math.Anal.Appl.***1999**, *233* , 790–515.

[2] A. Brett and M. R. S. Kulenović, Basins of attraction of equilibrium points of monotone difference equations. *Sarajevo J. Math.***2010**, *5(18)* , 211–233.

[3] E. Camouzis and G. Ladas, *Dynamics of third-order rational difference equations with open problems and conjectures*; Chapman and Hall/CRC Press, London/Boca Raton, FL 2008.

[4] E. Camouzis and G. Ladas, *When does local asymptotic stability imply global attractivity in rational equations?* **2006**,*12(8)*, 863–885.

[5] V.Hadžiabdić, M.R.S.Kulenović and E.Pilav, Dymanics of a Two-dimensional Competitive System of Rational Difference Equations with Quadratic Terms. *Adv.Difference Equ.***2014**, *2014:301*,32 p.

[6] M. Hirsch and H. Smith, *Monotone dynamical systems. Handbook of differential equations: ordinary differential equations.* Vol. II, 239–357, Elsevier B. V., Amsterdam, 2005.

[7] S. Kalabušić and M. R. S. Kulenović, Dynamics of Certain Anti-competitive Systems of Rational Difference Equations in the Plane. *J. Difference Equations Appl.* **2011**, *17*, 1599–1615,

[8] S. Kalabušić, M. R. S. Kulenović and E. Pilav, Global Dynamics of Anti-Competitive Systems in the Plane, *J. Difference Equ. Appl.***2013**, *19* 1849-1871.

[9] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica*, Chapman& Hall/CRC Press, Boca Raton, 2002.

[10] M. R. S. Kulenovic, E. Pilav and E. Silić, Local Dynamics and Global Attractivity of a Certain Second Order Quadratic Fractional Difference Equation, *Adv. Difference Equ.* **2014**, *32* pp.

[11] M. R. S. Kulenović and O. Merino, Competitive-Exclusion versus Competitive-Coexistence for Systems in the Plane. *Discrete Contin. Dyn. Syst. Ser. B* **2006**, *6*, 1141–1156.

[12] M. R. S. Kulenović and O. Merino, Global bifurcation for discrete competitive systems in the plane *Discrete Contin. Dyn. Syst. Ser.* B 12 (2009), 133–149.

[13] M. R. S. Kulenović and O. Merino, Invariant Manifolds for Competitive Discrete Systems in the Plane, *Int. J. of Bifurcations and Chaos* **2010**, *20*, 2471–2486.

[14] H. L. Smith, Planar Competitive and Cooperative Difference Equations,*J. Difference Equ. Appl.* **1998**, *3*, 335–357.

Fourier Approximation Schemes of Stochastic Pseudo-Hyperbolic Equations with Cubic Nonlinearity and Regular Noise

Tengjin Zhao[a], Chenzhong Wang[a], Quanyong Zhu[b], Zhiyue Zhang[a*]

a School of Mathematical Sciences, Jiangsu Key Laboratory for NSLSCS

Nanjing Normal University, Nanjing 210023, China,

b School of Technology, Lishui University, Lishui, 323000, China

November 13, 2018

Abstract

The pseudo-hyperbolic equation with cubic nonlinearity and additive space-time noise is discussed. The space-time noise is assumed to be Gaussian in time and possesses a Fourier series expansion in space. First, we prove the existence and uniqueness of the approximate strong solutions of the equation and show that the truncated Fourier solution which can be approximated by the truncated finite-dimensional system, is an approximate solution. Second, a new transformation is used to convert pseudo-hyperbolic equation into a system of equations, which can construct an infinitesimal generator with good properties. After analyzing the related total energy evolution, we obtain that the energy growth will not blow-up in the limited time. Finally, we present a Fourier scheme of a procedure for its numerical approximation and give the stability and convergence analysis of the scheme.

keyword: thermal convection equation, Fourier coefficients, cubic-type nonlinearities; stochastic; energy

1 Introduction

Stochastic differential equations (SDEs) can model many natural phenomena with white noise and engineering applications, such as epidemiology, economics and so on [15, 1, 14, 3, 43, 23, 22]. SDEs hold for the important original work

*Corresponding author. E-mail address: zhangzhiyue@njnu.edu.cn.

of Ito [12] as well as books [7, 27]. The shorter accounts of stochastic dynamic systems on stability, filtering, and control [18, 13] are rather unsuited for the study. Stability of SDEs has been well studied by researchers [25, 16, 19]. Since the analytical solution is difficult to obtain, different numerical methods have been introduced such as [30, 33, 8, 29]. The common theoretical basis is the stochastic Ito-Taylor expansion in terms of multiple Wiener integrals [15].

The analysis of the linear SDEs is well investigated such as [20, 6, 28, 24]. In the recent past, the nonlinear SDEs are researched. In [21] a class of fully nonlinear SDEs is studied by using the stochastic characteristic method. In [11] a strong convergence result under less restrictive conditions is proved by using Euler-Maruyama method. In [10], the exponential stability of the multidimensional nonlinear SDEs with variable delays is investigated. Nonlinear filtering equations have developed based on a classification where the measure term is either deterministic or random [39].

Consider the semi-linear stochastic pseudo-hyperbolic equation with cubic-type nonlinearities perturbed by additive space-time random noise W [40]:

$$\begin{cases} d(u+u_t) = \sigma^2 \dfrac{\partial^2 (u+u_t)}{\partial^2 x} dt + B(u+u_t)dt + b{\cdot}dW(t,x), \\ u(0,x) = u_0, \ u_t(0,x) = u_{t_0}, \quad 0 < x < L, \\ u(t,0) = u(t,L) = 0, \ u_t(t,0) = u_t(t,L) = 0, \quad 0 < t < T, \end{cases} \tag{1.1}$$

where $b \in \mathbb{R}^1$ is an overall noise intensity parameter. $B(u) = u(a_1 - a_2\|u\|_{L^2}^2)$ is cubic-type with real parameters $a_2 > 0$ and a_1 [38]. The space-time Q-regular noise $W(t,x)$ is as follows:

$$W(t,x) = \sum_{n=1}^{+\infty} \alpha_n W_n(t) \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{+\infty} \alpha_n W_n(t) e_n(x) \tag{1.2}$$

with independent and identically distributed Wiener process $W_n \in \mathcal{N}(0,t)$, where $trace(Q) = \sum_{n=1}^{+\infty} \alpha_n^2 < +\infty$. We know that $e_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), n \geq 1$ are the eigenfunctions of the Laplace operator which form an orthonormal system in $H = L^2(0,L)$ and satisfy in one-dimensional, $\Delta e_n(x) = -\frac{n^2\pi^2}{L^2} e_n$. The main contribution of this paper is to discuss the Fourier solution $u(t,x)$ and its numerical approximations by truncated Fourier series [41]. We construct an infinitesimal generator with good properties and convert into the equations which can be easily solved.

The rest of the paper is organized as follows. In section 2, we verify the existence and uniqueness of solution and give a finite-dimensional system of the SDEs. In section 3, we estimate the truncated total energy. In section 4 we show numerical methods to find those Fourier coefficients. In the last section 5 numerical experiments are provided which support our results.

2 Existence and Uniqueness of Approximate Strong Solutions and Fourier-Series Solutions

In general, it is difficult to solve nonlinear equations. However, taking the equations into system of equations can avoid lots of complex calculations in infinitesimal generators and energy estimation. Let $v = u + u_t$, Eqs (1.1) becomes

$$\begin{cases} v = u + u_t, \\ \dfrac{dv}{dt} = \left[\sigma^2 \dfrac{\partial^2 v}{\partial x^2} + v\left(a_1 - a_2\|v\|_{L^2}^2\right)\right] + b \cdot \dfrac{dW(t,x)}{dt}. \end{cases} \tag{2.1}$$

It can be rewritten as

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \left(\begin{array}{cc} -1 & 1 \\ 0 & \frac{\sigma^2 \partial^2}{\partial x^2} \end{array} \right)\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ v \end{pmatrix}(a_1 - a_2 \left\| \begin{pmatrix} 0 \\ v \end{pmatrix} \right\|_{L^2}^2) + \begin{pmatrix} 0 \\ b \end{pmatrix}\frac{dW}{dt}. \tag{2.2}$$

From the definitions of strong solution and approximate strong solution[36], we obtain that conditions of the strong solutions of (2.1) exist and the uniqueness is that all operators are globally Lipschitz-continuous. Under conditions weaker than global Lipschitz-continuity, we can also achieve a result of the strong solutions.

Lemma 2.1. *For all $a_2 \geq 0$, the mapping $v \in H \mapsto B(v) = v(a_1 - a_2\|v\|_{L^2}^2)$ satisfies the angle condition on H. In other words, for all $u, v \in H$, we have*

$$F(u,v) :=< B(u) - B(v), u - v >_H \leq a_1\|u-v\|_H^2, \tag{2.3}$$

specially

$$< B(v), v >_H \leq \left(a_1 - a_2\frac{\|v\|_H^2}{2}\right)\|v\|_H^2 \leq a_1\|v\|_H^2.$$

Proof. Denoting $f(u) := \|u\|_H^2 u$ and $g(u,v) :=< f(u) - f(v), u - v >_H$ which is symmetric. Then we obtain that

$$2g(u,v) = \left(\|u\|_H^2 + \|v\|_H^2\right)\|u-v\|_H^2 + \left(\|u\|_H^2 - \|v\|_H^2\right)^2\left(\|u\|_H^2 + \|v\|_H^2\right)\|u-v\|_H^2,$$
$$g(u,v) \geq \frac{\|u\|_H^2 + \|v\|_H^2}{2}\|u-v\|_H^2.$$

Now using (2.3), the above inequality and the definition of B, we have

$$F(u,v) \leq -a_2\frac{\|u\|_H^2 + \|v\|_H^2}{2}\|u-v\|_H^2 + a_1\|u-v\|_H^2 \leq a_1\|u-v\|_H^2,$$
$$< B(v), v >_H \leq -a_2\frac{\|v\|_H^2}{2}\|v\|_H^2 + a_1\|v\|_H^2 \leq a_1\|v\|_H^2, \text{by setting } u = (0,0).$$

Then the proof is completed. □

3

From Lemma 2.1 and Theorem 3 in [36], the unique approximate strong and continuous solution of Eqs (2.2) exists.

Theorem 2.2. *Assumptions of definitions of strong and approximate strong solution [36] are satisfied with* $\mathbb{E}\|v(0,\cdot)\|_H^2<\infty$*, for* $B(0,L)\times\mathcal{F}_0$*-measurable initial data* $v(0,\cdot)\in H$*. The approximate strong global solution* v *of Eqs (2.2) exists.*

Next, we propose our method to solve the SDEs. There are many methods such as Galerkin-type method [9, 26, 5], Monte-Carlo [4, 31], collocation method [42], projection methods [32]. The method presented in this paper is Fourier-series solutions. The existence of separated solutions is established in [17]. Solutions of this type are used in [2]. The difficulty lies in computing Fourier-series solutions and in finding a good infinitesimal generator to estimate the energy of the system.

Using the principle of linear superposition, the Fourier series is

$$u(t,x)=\sum_{n=1}^{+\infty}c_{un}(t)e_n(x),\ \ v(t,x)=\sum_{n=1}^{+\infty}c_{vn}(t)e_n(x). \tag{2.4}$$

We truncate the series as follows:

$$\widetilde{u}(t,x)=\sum_{n=1}^{N}c_{un}(t)e_n(x),\ \ \widetilde{v}(t,x)=\sum_{n=1}^{N}c_{vn}(t)e_n(x). \tag{2.5}$$

which form the strong solutions of Eqs (2.2).

Theorem 2.3. *The Fourier coefficients of Eqs (2.4) satisfy (P-a.s.) the infinite-dimensional system, for* $k=1,2,\cdots$ *and* $b_k=b\alpha_k$*,*

$$\begin{cases} c'_{uk}(t)=c_{vk}(t)-c_{uk}(t), \\ dc_{vk}=\left(-\sigma^2\dfrac{k^2\pi^2}{L^2}+a_1-a_2\displaystyle\sum_{n=1}^{+\infty}c_{vn}^2(t)\right)c_{vk}dt+b_kdW_k, \end{cases} \tag{2.6}$$

Proof. By plugging Eqs (2.4) into Eqs (2.1), we achieve that for $0\le t\le T$,

$$\int_0^L u'(t,x)e_k(x)dx=\sum_{n=1}^{+\infty}c'_{un}(t)\int_0^L e_n(x)e_k(x)dx=c_{vk}(t)-c_{uk}(t),$$

$$\int_0^L dv(t,x)e_k(x)dx=c_{vk}\left(-\frac{\sigma^2k^2\pi^2}{L^2}+a_1-a_2\sum_{n=1}^{+\infty}\left[c_{vn}(t)\right]^2\right)dt+b_kdW_k(t).$$

As we know that the u,v is the unique strong solution of (2.1) with

$$\|u(t,\cdot)\|_H^2=\sum_{k=1}^{\infty}\left[c_{uk}(t)\right]^2<\infty,\ \ \|v(t,\cdot)\|_H^2=\sum_{k=1}^{\infty}\left[c_{vk}(t)\right]^2<\infty,$$

and have Fourier coefficient $c_{uk},\ c_{vk}$ which can be approximated by the truncated finite-dimensional system. So the above computations work. □

We note that for stochastic systems with additive noise, the stochastic integration leads to the same type of stochastic integral. See more details in [37]. Therefore, we can calculate each $c_{uk},\ c_{vk}$ and $\widetilde{u}(t,x)=\sum_{n=1}^{N} c_{un}(t)e_n(x)$.

3 Total Energy Evolution

For the case of sufficiently strong diffusion with $\sigma^2\pi^2 > L^2(a_1+1)$, we investigate the behavior of related energy functional, which is defined at time $t \geq 0$ by

$$\mathcal{E}(t)=\frac{\sigma^2}{2}\|v_x(t,\cdot)\|_H^2-\frac{a_1+1}{2}\|v(t,\cdot)\|_H^2+\frac{a_2}{4}\|v(t,\cdot)\|_H^4. \tag{3.1}$$

This energy functional is indeed nonnegative and finite almost surely (a.s.) as one can see from the following theorem. For its proof, we take the functional in terms of its Fourier coefficients c_k by

$$V(t):=V(c_{vk}(t):k\in N)=\frac{1}{2}\sum_{n=1}^{+\infty}\left(\frac{\sigma^2n^2\pi^2}{L^2}-a_1-1\right)c_{vn}^2(t)+\frac{a_2}{4}\left(\sum_{n=1}^{+\infty}c_{vn}^2(t)\right)^2,$$

for $t \geq 0$. It is easy to know that $V \geq 0$ for all sequences $(c_{vk}(t))_k$ and acts as a Lyapunov functional. Besides, $\mathcal{E}(t)=V(t)$ for all $t\geq 0$.

Theorem 3.1. *Assume that $e(0)=\mathbb{E}V(c_{vk}(0):k\in N)<\infty$, $\sigma^2\pi^2\geq L^2(a_1+1)$ and trace$(Q)=\sum_{n=1}^{\infty}\alpha_n^2<\infty$. Then, the total expected energy of the original system (2.1) is linearly bounded in time by*

$$e(t)=\mathbb{E}V(c_{vk}(t):k\in N)\leq e(0)+2\left[\sum_{n=1}^{+\infty}[\frac{\sigma^2n^2\pi^2}{L^2}-a_1-1]c_{vn}^2(t)+\sqrt{a_2}(b^2\beta^2)^2\right]t,$$

where $\beta^2=\sum_{n=1}^{\infty}\alpha_n^2+2\max\limits_{n\in\mathbb{N}}\alpha_n^2$.

Proof. The truncated infinitesimal generator can be rewritten

$$L=\sum_{n=1}^{N}[c_{vn}-c_{un}]\frac{\partial}{\partial c_{un}}+\frac{b^2}{2}\sum_{n=1}^{N}\alpha_n^2\frac{\partial^2}{\partial c_{vn}^2}+\sum_{n=1}^{N}\left[-\frac{\sigma^2n^2\pi^2}{L}+a_1-a_2\sum_{k=1}^{N}c_{vk}^2\right]c_{vn}\frac{\partial}{\partial c_{vn}}.$$

We express Eqs (3.1) in terms of its truncated Fourier coefficients c_{vk} by

$$\widetilde{V}(t):=\widetilde{V}(c_{vk}(t):k\in N)=\frac{1}{2}\sum_{n=1}^{N}[\frac{\sigma^2n^2\pi^2}{L^2}-a_1-1]c_{vn}^2(t)+\frac{a_2}{4}\left(\sum_{n=1}^{N}c_{vn}^2(t)\right)^2,$$

for $t\geq 0$. Then, after calculating the infinitesimal generator, we estimate the

5

energy of the system (3.1) as follow:

$$L\widetilde{V}=L\widetilde{V}_1+L\widetilde{V}_2=\sum_{n=1}^{N}\left(c_{vn}-c_{un}\right)c_{un}-\left[\sum_{n=1}^{N}\left(\frac{\sigma^2n^2\pi^2}{L^2}-a_1+a_2\sum_{k=1}^{N}c_{vk}^2\right)c_{vn}\right]^2$$
$$+\frac{1}{2}\sum_{n=1}^{N}\alpha_n^2\sum_{n=1}^{N}\left(\frac{\sigma^2n^2\pi^2}{L^2}-a_1\right)+\frac{b^2}{2}\sum_{n=1}^{N}\alpha_n^2a_2\sum_{n=1}^{N}c_{vn}^2(t)+2c_{vn}^2(t).$$

$$L\widetilde{V}_1\leq\sum_{n=1}^{N}\frac{1}{4}\left(c_{un}^{'}+c_{un}\right)^2\leq\frac{5b^2}{12\sqrt{3a_2}}\sum_{n=1}^{+\infty}\alpha_n^2\left(\frac{\sigma^2n^2\pi^2}{L^2}-a_1+a_2\left(b^2\beta_N^2\right)^{\frac{3}{2}}\right).$$

From the estimate in [38] and denoting $\beta^2=\sum_{n=1}^{+\infty}\alpha_n^2+2\max\limits_{n\in\mathbb{N}}\alpha_n^2$, we obtain

$$L\widetilde{V}_2\leq b^2\sum_{n=1}^{+\infty}\alpha_n^2\left(\frac{\sigma^2n^2\pi^2}{L^2}-a_1+a_2\left(b^2\beta_N^2\right)^{\frac{3}{2}}\right)\left(\frac{1}{12a_2}\right)^{\frac{1}{2}}\frac{5}{6}.$$

Consequently, Dynkin formula says that

$$e_N(t)=\mathbb{E}\left[\widetilde{V}(t)\right]\leq e(0)+2b^2\sum_{n=1}^{+\infty}\alpha_n^2\left(\frac{\sigma^2n^2\pi^2}{L^2}-a_1\right)+2a_2\left(b^2\beta_N^2\right)^{\frac{3}{2}}\left(\frac{1}{12a_2}\right)^{\frac{1}{2}}\frac{5}{6},$$

for t ≥ 0. Since $e_N\geq 0$ is increasing in N and uniformly bounded in time t for any $t\in[0,T]$, we know that $lim_{N\to+\infty}e_n(t)=e(t)$, and

$$0\leq e(t)\leq e(0)+2b^2\sum_{n=1}^{+\infty}\alpha_n^2\left(\frac{\sigma^2n^2\pi^2}{L^2}-a_1\right)t+2a_2\left(b^2\beta_N^2\right)^{\frac{3}{2}}\left(\frac{1}{12a_2}\right)^{\frac{1}{2}}\frac{5}{6}t,$$

as $e(0)<\infty,\ \sigma^2\pi^2\geq L^2(a_1+1)$ and $trace(Q)=\sum_{n=1}^{\infty}\alpha_n^2<\infty$. □

More precisely, for $T<\infty,\forall\ 0\leq t\leq\ T,\exists\ K_1,K_2\geq 0$

$$(E\|v(t,\cdot)\|_H^2+K_0)e^{K_1T}\geq E\|v(t,\cdot)\|_H^2\geq E\|u(t,\cdot)\|_H^2.$$

In fact, if $\sigma^2\pi^2>L^2(a_1+1)$, we can know that the following mentioned estimates of second moments have linearly bounded ones (in time). For $T<\infty$, $\exists\ c\geq 0,\ 0\leq t\leq\ T,\ E\|u(t,\cdot)\|_H^2\ \leq\ E\|v(t,\cdot)\|_H^2\ \leq\ E\|v(0,\cdot)\|_H^2+ct.$

4 Numerical Methods for Fourier Coefficients

The truncated Fourier series $\widetilde{u},\ \widetilde{v}$ in Eqs (2.5) satisfy the Eqs (2.1). Since the explicit solution is unknown, we take advantage of numerical approximations.

Along partitions $0=t_0<t_1<t_2<\cdots<t_{n_T}=T$ of interval $[0,T]$ with the step sizes $h_n=t_{n+1}-t_n$, and $0=x_0<x_1<x_2<\cdots<x_{n_L}=L$ of interval $[0,L]$ with the step sizes $d_n=x_{n+1}-x_n$.

For each fixed x_m, let us consider the forward Euler method for c_{vk}

$$c_{vk}(n+1) = h_n c_{vk}(n)\left(\frac{-\sigma^2 k^2 \pi^2}{L^2} + a_1 - a_2 \sum_{l=1}^{N} c_{vl}^2(n)\right) + c_{vk}(n) + b_k \triangle W_n^k, \tag{4.1}$$

where $\triangle W_n^k = W_k(t_{n+1}) - W_k(t_n) \in \mathcal{N}(0, h_n)$. Other one is backward Euler method

$$c_{vk}(n+1) = h_n c_{vk}(n+1)\left(\frac{-\sigma^2 k^2 \pi^2}{L^2} + a_1 - a_2 \sum_{l=1}^{N} c_{vl}^2(n+1)\right) + c_{vk}(n) + b_k \triangle W_n^k. \tag{4.2}$$

In our opinion, the best approach is linear-implicit Euler-type method

$$c_{vk}(n+1) = h_n c_{vk}(n+1)\left(\frac{-\sigma^2 k^2 \pi^2}{L^2} + a_1 - a_2 \sum_{l=1}^{N} c_{vl}^2(n)\right) + c_{vk}(n) + b_k \triangle W_n^k. \tag{4.3}$$

After calculating the c_{vk}, we can obtain $\widetilde{v}(t_{n+1}, x_m) = \sum_{n=1}^{N} c_{vn}(t) e_n(x)$. Then u_N can be calculated

$$\widetilde{u}(t_{n+1}, x_{m+1}) = d_n\left(\widetilde{v}(t_{n+1}, x_m) - \widetilde{u}(t_{n+1}, x_m)\right) + \widetilde{u}(t_{n+1}, x_m).$$

We note that Eqs (4.1) has a disadvantage that is lacking of stability and monotonicity deficits. A slight disadvantage of Eqs (4.2) is that we have to solve locally implicit algebraic equations at each iteration step n, which results in a lot of calculation and time. An advantage of methods (4.2) and (4.3) is very well stability and moment dissipativity behavior, and they keep some monotonicity properties [34, 35].

Theorem 4.1. *Consider the forward Euler method that*

$$c_{vk}(n+1) = \frac{c_{vk}(n) + b_k \triangle W_n^k}{1 + h_n\left(\frac{\sigma^2 k^2 \pi^2}{L^2} - a_1 + a_2 \sum_{l=1}^{N} c_{vl}^2(n)\right)}, \tag{4.4}$$

where $n \in \mathbb{N}, b_k = b\alpha_k$ and $\triangle w_n^k \in N(0, h_n)$. If $\sigma^2\pi^2 \geq L^2(a_1+1)$, their second moments is linearly bounded in time,

$$\mathbb{E}\left[\|u(t_n, \cdot)\|_H^2\right] < +\infty.$$

Proof. Suppose that $1 + h_n\left[\frac{\sigma^2\pi^2}{L^2} - (a_1+1)\right] > 0$. The Eqs (4.4) is finite due to the linear-implicit character of method (4.3). From Eqs (2.6), it follows

$$\begin{cases} \dfrac{c_{uk}(n+1) - c_{uk}(n)}{h_n} = \left[c_{vk}(t) - c_{uk}(t)\right], \\ \dfrac{c_{vk}(n+1) - c_{vk}(n)}{h_n} = \left[\dfrac{-\sigma^2 k^2 \pi^2}{L^2} + a_1 - a_2 \displaystyle\sum_{l=1}^{+\infty} c_{vl}^2(t)\right] c_{vk} + b_k \dfrac{W_k(t_n+1) - W_k(t_n)}{h_n}. \end{cases}$$

7

It remains to consider the second moments. We estimate c_{vk} by Eqs (4.4)

$$c_{vk}(n+1) = \frac{c_{vk}(n) + b_k \triangle w_n^k}{1 + h_n \left[\frac{\sigma^2 k^2 \pi^2}{L^2} - a_1 + a_2 \sum_{l=1}^N c_{vl}^2(n)\right]},$$

$$\mathbb{E}\left[c_k(n+1)\right]^2 = \mathbb{E}\left[\frac{\left[c_k(n)\right]^2 + b_k^2 h_n}{\left[1 + h_n\left(\frac{\sigma^2 k^2 \pi^2}{L^2} - a_1 + a_2 \sum_{l=1}^N c_{vl}^2(n)\right)\right]^2}\right].$$

Since denominator is less than one, we have

$$\mathbb{E}\left[c_{vk}(n+1)\right]^2 \leq \mathbb{E}\left[c_{vk}(n)\right]^2 + (b_k)^2 h_n \leq \mathbb{E}\left[c_{vk}(0)\right]^2 + (b_k)^2 t_{n+1}.$$

From Eqs (2.5), we obtain

$$\sum_{k=1}^N \mathbb{E}\left[\|c_{vk}(n)\|^2\right] \leq \sum_{k=1}^N \mathbb{E}\left[\|c_{vk}(0)\|^2\right] + b^2 \sum_{k=1}^N \alpha_k^2 t_n.$$

Since $\sum_{k=1}^{+\infty} \alpha_k^2$, $\sum_{k=1}^{+\infty} \|c_{vk}(0)\|^2 < \infty$, we obtain that, as $N \to \infty$ and $h \to 0$

$$\mathbb{E}\left[\|v(t_n,\cdot)\|_H^2\right] = \sum_{k=1}^{+\infty} \mathbb{E}\left[\|c_{vk}(n)\|^2\right] \leq \sum_{k=1}^{+\infty} \mathbb{E}\left[\|c_{vk}(0)\|^2\right] + b^2 \sum_{k=1}^{+\infty} \alpha_k^2 t_n < \infty.$$

□

Recall the definition in [38], let c_k^h denote the numerical approximation of the k-th Fourier coefficients c_k. The numerical approximation $c_h = (c_k^h)_{k=1,2,\cdots,N}$ is said to be mean consistent with rate r_0 iff there are a constant $C_0 = C_0(T)$ and a positive continuous function or functional V such that

$$\forall n=0,1,\cdots,n_T-1: \|\mathbb{E}[c(n+1)] - \mathbb{E}[c^h(n+1)]\|_N \leq C_0 V(c(n)) h_n^{r_0}$$

along any (nonrandom) partitions with sufficiently small step sizes $h_n \leq \delta \leq 1$, where $\|\cdot\|$ is the Euclidean vector norm in $\mathbb{R}^N$, provided that one has nonrandom data $c(n) = c^h(n)$.

Lemma 4.2. *The method (LIM) governed by Eqs (4.3) is mean consistent with rate* $r_0 = 1.5$.

The similar results may be found in [38].

5 Numerical Experiments

Under the condition that $\sigma^2\pi^2 > L^2(a_1 + 1)$, we present the results of systematic numerical simulations for solutions of the SDEs. The order is defined by $order = \lg\left(\|\mathbb{E}[c(n+1)] - \mathbb{E}[c^h(n+1)]\|_N\right)$. The ratio is defined by

$ratio = \left|1 - \frac{\|E\|}{\|E+\epsilon\|}\right|$, where E is the total energy of the system at $t=0$ and ϵ is the noise.

Case 5.1. *We consider the simple initial data with*

$$u(0,x) = \begin{cases} x, & x < \dfrac{1}{2}L, \\ L - x, & x > \dfrac{1}{2}L. \end{cases}$$

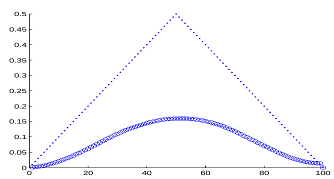


Figure 1: The numerical results at the times $t = 2$ with ratio $\approx 1\%$

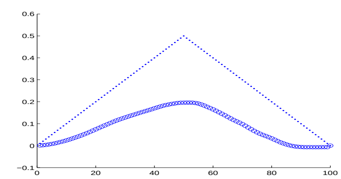


Figure 2: The numerical results at the times $t = 2$ with ratio$\approx 5\%$

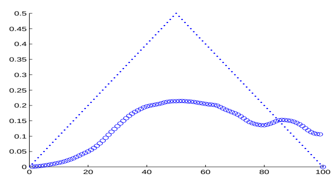


Figure 3: The numerical results at the times $t = 2$ with ratio $> 10\%$

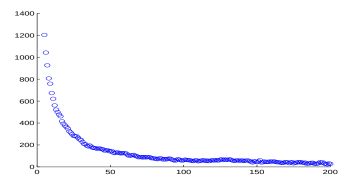


Figure 4: The total energy with different random terms at $t = 10$ with ratio $\approx 1\%$

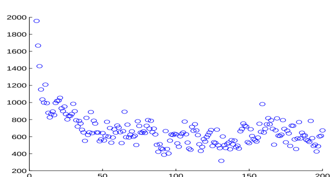


Figure 5: The total energy with different random terms at $t = 10$ with ratio$\approx 5\%$

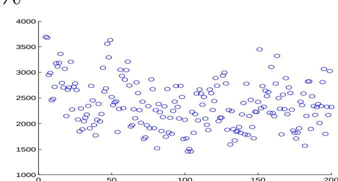


Figure 6: The total energy with different random terms at $t = 10$ with ratio $> 10\%$

The parameters $T = 2$, $\Delta t = 0.05$, $\Delta x = 0.01, L = 1$, $a_1 = 0.1, a_2 = 1$ and $\sigma = 9$ are chosen over the region $[0, 1]$. In Figure 1, 2 and 3, the lines of "$\cdot$" and "$\circ$" respectively denote the initial value $u(0, x)$ and the terminal value $u(2, x)$. Figure 1 shows that the wave dissipates at time $t = 2$. Figure 2, 3 show

9

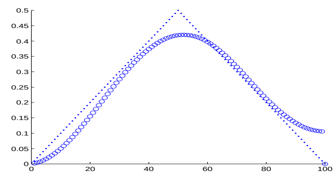


Figure 7: The numerical results at the times $t = 10$ with ratio $\approx 1\%$

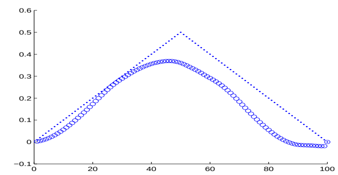


Figure 8: The numerical results at the times $t = 10$ with ratio$\approx 5\%$

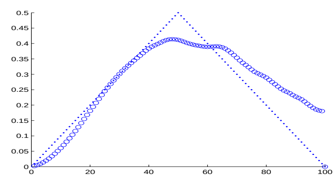


Figure 9: The numerical results at the times $t = 10$ with ratio $> 10\%$

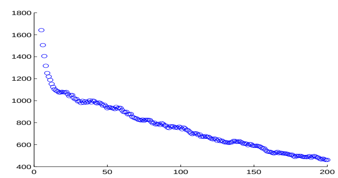


Figure 10: The total energy with different random terms at $t = 10$ with ratio $\approx 1\%$

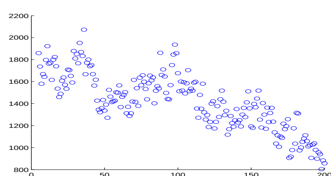


Figure 11: The total energy with different random terms at $t = 10$ with ratio$\approx 5\%$

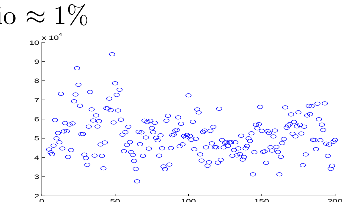


Figure 12: The total energy with different random terms at $t = 10$ with ratio $> 10\%$

Table 1: Order of convergence in space and time for the Euclidean vector norm

Δx	Δt	ratio	order	ratio	order
0.01	0.05	1%	4.1677	5%	3.0281
0.01	0.1	1%	4.014	5%	2.6964
0.05	0.05	1%	4.5498	5%	3.4972
0.05	0.1	1%	4.9379	5%	2.6441

10

the influence of noise enhancement on waveform. When the ratio $> 10\%$, the waveform was destroyed. Figure 4, 5 and 6 present the total energy evolution and show that the total energy stablely declines. However the total energy is linearly bounded in time. These results are in good agreement with the Theorem 3.1. Similarly, Table 1 presents the numerical results of the linear-implicit Euler-type schemes, which are in good agreement with Lemma 4.2.

Case 5.2. *In the second case, the initial data is same to the case 1. Using* $L = 1,\ a_1 = 1,\ a_2 = 1,\ b = 1$ *and* $\sigma = 9$*, we present the numerical solution at the terminal time* $T = 10$*.*

Table 2: Order of convergence in space and time for the Euclidean vector norm

Δx	Δt	ratio	order	ratio	order
0.01	0.05	1%	4.5093	5%	3.3822
0.01	0.1	1%	4.1886	5%	2.5853
0.05	0.05	1%	4.0957	5%	3.8375
0.05	0.1	1%	4.4245	5%	3.2689

Figure 7 shows that the wave dissipates at time $t = 10$. Figure 8, 9 show the influence of noise enhancement on waveform. When the ratio $> 10\%$, the waveform was destroyed. Figure 11, 10 and 12 present the numerical results of the total energy evolution and show that the total energy stablely declines. With the increase of the noise, the downward trend is not significant but vibrate. These results are in good agreement with the Theorem 4.2. Similarly, Table 2 presents the numerical results of the linear-implicit Euler-type schemes, which are in good agreement with Lemma 4.2.

Acknowledgements

The project is supported by the National Natural Science Foundation of China grants No.11471166 and No.11426134, Natural Science Foundation of Jiangsu Province grant No. BK20141443 and the Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD).

References

[1] A. Benveniste and M. Metivier. *Adaptive algorithms and stochastic approximations*. Springer Berlin Heidelberg, 1990.

[2] V. V. Bolotin. *Nonconservative problems of the theory of elastic stability.* A Pergamon Press Book, 1963.

[3] A. De Bouard, A. Debussche, and Y. Tsutsumi. Periodic solutions of the korteweg-de vries equation driven by white noise. *SIAM J. Math. Anal.*, 36(3):815–855, 2005.

[4] R. E. Caflisch. Monte carlo and quasi-monte carlo methods. *Acta Numerica*, 7:1–49, 1998.

[5] M. Deb and I. Babuška. Solution of stochastic partial differential equations using Galerkin finite element techniques. *Comput. Methods Appl. Mech. Engrg.*, 190:6359–6372, 2001.

[6] Q. Du. Numerical approximation of some linear stochastic partial differential equations driven by special additive noises. *SIAM J. Numer. Anal.*, 40(4):1421–1445, 2003.

[7] E. B. Dynkin. *Markov Processes*, volume 1,2. Springer, 1965.

[8] R. Erban, J. Haskovec, and Y. Sun. On cucker-smale model with noise and delay. *SIAM J. Appl. Math.*, 76(1):131–163, 2016.

[9] R. Ghanem and P. Spanos. *Stochastic finite elements: a spectral approach.* Springer Berlin, 1991.

[10] Q. Guo, X. Mao, and R. Yue. Almost sure exponential stability of stochastic differential delay equations. *SIAM J. Control Optim.*, 54(4):1919–1933, 2016.

[11] D. J. Higham, X. Mao, and A. M. Stuart. Strong convergence of euler-type methods for nonlinear stochastic differential equations. *SIAM J. Numer. Anal.*, 40(3):1041–1063, 2003.

[12] K. Ito. *On Stochastic Differential Equations.* American Mathematical Society, 1951.

[13] A. H. Jazwinski. *Stochastic processes and filtering theory.* Academic Press, 1970.

[14] J. U. Kim. On a stochastic hyperbolic system in linear elasticity. *SIAM J. Math. Anal.*, 32(2):304–322, 2000.

[15] P. E. Kloeden and E. Platen. *Numerical solution of SDE through computer experiments.* Springer Berlin Heidelberg, 1994.

[16] V. B. Kolmanovskii and A. Myshkis. *Applied Theory of Functional Differential Equations.* Kluwer Academic Publishers, 1992.

[17] S. W. Krieger. The effect of an axial force on the vibration of hinged bars. *J. Appl. Mech.*, 17(1):35–36, 1950.

[18] H. J. Kushner. *Stochastic stability and control.* Academic Press, 1967.

[19] B. Li. A note on stability of hybrid stochastic differential equations. *Appl. Math. Comput.*, 299:45–57, 2017.

[20] D. Liberzon and R. W. Brockett. Spectral analysis of Fokker-Planck and related operators arising from linear stochastic differential equations. *SIAM J. Control Optim.*, 38(5):1453–1467, 2000.

[21] P. L. Lions and P. E. Souganidis. Fully nonlinear stochastic partial differential equations. *SIAM J. Math. Anal.*, 27(1):40–55, 1998.

[22] F. Liu, M. Khan, and Y. Yan. Fourier spectral methods for stochastic space fractional partial differential equations driven by special additive noises. *J.Math.Anal.Appl.*, 24(2):290–309, 2018.

[23] N. Lu. Persistence of homoclinic orbits under white noise perturbation. *SIAM J. Appl. Dyn. Syst.*, 15(2):1124–1142, 2016.

[24] B. Mamporia. Linear stochastic differential equations in a banach space. *Theory Probab. Appl.*, 61(2):295–308, 2017.

[25] D. Mateos-nnez and J. Corts. pth moment noise-to-state stability of stochastic differential equations with persistent noise. *SIAM J. Control Optim.*, 52(4):2399–2421, 2015.

[26] H. G. Matthies and A. Keese. Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations. *Comput. Methods Appl. Mech. Engrg.*, 194(12-16):1295–1331, 2005.

[27] H.P McKean. *Stochastic Integrals*, volume 4. Academic Press, 1969.

[28] Yu. V. Mednitskii. On the existence of strong solutions of linear stochastic differential equations in $\mathbb{R}^\infty$. *Theory Probab. Appl.*, 42(4):826–831, 2012.

[29] E. Moro and H. Schurz. Boundary preserving semianalytic numerical algorithms for stochastic differential equations. *SIAM J. Sci. Comput.*, 29(4):1525–1549, 2007.

[30] N. Newton. Asymptotically efficient Runge-Kutta methods for a class of ito and stratonovich equations. *SIAM J. Appl. Math.*, 51:542–567, 1991.

[31] M. Papadrakakis and V. Papadopoulos. Robust and efficient methods for stochastic finite element analysis using Monte Carlo simulation. *Comput. Methods Appl. Mech. Engrg.*, 134:325–340, 1996.

[32] B. Puig and F. Poirion. Non-gaussian simulation using Hermite polynomial expansion: convergences and algorithms. *Probabilist. Engrg. Mech.*, 17:253–264, 2002.

[33] Y. Saito and T. Mitsui. Stability analysis of numerical schemes for stochastic differential equations. *SIAM J. Nnumer. Anal.*, 33(6):2254–2267, 1996.

[34] H. Schurz. Stability, stationarity, and boundedness of some implicit numerical methods for stochastic differential equations and applications. *Logos Verlag Berlin Berlin*, 12:264, 1997.

[35] H. Schurz. Preservation of asymptotical laws through euler methods for ornstein-uhlenbeck process. *Stochastic Analysis and Applications*, 3:463–486, 1999.

[36] H. Schurz. Existence and uniqueness of solutions of semilinear stochastic infinite-dimensional differential systems with h-regular noise. *J. Math. Anal. Appl.*, 332:334–345, 2007.

[37] H. Schurz. New stochastic integrals:oscillation theorems and energy identities. *Comm. Appl. Anal.*, 2:181–194, 2009.

[38] H. Schurz. Nonlinear stochastic heat equations with cubic nonlinearities and additive q-regular noise in r^1. *Electron.J.Differ.Equ.Conf.*, 19:221–233, 2010.

[39] S.Nevroz and P. E. Caines. Nonlinear filtering theory for mckean–vlasov type stochastic differential equations. *SIAM J. Control Optim.*, 54(1):153–174, 2016.

[40] H. C. Tuckwell and J. B. Walsh. Random currents through nerve membranes. *Biological Cybernetics*, 2:99–110, 1983.

[41] Q. Wang, Z. Zhang, and Z. Li. A fourier finite volume element method for solving two-dimensional quasi-geostrophic equations on a sphere. *Appl. Numer. Math.*, 71(1):1–13, 2013.

[42] Q. Zhang, Z.L. Li, and Z.Y. Zhang. A sparse grid stochastic collocation method for elliptic interface problems with random input. *J. Sci. Comput.*, 67:262–280, 2015.

[43] M. Zheng and G. E. Karniadakis. Numerical methods for spdes with tempered stable processes. *SIAM J. Sci. Comput.*, 37(3):A1197–A1217, 2015.

An iterative algorithm for solving split feasibility problems and fixed point problems in p-uniformly convex and smooth Banach spaces

P. Chuasuk[1], **A. Farajzadeh**[2] **and A. Kaewcharoen**[3*]

[2]Department of Mathematics, Faculty of Science, Razi University
Kermanshah 67419, Iran, E-mail: farajzadehali@gmail.com

[1,3] Department of Mathematics, Faculty of Science, Naresuan University,
Phitsanulok 65000, Thailand, E-mails: preeyanuchc56@nu.ac.th, anchaleeka@nu.ac.th

Abstract

In this paper, we introduce an iterative process for approximation of a common fixed point for a finite family of multi-valued Bregman relatively nonexpansive mappings with a solution of the split feasibility problems in p-uniformly convex and uniformly smooth Banach spaces. We prove the strong convergence theorems of the proposed iterative process in p-uniformly convex and uniformly smooth Banach spaces and present the numerical results to verify the efficiency and implementation of our results.

Keywords: Bregman relatively nonexpansive mappings; strong convergence theorems; uniformly convex Banach spaces; uniformly smooth Banach spaces; split feasibility problems.

1 Introduction

Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty closed convex subsets of E_1 and E_2 respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. The split feasibility problem (SFP) is to find a point

$$x \in C \ \text{ such that } \ Ax \in Q. \tag{1.1}$$

Note that the inverse image of the set Q under A is a convex set. Hence the problem 1.1 can be written in case that the intersection $C \cap A^{-1}(Q)$ is nonempty. We will denote the nonempty solution set of (1.1) by $\Omega = C \cap A^{-1}(Q)$. Therefore Ω is a closed convex subset of E_1.

In 1994, Censor and Elfving [8] introduced the SFP (1.1) in finite-dimensional Hilbert spaces for modelling inverse problems which arise from phase retrievals, medical image reconstruction. Various algorithms

*Corresponding author.

have been invented to solve the SFP (1.1) (see [2, 6, 11, 25, 28, 29] and the references therein). In particular, Byrne [6] introduced a so-called CQ algorithm, taking an initial point x_0 arbitrarily and construct the sequence $\{x_n\}$ by

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), n \geq 1,$$

where $0 < \gamma < \frac{2}{\|A\|^2}$, and P_C denotes the projection onto a set C. That is, $P_C(x) = \arg\min_{y \in C} \|x - y\|$. Recently, Schöpfer et al. [19] solved the SFP (1.1) in p-uniformly convex real Banach spaces which are also uniformly smooth using the following algorithm: for $x_1 \in E_1$ and $n \geq 1$, set

$$x_{n+1} = \Pi_C J^q_{E_1^*}[J^p_{E_1} x_n - t_n A^* J^p_{E_2}(Ax_n - P_Q(Ax_n))], \tag{1.2}$$

where Π_C denotes the the Bregman projection and J the duality mapping. Clearly the above algorithm covers the Byrne' CQ algorithm [6]. They used algorithm (1.2) for obtaining the weak convergence result in a p-uniformly convex real Banach spaces which are uniformly smooth with the condition that the duality mapping of E is sequentially weak-to-weak-continuous. In 2014, Wang [26] studied the following multiple-sets split feasibility problem (MSSFP) (see [11]): find $x \in E_1$ satisfying

$$x \in \bigcap_{i=1}^{r} C_i, \;\; Ax \in \bigcap_{j=r+1}^{r+s} Q_j \tag{1.3}$$

where r, s are two given integers, $C_i, i = 1, ..., r$, is a closed convex subset of E_1, and Q_j, $j = r+1, ..., r+s$, is a closed convex subset in E_2. Wang [26] modified the above algorithm (1.2) and proved the strong convergence theorem using an idea appeared in [13] and the following algorithm: for any initial guess x_0, define $\{x_n\}$ recursively by

$$\begin{cases} y_n = T_n x_n \\ D_n = \{u \in E : \Delta_p(y_n, u) \leq \Delta_p(x_n, u)\} \\ E_n = \{u \in E : \langle x_n - u, J^p_E(x_0) - J^p_E(x_n)\rangle \geq 0\} \\ x_{n+1} = \Pi_{D_n \cap E_n}(x_0), \end{cases} \tag{1.4}$$

where T_n is defined, for each $n \in \mathbb{N}$, by

$$T_n x = \begin{cases} \Pi_{C_{i(n)}}(x), \;\; 1 \leq i(n) \leq r \\ J^q_{E_1^*}[J^p_{E_1}(x) - t_n A^* J^p_{E_2}(I - P_{Q_{i(n)}})Ax], r+1 \leq i(n) \leq r+s, \end{cases} \tag{1.5}$$

$i : \mathbb{N} \to I$ is the cyclic control mapping

$$i(n) = n \mod (r+s) + 1,$$

and t_n satisfies

$$0 < t \leq t_n \leq \left(\frac{q}{C_q \|A\|^q}\right)^{\frac{1}{q-1}}.$$

For better comparison of (1.5) with (1.2), we state a version of (1.2) for solving problem (1.3):

$$x_{n+1} = \Pi_{C_{i(n)}} J^q_{E_1^*}[J^p_{E_1}(x_n) - t_n A^* J^p_{E_2}(Ax_n - P_{Q_{i(n)}}(Ax_n))], \tag{1.6}$$

where $i : \mathbb{N} \to I$ is the cyclic control mapping

$$i(n) = n \mod (r+s) + 1.$$

In 1967, Bregman [3] has discovered an elegant and effective technique for the use of the Bregman distance function Δ_p in the process of designing and analyzing feasibility and optimization algorithms.

This opened a growing area of research in which Bregman's technique is applied in various ways in order to design and analyze iterative algorithms for solving not only feasibility and optimization problems, but also algorithms for solving variational inequality problems, equilibrium problems, fixed point problems for nonlinear mappings, and so on (see [16, 4, 17] , and the references therein).

Recently, Shehu et al. [22] studied split feasibility problems and fixed point problems concerning left Bregman strongly nonexpansive mappings: find an element $x \in E_1$ satisfying

$$x \in C \cap F(T) \text{ such that } Ax \in Q. \tag{1.7}$$

Shehu et al. [22] proposed the following algorithm: for a fixed $u \in E_1$, let $\{x_n\}_{n=1}^{\infty}$ be iteratively generated by $u_1 \in E_1$,

$$\begin{cases} x_n = \Pi_C J^q_{E_1^*}[J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(Au_n - P_Q(Au_n))] \\ u_{n+1} = \Pi_C J^q_{E_1^*}(\alpha_n J^p_{E_1}(u) + (1-\alpha_n) J^p_{E_1}(Tx_n)), \ \ n \geq 1, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. Moreover Shehu et al. [22] proved the strong convergence of the sequence generated by (1.8) for solving problem (1.7) in p-uniformly convex real Banach spaces which are also uniformly smooth.

In 2014, Pang et al. [9] showed that the class of Bregman relatively nonexpansive mappings embraces properly the class of Bregman strongly nonexpansive mappings. Very recently, Shahzad and Zegeye [21] introduced the class of multi-valued Bregman relatively nonexpansive mappings which includes the class of single-valued Bregman relatively nonexpansive mappings. Hence, the class of multi-valued Bregman relatively nonexpansive mappings is a more general class of mappings and gave a example of a multi-valued Bregman relatively nonexpansive mappings. Moreover, Shahzad and Zegeye [21] proved that if C is a nonempty closed convex subset of $int(domf)$ where $f : E \to \mathbb{R}$ is a uniformly Frechet differentiable and totally convex on bounded subsets of E and $T : C \to CB(C)$ is a Bregman relatively nonexpansive mapping, then $F(T)$ is closed and convex.

Our aim in this paper is to construct an iterative scheme for solving problem (1.7) which is also a fixed point of a multi-valued Bregman relatively nonexpansive mapping T in p-uniformly convex real Banach spaces which are also uniformly smooth and then prove the strong convergence theorems of the sequences generated by our scheme under some suitable assumptions.

2 Preliminaries

Let $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. The modulus of smoothness of E is the function $\rho_E(\tau) : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) = \sup\Big\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \|y\| \leq 1\Big\}.$$

E is called to be uniformly smooth if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$$

and E is called to be q-uniformly smooth if there exists a $C_q > 0$ such that $\rho_E(\tau) \leq C_q \tau^q$ for any $\tau > 0$.

The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) := \inf\Big\{1 - \|\frac{x+y}{2}\| : \|x\| = \|y\| = 1; \epsilon = \|x-y\|\Big\}.$$

E is called to be uniformly convex if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$ and p-uniformly convex if there is a $C_p > 0$ so that $\delta_E(\varepsilon) \geq C_p\varepsilon^p$ for any $\varepsilon \in (0, 2]$. The L_p space is 2-uniformly convex for $1 < p \leq 2$ and p-uniformly convex for $p \geq 2$.

Lemma 2.1. *[27] Let $x, y \in E$. If E is q-uniformly smooth, then there exists a $C_q > 0$ such that*

$$\|x - y\|\|^q \leq \|x\|^q - q\langle y, J_E^q(x)\rangle + C_q\|y\|^q.$$

It is known that if E is p-uniformly convex and uniformly smooth, then its dual E^* is q-uniformly smooth and uniformly convex. Moreover the duality mapping J_E^p is one-to-one, single-valued and $J_E^p = (J_{E^*}^q)^{-1}$ where $J_{E^*}^q$ is the duality mapping of E^* (see [10, 14]).

Definition 2.2. The duality mapping $J_E^p : E \to 2^{E^*}$ is defined by

$$J_E^p(x) = \{\bar{x} \in E^* : \langle x, \bar{x}\rangle = \|x\|^p, \|\bar{x}\|^p = \|x\|^{p-1}\}.$$

The duality mapping J_E^p is said to be weak-to-weak continuous if

$$x_n \rightharpoonup x \Rightarrow \langle J_E^p x_n, y\rangle \to \langle J_E^p x, y\rangle$$

holds for any $y \in E$. We observe that $l_p(p > 1)$ has such a property, but $L_p(p > 2)$ does not have this property.

Let $f : E \to (-\infty, +\infty]$ be a convex function and $x \in int(dom)f$. The function f is said to be Gâteaux differentiable at x if

$$\lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t} \text{ exists for any } y \in E.$$

Definition 2.3. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable convex function. The Bregman distance with respect to f is defined as:

$$\Delta_f(x, y) = f(y) - f(x) - \langle f^{'}(x), y - x\rangle, \ \ x, y \in E.$$

It is worth noting that the duality mapping J_E^p is in fact the derivative of the function $f_p(x) = (\frac{1}{p})\|x\|^p$. Then the Bregman distance with respect to f_p is given by

$$\begin{aligned}\Delta_p(x, y) &= \frac{1}{q}\|x\|^p - \langle J_E^p x, y\rangle + \frac{1}{p}\|y\|^p \\ &= \frac{1}{p}(\|y\|^p - \|x\|^p) + \langle J_E^p x, x - y\rangle \\ &= \frac{1}{q}(\|x\|^p - \|y\|^p) - \langle J_E^p x - J_E^p y, x\rangle.\end{aligned}$$

In general, the Bregman distance is not a metric due to the absence of symmetry, but it has some distance-like properties.

The following are some of important properties of the Bregman distance which are needed in the sequel

$$\Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle z - y, J_E^p x - J_E^p z\rangle, \tag{2.1}$$

and

$$\Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J_E^p x - J_E^p y\rangle. \tag{2.2}$$

For the p-uniformly convex space, the metric and Bregman distance has the following relation (see [20]):

$$\tau\|x-y\|^p \leq \Delta_p(x,y) \leq \langle x-y, J_E^p x - J_E^p y\rangle, \tag{2.3}$$

where $\tau > 0$ is some fixed number.
Let C be a nonempty closed convex subset of E. The metric projection

$$P_C x = \arg\min_{y\in C} \|x-y\|, \ \ x \in E,$$

is the unique minimizer of the norm distance which can be characterized by a variational inequality:

$$\langle J_E^p(x - P_C x), z - P_C x\rangle \leq 0, \ \ \forall z \in C. \tag{2.4}$$

Similar to the metric projection, the Bregman projection is defined as

$$\Pi_C x = \arg\min_{y\in C} \Delta_p(x,y), \ \ x \in E,$$

which is well-defined and the minimizer of it is unique (for more details see [19]). The Bregman projection can also be characterized by a variational inequality:

$$\langle J_E^p(x) - J_E^p(\Pi_C x), z - \Pi_C x\rangle \leq 0, \ \ \forall z \in C, \tag{2.5}$$

from which one has

$$\Delta_p(\Pi_C x, z) \leq \Delta_p(x,z) - \Delta_p(x, \Pi_C x), \ \ \forall z \in C. \tag{2.6}$$

Following [1] and [7], we use of the function $V_p : E^* \times E \to [0,+\infty)$ associated with f_p which is defined by

$$V_p(\bar{x}, x) = \frac{1}{q}\|\bar{x}\|^q - \langle \bar{x}, x\rangle + \frac{1}{p}\|x\|^p, \ \ \forall x \in E, \ \bar{x} \in E^*.$$

Then V_p is nonnegative and $V_p(\bar{x}, x) = \Delta_p(J_{E^*}^q(\bar{x}), x)$ for all $x \in E^*$ and $y \in E$.
Moreover, by the subdifferential inequality,

$$\langle f^{'}(\bar{x}), x - \bar{x}\rangle \leq f(\bar{x}) - f(x). \tag{2.7}$$

With $f(x) = \frac{1}{q}\|x\|^q$, $x \in E^*$, then $f^{'}(x) = J_{E^*}^q$, we have

$$\langle J_{E^*}^q(x), y\rangle \leq \frac{1}{q}\|x-y\|^q - \frac{1}{q}\|x\|^q, \forall x, y \in E^*. \tag{2.8}$$

Using (2.8), we have for all $\bar{x}, \bar{y} \in E^*$ and $x \in E$ that

$$\begin{aligned}
V_p(\bar{x}+\bar{y}, x) &= \frac{1}{q}\|\bar{x}+\bar{y}\|^q - \langle \bar{x}+\bar{y}, x\rangle + \frac{1}{p}\|x\|^p \\
&\geq \frac{1}{q}\|\bar{x}\|^q + \langle \bar{y}, J_{E^*}^q(\bar{x})\rangle - \langle \bar{x}+\bar{y}, x\rangle + \frac{1}{p}\|x\|^p \\
&= \frac{1}{q}\|\bar{x}\|^q - \langle \bar{x}, x\rangle + \frac{1}{p}\|x\|^p + \langle \bar{y}, J_{E^*}^q(\bar{x})\rangle \\
&+ \langle \bar{y}, J_{E^*}^q(\bar{x})\rangle - \langle \bar{y}, x\rangle \\
&= \frac{1}{q}\|\bar{x}\|^q - \langle \bar{x}, x\rangle + \frac{1}{p}\|x\|^p + \langle \bar{y}, J_{E^*}^q(\bar{x}) - x\rangle \\
&= V_p(\bar{x}, x) + \langle \bar{y}, J_{E^*}^q(\bar{x}) - x\rangle.
\end{aligned}$$

In other words,

$$V_p(\bar{x}, x) + \langle \bar{y}, J^q_{E^*}(\bar{x}) - x\rangle \leq V_p(\bar{x} + \bar{y}, x), \tag{2.9}$$

for all $x \in E$ and $\bar{x}, \bar{y} \in E^*$ (see, for example, [23],[24]).

Let C be a nonempty closed convex subset of a smooth Banach space E and let T be a mapping from C into itself. A point $p \in C$ is said to be an asymptotic fixed point [16] of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$.

Definition 2.4. Let C be a nonempty convex subset of $int(domf)$. A mapping $T : C \to int(domf)$ with $F(T) \neq \emptyset$ is called to be

(i) Bregman quasi-nonexpansive if

$$\Delta_p(Tx, \bar{x}) \leq \Delta_p(x, \bar{x}),\ \forall x \in C, \bar{x} \in F(T);$$

(ii) Bregman relatively nonexpansive if $F(T) = \hat{F}(T)$,

$$\Delta_p(Tx, \bar{x}) \leq \Delta_p(x, \bar{x}),\ \forall x \in C, \bar{x} \in F(T);$$

(iii) left Bregman strongly nonexpansive with respect to a nonempty $\hat{F}(T)$ if

$$\Delta_p(Tx, \bar{x}) \leq \Delta_p(x, \bar{x}),\ \forall x \in C, \bar{x} \in \hat{F}(T),$$

and if whenever $\{x_n\} \subset C$ is bounded, $\bar{x} \in \hat{F}(T)$ and

$$\lim_{n\to\infty} (\Delta_p(x_n, \bar{x}) - \Delta_p(Tx_n, \bar{x})) = 0,$$

it follows that

$$\lim_{n\to\infty} \Delta_p(x_n, Tx_n) = 0.$$

It is obvious that any left Bregman strongly nonexpansive mapping is a Bregman relatively nonexpansive mapping, but the converse is not true in general. Pang et al. [9] showed that there exists a Bregman relatively nonexpansive mapping which is not a Bregman strongly nonexpansive mapping.

Let $N(C)$ and $CB(C)$ denote the families of nonempty subsets and nonempty closed bounded subsets of C, respectively. The Hausdorff metric on $CB(C)$ is defined by

$$H(A, B) = \max\{\sup_{x\in A} \operatorname{dist}(x, B), \sup_{y\in B} \operatorname{dist}(y, A)\},$$

for all $A, B \in CB(C)$ where $\operatorname{dist}(x, B) = \inf\{\|x - y\| : y \in B\}$ is the distance from a point x to a subset B.

Recall that a multi-valued mapping $T : C \to CB(C)$ is said to be

(i) nonexpansive if $H(Tx, Ty) \leq \|x - y\|$, for all $x, y \in C$;

(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \|x - p\|$, for all $x \in C$ and $p \in F(T)$.

Let $T : C \to CB(C)$. A point $p \in C$ is said to be a fixed point of T if $p \in F(T)$ where $F(T) = \{p \in T : p \in Tp\}$. A point $p \in C$ is said to be an asymptotic fixed point [16] of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges weakly to p and $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$.

Definition 2.5. *[21]* Let $T : C \to CB(C)$ is said to be Bregman relatively nonexpansive if the following conditions are satisfied:

(A1) $F(T)$ is nonempty;

(A2) $\Delta_p(z, \bar{x}) \leq \Delta_p(x, \bar{x})$ for $z \in Tx$, $x \in C$ and $\bar{x} \in F(T)$;

(A3) $F(T) = \hat{F}(T)$.

The following is the example of a multi-valued Bregman relatively nonexpansive mapping appeared in [21]:

Example 2.6. [21] Let $I = [0, 1]$, $X = L^p(I)$, $1 < p < \infty$ and $C = \{f \in X : f(x) \geq 0, \forall x \in I\}$. Let $T : C \to CB(C)$ be defined by

$$\begin{cases} \{h \in C : f(x) - \frac{1}{2} \leq h(x) \leq f(x) - \frac{1}{4}, \forall x \in I\} \text{ if } f(x) > 1, \forall x \in I \\ \{0\}, \text{ otherwise.} \end{cases} \tag{2.10}$$

Then T is defined by (2.10) is a multi-valued Bregman relatively nonexpansive mapping.

We next state the following lemmas which will be used in the sequel.

Lemma 2.7. *[5] Let E be a Banach space and $f : E \to \mathbb{R}$ a Gâteaux differentiable function which is locally uniformly convex on E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then the following assertions are equivalent*

(i) $\lim_{n\to\infty} D_f(x_n, y_n) = 0$;

(ii) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.8. *[12] Let E be a Banach space, let $r > 0$ be a constant, and let $f : E \to \mathbb{R}$ be a uniformly convex function on bounded subsets of E. Then*

$$f\Big(\sum_{k=0}^{n} \alpha_k x_k\Big) \leq \sum_{k=0}^{n} \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r(\|x_i - y_i\|),$$

for all $i, j \in \{0, 1, 2, ..., n\}$, $x_k \in B_r$, $\alpha_k \in (0, 1)$, and $k = 0, 1, 2, ..., n$ with $\sum_{k=0}^{n} \alpha_k = 1$,where ρ_r is the gauge of uniform convexity of f .

Lemma 2.9. *[27] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \ \ n \geq 1,$$

where (i) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 1)$, $\sum \gamma_n < 0$. Then, $a_n \to 0$ as $n \to \infty$.

3 Main results

In this section, we introduce an iterative process for approximation of a common fixed point for a finite family of multi-valued Bregman relatively nonexpansive mappings with a solution of the split feasibility problems in p-uniformly convex and uniformly smooth Banach spaces and prove the strong convergence theorems of the proposed iterative process in p-uniformly convex and uniformly smooth Banach spaces

Theorem 3.1. *Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty closed convex subsets of E_1 and E_2, respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that SFP has a nonempty solution set Ω. Let $\{T_i\}_{i=1}^N$ be a finite family of multi-valued bregman relative nonexpansive mappings of C into $CB(C)$ such that $\mathcal{F} = \cap_{i=1}^N F(T_i) \cap \Omega \neq \emptyset$. Let $u_1 \in E_1$ and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_n = \Pi_c J_{E_1^*}^q [J_{E_1}^p(u_n) - t_n A^* J_{E_2}^p(Au_n - P_Q(Au_n))] \\ u_{n+1} = \Pi_c J_{E_1^*}^q (\alpha_n^{(0)} J_{E_1}^p(x_n) + \sum_{i=1}^N \alpha_n^{(i)} J_{E_1}^p(z_n^{(i)})) \ , z_n^{(i)} \in T_i x_n, \end{cases} \tag{3.1}$$

where $\{\alpha_n^{(i)}\} \subset [a.b] \subset (0,1)$ for all $i = 0,1,...,N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$. Suppose the following conditions are satisfied:

(i) $\Sigma_{n=1}^\infty \alpha_n^{(i)} = 0$ for all $i = 0,1,...,N$.

(ii) $0 < t \leq t_n \leq k < \left(\frac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}$.

Then the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to an element $x^* \in \mathcal{F}$.

Proof. Let $x^* \in \Omega$. Suppose that $w_n = Au_n - P_Q(Au_n)$ and $v_n = J_{E_1^*}^q[J_{E_1}^p(u_n) - t_n A^* J_{E_2}^p(Au_n - P_Q(Au_n))]$, $\forall n \geq 1$. Therefore $x_n = \Pi_c v_n$, $\forall n \geq 1$. It follows that

$$\begin{aligned} \langle J_{E_2}^p(w_n), Au_n - Ax^* \rangle &= \|Au_n - P_Q(Au_n)\|^p + \langle J_{E_2}^p(w_n), P_Q(Au_n) - Ax^* \rangle \\ &\geq \|Au_n - P_Q(Au_n)\|^p = \|w_n\|^p. \end{aligned} \tag{3.2}$$

By Lemma 2.1, we obtain that

$$\begin{aligned} \Delta_p(x_n, x^*) &\leq \Delta_p(v_n, x^*) \\ &= \Delta_p(J_{E_1^*}^q[J_{E_1}^p(u_n) - t_n A^* J_{E_2}^p(w_n)], x^*) \\ &= \frac{1}{q}\|J_{E_1}^p(u_n) - t_n A^* J_{E_2}^p(w_n)\|^q - \langle J_{E_1}^p(u_n), x^* \rangle + t_n \langle J_{E_2}^p(w_n), Ax^* \rangle + \frac{1}{p}\|x^*\|^p \\ &\leq \frac{1}{q}\|J_{E_1}^p(u_n)\|^q - t_n \langle Au_n, J_{E_2}^p(w_n) \rangle + \frac{C_q(t_n\|A\|)^q}{q}\|J_{E_2}^p(w_n)\|^q \\ &- \langle J_{E_1}^p(u_n), x^* \rangle + t_n \langle J_{E_2}^p(w_n), Ax^* \rangle + \frac{1}{p}\|x^*\|^p \\ &= \frac{1}{q}\|u_n\|^p - \langle J_{E_1}^p(u_n), x^* \rangle + \frac{1}{p}\|x^*\|^p + t_n \langle Au_n, J_{E_2}^p(w_n) \rangle \\ &+ \frac{C_q(t_n\|A\|)^q}{q}\|J_{E_2}^p(w_n)\|^q \\ &= \Delta_p(u_n, x^*) + t_n \langle J_{E_2}^p(w_n), Ax^* - Au_n \rangle + \frac{C_q(t_n\|A\|)^q}{q}\|J_{E_2}^p(w_n)\|^q \\ &= \Delta_p(u_n, x^*) + \Big(t_n - \frac{C_q(t_n\|A\|)^q}{q}\Big)\|w_n\|^p. \end{aligned} \tag{3.3}$$

Using the condition(ii), we have

$$\Delta_p(x_n, x^*) \leq \Delta_p(u_n, x^*) \ \ \forall n \geq 1.$$

Now, using (3.1), we have

$$\begin{aligned}
\Delta_p(x_{n+1}, x^*) \leq \Delta_p(u_{n+1}, x^*) &\leq \alpha_n^{(0)} \Delta_p(x_n, x^*) + \sum_{i=1}^{N} \alpha_n^{(i)} \Delta_p(z_n^{(i)}, x^*) \\
&\leq \alpha_n^{(0)} \Delta_p(x_n, x^*) + \sum_{i=1}^{N} \alpha_n^{(i)} \Delta_p(x_n, x^*) \\
&= \Delta_p(x_n, x^*). \qquad (3.4)
\end{aligned}$$

This shows that $\{\Delta_p(x_n, x^*)\}$ is a bounded decreasing sequence. Hence the $\lim_{n\to\infty} \Delta_p(x_n, x^*)$ exists and thus $\lim_{n\to\infty}(\Delta_p(x_n, x^*) - \Delta_p(x_{n+1}, x^*)) = 0$. Let $y_n = J_{E_1^*}^q(\alpha_n^{(0)} J_{E_1}^p(x_n) + \sum_{i=1}^N \alpha_n^{(i)} J_{E_1}^p(z_n^{(i)}))$, $n \geq 1$. Therefore

$$\begin{aligned}
\Delta_p(x_{n+1}, x^*) &\leq \Delta_p(u_{n+1}, x^*) \\
&= V_p(\alpha_n^{(0)} J_{E_1}^p(x_n) + \sum_{i=1}^{N} \alpha_n^{(i)} J_{E_1}^p(z_n^{(i)}), x^*) \\
&\leq V_p(\alpha_n^{(0)} J_{E_1}^p(x_n) + \sum_{i=1}^{N} \alpha_n^{(i)} J_{E_1}^p(z_n^{(i)}) - \alpha_n^{(0)}(J_{E_1}^p(x_n) - J_{E_1}^p(x^*)), x^*) \\
&+ \alpha_n^{(0)} \langle J_{E_1}^p(x_n) - J_{E_1}^p(x^*), y_n - x^* \rangle \\
&= V_p(\alpha_n^{(0)} J_{E_1}^p(x^*) + \sum_{i=1}^{N} \alpha_n^{(i)} J_{E_1}^p(z_n^{(i)}), x^*) + \alpha_n^{(i)} \langle J_{E_1}^p(x_n) - J_{E_1}^p(x^*), y_n - x^* \rangle \\
&= \alpha_n^{(0)} V_p(J_{E_1}^p(x^*), x^*) + \sum_{i=1}^{N} \alpha_n^{(i)} V_p(J_{E_1}^p(z_n^{(i)}), x^*) \\
&+ \alpha_n^{(0)} \langle J_{E_1}^p(x_n) - J_{E_1}^p(x^*), y_n - x^* \rangle \\
&= \sum_{i=1}^{N} \alpha_n^{(i)} \Delta_p(z_n^{(i)}, x^*) + \alpha_n^{(0)} \langle J_{E_1}^p(x_n) - J_{E_1}^p(x^*), y_n - x^* \rangle \\
&\leq \sum_{i=1}^{N} \alpha_n^{(i)} \Delta_p(x_n, x^*) + \alpha_n^{(0)} \langle J_{E_1}^p(x_n) - J_{E_1}^p(x^*), y_n - x^* \rangle. \qquad (3.5)
\end{aligned}$$

By Lemma 2.8, we obtain that

$$\begin{aligned}
\Delta_p(x_{n+1}, x^*) &\leq \Delta_p(u_{n+1}, x^*) \\
&= V_p(\alpha_n^{(0)} J_{E_1}^p(x_n) + \sum_{i=1}^N \alpha_n^{(i)} J_{E_1}^p(z_n^{(i)}), x^*) \\
&= \frac{1}{q}\|x^*\|^q - \langle \alpha_n^{(0)} J_{E_1}^p(x_n) + \sum_{i=1}^N \alpha_n^{(i)} J_{E_1}^p(z_n^{(i)}), x^* \rangle \\
&+ \frac{1}{p}\|\alpha_n^{(0)} J_{E_1}^p(x_n) + \sum_{i=1}^N \alpha_n^{(i)} J_{E_1}^p(z_n^{(i)})\|^p \\
&= \frac{1}{q}\|x^*\|^q - \alpha_n^{(0)} \langle J_{E_1}^p(x_n), x^* \rangle - \sum_{i=1}^N \alpha_n^{(i)} \langle J_{E_1}^p(z_n^{(i)}), x^* \rangle \\
&+ \frac{1}{p}\|\alpha_n^{(0)} J_{E_1}^p(x_n) + \sum_{i=1}^N \alpha_n^{(i)} J_{E_1}^p(z_n^{(i)})\|^p
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{q}\|x^*\|^q - \alpha_n^{(0)} \langle J_{E_1}^p(x_n), x^* \rangle - \sum_{i=1}^N \alpha_n^{(i)} \langle J_{E_1}^p(z_n^{(i)}), x^* \rangle \\
&+ \alpha_n^{(0)} \frac{1}{p}\|J_{E_1}^p(x_n)\|^p + \sum_{i=1}^N \alpha_n^{(i)} \frac{1}{p}\|J_{E_1}^p(z_n^{(i)})\|^p - \alpha_n^{(i)}\alpha_n^{(j)} \rho_r(\|J_{E_1}^p(x_n) - J_{E_1}^p(z_n^{(i)})\|) \\
&= \alpha_n^{(0)} V_p(J_{E_1}^p(x_n), x^*) + \sum_{i=1}^N \alpha_n^{(i)} V_p(J_{E_1}^p(z_n^{(i)}), x^*) - \alpha_n^{(i)}\alpha_n^{(j)} \rho_r(\|J_{E_1}^p(x_n) - J_{E_1}^p(z_n^{(i)})\|) \\
&= \alpha_n^{(0)} \Delta_p(x_n, x^*) + \sum_{i=1}^N \alpha_n^{(i)} \Delta_p(z_n^{(i)}, x^*) - \alpha_n^{(i)}\alpha_n^{(j)} \rho_r(\|J_{E_1}^p(x_n) - J_{E_1}^p(z_n^{(i)})\|) \\
&\leq \alpha_n^{(0)} \Delta_p(x_n, x^*) + \sum_{i=1}^N \alpha_n^{(i)} \Delta_p(x_n, x^*) - \alpha_n^{(i)}\alpha_n^{(j)} \rho_r(\|J_{E_1}^p(x_n) - J_{E_1}^p(z_n^{(i)})\|) \\
&= \Delta_p(x_n, x^*) - \alpha_n^{(i)}\alpha_n^{(j)} \rho_r(\|J_{E_1}^p(x_n)^p - J_{E_1}^p(z_n^{(i)})\|).
\end{aligned}$$

Thus

$$\alpha_n^{(i)}\alpha_n^{(j)} \rho_r(\|J_{E_1}^p(x_n)\|^p - J_{E_1}^p(z_n^{(i)})\|) \leq \Delta_p(x_n, x^*) - \Delta_p(x_{n+1}, x^*). \tag{3.6}$$

Then, from (3.6), we have

$$\alpha_n^{(i)}\alpha_n^{(j)} \rho_r(\|J_{E_1}^p(x_n) - J_{E_1}^p(z_n^{(i)})\|) \to 0, \ \ n \to \infty.$$

By the property of ρ_r, we have

$$\lim_{n\to\infty} \|J_{E_1}^p(x_n) - J_{E_1}^p(z_n^{(i)})\| = 0.$$

Since $J_{E_1^*}^q$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty} \|x_n - z_n^{(i)}\| = 0.$$

Since $d(x_n, T_i x_n) \leq \|x_n - z_n^{(i)}\|$, we have

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0,$$

for each $i = \{1, 2, ..., N\}$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to z. Since T_i is a multi-valued Bregman relative nonexpansive mapping, we obtain $z \in F(T_i)$, for each $i \in \{1, 2, ..., N\}$ and hence $z \in \cap_{i=1}^{N} F(T_i)$.

We now show that $z \in \Omega$. From (3.3), we obtain that

$$\Big(\frac{C_q(t_n\|A\|)^q}{q}\Big)\|Au_n - P_Q(Au_n)\|^p \leq \Delta_p(u_n, x^*) - \Delta_p(x_n, x^*). \tag{3.7}$$

From (3.4), we have

$$\Delta_p(u_{n+1}, x^*) \leq \Delta_p(x_n, x^*). \tag{3.8}$$

Putting (3.7) into (3.8), we have

$$\Big(\frac{C_q(t_n\|A\|)^q}{q}\Big)\|Au_n - P_Q(Au_n)\|^p \leq \Delta_p(x_{n-1}, x^*) - \Delta_p(x_n, x^*). \tag{3.9}$$

By condition (ii) and (3.9), we have

$$\begin{aligned} 0 &< t\Big(1 - \frac{C_q k^{q-1}\|A\|^q}{q}\Big)\|Au_n - P_Q(Au_n)\|^p \\ &\leq \Big(t_n - \frac{C_q(t_n\|A\|)^q}{q}\Big)\|Au_n - P_Q(Au_n)\|^p \\ &\leq \Delta_p(x_{n-1}, x^*) - \Delta_p(x_n, x^*). \end{aligned}$$

Hence, we obtain that

$$\lim_{n\to\infty} \|Au_n - P_Q(Au_n)\| = 0. \tag{3.10}$$

Since $v_n = J^q_{E_1^*}[J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(Au_n - P_Q(Au_n))]$, $\forall n \geq 1$, then we have

$$\begin{aligned} 0 \leq \|J^p_{E_1}(v_n) - J^p_{E_1}(u_n)\| &\leq t_n\|A^*\|\|J^p_{E_2}(Au_n - P_Q(Au_n))\| \\ &\leq \Big(\frac{q}{C_q\|A\|^q}\Big)^{q-1}\|A^*\|\|Au_n - P_Q(Au_n)\|^{p-1}. \end{aligned} \tag{3.11}$$

It follows that

$$\lim_{n\to\infty} \|J^p_{E_1}(v_n) - J^p_{E_1}(u_n)\| = 0.$$

Since $J^q_{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty} \|v_n - u_n\| = 0.$$

Furthermore,

$$\|J^q_{E_1^*}[J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(Au_n - P_Q(Au_n))] - u_n\| = \|v_n - u_n\| \to 0, n \to \infty.$$

Since J_{E_1} is norm-to-norm uniformly continuous on bounded subsets of E_1, then

$$\begin{aligned} t\|A^* J^p_{E_2}(Au_n - P_Q(Au_n))\| &\leq t_n\|A^* J^p_{E_2}(Au_n - P_Q(Au_n))\| \\ &= \|J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(Au_n - P_Q(Au_n)) - J^p_{E_1}(u_n)\|. \end{aligned}$$

Thus

$$\lim_{n\to\infty} \|A^* J^p_{E_2}(Au_n - P_Q(Au_n))\| = 0.$$

From (2.6) and (3.4), we obtain that

$$\begin{aligned}\Delta_p(v_n, x_n) &= \Delta_p(v_n, \Pi_c v_n)\\ &\leq \Delta_p(v_n, x^*) - \Delta_p(x_n, x^*)\\ &\leq \Delta_p(u_n, x^*) - \Delta_p(x_n, x^*)\\ &\leq \Delta_p(x_{n-1}, x^*) - \Delta_p(x_n, x^*).\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \|v_n - x_n\| = 0.$$

Hence

$$\|x_n - u_n\| = \|v_n - u_n\| + \|v_n - x_n\| \to 0 \text{ as } n \to \infty.$$

Since $\{x_n\}$ is bounded, there exists $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z \in \omega_\omega(x_n)$. Since $x_{n_j} \rightharpoonup z$ and $\lim_{n\to\infty} \|x_n - u_n\| = 0$, we obtain that $u_{n_j} \rightharpoonup z$. From (2.2), (2.5) and (2.3), we have

$$\begin{aligned}\Delta_p(z, \Pi_c z) &\leq \langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_c z), z - \Pi_c z\rangle\\ &= \langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_c z), z - u_{n_j}\rangle + \langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_c z), u_{n_j} - \Pi_c u_{n_j}\rangle\\ &+ \langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_c z), \Pi_c u_{n_j} - \Pi_c z\rangle\\ &\leq \langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_c z), z - u_{n_j}\rangle + \langle J^p_{E_1}(z) - J^p_{E_1}(\Pi_c z), u_{n_j} - \Pi_c u_{n_j}\rangle.\end{aligned}$$

As $j \to \infty$, we obtain that $\Delta_p(z, \Pi_c z) = 0$. Thus $z \in C$. Let us now fix $x \in C$. Then $Ax \in Q$ and

$$\begin{aligned}\|(I - P_Q)Au_{n_j}\|^p &= \langle J^p_{E_2}(Ax_n - P_Q(Au_{n_j})), Ax_n - P_Q(Au_{n_j})\rangle\\ &= \langle J^p_{E_2}(Ax_n - P_Q(Au_{n_j})), Ax_n - Ax\rangle + \langle J^p_{E_2}(Ax_n - P_Q(Au_{n_j})), Ax_n - P_Q(Au_{n_j})\rangle\\ &\leq \langle J^p_{E_2}(Ax_n - P_Q(Au_{n_j})), Au_{n_j} - Ax\rangle\\ &\leq M\|A^*(I - P_Q)Au_{n_j}\|^{p-1} \to 0, n \to \infty,\end{aligned}$$

where $M > 0$ is sufficiently large number. It then follows from (2.4) that

$$\begin{aligned}\|(I - P_Q)Az\|^p &= \langle J^p_{E_2}(Az - P_Q(Az)), Az - P_Q(Az)\rangle\\ &= \langle J^p_{E_2}(Az - P_Q(Az)), Az - Au_{n_j}\rangle + \langle J^p_{E_2}(Az - P_Q(Az)), Au_{n_j} - P_Q(Au_{n_j})\rangle\\ &+ \langle J^p_{E_2}(Az - P_Q(Az)), P_Q(Au_{n_j}) - P_Q(Az)\rangle\\ &\leq \langle J^p_{E_2}(Az - P_Q(Az)), Az - Au_{n_j}\rangle + \langle J^p_{E_2}(Az - P_Q(Az)), Au_{n_j} - P_Q(Au_{n_j})\rangle.\end{aligned}$$

Also, since $Au_{n_j} \rightharpoonup Az$, we have that

$$\lim_{n\to\infty} \|(I - P_Q)Az\| = 0.$$

Thus $Az \in Q$. This implies that $z \in \Omega$ and hence $z \in F(T) \cap \Omega$.Furthermore, we have

$$\Delta_p(x_n, y_n) \leq \alpha_n^{(0)} \Delta_p(x_n, x_n) + \sum_{i=1}^{N} \alpha_n^{(i)} \Delta_p(x_n, z_n^{(i)}). \tag{3.12}$$

Since $\|x_n - z_n^{(i)}\| \to 0$ as $n \to \infty$ and $\{z_n^i\}$ is a bounded sequence. By Lemma 2.7, we obtain that $\lim_{n\to\infty} \Delta_p(x_n, z_n^{(i)}) = 0$. From (3.12), it follows that $\|x_n - y_n\| \to 0, n \to \infty$.

Let $p \in F(T) \cap \Omega$. We next show that $\limsup_{n\to\infty}\langle J^p_{E_1}(x_n) - J^p_{E_1}(p), y_n - p\rangle \le 0$. To show the inequality $\limsup_{n\to\infty}\langle J^p_{E_1}(x_n) - J^p_{E_1}(p), y_n - p\rangle \le 0$, we choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle J^p_{E_1}(x_n) - J^p_{E_1}(p), x_n - p\rangle = \lim_{n\to\infty}\langle J^p_{E_1}(x_n) - J^p_{E_1}(p), x_{n_j} - p\rangle = 0.$$

Since $\|x_n - y_n\| \to 0$ as $n \to \infty$ and (2.5), we obtain that

$$\limsup_{n\to\infty}\langle J^p_{E_1}(x_n) - J^p_{E_1}(p), y_n - p\rangle \le \limsup_{n\to\infty}\langle J^p_{E_1}(x_n) - J^p_{E_1}(p), x_n - p\rangle = 0. \tag{3.13}$$

Using (3.13), (3.5) and Lemma 2.9, we obtain that $\Delta_p(x_n, p) \to 0, n \to \infty$. Hence, $x_n \to p$ as $n \to \infty$. □

Corollary 3.2. *Let E_1 and E_2 be two L_p spaces with $2 \le p < \infty$. Let C and Q be nonempty closed convex subsets of E_1 and E_2, respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that SFP has a nonempty solution set Ω. Let $\{T_i\}_{i=1}^N$ be a finite family of multi-valued Bregman relative nonexpansive mappings of C into $CB(C)$ such that $\mathcal{F} = \cap_{i=1}^N F(T_i) \cap \Omega \ne \emptyset$. Let $u_1 \in E_1$ and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_n = \Pi_c J^q_{E_1^*}[J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(Au_n - P_Q(Au_n))] \\ u_{n+1} = \Pi_c J^q_{E_1^*}(\alpha_n^{(0)} J^p_{E_1}(x_n) + \sum_{i=1}^N \alpha_n^{(i)} J^p_{E_1}(z_n^{(i)})) \ , z_n^{(i)} \in T_i x_n, \end{cases} \tag{3.14}$$

where $\{\alpha_n^{(i)}\} \subset [a.b] \subset (0,1)$ for all $i = 0,1,...,N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$. Suppose the following conditions are satisfied:

(i) $\Sigma_{n=1}^\infty \alpha_n^{(i)} = 0$ for all $i = 0,1,...,N$

(ii) $0 < t \le t_n \le k < \left(\frac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}$.

Then the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to an element $x^* \in \mathcal{F}$.

If we assume that each T_i, $i = 1,2,...,N$, in Theorem 3.1 is a Bregman relative nonexpansive single-valued mapping, we obtain the following corollary:

Corollary 3.3. *Let E_1 and E_2 be two p-uniformly convex real Banach spaces which are also uniformly smooth. Let C and Q be nonempty closed convex subsets of E_1 and E_2, respectively, $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be the adjoint of A. Suppose that SFP has a nonempty solution set Ω. Let $\{T_i\}_{i=1}^N$ be a finite family of single-valued Bregman relative nonexpansive mappings of C into C such that $\mathcal{F} = \cap_{i=1}^N F(T_i) \cap \Omega \ne \emptyset$. Let $u_1 \in E_1$ and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_n = \Pi_c J^q_{E_1^*}[J^p_{E_1}(u_n) - t_n A^* J^p_{E_2}(Au_n - P_Q(Au_n))] \\ u_{n+1} = \Pi_c J^q_{E_1^*}(\alpha_n^{(0)} J^p_{E_1}(x_n) + \sum_{i=1}^N \alpha_n^{(i)} J^p_{E_1}(T_i x_n)), \end{cases} \tag{3.15}$$

where $\{\alpha_n^{(i)}\} \subset [a.b] \subset (0,1)$ for all $i = 0,1,...,N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$. Suppose the following conditions are satisfied:

(i) $\Sigma_{n=1}^\infty \alpha_n^{(i)} = 0$ for all $i = 0,1,...,N$

(ii) $0 < t \le t_n \le k < \left(\frac{q}{C_q\|A\|^q}\right)^{\frac{1}{q-1}}$.

Then the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to an element $x^* \in \mathcal{F}$.

4 Numerical Example

In this section, we present the numerical example supporting our main result. All codes are written in Matlab2013b.

Example 4.1. Let $E_1 = L_2([0,1]) = E_2$ with the inner product given by

$$\langle f, g\rangle = \int_0^1 f(t)(g(t)dt.$$

Suppose that

$$C := \{x \in L_2([0,1]) : \langle x, a\rangle = b\},$$

where $a = 2t^2$ and $b = 0$. Therefore

$$P_C(x) = \max\Big\{0, \frac{b - \langle a, x\rangle}{\|a\|_2^2}\Big\}a + x.$$

Let

$$Q := \{x \in L_2([0,1]) : \langle x, c\rangle \geq d\},$$

where $c = \frac{t}{3}$ and $d = -2$. It follows that

$$P_Q(x) := \frac{d - \langle c, x\rangle}{\|c\|_2^2}c + x.$$

Define

$$A : L_2([0,1] \to L_2([0,1] \text{ by } (Ax)(t) = \frac{x(t)}{2}.$$

Then A is a bounded linear operator with $\|A\| = 2$ and $A^* = A$. Suppose that

$$T_1(f)\begin{cases} \{h \in C : f(x) - \frac{3}{4} \leq h(x) \leq f(x) - \frac{1}{3}, \forall x \in I\} \text{ if } f(x) > 1, \forall x \in I \\ \{0\}, \text{ otherwise,} \end{cases} \tag{4.1}$$

and

$$T_2(f)\begin{cases} \{g \in C : f(x) - \frac{1}{2} \leq g(x) \leq f(x) - \frac{1}{4}, \forall x \in I\} \text{ if } f(x) > 1, \forall x \in I \\ \{0\}, \text{ otherwise.} \end{cases} \tag{4.2}$$

In [21], we obtain that T_1 and T_2 are multi-valued Bregman relative nonexpansive mappings. Consider the problem:

$$\text{find } x \in F(T) \cap C \text{ such that } Ax \in Q. \tag{4.3}$$

We see that the set of solutions of problem (4.3) is nonempty, since $x = 0$ is in the set of solutions. Let $\alpha_n^{(0)} = \frac{1}{12n}$, $\alpha_n^{(1)} = \frac{12n-1}{36n}$, and $\alpha_n^{(2)} = \frac{12n-1}{18n}$ for all $n \geq 1$. Put $z_n^{(1)} = x_n - \frac{3}{4}$ and $z_n^{(2)} = x_n - \frac{1}{2}$. Using the iterative method (3.1), we obtain that

$$\begin{cases} x_n = \Pi_c[u_n - t_n A^*(Au_n - P_Q(Au_n))] \\ u_{n+1} = \Pi_c(\frac{1}{12n}(x_n) + \frac{12n-1}{36n}(x_n - \frac{3}{4}) + \frac{12n-1}{18n}(x_n - \frac{1}{2})), \ \ n \geq 1. \end{cases} \tag{4.4}$$

We make different choices of u_1 and t_n and take $\frac{\|x_{n+1} - x_n\|}{\|x_2 - x_1\|} < 10^{-6}$ as our stopping criterion.

Case 1 $t_n = 0.001$ and $u_1 = t$. We have the numerical analysis tabulated in Table 1 and show in Figure 1.

Table 1 Example 4.1: Case 1

No.of iteration	$\|x_{n+1} - x_n\|_2$	$\|u_{n+1} - u_n\|_2$
2	0.45960659	0.45871914
3	0.03706339	0.03979071
4	0.00089775	0.00150921
5	0.00002339	0.00002339
6	0.00000070	0.00000210
7	0.00000043	0.00000141
8	0.00000030	0.00000100
9	0.00000023	0.00000075

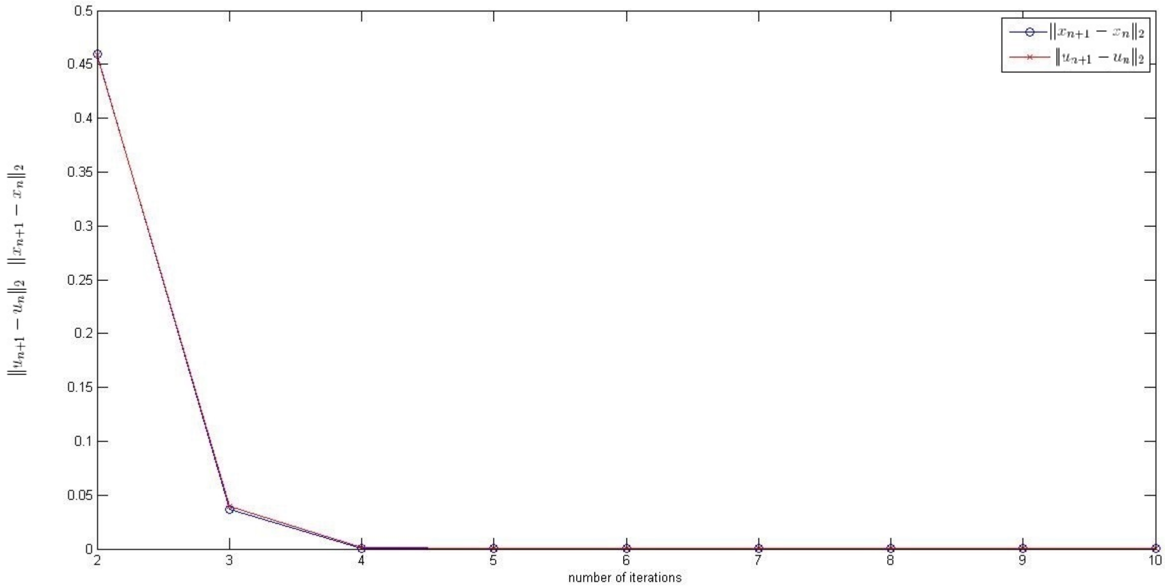
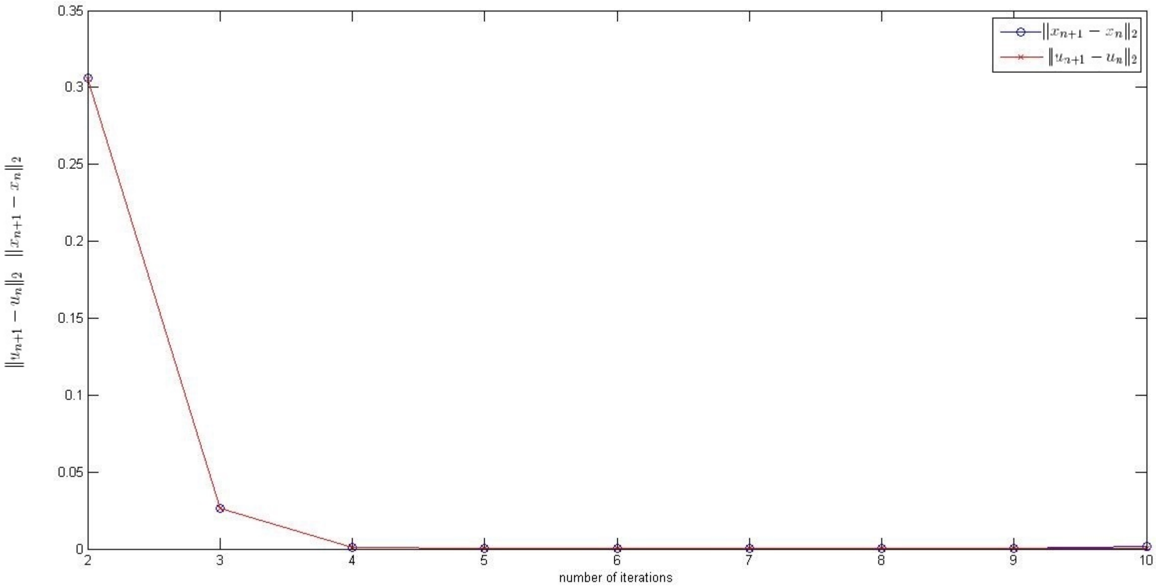


Figure 1. Example 4.1: Case 1.

Case 2 $t_n = 0.0002$ and $u_1 = t^2$. We have the numerical analysis tabulated in Table 2 and show in Figure 2.

Table 2 Example 4.1: Case 2

No.of iteration	$\|x_{n+1} - x_n\|_2$	$\|u_{n+1} - u_n\|_2$
2	0.30581518	0.30563219
3	0.02659138	0.02659287
4	0.0008344	0.00110931
5	0.00002409	0.00002388
6	0.00000053	0.00000064
7	0.00000009	0.00000028
8	0.00000006	0.00000020
9	0.00000005	0.00000015



4.jpg 4.bb

Figure 2. Example 4.1: Case 2.

References

[1] Y.I Alber, Metric and generalized projection operator in Banach spaces: properties and applications,in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type vol 178 of Lecture Notes in Pure and Applied Mathematics, pp, vol. 15-50. USA, Dekker, New York, NY (1996)

[2] H.H. Bauschke, P. L. Combettes, and D. R. Luke, A strongly convergent reflection method for finding the projection onto the intersection of two closed convex sets in a Hilbert space, J. Approx. Theory.(141) 63-69 (2006)

[3] L.M. Bregman, The relaxation method for finding the common point of convex sets and its application to the solution of problems in convex programming, USSR Comput. Math. Math. Phys. (7) 200-217 (1967)

[4] R.E. Bruck, S. Reich, Nonexpansive projections and resolvents of accretive operators in Banach spaces. Houst. J. Math. (3) 459–470 (1977)

[5] D. Butnariu and A.N. Iusem, Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization, vol. 40, Kluwer Academic Publishers, Dordrecht,The Netherlands, 2000.

[6] C. Byrne, Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Problems, 18 (2) 441- 453 (2002)

[7] Y. Censor, A. Lent, An iterative row-action method for interval convex programming, J. Optim. Theory Appl. (34) 321-353 (1981)

[8] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space. Numerical Algorithms 8(4), 221-239 (1994)

[9] C.T. Pang, E. Naraghirad and C. F. Wen, Weak convergence theorems for Bregman Relatively nonexpansive mappings in Banach spaces. J. Appl. Math. Volume 2014, Article ID 573075, 9 pages

[10] I. Cioranescu, Geometry of banach spaces, duality mappings and nonlinear problems. Kluwer Academic Dordrecht (1990)

[11] E. Masad and S. Reich, A note on the multiple-set split convex feasibility problem in Hilbert space. J. Nonlinear. Convex Anal. 8 (3) 367-371 (2007)

[12] E. Naraghirad, J.C. Yao, Bregman weak relatively nonexpansive mappings in Banach spaces. Fixed Point Theory Appl. (2013) doi:10.1186/1687-1812-2013-141.

[13] K. Nakajo, W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups. J. Math. Anal. Appl. 279 372-379 (2003)

[14] S.Reich,Book Review: Geometry of Banach spaces, duality mappings and nonlinear problems, Bull. Amer. Math. Soc. 26 367-370 (1992)

[15] S. Reich, A weak convergence theorem for the alternating method with Bergman distance, In: Kartsatos, AG (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type. Lect. Notes Pure Appl. Math. 178 (313–318. Dekker, New York (1996)

[16] S. Reich, A weak convergence theorem for the alternating method with Bregman distances, in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type. 17 313-318, Marcel Dekker, New York, NY, USA, 1996.

[17] S. Reich, S. Sabach, Two strong convergence theorems for Bregman strongly nonexpansive operators in reflexive Banach spaces, Nonlinear Anal. 73 122–135 (2010)

[18] V. Martín-Márquez, S. Reich, S. Sabach, Bregman strongly nonexpansive operators in reflexive Banach spaces. J. Math. Anal. Appl. 400 597-614 (2013)

[19] F. Schöpfer, Iterative regularization method for the solution of the split feasibility problem in Banach spaces, PhD thesis, Saarbrucken (2007)

[20] F. Schpfer, T. Schuster, A.K. Louis, An iterative regularization method for the solution of the split feasibility problem in Banach spaces, Inverse Probl. 2008;24:055008.

[21] N. Shahzad, H. Zegeye, Convergence theorem for common fixed points of a finite family of multi-valued Bregman relatively nonexpansive mappings. Fixed Point Theory Appl. 2014, 2014:152

[22] Y. Shehu, O. S. Iyiola, C. D. Enyi, An iterative algorithm for solving split feasibility problems and fixed point problems in Banach spaces.Numer. Algor. DOI 10.1007/s11075-015-0069-4.

[23] Y. Shehu, Strong convergence theorem for Multiple Sets Split Feasibility Problems in Banach Spaces, Under review: Numerical Functional Analysis and Optimization

[24] Y. Shehu, A cyclic iterative method for solving Multiple Sets Split Feasibility Problems in Banach Spaces, Under review: Quaestiones Mathematicae

[25] F. Wang and H.K. Xu, Approximating curve and strong convergence of the CQ algorithm for the split feasibility problem, Journal of Inequalities and Applications, vol. 2010, Article ID102085, 13 pages, 2010.

[26] F. Wang, A new algorithm for solving the multiple-sets split feasibility problem in Banach spaces," Numer. Funct. Anal. Optim. 35 99-110 (2014)

[27] H.K. Xu, Inequalities in Banach spaces with applications," Nonlinear Anal. 16(2), 1127-1138 (1991)

[28] H.K. Xu, A variable Krasnoselskii-Mann algorithm and the multiple-set split feasibility problem, Inverse Problems. 22 (6) 2021-2034 (2006)

[29] J. Zhao and Q. Yang, Several solution methods for the split feasibility problem, Inverse Problems. 21 (5) 1791- 1799 (2005)

Expressions and Dynamical Behavior of Rational Recursive Sequences

E. M. Elsayed[1,2] and Marwa M. Alzubaidi[1,3]
[1]King Abdulaziz University, Faculty of Science,
Mathematics Department, P.O. Box 80203,
Jeddah 21589,Saudi Arabia.
[2]Department of Mathematics, Faculty of Science,
Mansoura University,Mansoura35516, Egypt.
[3]Mathematics Department , The University College of Duba,
University of Tabuk, Tabuk, Saudi Arabia.
E-mail: emmelsayed@yahoo.com, mmialzubaidi@hotmail.com.

ABSTRACT

In this paper, we study the qualitative behavior of the rational recursive sequences

$$x_{n+1} = \frac{x_{n-11}}{\pm 1 \pm x_{n-2}x_{n-5}x_{n-8}x_{n-11}}, \quad n = 0, 1, 2, ...,$$

where the initial conditions are arbitrary real numbers. Also, we give the numerical examples of some cases of difference equations and obtained some related graphs and figures using by Matlab.

Keywords: Difference Equation, Recursive sequence, Local stability, Periodicity.

Mathematics Subject Classification: 39A10.

1. INTRODUCTION

Difference equations and dynamic equations on time scales have an immense possibility for applications in engineering, physics, biology, economics, etc. Lately, considerable attentiveness has been devoted to the oscillation theory of the various classes of equations,see e.g. [1]-[42] and the references cited therein.

In this study, we are interested with the behavior of the solution of difference equations

$$x_{n+1} = \frac{x_{n-11}}{\pm 1 \pm x_{n-2}x_{n-5}x_{n-8}x_{n-11}}, \quad n = 0, 1, 2, ..., \tag{1}$$

where the initial conditions are arbitrary real numbers. For some outcome in this study for examples: Cinar [8-10] obtained the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}}, \quad x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

Cinar et al. [11] gave the form of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{-1 + x_n x_{n-1} x_{n-2} x_{n-3}}.$$

Elabbasy et al. [13] solved the following problem

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

In [14] Elsayed studied the difference equation

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-2}x_{n-5}}.$$

Elsayed [21-22] obtained the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-7}}{\pm 1 \pm x_{n-1}x_{n-3}x_{n-5}x_{n-7}}, \quad x_{n+1} = \frac{x_{n-9}}{\pm 1 \pm x_{n-4}x_{n-9}}.$$

Elsayed [23] investigated the Solution of difference equations

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-1}x_{n-3}}.$$

Elsayed and Iricanin [24] has got the solution of the difference equation

$$x_{n+1} = \max\left\{A_n/x_n, x_{n-1}\right\}.$$

Ibrahim [26] studied the third order rational difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}\left(a + b x_n x_{n-2}\right)}.$$

In [30] Kent et al studied the Behavior of solutions of the difference equation

$$x_{n+1} = x_n x_{n-2} - 1.$$

Let I be some interval of real numbers and let $F : I^{k+1} \to I$, be a continuously differentiable function. Then for every set of initial condition $x_{-k},\ x_{-k+1},\ ...,\ x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, x_{n-2}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if $\overline{x} = F(\overline{x})$, that is,

$$x_n = \overline{x} \ for\ all \quad n \geq -k.$$

is a solution of Eq.(2), or equivalently, $\overline{x}$ is **a fixed point** of F.

Definition 2. **(Periodicity)** A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Linearized Stability Analysis

Suppose that the function F is continuously differentiable in some open neighborhood of an equilibrium point x^*. Let

$$p_i = \frac{\partial F}{\partial u_i}\left(\overline{x}, \overline{x}, ..., \overline{x}\right) \quad for \quad i = 0, 1, ..., k,$$

denote the partial derivatives of $F(u_0, u_1, ...u_k)$ evaluated at the equilibrium $\overline{x}$ of Eq.(2).

Then the equation

$$y_{n+1} = p_0 y_n + p_1 y_{n-1} + ... + p_k y_{n-k}\ , \quad n = 0, 1, ..., \tag{3}$$

is called **the linearized equation associated** of Eq.(2) about the equilibrium point $\overline{x}$ and the equation

$$\lambda^{k+1} - p_0\lambda^k - ... - p_{k-1}\lambda - p_k = 0, \tag{4}$$

is called the characteristic equation of Eq.(3) about $\overline{x}$.

The following result known as the Linear Stability Theorem is very useful in determining the local stability character of the equilibrium point $\overline{x}$ of Eq.(2).

Definition 3. The equilibrium point $\overline{x}$ is said to be **hyperbolic** if $|F(\overline{x})| \neq 1$.

If $|F(\overline{x})| = 1$, $\overline{x}$ is **non hyperbolic.**

Theorem A. [31] Assume that $p_0, p_2, ..., p_k$ are real numbers such that

$$|p_0| + |p_1| + ... + |p_k| < 1, \qquad \text{or} \quad \sum_{i=1}^{k} |p_i| < 1.$$

Then all roots of Eq.(4) lie inside the unit disk.

2. THE FIRST EQUATION $X_{N+1} = \frac{X_{N-11}}{1+X_{N-2}X_{N-5}X_{N-8}X_{N-11}}$

In this part, we obtain the following special case of Eq.(1) in the form:

$$x_{n+1} = \frac{x_{n-11}}{1 + x_{n-2}x_{n-5}x_{n-8}x_{n-11}}, \tag{5}$$

where the initial values are arbitrary real numbers.

Theorem 2.1. Let $\{x_n\}_{n=-11}^{\infty}$ be a solution of difference equation (5). Then for $n = 0, 1, ...$

$$\begin{aligned}
x_{12n-11} &= p\prod_{i=0}^{n-1}\frac{1+4ipkfc}{1+(4i+1)\,pkfc}, \; x_{12n-10} = m\prod_{i=0}^{n-1}\frac{1+4imheb}{1+(4i+1)\,mheb}, \; x_{12n-9} = l\prod_{i=0}^{n-1}\frac{1+4ildga}{1+(4i+1)\,ldga}, \\
x_{12n-8} &= k\prod_{i=0}^{n-1}\frac{1+(4i+1)\,pkfc}{1+(4i+2)\,pkfc}, \; x_{12n-7} = h\prod_{i=0}^{n-1}\frac{1+(4i+1)\,mheb}{1+(4i+2)\,mheb}, \; x_{12n-6} = g\prod_{i=0}^{n-1}\frac{1+(4i+1)\,ldga}{1+(4i+2)\,ldga}, \\
x_{12n-5} &= f\prod_{i=0}^{n-1}\frac{1+(4i+2)\,pkfc}{1+(4i+3)\,pkfc}, \; x_{12n-4} = e\prod_{i=0}^{n-1}\frac{1+(4i+2)\,mheb}{1+(4i+3)\,mheb}, \; x_{12n-3} = d\prod_{i=0}^{n-1}\frac{1+(4i+2)\,ldga}{1+(4i+3)\,ldga}, \\
x_{12n-2} &= c\prod_{i=0}^{n-1}\frac{1+(4i+3)\,pkfc}{1+(4i+4)\,pkfc}, \; x_{12n-1} = b\prod_{i=0}^{n-1}\frac{1+(4i+3)\,mheb}{1+(4i+4)\,mheb}, \; x_{12n} = a\prod_{i=0}^{n-1}\frac{1+(4i+3)\,ldga}{1+(4i+4)\,ldga},
\end{aligned}$$

where $x_{-11} = p,\ x_{-10} = m,\ x_{-9} = l,\ x_{-8} = k,\ x_{-7} = h,\ x_{-6} = g,\ x_5 = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,$ $x_{-1} = b,\ x_0 = a$ and $\prod_{i=0}^{-1}\alpha_i = 1.$

Proof. For $n = 0$, the result holds. Now, assume that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$\begin{aligned}
x_{12n-23} &= p\prod_{i=0}^{n-2}\frac{1+4ipkfc}{1+(4i+1)\,pkfc}, \; x_{12n-22} = m\prod_{i=0}^{n-2}\frac{1+4imheb}{1+(4i+1)\,mheb}, \; x_{12n-21} = l\prod_{i=0}^{n-2}\frac{1+4ildga}{1+(4i+1)\,ldga}, \\
x_{12n-20} &= k\prod_{i=0}^{n-2}\frac{1+(4i+1)\,pkfc}{1+(4i+2)\,pkfc}, \; x_{12n-19} = h\prod_{i=0}^{n-2}\frac{1+(4i+1)\,mheb}{1+(4i+2)\,mheb}, \; x_{12n-18} = g\prod_{i=0}^{n-2}\frac{1+(4i+1)\,ldga}{1+(4i+2)\,ldga}, \\
x_{12n-17} &= f\prod_{i=0}^{n-2}\frac{1+(4i+2)\,pkfc}{1+(4i+3)\,pkfc}, \; x_{12n-16} = e\prod_{i=0}^{n-2}\frac{1+(4i+2)\,mheb}{1+(4i+3)\,mheb}, \; x_{12n-15} = d\prod_{i=0}^{n-2}\frac{1+(4i+2)\,ldga}{1+(4i+3)\,ldga},
\end{aligned}$$

$$x_{12n-14} = c\prod_{i=0}^{n-2}\frac{1+(4i+3)\,pkfc}{1+(4i+4)\,pkfc},\ x_{12n-13} = b\prod_{i=0}^{n-2}\frac{1+(4i+3)\,mheb}{1+(4i+4)\,mheb},\ x_{12n-12} = a\prod_{i=0}^{n-2}\frac{1+(4i+3)\,ldga}{1+(4i+4)\,ldga}.$$

Now, it follows from Eq. (5) that

$$\begin{aligned}
x_{12n-11} &= \frac{x_{12n-23}}{1+x_{12n-14}x_{12n-17}x_{12n-20}x_{12n-23}} \\
&= \frac{p\prod_{i=0}^{n-2}\frac{1+4ipkfc}{1+(4i+1)pkfc}}{1+c\prod_{i=0}^{n-2}\frac{1+(4i+3)pkfc}{1+(4i+4)pkfc}f\prod_{i=0}^{n-2}\frac{1+(4i+2)pkfc}{1+(4i+3)pkfc}k\prod_{i=0}^{n-2}\frac{1+(4i+1)pkfc}{1+(4i+2)pkfc}p\prod_{i=0}^{n-2}\frac{1+4ipkfc}{1+(4i+1)pkfc}} \\
&= \frac{p\prod_{i=0}^{n-2}\frac{1+4ipkfc}{1+(4i+1)pkfc}}{1+pkfc\prod_{i=0}^{n-2}\frac{1+4ipkfc}{1+(4i+4)pkfc}} = p\prod_{i=0}^{n-2}\frac{1+4ipkfc}{1+(4i+1)pkfc}\left(\frac{1}{1+\frac{pkfc}{1+(4n-4)pkfc}}\right) \\
&= p\prod_{i=0}^{n-2}\frac{1+4ipkfc}{1+(4i+1)\,pkfc}\left(\frac{1+(4n-4)\,pkfc}{1+(4n-3)\,pkfc}\right)
\end{aligned}$$

Therefore, we have

$$x_{12n-11} = p\prod_{i=0}^{n-1}\frac{1+4ipkfc}{1+(4i+1)\,pkfc}.$$

Similarly

$$\begin{aligned}
x_{12n-7} &= \frac{x_{12n-19}}{1+x_{12n-10}x_{12n-13}x_{12n-16}x_{12n-19}} \\
&= \frac{h\prod_{i=0}^{n-2}\frac{1+(4i+1)mheb}{1+(4i+2)mheb}}{1+m\prod_{i=0}^{n-1}\frac{1+4imheb}{1+(4i+1)mheb}b\prod_{i=0}^{n-2}\frac{1+(4i+3)mheb}{1+(4i+4)mheb}e\prod_{i=0}^{n-2}\frac{1+(4i+2)mheb}{1+(4i+3)mheb}h\prod_{i=0}^{n-2}\frac{1+(4i+1)mheb}{1+(4i+2)mheb}} \\
&= \frac{h\prod_{i=0}^{n-2}\frac{1+(4i+1)mheb}{1+(4i+2)mheb}}{1+mheb\prod_{i=0}^{n-1}\frac{1+4imheb}{1+(4i+1)mheb}\prod_{i=0}^{n-2}\frac{1+(4i+1)mheb}{1+(4i+4)mheb}} \\
&= h\prod_{i=0}^{n-2}\frac{1+(4i+1)\,mheb}{1+(4i+2)\,mheb}\left(\frac{1}{1+\frac{mheb}{1+(4n-3)mheb}}\right) \\
&= h\prod_{i=0}^{n-2}\frac{1+(4i+1)\,mheb}{1+(4i+2)\,mheb}\left(\frac{1+(4n-3)\,mheb}{1+(4n-2)\,mheb}\right).
\end{aligned}$$

Hence, we have

$$x_{12n-7} = h\prod_{i=0}^{n-1}\frac{1+(4i+1)\,mheb}{1+(4i+2)\,mheb}.$$

Similarly, other relations can be obtained and thus, the proof has been proved.

Theorem 2.2. Eq.(5) has unique equilibrium point which is the number zero and this equilibrium is not locally asymptotically stable. Also, $\overline{x}$ is non hyperbolic.

Proof. For the equilibrium points of Eq.(5), we can write

$$\overline{x} = \frac{\overline{x}}{1+\overline{x}^4},$$

Then

$$\overline{x} + \overline{x}^5 = \overline{x},$$

or $\overline{x}^5 = 0$. Then the unique equilibrium point of Eq.(5) is $\overline{x} = 0$.

Let $f : (0,\infty)^4 \to (0,\infty)$ be a function defined by

$$F(u,v,w,t) = \frac{u}{1+uvwt}.$$

Then it follows that,

$$\begin{aligned} F_u(u,v,w,t) &= \frac{1}{(1+uvwt)^2},\ F_v(u,v,w,t) = \frac{-u^2wt}{(1+uvwt)^2}, \\ F_w(u,v,w,t) &= \frac{-u^2vt}{(1+uvwt)^2}, F_t(u,v,w,t) = \frac{-u^2vw}{(1+uvwt)^2}, \end{aligned}$$

we see that

$$F_u(\overline{x},\overline{x},\overline{x},\overline{x}) = 1,\ F_v(\overline{x},\overline{x},\overline{x},\overline{x}) = 0,\ F_w(\overline{x},\overline{x},\overline{x},\overline{x}) = 0,\ F_t(\overline{x},\overline{x},\overline{x},\overline{x}) = 0.$$

The proof follows by using Theorem A. By Definition 3, $\overline{x}$ is non hyperbolic.

Theorem 2.3. Every positive solution of Eq.(5) is bounded and $\lim\limits_{n\to\infty} x_n = 0$.

Proof. It is following by Eq.(5) that

$$x_{n+1} = \frac{x_{n-11}}{1+x_{n-2}x_{n-5}x_{n-8}x_{n-11}} \le x_{n-11}.$$

Then

$$x_{n+1} < x_{n-11}, \quad for\ all \quad n \ge 0$$

Then the subsequences $\{x_{12n-11}\}_{n=0}^{\infty}$, $\{x_{12n-10}\}_{n=0}^{\infty}$, $\{x_{12n-9}\}_{n=0}^{\infty}$, ..., $\{x_{12n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by

$$M = \max\{x_{-11}, x_{-10}, x_{-9}, x_{-8}, x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0\}.$$

3. THE SECOND EQUATION $X_{N+1} = \frac{X_{N-11}}{-1+X_{N-2}X_{N-5}X_{N-8}X_{N-11}}$

In this part, we give the solution of the recursive equation in the form:

$$x_{n+1} = \frac{x_{n-11}}{-1+x_{n-2}x_{n-5}x_{n-8}x_{n-11}}, \tag{6}$$

where the initial values are arbitrary real numbers with $x_{-2}x_{-5}x_{-8}x_{-11} \neq 1$, $x_{-1}x_{-4}x_{-7}x_{-10} \neq 1$, $x_0x_{-3}x_{-6}x_{-9} \neq 1$.

Theorem 3.1. Let $\{x_n\}_{n=-11}^{\infty}$ be a solution of difference equation (6). Then for $n = 0, 1, ...$

$$\begin{aligned}
x_{12n-11} &= \frac{p}{(-1+pkfc)^n},\ x_{12n-10} = \frac{m}{(-1+mheb)^n},\ x_{12n-9} = \frac{l}{(-1+ldga)^n}, \\
x_{12n-8} &= k\left(-1+pkfc\right)^n,\ x_{12n-7} = h\left(-1+mheb\right)^n,\ x_{12n-6} = g\left(-1+ldga\right)^n, \\
x_{12n-5} &= \frac{f}{(-1+pkfc)^n},\ x_{12n-4} = \frac{e}{(-1+mheb)^n},\ x_{12n-3} = \frac{d}{(-1+ldga)^n}, \\
x_{12n-2} &= c\left(-1+pkfc\right)^n,\ x_{12n-1} = b\left(-1+mheb\right)^n,\ x_{12n} = a\left(-1+ldga\right)^n,
\end{aligned}$$

where $x_{-11} = p,\ x_{-10} = m,\ x_{-9} = l,\ x_{-8} = k,\ x_{-7} = h,\ x_{-6} = g,\ x_5 = f,\ x_{-4} = e,\ x_{-3} = d,\ x_{-2} = c,$ $x_{-1} = b,\ x_0 = a.$

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. That is

$$\begin{aligned}
x_{12n-23} &= \frac{p}{(-1+pkfc)^{n-1}},\ x_{12n-22} = \frac{m}{(-1+mheb)^{n-1}},\ x_{12n-21} = \frac{l}{(-1+ldga)^{n-1}}, \\
x_{12n-20} &= k\left(-1+pkfc\right)^{n-1},\ x_{12n-19} = h\left(-1+mheb\right)^{n-1},\ x_{12n-18} = g\left(-1+ldga\right)^{n-1}, \\
x_{12n-17} &= \frac{f}{(-1+pkfc)^{n-1}},\ x_{12n-116} = \frac{e}{(-1+mheb)^{n-1}},\ x_{12n-15} = \frac{d}{(-1+ldga)^{n-1}}, \\
x_{12n-14} &= c\left(-1+pkfc\right)^{n-1},\ x_{12n-13} = b\left(-1+mheb\right)^{n-1},\ x_{12n-12} = a\left(-1+ldga\right)^{n-1}.
\end{aligned}$$

Now, it follows from Eq.(6) that

$$\begin{aligned}
x_{12n-11} &= \frac{x_{12n-23}}{-1+x_{12n-14}x_{12n-17}x_{12n-20}x_{12n-23}} \\
&= \frac{\frac{p}{(-1+pkfc)^{n-1}}}{-1+c\left(-1+pkfc\right)^{n-1}\frac{f}{(-1+pkfc)^{n-1}}k\left(-1+pkfc\right)^{n-1}\frac{p}{(-1+pkfc)^{n-1}}} \\
&= \frac{p}{(-1+pkfc)^{n-1}\left(-1+pkfc\right)}.
\end{aligned}$$

Then

$$x_{12n-11} = \frac{p}{(-1+pkfc)^n}.$$

Similarly

$$\begin{aligned}
x_{12n-6} &= \frac{x_{12n-18}}{-1+x_{12n-9}x_{12n-12}x_{12n-15}x_{12n-18}} \\
&= \frac{g\left(-1+ldga\right)^{n-1}}{-1+\frac{l}{(-1+ldga)^n}a\left(-1+ldga\right)^{n-1}\frac{d}{(-1+ldga)^{n-1}}g\left(-1+ldga\right)^{n-1}} \\
&= \frac{g\left(-1+ldga\right)^{n-1}}{-1+ldga\left(-1+ldga\right)^{-1}}.
\end{aligned}$$

Therefore, we have

$$x_{12n-6} = g\left(-1+ldga\right)^n.$$

The same other relations can be proved and thus, the proof has been completed.

Theorem 3.2. Eq.(6) has three equilibrium points which are $0, \pm\sqrt[4]{2}$ and these equilibrium points are not locally asymptotically stable.

Proof. The proof is the same as Theorem 2.2.

Theorem 3.3. Eq.(6) has a periodic solutions of period twelve iff $pkfc = mheb = ldga = 2$ and will be take the form

$$\{p, m, l, k, h, g, f, e, d, c, b, a, p, m, l, k, h, g, f, e, d, c, b, a, ...\}.$$

Proof. Assume that there exists a prime twelve solutions

$$p, m, l, k, h, g, f, e, d, c, b, a, p, m, l, k, h, g, f, e, d, c, b, a, ...,$$

of Eq.(6) ,we have from Eq.(6) that

$$\begin{aligned} p &= \frac{p}{(-1+pkfc)^n},\ m = \frac{m}{(-1+mheb)^n},\ l = \frac{l}{(-1+ldga)^n}, \\ k &= k(-1+pkfc)^n,\ h = h(-1+mheb)^n,\ g = g(-1+ldga)^n, \\ f &= \frac{f}{(-1+pkfc)^n},\ e = \frac{e}{(-1+mheb)^n},\ d = \frac{d}{(-1+ldga)^n}, \\ c &= c(-1+pkfc)^n,\ b = b(-1+mheb)^n,\ a = a(-1+ldga)^n, \end{aligned}$$

or

$$(-1+pkfc)^n = 1, \qquad (-1+mheb)^n = 1, \qquad (-1+ldga)^n = 1$$

Then

$$pkfc = mheb = ldga = 2.$$

Second let $pkfc = mheb = ldga = 2$. Then we have from Eq.(6) that

$$\begin{aligned} x_{12n-11} &= p,\ x_{12n-10} = m,\ x_{12n-9} = l,\ x_{12n-8} = k, \\ x_{12n-7} &= h,\ x_{12n-6} = g,\ x_{12n-5} = f,\ x_{12n-4} = e, \\ x_{12n-3} &= d,\ x_{12n-2} = c,\ x_{12n-1} = b, x_{12n} = a. \end{aligned}$$

Therefore we have a period twelve solutions and the proof is complete.

4. THE THIRD EQUATION $X_{N+1} = \frac{X_{N-11}}{1-X_{N-2}X_{N-5}X_{N-8}X_{N-11}}$

In this section we examine the following equation

$$x_{n+1} = \frac{x_{n-11}}{1 - x_{n-2}x_{n-5}x_{n-8}x_{n-11}}, \tag{7}$$

where the initial conditions are arbitrary positive real numbers.

Theorem 4.1. Let $\{x_n\}_{n=-11}^{\infty}$ be a solution of difference equation (7). Then for $n = 0, 1, ...$

$$\begin{aligned} x_{12n-11} &= p\prod_{i=0}^{n-1}\frac{1-4ipkfc}{1-(4i+1)pkfc},\ x_{12n-10} = m\prod_{i=0}^{n-1}\frac{1-4imheb}{1-(4i+1)mheb},\ x_{12n-9} = l\prod_{i=0}^{n-1}\frac{1-4ildga}{1-(4i+1)ldga}, \\ x_{12n-8} &= k\prod_{i=0}^{n-1}\frac{1-(4i+1)pkfc}{1-(4i+2)pkfc},\ x_{12n-7} = h\prod_{i=0}^{n-1}\frac{1-(4i+1)mheb}{1-(4i+2)mheb},\ x_{12n-6} = g\prod_{i=0}^{n-1}\frac{1-(4i+1)ldga}{1-(4i+2)ldga}, \\ x_{12n-5} &= f\prod_{i=0}^{n-1}\frac{1-(4i+2)pkfc}{1-(4i+3)pkfc},\ x_{12n-4} = e\prod_{i=0}^{n-1}\frac{1-(4i+2)mheb}{1-(4i+3)mheb},\ x_{12n-3} = d\prod_{i=0}^{n-1}\frac{1-(4i+2)ldga}{1-(4i+3)ldga}, \\ x_{12n-2} &= c\prod_{i=0}^{n-1}\frac{1-(4i+3)pkfc}{1-(4i+4)pkfc},\ x_{12n-1} = b\prod_{i=0}^{n-1}\frac{1-(4i+3)mheb}{1-(4i+4)mheb},\ x_{12n} = a\prod_{i=0}^{n-1}\frac{1-(4i+3)ldga}{1-(4i+4)ldga}, \end{aligned}$$

where $x_{-11} = p, x_{-10} = m, x_{-9} = l$, $x_{-8} = k$, $x_{-7} = h$, $x_{-6} = g$, $x_5 = f, x_{-4} = e$, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b, x_0 = a$ and $\delta pkfc \neq 1, \delta mheb \neq 1, \delta ldga \neq 1$ for $\delta = 1, 2, 3, ...$

Proof. The proof is similar as the proof of the Theorem 2.1.

Theorem 4.2. Eq.(7) has unique equilibrium point which is the number zero and this equilibrium is not locally asymptotically stable.

5. THE FOURTH EQUATION $X_{N+1} = \frac{X_{N-11}}{-1-X_{N-2}X_{N-5}X_{N-8}X_{N-11}}$

Here we obtain a form of the solutions of the equation

$$x_{n+1} = \frac{x_{n-11}}{-1 - x_{n-2}x_{n-5}x_{n-8}x_{n-11}} \tag{8}$$

where the initial values are arbitrary non zero real numbers with $x_{-2}x_{-5}x_{-8}x_{-11} \neq -1$, $x_{-1}x_{-4}x_{-7}x_{-10} \neq -1$, $x_0x_{-3}x_{-6}x_{-9} \neq -1$.

Theorem 5.1. Suppose $\{x_n\}_{n=-11}^{\infty}$ be a solution of difference equation $x_{n+1} = \frac{x_{n-11}}{-1-x_{n-2}x_{n-5}x_{n-8}x_{n-11}}$,Then for $n = 0, 1, ...$

$$\begin{aligned}
x_{12n-11} &= \frac{p}{(-1 - pkfc)^n},\ x_{12n-10} = \frac{m}{(-1 - mheb)^n},\ x_{12n-9} = \frac{l}{(-1 - ldga)^n}, \\
x_{12n-8} &= k(-1 - pkfc)^n,\ x_{12n-7} = h(-1 - mheb)^n,\ x_{12n-6} = g(-1 - ldga)^n, \\
x_{12n-5} &= \frac{f}{(-1 - pkfc)^n},\ x_{12n-4} = \frac{e}{(-1 - mheb)^n},\ x_{12n-3} = \frac{d}{(-1 - ldga)^n}, \\
x_{12n-2} &= c(-1 - pkfc)^n,\ x_{12n-1} = b(-1 - mheb)^n,\ x_{12n} = a(-1 - ldga)^n,
\end{aligned}$$

where $x_{-11} = p, x_{-10} = m, x_{-9} = l$, $x_{-8} = k$, $x_{-7} = h$, $x_{-6} = g$, $x_5 = f, x_{-4} = e$, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, and $x_0 = a$.

Theorem 5.2 Eq.(8) has three equilibrium points which are $0, \pm\sqrt[4]{-2}$ and these equilibrium points are not locally asymptotically stable.

Proof. The proof as the proof of Theorem 3.3.

Theorem 5.3. Eq.(8) has a periodic solutions of period twelve iff $pkfc = mheb = ldga = -2$ and will be take the form

$$\{p, m, l, k, h, g, f, e, d, c, b, a, p, m, l, k, h, g, f, e, d, c, b, a, ...\}.$$

6. NUMERICAL EXAMPLES

To verify the results of this paper, we consider some numerical examples as follows.

Example 6.1 The graph of the difference equation (5) and the case when $x_{-11} = 3.3$, $x_{-10} = 1.7$, $x_{-9} = 2.6$, $x_{-8} = 5$, $x_{-7} = 3$, $x_{-6} = 11$, $x_5 = 6$, $x_{-4} = 2$, $x_{-3} = 7$, $x_{-2} = 9$, $x_{-1} = 4.6$ and $x_0 = 1.6$.shown in Figure 1.

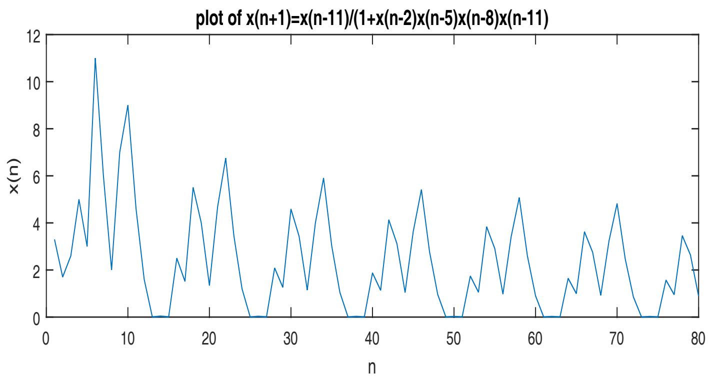


Figure 1.

Example 6.2. In Figure 2, we show that for Eq.(5) that $x_{-11} = 4.1,\ x_{-10} = 2,\ x_{-9} = 3.2,\ x_{-8} = 6,\ x_{-7} = -1,$ $x_{-6} = 2.4,\ x_5 = 1,\ x_{-4} = 4.2,\ x_{-3} = 7,\ x_{-2} = 11,\ x_{-1} = 4$ and $x_0 = -2$.

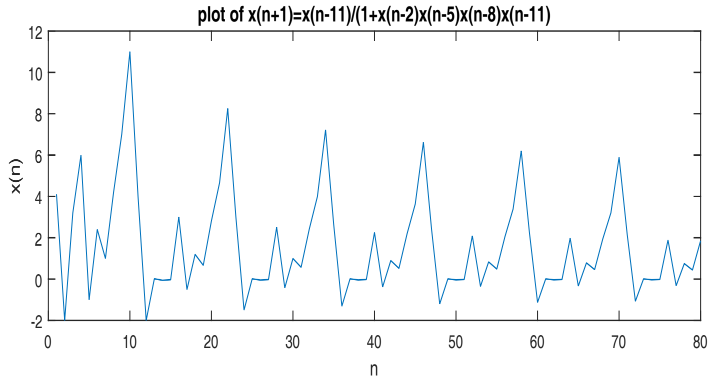


Figure 2.

Example 6.3. The graph is shown of the solutions of Eq.(6) where $x_{-11} = 3,\ x_{-10} = -2,\ x_{-9} = 9,\ x_{-8} = -5,$ $x_{-7} = 8,\ x_{-6} = 2,\ x_5 = 4,\ x_{-4} = 4,\ x_{-3} = -4,\ x_{-2} = -1/30,\ x_{-1} = -1/32$ and $x_0 = -1/36$.in Figure 3.

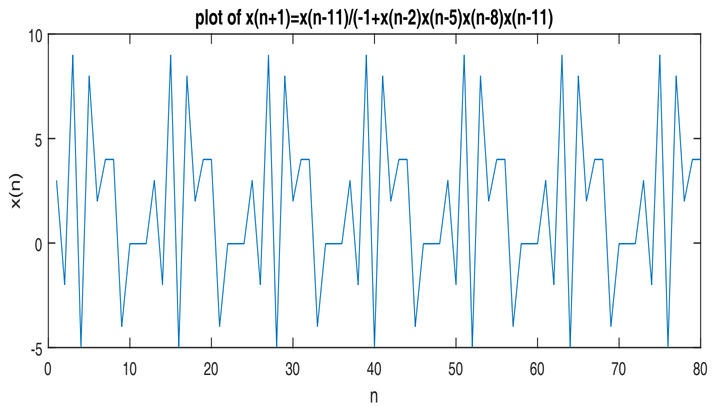


Figure 3.

Example 6.4. Figure 4 shows the behavior of difference equation.(6) when we choose $x_{-11} = 5,\ x_{-10} = -2,\ x_{-9} = 6,\ x_{-8} = -1,\ x_{-7} = 4,\ x_{-6} = -11,\ x_5 = 6,\ x_{-4} = 2,\ x_{-3} = 7,\ x_{-2} = -1/15,\ x_{-1} = -1/8$ and $x_0 = -1/231$.

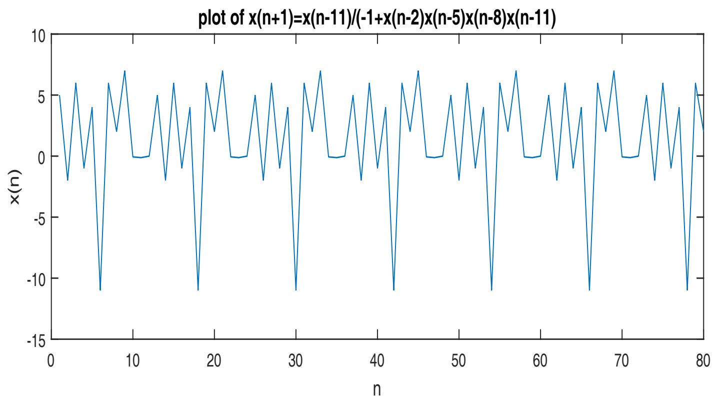


Figure 4.

Example 6.5. The diagram of the difference equation defined by $x_{n+1} = \frac{x_{n-11}}{1-x_{n-2}x_{n-5}x_{n-8}x_{n-11}}$ shows the period thirty six solutions since $x_{-11} = 1,\ x_{-10} = 3.5,\ x_{-9} = -4,\ x_{-8} = 6,\ x_{-7} = -2,\ x_{-6} = 2.4,\ x_5 = -1,\ x_{-4} = 1.2,$ $x_{-3} = 8,\ x_{-2} = 10,\ x_{-1} = -3$ and $x_0 = 4$. in Figure 5

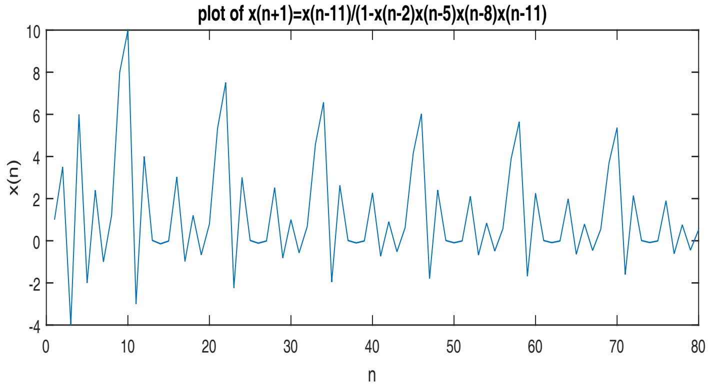


Figure 5.

Example 6.6. See Figure 6, we suppose for Eq.(7), that $x_{-11} = 4.3,\ x_{-10} = 8.1,\ x_{-9} = -3,\ x_{-8} = 2.7,$ $x_{-7} = -1,\ x_{-6} = 2.4,\ x_5 = 3,\ x_{-4} = 1.5,\ x_{-3} = 11,\ x_{-2} = -2,\ x_{-1} = 5$ and $x_0 = -2$.

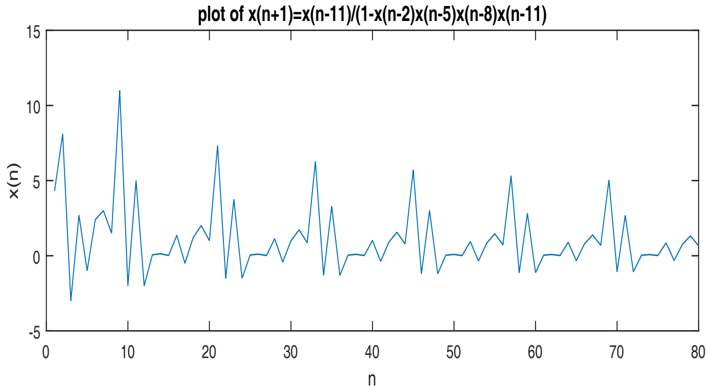


Figure 6.

Example 6.7.(see Figure 7) shows the period thirty six solutions of Eq.(8) since $x_{-11} = 3,\ x_{-10} = 9,\ x_{-9} = -6,$ $x_{-8} = 2,\ x_{-7} = 1,\ x_{-6} = 4,\ x_5 = 5,\ x_{-4} = -4,\ x_{-3} = 3,\ x_{-2} = -1/15,\ x_{-1} = 1/18$ and $x_0 = 1/36$.

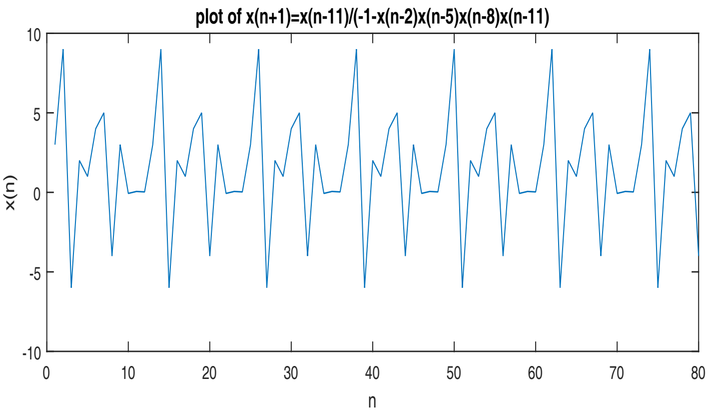


Figure 7.

Example 6.8. (See Figure 8) , we suppose for difference equation (8), that $x_{-11} = 11$, $x_{-10} = 3$, $x_{-9} = -5$, $x_{-8} = -2$, $x_{-7} = 4$, $x_{-6} = 2$, $x_5 = 9$, $x_{-4} = -2$, $x_{-3} = 7$, $x_{-2} = 1/99$, $x_{-1} = 1/12$ and $x_0 = 1/35$.

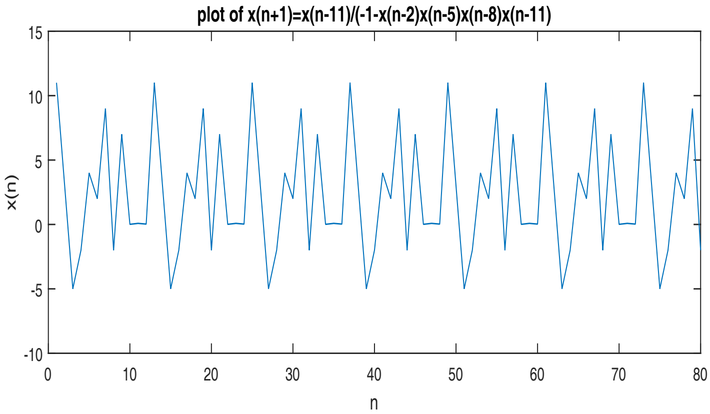


Figure 8.

REFERENCES

1. M. A. E. Abdelrahman and O. Moaaz,ON The New Class of The Nonlinear Rational Difference Equations, Electronic Journal of Mathematical Analysis and Applications, 6 (1) (2018), 117-125.
2. R P Agarwal, Difference Equations and Inequalities. 1^{st} edition, Marcel Dekker, New York, 1992, 2^{nd} edition, 2000.
3. R. P. Agarwal,and E. M. Elsayed, Periodicity and stability of solutions of higher order rational difference equation, Advanced Studies in Contemporary Mathematics, 17(2) (2008), 181-201.
4. M. Aloqeili, Dynamics of a Rational Difference Equation, Appl. Math. Comput., 176 (2006), 768-774.
5. A. M. Amleh, J. Hoag and G. Ladas, A difference equation with eventually periodic solutions, Comput. Math. Appl., 36 (10-12), 1998, 401-404.
6. L. Berezansky, E. Braverman, and E. Liz. Sufficient conditions for the global stability of nonautonomous higher order difference equations, Journal of Difference Equations and Applications,vol. 11, no. 9, pp. 785–798, 2005.
7. L. Berg, Oscillating solutions of rational difference equations, Rostocker Mathematisches Kolloquium, vol. 58, pp. 31–35, 2004.
8. C. Cinar, On the positive solutions of the difference equation $x_{n+1} = x_{n-1}/1 + x_n x_{n-1}$, Appl. Math. Comput., 150 (2004), 21–24.
9. C. Cinar, On the difference equation $x_{n+1} = x_{n-1}/-1+x_n x_{n-1}$, Appl. Math. Comput., 158 (2004), 813–816
10. C. Cinar, On the positive solutions of the difference equation $x_{n+1} = ax_{n-1}/1+bx_n x_{n-1}$ Appl.Math. Comp., 156:587-590, 2004.
11. C. Cinar, R Karatas and I Yalcinkaya, On solutions of the difference equation $x_{n+1} = x_{n-3}/-1 + x_n x_{n-1} x_{n-2} x_{n-3}$ Mathematica Bohemica, 132(3):257-261, 2007.
12. R. Devault, V. L. Kocic, and D. Stutson, Global behavior of solutions of the nonlinear difference equation $x_{n+1} = p_n + x_{n-1}/x_n$, J. Differ. Equ. Appl., 11 (8) (2005), 707-719.
13. E. M. Elabbasy, H. El-Metwally, E. M. Elsayed, On the difference equation $x_{n+1} = \frac{\alpha x_{n-k}}{\beta+\gamma \prod_{i=0}^{k} x_{n-i}}$, Journal of Concrete and Applicable Mathematics, 5(2) (2007),101 113.
14. E. M. Elsayed, On The Difference Equation $x_{n+1} = x_{n-5}/-1 + x_{n-2}x_{n-5}$, Inter. J. Contemp. Math. Sci., 3 (33) (2008), 1657-1664.
15. E. M. Elsayed, A Solution Form of a Class of Rational Difference Equations, International Journal of Nonlinear Science, 8 (4) (2009), 402-411.
16. E. M. Elsayed, On the Global attractivity and the solution of recursive sequence. Studia Scientiarum Mathematicarum Hungarica, 47 (3) (2010), 401-418.

17. E. M. Elsayed, On the global attractivity and the periodic character of a recursive sequence, Opuscula Mathematica, 30 (4) (2010), 431–446.
18. E. M. Elsayed, On the Solutions of a Rational System of Difference Equations, Fasciculi Mathematici, 45 (2010), 25-36.
19. E. M. Elsayed, Solution and Behavior of a Rational Difference Equations, Acta Universitatis Apulensis, 23 (2010), 233-249.
20. E. M. Elsayed, Dynamics of Recursive Sequence of Order Two, Kyung-pook Mathematical Journal, 50 (2010), 483-497.
21. E. M. Elsayed, Behavior of a rational recursive sequences, Stud. Univ. Babeş-Bolyai Math.vol LVI (1) (2011), 27–42.
22. E. M. Elsayed, Solution of a Recursive Sequence of Order Ten, General Mathematics vol. 19, no. 1 (2011), 145-162.
23. E. M. Elsayed, On the Solution of Some Difference Equations, Eur. J. Pure Appl. Math., 4 (3) (2011), 287-303.
24. E. M. Elsayed, B. Iricanin and S. Stevic, On The Max-Type Equation, Ars Combinatoria, 95 (2010), 187–192
25. E. A. Grove , G. Ladas, Periodicities in Nonlinear Difference Equations. Chapman & Hall / CRC Press (2005).
26. T. Ibrahim, On the third order rational difference equation $x_{n+1} = x_n x_{n-2}/x_{n-1}\left(a + bx_n x_{n-2}\right)$, Int. J. Contemp. Math. Sciences, 4 (27) (2009), 1321-1334.
27. R. Karatas: On solutions of the difference equation $x_{n+1} = \left(-1\right)^n x_{n-4}/1 + \left(-1\right)^n x_n x_{n-1} x_{n-2} x_{n-3} x_{n-4}$, Selcuk J. Appl. Math., 8 (1) (2007), 51- 56.
28. R. Karatas, C. Cinar and D. Simsek, On Positive Solutions of The Difference Equation $x_{n+1} = x_{n-5}/1 + x_{n-2}x_{n-5}$,Int. J. Contemp. Math. Sci., 1 (2006), 495-500.
29. R. Karatas and C. Cinar, On the solutions of the difference equation $x_{n+1} = \alpha x_{n-(2k+2)}/ - a + \prod_{i=0}^{2k+2} x_{n-i}$,Int. J. Contemp. Math. Sciences. 2 (31) (2007), 1505-1509.
30. C. Kent, W. Kosmala and S. Stevic, Long-Term Behavior of Solutions of the Difference Equation, Abstract and Applied Analysis vol 2010 (2010), Article ID 152378, 17 pages.
31. V. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht,1993.
32. K. Liu, P. Li, F. Han and W. Zhong, Global Dynamics of Nonlinear Difference Equation $x_{n+1} = A + x_n/x_{n-1}x_{n-2}$, Journal of Computational Analysis and Applications,24 (6) (2018), 1125-1132.
33. H. Ma, H. Feng, J. Wang, W. Ding, Boundedness and Asymptotic Behavior of Positive Solutions for Difference Equations of Exponential form, J. Nonlinear Sci. Appl., 8 (2015), 893-899.
34. R. Mostafaei and N. Rastegar,On a recurrence relation, QScience Connect,10 (2014),1-11.
35. W. T. Patula and H. D. Voulov, On a Max Type Recurrence Relation with Periodic Coefficients, Journal of Difference Equations and Applications 10 (3) (2004), 329-338.
36. M. Saleh, and M. Aloqeili, On the rational difference equation $y_{n+1} = A + y_{n-k}/y_n$,, Appl. Math. Comp., 171 (2) (2005), 862-869.
37. H. Sedaghat, Global behaviours of rational difference equations of orders two and three with quadratic terms, J. Diff. Eq. Appl., 15(3) (2009), 215–224.
38. S. Stevic, More on a rational recurrence relation, Appl. Math. E-Notes, 4 (2004), 80- 84.
39. S. Stevic, A note on periodic character of a higher order difference equation, Rostock. Math. Kolloq., 61 (2006), 21–30.
40. N. Touafek and E. M. Elsayed, On The Solution of some Difference Equations, Hokkaido Math. J., 44 (1) (2015), 29-45.
41. I. Yalçınkaya, B. D. Iricanin and C. Cinar, On a max-type difference equation. Discrete Dynamics in Nature and Society, 2007 (2007), 10 pages, Article ID 47264, doi: 1155/2007/47264.
42. Y. Yazlik, On the solutions and behavior of rational difference equations, J. Comp. Anal. Appl., 17 (2014), 584–594.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Some fixed point theorems of non-self contractive mappings in complete metric spaces

Dangdang Wang, Chuanxi Zhu*, Zhaoqi Wu

Department of Mathematics, Nanchang University, Nanchang, 330031, P. R. China

Abstract In this paper, we establish some fixed point theorems for non-self mappings, which solve the problem 1 in [1], satisfying special contractive conditions in complete metric spaces.

Keyword Fixed point; non-self mapping; contractive mapping; complete metric spaces

1 Introduction

The aim of this paper is to answer an open problem of Rus [1]. We give a non-self mapping T satisfying receptively four contractive conditions such that T has a unique fixed point. This is a solution for the open problem.

An open problem in [1] as following:

Let (X, d) be a metric space, Y a non-empty bounded and closed subset of X and $T : Y \to X$ a non-self operator. We suppose that there exists a sequence $(x_n)_{n \in N^*}$ such that $T^n(x_n)$ is defined for all $n \in N^*$. In which additional conditions on T we have:

(a) $F_T \neq \emptyset$?

(b) $F_T = \{x^*\}$?

where $F_T := \{x \in X | x = Tx\}$.

In this paper, we give following marks.

(1) $M_T(Y) = \sup\{d(x, y) | x, y \in Y\}$;

(2) $E_T(Y) = \sup\{d(x, Tx) | x \in Y\}$;

(3) $N_T(y) = \sup\{d(x, Ty) | x, y \in Y\}$.

In (2), we can easy to obtain: i) if $X \subset Y$, then $E_T(X) \leq E_T(Y)$; ii) $E_T(Y) = E_T(\overline{Y})$.

Lemma 1 [4] Let $a_n, b_n \in R_+, n \in N$. We suppose that:

(i)$\Sigma_{k=0}^{\infty} a_k < \infty$;

(ii)$b_n \to 0$ as $n \to \infty$.

Then

$$\Sigma_{k=0}^{\infty} a_{n-k} b_k \to 0 \text{ as } n \to \infty.$$

1*Correspondence author. Chuanxi Zhu. Email address: chuanxizhu@126.com. Tel:+8613970815298.

2 Fixed point theorems

In this section, we give some non-self contractions as follows.

Let (X, d) be a metric space, Y be a non-empty bounded and closed subset of X. Suppose that $T : Y \to X$ be a non-self mapping satisfied following condition:

(W1) $d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty)$, for all $x, y \in Y$, where $a, b, c \in R_+$ and $a + b + c < 1$;

(W2) $d(Tx, Ty) \leq bd(x, Ty) + cd(y, Tx)$, for all $x, y \in Y$, where $b, c \in R_+$;

(W3) $d(Tx, Ty) \leq ad(x, y) + bd(x, Ty) + cd(y, Tx)$, for all $x, y \in Y$, where $a, b, c \in R_+$ and $a < 1$;

(W4) $d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty)$, for all $x, y \in Y$, where $a_1, a_2, a_3, a_4 \in R_+$ and $a_1 + a_2 + a_3 < 1$.

Lemma 2 Let (X, d) be a metric space, Y be a bounded and non-empty closed subset of X. If $T : Y \to X$ satisfying (W1), then T is a non-self α-graphic contraction with $\alpha = a + c$.

Proof Let $x \in Y$ such that $Tx \in Y$, we get

$$d(T^2x, Tx) \leq ad(Tx, x) + bd(Tx, T^2x) + cd(x, Tx),$$

so

$$d(T^2x, Tx) \leq \frac{a + c}{1 - b} d(x, Tx).$$

Theorem 1 Let (X, d) be a metric space, Y be a non-empty bounded and closed subset of X. $T : Y \to X$ be a non-self mapping satisfying (W1). We suppose that there exists a sequence $(x_n)_{n\in N^*}$ such that $T^n(x_n)$ is defined for all $n \in N^*$. Then

(i) T has a unique fixed point;

(ii) $T^{n-1}(x_n) \to x^*$ *and* $T^n(x_n) \to x^*$ *as* $n \to +\infty$;

(iii) $d(x, x^*) \leq \frac{1+b}{1-a} d(x, Tx),\ \forall\ x \in\ Y$, i.e. $M_T(Y) \leq \frac{1+b}{1-a} E_T(Y)$.

Proof (i)+(ii) Let $Y_1 := \overline{T(Y)}$, $Y_2 := \overline{T(Y_1 \cap Y)}$, $\cdots$, $Y_{n+1} := \overline{T(Y_n \cap Y)}$, $n \in N^*$. We remark that:

(1) $Y_{n+1} \subset Y_n$, $\forall\ n \in N^*$;

(2) $T^n(x_n) \in Y_n$, $\forall\ n \in N^*$, so $Y_n \neq \varnothing$.

Since T satisfying (W1), we have that:

$$\begin{aligned} M(Y_{n+1}) &= M(\overline{T(Y_n \cap Y)}) = M(T(Y_n \cap Y)) \\ &\leq aM(Y_n \cap Y) + (b + c)E_T(Y_n \cap Y) \leq \cdots \\ &\leq a^{n+1}M(Y) + a^n(b + c)E_T(Y) + \cdots + a(b + c)E_T(Y_{n-1} \cap Y) + (b + c)E_T(Y_n \cap Y). \end{aligned} \tag{2.1}$$

On the other hand, from Lemma 2, we get

$$\begin{aligned} E_T(Y_n \cap Y) &= E_T(\overline{T(Y_{n-1} \cap Y} \cap Y) = E_T(T(Y_{n-1} \cap Y) \cap Y) \\ &= \sup\{d(Tx, T^2x) | x \in Y_{n-1} \cap Y, Tx \in Y\} \leq \frac{a+c}{1-b} E_T(Y_{n-1} \cap Y) \\ &\leq \cdots \leq (\frac{a+c}{1-b})^n E_T(Y),\ n \in N^*. \end{aligned}$$

Because of $a+b+c<1$, so $(\frac{a+c}{1-b})^n \to 0,\ n \to +\infty$, i.e. $E_T(Y_n \cap Y) \to 0,\ n \to +\infty$.

Let $a_n = a^n$ and $b_n = (b+c)E_T(Y_n \cap Y)$, by lemma 1, we have

$$M(Y_{n+1}) \to 0\ as\ n \to +\infty.$$

From Cantor intersection lemma, we get

$$Y_\infty := \cap_{n \in N} Y_n \neq \emptyset,\ M(Y_\infty) = 0 \text{ and } T(Y_\infty \cap Y) \subset Y_\infty.$$

From $Y_\infty \neq \emptyset$ and $M(Y_\infty) = 0$, we have that $Y_\infty = x^*$, i.e. Y_∞ be a single point set. Otherwise, $T^n(x_n) \in Y_n$ and $T^{n-1}(x_n) \in Y_{n-1} \cap Y$, this implies that $\{T^n(x_n)\}_{n \in N}$ and $\{T^{n-1}(x_n)\}_{n \in N}$ are fundamental sequences.

Since $Y_n, n \in N$ are closed, so we get

$$T^{n-1}(x_n) \to x^*\ and\ T^n(x_n) \to x^*\ as\ n \to +\infty.$$

Also because of T is continuous, then $T^n(x_n) \to T(x^*)$. Therefore, $T(x^*) = x^*$.

(iii) Let $x \in Y$, by using (W1) we have

$$d(x, x^*) \leq d(x, Tx) + d(Tx, x^*) \leq d(x, Tx) + ad(x, x^*) + bd(x, Tx) + cd(d(x, Tx), Td(x, Tx)),$$

so

$$d(x, x^*) \leq \frac{1+b}{1-a} d(x, Tx),\ \forall\ x \in\ Y.$$

Remark 1 Let $b = c$ in Theorem 1, then T is a non-self $\acute{C}iri\acute{c} - Reich - Rus$ operator. And then, Theorem 1 generalizes Theorem 5 in Rus [1]. At the same time, this theorem gives an answer to the Problem 1 of [1].

For (W4), we give a Lemma as following:

Lemma 3 Let (X, d) be a metric space, Y be a non-empty bounded and closed subset of X. Define $T: Y \to X$ be a non-self mapping. Then $N_T(Y_n \cap Y) \to 0,\ as\ n \to \infty$, where $Y_n = \overline{T(Y_{n-1} \cap Y)}$.

Proof From the definitions of N_T and Y_n, we have

$$\begin{aligned} \sup\{d(x, Ty) | x, y \in Y_n \cap Y\} &= N_T(Y_n \cap Y) = N_T(\overline{T(Y_{n-1} \cap Y)} \cap Y) \\ &= N_T(T(Y_{n-1} \cap Y) \cap Y) = \sup\{d(Tx, T^2y) | x, y \in Y_{n-1} \cap Y\}. \end{aligned}$$

Since $Y_{n-1} \cap Y \subset Y_n \cap Y$, so $d(Tx, T^2y) \leq d(x, Ty)$, for all $x, y \in Y_{n-1} \cap Y$. Hence, $N_T(Y_n \cap Y) \leq N_T(Y_{n-1} \cap Y)$.

By the density of real numbers we can get, there exists $k \in R^+$ and $k < 1$, such that $N_T(Y_n \cap Y) \leq kN_T(Y_{n-1} \cap Y)$.

And then,

$$N_T(Y_n \cap Y) \leq kN_T(Y_{n-1} \cap Y) \leq \cdots \leq k^n N_T(Y) \to 0, \text{ as } n \to +\infty.$$

Theorem 2 Let (X, d) be a metric space, Y be a non-empty bounded and closed subset of X. $T : Y \to X$ be a non-self mapping satisfying (W3). We suppose that there exists a sequence $(x_n)_{n\in N^*}$ such that $T^n(x_n)$ is defined for all $n \in N^*$. Then

(i) T has a unique fixed point;

(ii) $T^{n-1}(x_n) \to x^*$ and $T^n(x_n) \to x^*$ as $n \to +\infty$.

Proof Let $Y_1 := \overline{T(Y)}$, $Y_2 := \overline{T(Y_1 \cap Y)}$, $\cdots$, $Y_{n+1} := \overline{T(Y_n \cap Y)}$, $n \in N^*$. We remark that:

(1) $Y_{n+1} \subset Y_n$, $\forall\ n \in N^*$;

(2) $T^n(x_n) \in Y_n$, $\forall\ n \in N^*$, so $Y_n \neq \varnothing$.

Since T satisfying (W3), we have that:

$$\begin{aligned}
&M(Y_{n+1}) = M(\overline{T(Y_n \cap Y)}) = M(T(Y_n \cap Y)) \\
&\leq aM(Y_n \cap Y) + (b + c)N_T(Y_n \cap Y) \\
&\leq aM(Y_n) + (b + c)N_T(Y_n \cap Y) \leq \cdots \\
&\leq a^{n+1}M(Y) + a^n(b + c)N_T(Y) + \cdots + a(b + c)N_T(Y_{n-1} \cap Y) + (b + c)N_T(Y_n \cap Y).
\end{aligned}$$

Let $a_n = a^n$ and $b_n = (b + c)N_T(Y_n \cap Y)$, by lemma 3 we have

$$M(Y_{n+1}) \to 0 \ as\ n \to +\infty,$$

and the proof is similar with the proof of Theorem 1. This is the complete proof.

For (W3), when $a = 0$, it becomes condition (W2). Thence, we have the following Corollary:

Corollary 1 Let (X, d) be a metric space, Y be a non-empty bounded and closed subset of X. Define $T : Y \to X$ be a non-self mapping satisfying (W3), the conclusions of Theorem 2 remain holds.

Theorem 3 Let (X, d) be a metric space, Y be a non-empty bounded and closed subset of X. $T : Y \to X$ be a non-self mapping satisfying (W4). We suppose that there exists a sequence $(x_n)_{n\in N^*}$ such that $T^n(x_n)$ is defined for all $n \in N^*$. Then

(i) T has a unique fixed point;

(ii) $T^{n-1}(x_n) \to x^*$ and $T^n(x_n) \to x^*$ as $n \to +\infty$;

(iii) $d(x, x^*) \leq \frac{1+b}{1-a} d(x, Tx)$, $\forall\ x \in\ Y$, i.e. $M_T(Y) \leq \frac{1+b}{1-a} E_T(Y)$.

Proof (i)+(ii) Let $Y_1 := \overline{T(Y)}$, $Y_2 := \overline{T(Y_1 \cap Y)}$, $\cdots$, $Y_{n+1} := \overline{T(Y_n \cap Y)}$, $n \in N^*$. We remark that:

(1) $Y_{n+1} \subset Y_n$, $\forall\ n \in N^*$;

(2) $T^n(x_n) \in Y_n$, $\forall\ n \in N^*$, so $Y_n \neq \varnothing$.

Since T satisfying (W1), we have that:

$$\begin{aligned}
&M(Y_{n+1}) = M(\overline{T(Y_n \cap Y)}) = M(T(Y_n \cap Y))\\
&\leq a_1 M(Y_n \cap Y) + (a_2 + a_3)E_T(Y_n \cap Y) + a_4 N_T(Y_n \cap Y)\\
&\leq \cdots \leq\\
&\leq a_1^{n+1} M(Y) + [a_1^n (a_2 + a_3)E_T(Y) + \cdots + a_1(a_2 + a_3)E_T(Y_{n-1} \cap Y) + (a_2 + a_3)E_T(Y_n \cap Y)]\\
&+ [a_1^n \cdot a_4 N_T(Y) + \cdots + a_1 \cdot a_4 N_T(Y_{n-1} \cap Y) + a_4 N_T(Y_n \cap Y)]\\
&= a_1^{n+1} M(Y) + \Phi_{E_T} + \Phi_{N_T},
\end{aligned}$$

where $\Phi_{E_T} = a_1^n(a_2 + a_3)E_T(Y) + \cdots + a_1(a_2 + a_3)E_T(Y_{n-1} \cap Y) + (a_2 + a_3)E_T(Y_n \cap Y)$, $\Phi_{N_T} = a_1^n \cdot a_4 N_T(Y) + \cdots + a_1 \cdot a_4 N_T(Y_{n-1} \cap Y) + a_4 N_T(Y_n \cap Y)$.

For $x \in Y$, such that $Tx \in T(Y)$, we have

$$d(T^2x, Tx) \leq a_1 d(Tx, x) + a_2 d(Tx, T^2x) + a_3 d(x, Tx) + a_4 d(Tx, Tx),$$

so

$$d(T^2x, Tx) \leq \frac{a_1 + a_3}{1 - a_2} d(x, Tx). \tag{2.2}$$

Thence

$$\begin{aligned}
E_T(Y_n \cap Y) &= E_T(\overline{T(Y_{n-1} \cap Y} \cap Y) = E_T(T(Y_{n-1} \cap Y) \cap Y)\\
&= \sup\{d(Tx, T^2x) | x \in Y_{n-1} \cap Y, Tx \in Y\} \leq \frac{a_1 + a_3}{1 - a_2} E_T(Y_{n-1} \cap Y)\\
&\leq \cdots \leq (\frac{a_1 + a_3}{1 - a_2})^n E_T(Y),\ n \in N^*.
\end{aligned}$$

Because of $a + b + c < 1$, so $(\frac{a_1+a_3}{1-a_2})^n \to 0,\ n \to +\infty$, i.e. $E_T(Y_n \cap Y) \to 0,\ n \to +\infty$.

Let $a_n = a_1^n$ and $b_n = (a_2 + a_3)E_T(Y_n \cap Y)$, by lemma 1 we have $\Phi_{E_T} \to 0$ as $n \to +\infty$.

From Lemma 3, we know, $N_T(Y_n \cap Y) \to 0$ as $n \to +\infty$.

Let $a_n^{'} = a_1^n$ and $b_n^{'} = a_4 N_T(Y_n \cap Y)$, by lemma 1 we obtain $\Phi_{N_T} \to 0$ as $n \to +\infty$.

In summary, we get $M(Y_{n+1}) \to 0$ as $n \to +\infty$, and the proof is similar with the proof of Theorem 1.

Although let $a = 0$ in (W4), it becomes (W2), because different proof process details of the transformation, so we give separately the proof of Theorem 1 and Theorem 3.

Acknowledgements

The author would like to thank the editors and the referees for their constructive comments and suggestions. The research was supported by the Natural Science Foundation of China (11771198,11361042,11071108, 11461045,11701259), the Natural Science Foundation of Jiangxi Province of China (20132BAB201001, 20142BAB211016) and the Scientific Program of the Provincial Education Department of Jiangxi (GJJ150008) and the Innovation Program of the Graduate student of Nanchang University(colonel-level project)(cx2016148).

References

[1] I. A., Rus, M. A., Serban, *Some fixed point theorems for nonself generalized contraction.* Miskolc Math. Notes, **17** (2017), 1021-1031.

[2] S., Reich, A. J., Zasavski, *A fixed point theorem for Matkowski contractions*, Fixed Point Theory, **8** (2007), 303-307.

[3] A. Amini-Harandi, M., Fakhar, M., Goli, *Some fixed point theorems for non-self mappings of contractive type with applicatons to endpoint theory*, J. Fixed Point Theory Appl., **2017** (2017), 10pages.

[4] I. A., Rus, M. A., Serban, *Some generalizations of a Cauchy lemma and applizations*, Topicsnin Mathematicas, computer sciece and philosophy, **2008** (2008), 173-181.

[5] A., Amini-Harandi, *Endpoints of set-valued contractions in metric spaces*, Nonlinear Anal., **72** (2010), 132-134.

[6] A., Petrusel, I. A., Rus, M. A., Serban,*Fixed points, fixed sets and iterated multifunction systems for nonself multivalued operators*, Set-Valued Var. Anal., **23** (2015), 223-237.

[7] B. E., Rhoades, *A fixed point theorem for non-self-mappings*, Math. Japon., **23** (1979), 457-459.

[8] X. H., Mu, C. X., Zhu, Z. Q., Wu, *New multipled common fixed point theorems in Menger PM-space*, Fixed Point Theory Appl., **2015** (2015), 136 pages.

[9] C. R., Ji, C. X., Zhu, Z. Q., Wu, *Some new fixed point theorems in generalized probabilistic metric spaces*, J. of Nonlinear Science and Appl., **9** (2016), 3735-3743.

[10] Q., Tu, C. X., Zhu, Z. Q., Wu, *Some new coupled fixed point theorems in partially ordered complete probabilistic metric spaces*, J. of Nonlinear Science and Appl., **9** (2016), 1116-1128.

[11] C. X. Zhu, Z. B., Xu, *Inequalities and solution of an operator equation*, Appl. Math. Lett., **21** (2008), 607-611.

[12] C. X., Zhu, *Research on some problems for nonlinear operators*, Nonlinear Anal., **71** (2009), 4568-4571.

[13] C. X.,Zhu, C. F., Chen, *Calculations of random fixed point intex*, J. Math. Anal. Appl., **339** (2008), 839-844.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Double-framed soft sets in B-algebras

Jung Mi Ko[1] and Sun Shin Ahn[2,*]

[1] *Department of Mathematics, Gangneung-Wonju National University, Gangneung 25457, Korea*

[2] *Department of Mathematics Education, Dongguk University, Seoul 04620, Korea*

Abstract. The notion of a double-framed soft (normal) subalgebra in a B-algebra is introduced and related properties are investigated. We consider characterizations of a double-framed soft (normal) subalgebra and establish a new double-framed soft subalgebra from old one. Also, we show that the int-uni double-framed soft of two double framed soft subalgebras is a double framed soft subalgebra.

1. Introduction

Molodtsov [11] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [10] described the application of soft set theory to a decision making problem. Jun [5] discussed the union soft sets with applications in BCK/BCI-algebras. Jun et al. [6] introduced the notion of double-framed soft sets, and applied it to BCK/BCI-algebras. They discussed double-frame soft algebras and investigated some related properties.

We refer the reader to the papers [3, 4, 14] for further information regarding algebraic structures/properties of soft set theory. On the while, Y. B. Jun, E. H. Roh and H. S. Kim [7] introduced a new notion, called a BH-algebra. J. Neggers and H. S. Kim [12] introduced a new notion, called a B-algebra. C. B. Kim and H. S. Kim [9] introduced the notion of a BG-algebra which is a generalization of B-algebras. S. S. Ahn and H. D. Lee [1] classified the subalgebras by their family of level subalgebras in BG-algebras.

In this paper, we introduce the notion of a double-framed soft (normal) subalgebra in a B-algebra and investigate some related properties. We consider characterizations of a double-framed soft (normal) subalgebra and establish a new double-framed soft subalgebra from old one. Also, we show that the int-uni double-framed soft of two double framed soft subalgebras is a double framed soft subalgebra.

2. Preliminaries

A *B-algebra* [12] is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying axioms:

[0] **2010 Mathematics Subject Classification**: 06F35; 03G25; 06D72.

[0] **Keywords**: γ-exclusive set, double-framed soft (normal) subalgebra, B-algebra.

* Correspondence: Tel: +82 2 2260 3410, Fax: +82 2 2266 3409 (S. S. Ahn).

[0] **E-mail**: jmko@gwnu.ac.kr (J. M. Ko); sunshine@dongguk.edu (S. S. Ahn).

[0] This study was supported by Gangneung-Wonju National University.

(B1) $x * x = 0$,

(B2) $x * 0 = x$,

(B) $(x * y) * z = x * (z * (0 * y))$

for any x, y, z in X. For brevity we call X a *B-algebra.* In X we can define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BH-algebra* if it satisfies (B1), (B2) and

(BH) $x * y = y * x = 0$ imply $x = y$ for any $x, y \in X$.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BG-algebra* if it satisfies (B1), (B2) and

(BG) $(x * y) * (0 * y) = x$ for any $x, y \in X$.

Proposition 2.1. [2, 12] *Let $(X; *, 0)$ be a B-algebra. Then*

(i) *the left cancellation law holds in X, i.e., $x * y = x * z$ implies $y = z$,*

(ii) *if $x * y = 0$, then $x = y$ for any $x, y \in X$,*

(iii) *if $0 * x = 0 * y$, then $x = y$ for any $x, y \in X$,*

(iv) *$0 * (0 * x) = x$, for all $x \in X$,*

(v) *$x * (y * z) = (x * (0 * z)) * y$ for all $x, y, z \in X$.*

A non-empty subset S of a B-algebra X is called a *subalgebra* of X if $x * y \in S$ for any $x, y \in S$. A non-empty subset N of X is said to be *normal* if $(x * a) * (y * b) \in N$ for any $x * y, a * b \in N$. Then any normal subset N of a B-algebra X is a subalgebra of X, but the converse need not be true ([13]). A non-empty subset X of a B-algebra X is a called a *normal subalgebra* of X if it is both a subalgebra and normal.

Molodtsov [11] defined the soft set in the following way: Let U be an initial universe set and let E be a set of parameters. We say that the pair (U, E) is a *soft universe.* Let $\mathscr{P}(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq E$.

A fair $(\tilde{f}, A)$ is called a *soft set* over U, where $\tilde{f}$ is a mapping given by $\tilde{f} : X \to \mathscr{P}(U)$. In other words, a soft set over U is parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\tilde{f}(\varepsilon)$ may be considered as the set of ε-approximate elements of the set $(\tilde{f}, A)$. A soft set over U can be represented by the set of ordered pairs:

$$(\tilde{f}, A) = \{(x, \tilde{f}(x)) | x \in A, \tilde{f}(x) \in \mathscr{P}(U)\},$$

where $\tilde{f} : X \to \mathscr{P}(U)$ such that $\tilde{f}(x) = \emptyset$ if $x \notin A$. Clearly, a soft set is not a set.

3. Double-framed soft normal subalgebras

In what follows let X denote a B-algebra unless otherwise specified.

Definition 3.1. A double-framed pair $\langle(\alpha, \beta); X\rangle$ is called a *double-framed soft set* over U, where α and β are mappings from X to $\mathscr{P}(U)$.

Definition 3.2. A double-framed soft set $\langle(\alpha, \beta); X\rangle$ over U is called a *double-framed soft subalgebra* over U if it satisfies :

(3.1) $(\forall x, y \in X)\,(\alpha(x*y) \supseteq \alpha(x) \cap \alpha(y),\ \beta(x*y) \subseteq \beta(x) \cup \beta(y))\,.$

Example 3.3. Let X be the set of parameters where $X := \{0,1,2,3\}$ is a B-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Let $\langle(\alpha,\beta);X\rangle$ be a double-framed soft set over U defined as follows:

$$\alpha : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \tau_3 & \text{if } x = 0, \\ \tau_1 & \text{if } x = 3, \\ \tau_2 & \text{if } x = \{1,2\}, \end{cases}$$

and

$$\beta : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_3 & \text{if } x = 0, \\ \gamma_2 & \text{if } x = 3, \\ \gamma_1 & \text{if } x = \{1,2\} \end{cases}$$

where $\tau_1, \tau_2, \tau_3, \gamma_1, \gamma_2$ and γ_3 are subsets of U with $\tau_1 \subsetneq \tau_2 \subsetneq \tau_3$ and $\gamma_1 \supsetneq \gamma_2 \supsetneq \gamma_3$ It is easy to show that $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U.

Lemma 3.4. *Every double-framed soft subalgebra $\langle(\alpha,\beta);X\rangle$ over U satisfies the following condition:*

(3.2) $(\forall x \in X)\,(\alpha(x) \subseteq \alpha(0),\ \beta(x) \supseteq \beta(0))\,.$

Proof. Straightforward. □

Proposition 3.5. *For a double-framed soft subalgebra $\langle(\alpha,\beta);X\rangle$ over U, the following are equivalent:*

(i) $(\forall x \in X)\,(\alpha(x) = \alpha(0),\ \beta(x) = \beta(0))\,.$
(ii) $(\forall x, y \in X)\,(\alpha(y) \subseteq \alpha(x*y),\ \beta(y) \supseteq \beta(x*y))\,.$

Proof. Assume that (ii) is valid. Taking $y := 0$ in (ii) and using (B2), we have $\alpha(0) \subseteq \alpha(x*0) = \alpha(x)$ and $\beta(0) \supseteq \beta(x*0) = \beta(x)$. It follows from Lemma 3.4 that $\alpha(x) = \alpha(0)$ and $\beta(x) = \beta(0)$.

Conversely, suppose that $\alpha(x) = \alpha(0)$ and $\beta(x) = \beta(0)$ for all $x \in X$. Using (3.1), we have

$$\alpha(y) = \alpha(0) \cap \alpha(y) = \alpha(x) \cap \alpha(y) \subseteq \alpha(x*y),$$
$$\beta(y) = \beta(0) \cup \beta(y) = \beta(x) \cup \beta(y) \supseteq \beta(x*y)$$

for all $x, y \in X$. This completes the proof. □

For two double-framed soft sets $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ over U, the *double-framed soft int-uni set* of $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ is defined to be a double-framed soft set $\langle(\alpha\tilde{\cap}f, \beta\tilde{\cup}g);X\rangle$ where

$$\alpha\tilde{\cap}f : X \to \mathscr{P}(U),\ x \mapsto \alpha(x) \cap f(x),$$
$$\beta\tilde{\cup}g : X \to \mathscr{P}(U),\ x \mapsto \beta(x) \cup g(x).$$

It is denoted by $\langle(\alpha,\beta);X\rangle \sqcap \langle(f,g);X\rangle = \langle(\alpha\tilde{\cap}f, \beta\tilde{\cup}g);X\rangle\,.$

Theorem 3.6. *The double-framed soft int-uni set of two double-framed soft subalgebras $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ over U is a double-framed soft subalgebra over U.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned}(\alpha\tilde{\cap}f)(x*y) =& \alpha(x*y)\cap f(x*y) \supseteq (\alpha(x)\cap\alpha(y))\cap(f(x)\cap f(y))\\ =&(\alpha(x)\cap f(x))\cap(\alpha(y)\cap f(y)) = (\alpha\tilde{\cap}f)(x)\cap(\alpha\tilde{\cap}f)(y)\end{aligned}$$

and

$$\begin{aligned}(\beta\tilde{\cup}g)(x*y) =& \beta(x*y)\cup g(x*y) \subseteq (\beta(x)\cup\beta(y))\cup(g(x)\cup g(y))\\ =&(\beta(x)\cup g(x))\cup(\beta(y)\cup g(y)) = (\beta\tilde{\cup}g)(x)\cup(\beta\tilde{\cup}g)(y).\end{aligned}$$

Therefore $\langle(\alpha,\beta);X\rangle \sqcap \langle(f,g);X\rangle$ is a double-framed soft subalgebra over U. $\square$

For two double-framed soft sets $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ over U, the *double-framed soft uni-int set* of $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ is defined to be a double-framed soft set $\langle(\alpha\tilde{\cup}f,\beta\tilde{\cap}g);X\rangle$ where

$$\alpha\tilde{\cup}f : X \to \mathscr{P}(U),\ x \mapsto \alpha(x)\cup f(x),$$
$$\beta\tilde{\cap}g : X \to \mathscr{P}(U),\ x \mapsto \beta(x)\cap g(x).$$

It is denoted by $\langle(\alpha,\beta);X\rangle \sqcup \langle(f,g);X\rangle = \langle(\alpha\tilde{\cup}f,\beta\tilde{\cap}g);X\rangle$.

The following example shows that the double-framed soft uni-int set of two double-framed soft subalgebras $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ over U may not be a double-framed soft subalgebra over U.

Example 3.7. Let $E = X$ be the set of parameters, and let $U = \mathbb{Z}$ be the initial universe set, where $X = \{0,1,2,3,4,5\}$ is a B-algebra [12] with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Let $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ be double-framed soft sets over U defined, respectively, as follows:

$$\alpha : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x \in \{0,4\}, \\ 9\mathbb{Z} & \text{if } x \in \{1,2,3,5\}, \end{cases}$$

$$\beta : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} 7\mathbb{Z} & \text{if } x \in \{0,4\}, \\ \mathbb{Z} & \text{if } x \in \{1,2,3,5\}, \end{cases}$$

$$f : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x \in \{0,5\}, \\ 3\mathbb{Z} & \text{if } x \in \{1,2,3,4\}, \end{cases}$$

and

$$g : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} 2\mathbb{Z} & \text{if } x \in \{0,5\}, \\ \mathbb{Z} & \text{if } x \in \{1,2,3,4\}, \end{cases}$$

It is routine to verify that $\langle(\alpha,\beta);X\rangle$ and $\langle(f,g);X\rangle$ are double-framed soft subalgebras over U. But $\langle(\alpha,\beta);X\rangle \sqcup \langle(f,g);X\rangle = \langle(\alpha\tilde{\cup}f, \beta\tilde{\cap}g);X\rangle$ is not a double-framed soft subalgebra over U, since $(\alpha\tilde{\cup}f)(4*5) = (\alpha\tilde{\cup}f)(2) = \alpha(2)\cup f(2) = 9\mathbb{Z}\cup 3\mathbb{Z} = 3\mathbb{Z} \not\supseteq \mathbb{Z} = (\alpha\tilde{\cup}f)(4)\cap(\alpha\tilde{\cup}f)(5)$ and/or $(\beta\tilde{\cap}g)(4*5) = (\beta\tilde{\cap}g)(2) = \mathbb{Z}\cap\mathbb{Z} = \mathbb{Z} \not\subseteq 7\mathbb{Z}\cup 2\mathbb{Z} = (\beta\tilde{\cap}g)(4)\cup(\beta\tilde{\cap}g)(5)$.

Let $\langle(\alpha,\beta);A\rangle$ and $\langle(f,g);B\rangle$ be double-framed soft sets over a common universe U. Then $\langle(\alpha,\beta);A\rangle$ is called a *double-framed soft subset* of $\langle(f,g);B\rangle$, denoted by $\langle(\alpha,\beta);A\rangle \,\tilde{\subseteq}\, \langle(f,g);B\rangle$, if

(i) $A \subseteq B$,

(ii) $(\forall e \in A)\left(\begin{array}{l}\alpha(e) \text{ and } f(e) \text{ are identical approximations,}\\ \beta(e) \text{ and } g(e) \text{ are identical approximations.}\end{array}\right)$.

Theorem 3.8. *Let $\langle(\alpha,\beta);A\rangle$ be a double-framed soft subset of a double-framed soft set $\langle(f,g);B\rangle$. If $\langle(f,g);B\rangle$ is a double-framed soft subalgebra over U, then so is $\langle(\alpha,\beta);A\rangle$.*

Proof. Let $x, y \in A$. Then $x, y \in B$, and so

$$\alpha(x)\cap\alpha(y) = f(x)\cap f(y) \subseteq f(x*y) = \alpha(x*y),$$
$$\beta(x)\cup\beta(y) = g(x)\cup g(y) \supseteq g(x*y) = \beta(x*y).$$

Hence $\langle(\alpha,\beta);A\rangle$ is a double-framed soft subalgebra over U. □

The converse of Theorem 3.8 is not true as seen in the following example.

Example 3.9. Let $(U=\mathbb{Z}, X)$ where $X = \{0,1,2,3\}$ is a B-algebra as in Example 3.3. For a subalgebra $\{0,3\}$, define a double-framed soft set $\langle(\alpha,\beta);\{0,3\}\rangle$ over U as follows:

$$\alpha : \{0,3\} \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x = 0,\\ 2\mathbb{Z} & \text{if } x = 3,\end{cases}$$

and

$$\beta : \{0,3\} \to \mathscr{P}(U),\ x \mapsto \begin{cases} 27\mathbb{Z} & \text{if } x = 0,\\ 9\mathbb{Z} & \text{if } x = 3,\end{cases}$$

Then $\langle(\alpha,\beta);\{0,3\}\rangle$ is a double-framed soft subalgebra over U. Take $B := X$ and define a double-framed soft set $\langle(f,g);B\rangle$ over U as follows:

$$f : B \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x = 0,\\ 72\mathbb{Z} & \text{if } x = 1,\\ 4\mathbb{Z} & \text{if } x = 2,\\ 2\mathbb{Z} & \text{if } x = 3,\end{cases}$$

and

$$g : B \to \mathscr{P}(U),\ x \mapsto \begin{cases} 27\mathbb{Z} & \text{if } x = 0,\\ 3\mathbb{Z} & \text{if } x = 1,\\ \mathbb{Z} & \text{if } x = 2,\\ 9\mathbb{Z} & \text{if } x = 3.\end{cases}$$

Then $\langle(f,g);B\rangle$ is not a double-framed soft subalgebra over U since $f(0*2) = f(1) = 72\mathbb{Z} \not\supseteq f(0)\cap f(2) = \mathbb{Z}\cap 4\mathbb{Z} = 4\mathbb{Z}$ and/or $g(1*3) = g(2) = \mathbb{Z} \not\subseteq g(1)\cup g(3) = 3\mathbb{Z}\cup 9\mathbb{Z} = 3\mathbb{Z}$.

For a double-framed soft set $\langle(\alpha,\beta);X\rangle$ over U and two subsets γ and δ of U, the *γ-inclusive set* and the *δ-exclusive set* of $\langle(\alpha,\beta);X\rangle$, denoted by $i_X(\alpha;\gamma)$ and $e_X(\beta;\delta)$, respectively, are defined as follows: $i_X(\alpha;\gamma) := \{x\in X \mid \gamma\subseteq\alpha(x)\}$ and $e_X(\beta;\delta) := \{x\in X \mid \delta\supseteq\beta(x)\}$, respectively. The set $DF_X(\alpha,\beta)_{(\gamma,\delta)} := \{x\in X \mid \gamma\subseteq\alpha(x),\ \delta\supseteq\beta(x)\}$ is called a *double-framed including set* of $\langle(\alpha,\beta);X\rangle$. It is clear that $DF_X(\alpha,\beta)_{(\gamma,\delta)} = i_X(\alpha;\gamma)\cap e_X(\beta;\delta)$.

Theorem 3.10. *For a double-framed soft set $\langle(\alpha,\beta);X\rangle$ over U, the following are equivalent:*

(i) *$\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U.*
(ii) *For every subsets γ and δ of U with $\gamma\in Im(\alpha)$ and $\delta\in Im(\beta)$, the γ-inclusive set and the δ-exclusive set of $\langle(\alpha,\beta);X\rangle$ are subalgebras of X.*

Proof. Assume that $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U. Let $x,y\in X$ be such that $x,y\in i_X(\alpha;\gamma)$ and $x,y\in e_X(\beta;\delta)$ for every subsets γ and δ of U with $\gamma\in Im(\alpha)$ and $\delta\in Im(\beta)$. It follows from (3.1) that

$$\alpha(x*y)\supseteq\alpha(x)\cap\alpha(y)\supseteq\gamma \text{ and } \beta(x*y)\subseteq\beta(x)\cup\beta(y)\subseteq\delta.$$

Hence $x*y\in i_X(\alpha;\gamma)$ and $x*y\in e_X(\beta;\delta)$, and therefore $i_X(\alpha;\gamma)$ and $e_X(\beta;\delta)$ are subalgebras of X.

Conversely, suppose that (ii) is valid. Let $x,y\in X$ be such that $\alpha(x)=\gamma_x$, $\alpha(y)=\gamma_y$, $\beta(x)=\delta_x$ and $\beta(y)=\delta_y$. Taking $\gamma=\gamma_x\cap\gamma_y$ and $\delta=\delta_x\cup\delta_y$ imply that $x,y\in i_X(\alpha;\gamma)$ and $x,y\in e_X(\beta;\delta)$. Hence $x*y\in i_X(\alpha;\gamma)$ and $x*y\in e_X(\beta;\delta)$, which imply that $\alpha(x*y)\supseteq\gamma=\gamma_x\cap\gamma_y=\alpha(x)\cap\alpha(y)$ and $\beta(x*y)\subseteq\delta=\delta_x\cup\delta_y=\beta(x)\cup\beta(y)$. Therefore $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U. □

Corollary 3.11. *If $\langle(\alpha,\beta);X\rangle$ is a double-framed soft algebra over U, then the double-framed including set of $\langle(\alpha,\beta);X\rangle$ is a subalgebra X.*

For any double-framed soft set $\langle(\alpha,\beta);X\rangle$ over U, let $\langle(\alpha^*,\beta^*);X\rangle$ be a double-framed soft set over U defined by

$$\alpha^*: X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\alpha(x) & \text{if } x\in i_X(\alpha;\gamma),\\ \eta & \text{otherwise,}\end{cases}$$

$$\beta^*: X\to\mathscr{P}(U),\ x\mapsto\begin{cases}\beta(x) & \text{if } x\in e_X(\beta;\delta),\\ \rho & \text{otherwise,}\end{cases}$$

where γ,δ,η and ρ are subsets of U with $\eta\subsetneq\alpha(x)$ and $\rho\supsetneq\beta(x)$.

Theorem 3.12. *If $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U, then so is $\langle(\alpha^*,\beta^*);X\rangle$.*

Proof. Assume that $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U. Then $i_X(\alpha;\gamma)$ and $e_X(\beta;\delta)$ are subalgebras of X for every subsets γ and δ of U with $\gamma\in Im(\alpha)$ and $\delta\in Im(\beta)$, by Theorem 3.10. Let $x,y\in X$. If $x,y\in i_X(\alpha;\gamma)$, then $x*y\in i_X(\alpha;\gamma)$. Thus

$$\alpha^*(x*y)=\alpha(x*y)\supseteq\alpha(x)\cap\alpha(y)=\alpha^*(x)\cap\alpha^*(y).$$

If $x\notin i_X(\alpha;\gamma)$ or $y\notin i_X(\alpha;\gamma)$, then $\alpha^*(x)=\eta$ or $\alpha^*(y)=\eta$. Hence

$$\alpha^*(x*y)\supseteq\eta=\alpha^*(x)\cap\alpha^*(y).$$

Now, if $x, y \in e_X(\beta;\delta)$, then $x * y \in e_X(\beta;\delta)$. Thus

$$\beta^*(x * y) = \beta(x * y) \subseteq \beta(x) \cup \beta(y) = \beta^*(x) \cup \beta^*(y).$$

If $x \notin e_X(\beta;\delta)$ or $y \notin e_X(\beta;\delta)$, then $\beta^*(x) = \rho$ or $\beta^*(y) = \rho$. Hence

$$\beta^*(x * y) \subseteq \rho = \beta^*(x) \cup \beta^*(y).$$

Therefore $\langle(\alpha^*, \beta^*); X\rangle$ is a double-framed soft subalgebra over U. □

Let $\langle(\alpha, \beta); X\rangle$ and $\langle(\alpha, \beta); Y\rangle$ be double-framed soft sets over U, where X, Y are B-algebras. The $(\alpha_\wedge,\ \beta_\vee)$-*product* of $\langle(\alpha, \beta); X\rangle$ and $\langle(\alpha, \beta); Y\rangle$ is defined to be a double-framed soft set $\langle(\alpha_{X\wedge Y}, \beta_{X\vee Y}); X \times Y\rangle$ over U in which

$$\alpha_{X\wedge Y} : X \times Y \to \mathscr{P}(U),\ (x, y) \mapsto \alpha(x) \cap \alpha(y),$$
$$\beta_{X\vee Y} : X \times Y \to \mathscr{P}(U),\ (x, y) \mapsto \beta(x) \cup \beta(y).$$

Theorem 3.13. *For any B-algebras X and Y as sets of parameters, let $\langle(\alpha, \beta); X\rangle$ and $\langle(\alpha, \beta); Y\rangle$ be double-framed soft subalgebras over U. Then the $(\alpha_\wedge,\ \beta_\vee)$-product of $\langle(\alpha, \beta); X\rangle$ and $\langle(\alpha, \beta); Y\rangle$ is also a double-framed soft subalgebra over U.*

Proof. Note that $(X \times Y, \circledast, (0, 0))$ is a B-algebra. For any $(x, y), (a, b) \in X \times Y$, we have

$$\begin{aligned}
&\alpha_{X\wedge Y}\left((x, y) \circledast (a, b)\right) = \alpha_{X\wedge Y}\left(x * a, y * b\right)\\
&= \alpha(x * a) \cap \alpha(y * b) \supseteq (\alpha(x) \cap \alpha(a)) \cap (\alpha(y) \cap \alpha(b))\\
&= (\alpha(x) \cap \alpha(y)) \cap (\alpha(a) \cap \alpha(b))\\
&= \alpha_{X\wedge Y}\left(x, y\right) \cap \alpha_{X\wedge Y}\left(a, b\right)
\end{aligned}$$

and

$$\begin{aligned}
&\beta_{X\vee Y}\left((x, y) \circledast (a, b)\right) = \beta_{X\vee Y}\left(x * a, y * b\right)\\
&= \beta(x * a) \cup \beta(y * b) \subseteq (\beta(x) \cup \beta(a)) \cup (\beta(y) \cup \beta(b))\\
&= (\beta(x) \cup \beta(y)) \cup (\beta(a) \cup \beta(b))\\
&= \beta_{X\vee Y}\left(x, y\right) \cup \beta_{X\vee Y}\left(a, b\right)
\end{aligned}$$

Hence $\langle(\alpha_{X\wedge Y}, \beta_{X\vee Y}); E \times F\rangle$ is a double-framed soft subalgebra over U. □

Definition 3.14. A double-framed soft set $\langle(\alpha, \beta); X\rangle$ over U is said to be *double-framed soft normal* of a B-algebra X if it satisfies:

(3.3) $(\forall x, y, a, b \in X)(\alpha((x * a) * (y * b)) \supseteq \alpha(x * y) \cap \alpha(a * b), \beta((x * a) * (y * b)) \subseteq \beta(x * y) \cup \beta(a * b)).$

A double-framed soft $\langle(\alpha, \beta); X\rangle$ over U is called a *double-framed soft normal subalgebra* of a B-algebra X if it satisfies (3.1) and (3.3).

Example 3.15. Let $(U=\mathbb{Z},X)$ where $X=\{0,1,2,3\}$ is a B-algebra as in Example 3.3. Let $\langle(\alpha,\beta);X\rangle$ be a double-framed soft set over U defined as follows:

$$\alpha: X\to \mathscr{P}(U),\ x\mapsto \begin{cases} \mathbb{Z} & \text{if } x\in\{0,3\},\\ 2\mathbb{Z} & \text{if } x\in\{1,2\},\end{cases}$$

and

$$\beta: X\to \mathscr{P}(U),\ x\mapsto \begin{cases} 3\mathbb{Z} & \text{if } x\in\{0,3\},\\ \mathbb{Z} & \text{if } x\in\{1,2\},\end{cases}$$

It is easy to check that $\langle(\alpha,\beta);X\rangle$ is a double-framed soft normal over U.

Proposition 3.16. *Every double-framed soft normal $(\tilde{f},X)$ of a B-algebra X is a double-framed soft subalgebra of X.*

Proof. Put $y:=0, b:=0$ and $a:=y$ in (3.3). Then $\alpha((x*y)*(0*0))\supseteq\alpha(x*0)\cap\alpha(y*0)$ and $\beta((x*y)*(0*0))\subseteq\beta(x*0)\cup\beta(y*0)$ for any $x,y\in X$. Using (B2) and (B1), we have $\alpha(x*y)\supseteq\alpha(x)\cap\alpha(y)$ and $\beta(x*y)\subseteq\beta(x)\cup\beta(y)$. Hence $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U. □

The converse of Proposition 3.16 may not be true in general (Example 3.17).

Example 3.17. Let $E=X$ be the set of parameters, and let $U=X$ be the initial universe set, where $X=\{0,1,2,3,4,5\}$ is a B-algebra as in Example 3.7. Let $\langle(\alpha,\beta);X\rangle$ be double-framed soft set over U defined as follows:

$$\alpha: X\to \mathscr{P}(U),\ x\mapsto \begin{cases} \gamma_3 & \text{if } x=0,\\ \gamma_2 & \text{if } x=5,\\ \gamma_1 & \text{if } x\in\{1,2,3,4\},\end{cases}$$

and

$$\beta: X\to \mathscr{P}(U),\ x\mapsto \begin{cases} \tau_1 & \text{if } x=0,\\ \tau_2 & \text{if } x=5,\\ \tau_3 & \text{if } x\in\{1,2,3,4\},\end{cases}$$

where $\gamma_1,\gamma_2,\gamma_3,\tau_1,\tau_2$ and τ_3 are subsets of U with $\gamma_1\subsetneq\gamma_2\subsetneq\gamma_3$ and $\tau_1\subsetneq\tau_2\subsetneq\tau_3$. It is routine to verify that $\langle(\alpha,\beta);X\rangle$ is a double-framed soft subalgebra over U. But it is not double-framed soft normal over U since since $\alpha(1)=\alpha((1*3)*(4*2))=\gamma_1\not\supseteq\alpha(1*4)\cap\alpha(3*2)=\alpha(5)\cap\alpha(5)=\gamma_2$ and/or $\beta(1)=\beta((1*3)*(4*2))=\tau_3\not\subseteq\beta(1*4)\cup\beta(3*2)=\beta(5)\cup\beta(5)=\tau_2$.

Theorem 3.18. *For a double-framed soft set $\langle(\alpha,\beta);X\rangle$ over U, the following are equivalent:*

(i) *$\langle(\alpha,\beta);X\rangle$ is a double-framed soft normal subalgebra over U.*
(ii) *For every subsets γ and δ of U with $\gamma\in Im(\alpha)$ and $\delta\in Im(\beta)$, the γ-inclusive set and the δ-exclusive set of $\langle(\alpha,\beta);X\rangle$ are normal subalgebras of X.*

Proof. Similar to Theorem 3.10. □

Proposition 3.19. *Let a double-framed soft set $\langle(\alpha,\beta);X\rangle$ over U of a B-algebra X be double-framed soft normal. Then $\alpha(x*y)=\alpha(y*x)$ and $\beta(x*y)=\beta(y*x)$ for any $x,y\in X$.*

Proof. Let $x, y \in X$. By (B1) and (B2), we have $\alpha(x*y) = \alpha((x*y)*(x*x)) \supseteq \alpha(x*x) \cap \alpha(y*x) = \alpha(0) \cap \alpha(y*x) = \alpha(y*x)$. Interchanging x with y, we obtain $\alpha(y*x) \supseteq \alpha(x*y)$.

By (B1) and (B2), we have $\beta(x*y) = \beta((x*y)*(x*x)) \subseteq \beta(x*x) \cup \beta(y*x) = \beta(0) \cup \beta(y*x) = \beta(y*x)$. Interchanging x with y, we obtain $\beta(y*x) \subseteq \beta(x*y)$. □

Theorem 3.20. *Let $\langle(\alpha, \beta); X\rangle$ be a double-framed soft normal subalgebra of a B-algebra X. Then the set $X_{(\alpha,\beta)} := \{x \in X | \alpha(x) = \alpha(0), \beta(x) = \beta(0)\}$ is a normal subalgebra of X.*

Proof. It is sufficient to show that $X_{(\alpha,\beta)}$ is normal. Let $a, b, x, y \in X$ be such that $x*y \in X_{(\alpha,\beta)}$ and $a*b \in X_{(\alpha,\beta)}$. Then $\alpha(x*y) = \alpha(0) = \alpha(a*b), \beta(x*y) = \beta(0) = \beta(a*b)$. Since $\langle(\alpha, \beta); X\rangle$ is a double-framed soft normal subalgebra of X, we have $\alpha((x*a)*(y*b)) \supseteq \alpha(x*y) \cap \alpha(a*b) = \alpha(0)$ and $\beta((x*a)*(y*b)) \subseteq \beta(x*y) \cup \beta(a*b) = \beta(0)$. Using (3.2), we conclude that $\alpha((x*a)*(y*b)) = \alpha(0)$ and $\beta((x*a)*(y*b)) = \beta(0)$. Hence $(x*a)*(y*b) \in X_{(\alpha,\beta)}$. This completes the proof. □

References

[1] S. S. Ahn and H. D. Lee, *Fuzzy subalgebras of BG-algebras*, Comm. Kore. Math. Soc. **19** (2004), 243-251.

[2] J. R. Cho and H. S. Kim, *On B-algebras and quasi groups,* Quasigroups and related systems **8**(2001), 1-6.

[3] F. Feng, Y. B. Jun and X. Zhao, *Soft semirings,* Comput. Math. Appl. **56** (2008) 2621-2628.

[4] Y. B. Jun, *Soft BCK/BCI-algebras,* Comput. Math. Appl. **56** (2008) 1408-1413.

[5] Y. B. Jun, *Union soft sets with applications in BCK/BCI-algebras,* Bull. Korean Math. Soc. **50** (2013), 1937-1956.

[6] Y. B Jun and S. S. Ahn, *Double-framed soft sets with applications in BCK/BCI-algebras*, J. Appl. Math. (2012), 1-15.

[7] Y. B. Jun, E. H. Roh and H. S. Kim, *On BH-algebras*, Sci. Mathematica **1** (1998), 347-354.

[8] Y. B. Jun, E. H. Roh and H. S. Kim, *On fuzzy B-algebras*, Czech. Math. J. **52** (2002), 375-384.

[9] C. B. Kim and H. S. Kim, *On BG-algebras*, Demon. Math. **41** (2008), 497-505.

[10] P. K. Maji, A. R. Roy and R. Biswas, An application of soft sets in a decision making problem, Comput. Math. Appl. **44** (2002) 1077-1083.

[11] D. Molodtsov, Soft set theory - First results, Comput. Math. Appl. **37** (1999) 19-31.

[12] J. Neggers and H. S. Kim, *On B-algebras*, Mate. Vesnik **54**(2002), 21-29.

[13] J. Neggers and H. S. Kim, *A fundamental theorem of B-homomorphism for B-algebras*, Intern. Math. J. **2**(2002), 207-214.

[14] K. S. Yang and S. S. Ahn, Union soft q-ideals in BCI-algebras, Applied Mathematical Scineces **8**(2014), 2859-2869.

APPLICATIONS OF DOUBLE DIFFERENCE FRACTIONAL ORDER OPERATORS TO ORIGINATE SOME SPACES OF SEQUENCES

ANU CHOUDHARY AND KULDIP RAJ

Abstract. In the present article, we introduce and study some sequence spaces by means of double difference fractional order operators, Orlicz function and four dimensional bounded regular matrix. We make an effort to study some topological and algebraic properties of these sequence spaces. Some inclusion relations between newly formed sequence spaces are also establish. Finally, we study several results under the suitable choice of order γ.

1. Introduction and Preliminaries

Let $(\varpi_{k,l}, \nu_{k,l})$ be a double sequence of seminormed spaces such that $\varpi_{k-1,l-1} \subseteq \varpi_{k,l}$ for all non-negative integers k and l. A sequence space X is called solid or normal if and only if it contains all such sequences $y = (y_{k,l})$ corresponding to each of which there is a sequence $x = (x_{k,l}) \in X$ such that $|y_{k,l}| \leq |x_{k,l}|$ for all non negative integers k and l. Let Q be a normal sequence space and Ω^2 denotes the set of all double complex sequences. Define a linear space

$$\Omega^2(\varpi_{k,l}) = \{x = (x_{k,l}) \in \Omega^2 : x_{k,l} \in \varpi_{k,l} \text{ for all non-negative integers } k \text{ and } l\}.$$

Let ν and ν' be seminorms on a linear space X. Then ν is said to be stronger than ν' if whenever $(x_{k,l})$ is a sequence such that $\nu(x_{k,l}) \to 0$, then also $\nu'(x_{k,l}) \to 0$. If each is stronger than the other, then ν and ν' are said to be equivalent.

A double sequence has Pringsheim limit L (denoted by $P-\lim x = L$) provided that given $\epsilon > 0$ there exist $n \in \mathbb{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n$ (see [11]). A double sequence $x = (x_{k,l})$ is bounded if there exists a positive number n such that $|x_{k,l}| < n$ for all k and l.

Some initial works on double sequences is due to Bromwich [5]. Later on, the double sequences were studied in (see [12], [13]) and operators on sequence spaces were studied in (see [1], [9]).

The fractional difference operator $\Delta^{(\gamma)}$ for a positive proper fraction γ on single sequence is defined as

$$\Delta^{(\gamma)}(x_k) = \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(\gamma+1)}{m!\Gamma(\gamma-m+1)} x_{k-m},$$

where $\Gamma(\gamma)$ denotes the Euler gamma function of a real number γ or generalized factorial function (see [2], [3]). For $\gamma \notin \{0, -1, -2, -3, \cdots\}$, $\Gamma(\gamma)$ can be expressed as an improper integral,

$$\Gamma(\gamma) = \int_0^{\infty} e^{-s} s^{\gamma-1} ds.$$

2010 *Mathematics Subject Classification.* 40A05, 40A30.

Key words and phrases. Orlicz function, fractional double difference operator, double sequence, paranormed space.

For $x \in \Omega^2$ and a positive proper fraction γ, the double difference operator of fractional order γ is defined as

$$\Delta_2^{(\gamma)}(x_{k,l}) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}\frac{\Gamma(\gamma+1)^2}{m!n!\Gamma(\gamma-m+1)\Gamma(\gamma-n+1)}x_{k-m,l-n}. \tag{1.1}$$

The above defined infinite series can be reduced to finite series if γ is a positive integer (see [4]). Throughout the text it is assumed that $(x_{k,l}) = 0$ for any negative integers k and l.

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists $R > 0$ such that $M(2u) \leq RM(u), u \geq 0$.

The idea of Orlicz function was used by Lindenstrauss and Tzafriri [7] to define the following sequence space:

$$\ell_M = \Big\{x = (x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

known as an *Orlicz sequence space*. The space l_M is a Banach space with the norm,

$$\|x\| = \inf\Bigg\{\rho > 0 : \sum_{k=1}^{\infty} M\Bigg(\frac{|x_k|}{\rho}\Bigg) \leq 1\Bigg\}.$$

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a *Musielak-Orlicz function* (see [8], [10]).

Remark 1.1. (1) Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function and q be a non-negative integer. Then for a real number $d \in [0, \infty)$, we have

(i)$\mathcal{M}(qx) \leq q\mathcal{M}(x)$

(ii) $\mathcal{M}(dx) \leq (1 + \lfloor d \rfloor)\mathcal{M}(x)$, where $\lfloor . \rfloor$ denotes the greatest integer function.

(2) For a complex number α,

$$|\alpha|^{p_{k,l}} \leq \max\{1, |\alpha|^L\}$$

and

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq D(|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}),$$

where $L = \sup_{k,l} p_{k,l} < \infty$ and $D = \max(1, 2^{L-1})$.

Let $\mathcal{A} = (a_{ijkl})$ be a four-dimensional infinite matrix of scalars. For all $i, j \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the sum

$$y_{i,j} = \sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}x_{k,l}$$

is called the $\mathcal{A}$-*means* of the double sequence $(x_{k,l})$. A double sequence $(x_{k,l})$ is said to be $\mathcal{A}$-*summable* to the limit L if the $\mathcal{A}$-means exist for all i, j in the sense of Pringsheim's convergence

$$P\text{-}\lim_{p,q\to\infty}\sum_{k,l=0,0}^{p,q} a_{ijkl}x_{k,l} = y_{i,j} \text{ and } P\text{-}\lim_{i,j\to\infty} y_{i,j} = L.$$

A four-dimensional matrix $\mathcal{A}$ is said to be *bounded-regular* (or *RH-regular*) if every bounded P-convergent sequence is $\mathcal{A}$-summable to the same limit and the $\mathcal{A}$-means are also bounded.

Theorem 1.2. *(Robison* [14] *and Hamilton* [6]*) The four dimensional matrix* $\mathcal{A}$ *is RH-regular if and only if*

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

(RH_1) $P\text{-}\lim_{i,j} a_{ijkl} = 0$ *for each* k *and* l,

(RH_2) $P\text{-}\lim_{i,j} \sum_{k,l=1,1}^{\infty,\infty} |a_{ijkl}| = 1$,

(RH_3) $P\text{-}\lim_{i,j} \sum_{k=1}^{\infty} |a_{ijkl}| = 0$ *for each* l,

(RH_4) $P\text{-}\lim_{i,j} \sum_{l=1}^{\infty} |a_{ijkl}| = 0$ *for each* k,

(RH_5) $\sum_{k,l=1,1}^{\infty,\infty} |a_{ijkl}| < \infty$ *for all* $i, j \in \mathbb{N}$.

A real valued function g defined on a linear space X is called a paranorm, if it satisfies the following conditions for all $x, y \in X$ and for all scalars β
(i) $g(\lambda) = 0$, where λ is the zero element of X
(ii) $g(-x) = g(x)$
(iii) $g(x + y) \leq g(x) + g(y)$
(iv) If (β_n) is a sequence of scalars with $\beta_n \to 0$ as $n \to 0$ and x_n is a sequence in X such that $g(x_n - x) \to 0$ as $n \to \infty$ for some $x \in X$, then $g(\beta_n x_n - \beta x) \to 0$ as $n \to \infty$.
Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function, $\mathcal{A} = (a_{ijkl})$ be a nonnegative four-dimensional bounded-regular matrix, $u = (u_{k,l})$ be any double sequence of strictly positive real numbers, $p = (p_{k,l})$ be a bounded double sequence of positive real numbers, $\Delta_2^{(\gamma)}$ denotes the double difference operator of fractional order γ. In this paper we define the following sequence space
$Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}] =$

$$\Big\{x = (x_{k,l}) \in \Omega^2(\varpi_{k,l}) : \Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)_{i,j} \in Q, \text{ for some } \rho > 0\Big\}.$$

Remark 1.3. (1) Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function and $\rho = \rho_1 + \rho_2$. Then for $x = (x_{k,l})$ and $y = (y_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}]$, we have

$$\begin{aligned}
&\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l}) + u_{k,l}\Delta_2^{(\gamma)}(y_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}} \\
&\qquad \leq D\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho_1}\Big)\Big)\Big]^{p_{k,l}} \\
&\qquad + \sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)} y_{k,l}}{\rho_2}\Big)\Big)\Big]^{p_{k,l}}\Big)
\end{aligned}$$

for all non-negative integers i and j and for some $\rho_1, \rho_2 > 0$.

(2) Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function and $d \in \mathbb{C}$. Then for $L = \sup_{k,l} p_{k,l} < \infty$, we have

$$\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(dx_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}$$
$$\leq \max\{1,(1+\lfloor|d|\rfloor)^L\}\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)$$

for all non-negative integers i and j and for some $\rho > 0$.

(3) Let $\mathcal{M} = (M_{k,l})$ and $\mathcal{M}' = (M'_{k,l})$ be two Musielak Orlicz functions. Then

$$\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[(M_{k,l}+M'_{k,l})\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}$$
$$\leq D\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}$$
$$+ \sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M'_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)$$

for all non-negative integers i and j and for some $\rho > 0$.

(4) Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function. Let $\nu = (\nu_{k,l})$ and $\nu' = (\nu'_{k,l})$ be two sequences of seminorms. Then

$$\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big((\nu_{k,l}+\nu'_{k,l})\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}$$
$$\leq D\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}$$
$$+ \sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu'_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)$$

for all non-negative integers i and j and for some $\rho > 0$.

The main goal of this paper is to introduce the double difference operator $\Delta_2^{(\gamma)}$ of fractional order γ. In this study, being an application of double difference operator $\Delta_2^{(\gamma)}$, some new difference double sequence spaces of fractional order have been introduced and subsequently, their topological and algebraic properties have been discussed in detail. Infact, this study involves new results obtained under different suitable choice of γ.

2. **Main Results**

Theorem 2.1. *Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function, $\nu = (\nu_{k,l})$ be a sequence of seminorms and $u = (u_{k,l})$ be a double sequence of strictly positive real numbers. Then the sequence space $Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}]$ is a linear space over the complex field $\mathbb{C}$.*

Proof. This is a routine matter, so we omit it. ☐

Theorem 2.2. *Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function, $\nu = (\nu_{k,l})$ be a sequence of seminorms and $u = (u_{k,l})$ be a double sequence of strictly positive real numbers. Then the*

sequence space $Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}]$ *is a paranormed space with paranorm* g *defined by*

$$g(x) = \inf\Bigg\{(\rho)^{\frac{p_{k,l}}{L}} : \Bigg(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Bigg[M_{k,l}\Bigg(\nu_{k,l}\Bigg(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\Bigg)\Bigg)\Bigg]^{p_{k,l}}\Bigg)^{\frac{1}{N}} \leq 1, \ \textit{for some } \rho > 0\Bigg\},$$

where $N = \max\{1, L\}$ *and* $L = \sup_{k,l} p_{k,l} < \infty$.

Proof. (i) Clearly $g(x) \geq 0$, for $x = (x_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}]$. Since $M_{k,l}(0) = 0$, we get $g(0) = 0$.
(ii) $g(-x) = g(x)$.
(iii) Let $x = (x_{k,l}), y = (y_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}]$, then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\Bigg(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Bigg[M_{k,l}\Bigg(\nu_{k,l}\Bigg(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho_1}\Bigg)\Bigg)\Bigg]^{p_{k,l}}\Bigg)^{\frac{1}{N}} \leq 1$$

and

$$\Bigg(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Bigg[M_{k,l}\Bigg(\nu_{k,l}\Bigg(\frac{u_{k,l}\Delta_2^{(\gamma)}(y_{k,l})}{\rho_2}\Bigg)\Bigg)\Bigg]^{p_{k,l}}\Bigg)^{\frac{1}{N}} \leq 1.$$

Now for $\rho = \rho_1 + \rho_2$ and by using Minkowski's inequality, we have

$$\begin{aligned}
&\Bigg(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Bigg[M_{k,l}\Bigg(\nu_{k,l}\Bigg(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l}) + u_{k,l}\Delta_2^{(\gamma)}(y_{k,l})}{\rho_1+\rho_2}\Bigg)\Bigg)\Bigg]^{p_{k,l}}\Bigg)^{\frac{1}{N}} \\
\leq\ & \Bigg(\frac{\rho_1}{\rho_1+\rho_2}\Bigg)\Bigg(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Bigg[M_{k,l}\Bigg(\nu_{k,l}\Bigg(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho_1}\Bigg)\Bigg)\Bigg]^{p_{k,l}}\Bigg)^{\frac{1}{N}} \\
+\ & \Bigg(\frac{\rho_2}{\rho_1+\rho_2}\Bigg)\Bigg(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Bigg[M_{k,l}\Bigg(\nu_{k,l}\Bigg(\frac{u_{k,l}\Delta_2^{(\gamma)}y_{k,l}}{\rho_2}\Bigg)\Bigg)\Bigg]^{p_{k,l}}\Bigg)^{\frac{1}{N}} \\
\leq\ & 1.
\end{aligned}$$

Hence, $g(x+y)$

$$\begin{aligned}
&= \inf\Bigg\{(\rho_1+\rho_2)^{\frac{p_{k,l}}{L}} : \Bigg(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Bigg[M_{k,l}\Bigg(\nu_{k,l}\Bigg(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l}) + u_{k,l}\Delta_2^{(\gamma)}(y_{k,l})}{\rho}\Bigg)\Bigg)\Bigg]^{p_{k,l}}\Bigg)^{\frac{1}{N}} \leq \\
&\qquad 1, \text{ for some } \rho > 0\Bigg\} \\
\leq\ & \inf\Bigg\{(\rho_1)^{\frac{p_{k,l}}{L}} : \Bigg(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Bigg[M_{k,l}\Bigg(\nu_{k,l}\Bigg(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho_1}\Bigg)\Bigg)\Bigg]^{p_{k,l}}\Bigg)^{\frac{1}{N}} \leq 1, \text{ for some } \rho_1 > 0\Bigg\} \\
+\ & \inf\Bigg\{(\rho_2)^{\frac{p_{k,l}}{L}} : \Bigg(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Bigg[M_{k,l}\Bigg(\nu_{k,l}\Bigg(\frac{u_{k,l}\Delta_2^{(\gamma)}(y_{k,l})}{\rho_2}\Bigg)\Bigg)\Bigg]^{p_{k,l}}\Bigg)^{\frac{1}{N}} \leq 1, \text{ for some } \rho_2 > 0\Bigg\} \\
=\ & g(x) + g(y).
\end{aligned}$$

(iv) Finally, we show that scalar multiplication is continuous. In order to show this, let us consider a complex number σ. Then by definition we have

$$
\begin{aligned}
&g(\sigma x)\\
&= \inf\Big\{(\rho)^{\frac{p_{k,l}}{L}} : \Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{\sigma u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{N}} \leq 1, \rho > 0\Big\},\\
&= \inf\Big\{(|\sigma|t)^{\frac{p_{k,l}}{L}} : \Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}(x_{k,l})}{t}\Big)\Big)\Big]^{p_{k,l}}\Big)^{\frac{1}{N}} \leq 1, t > 0\Big\},
\end{aligned}
$$

where $t = \frac{\rho}{|\sigma|}$. Hence the proof. □

Theorem 2.3. *Let $\mathcal{M} = (M_{k,l})$ and $\mathcal{M}' = (M'_{k,l})$ be two Musielak Orlicz functions, $u = (u_{k,l})$ be a double sequence of strictly positive real numbers and $\mathcal{A} = (a_{ijkl})$ be a nonnegative four-dimensional bounded-regular matrix. Then*

$$Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}] \cap Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}'] \subseteq Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M} + \mathcal{M}'].$$

Proof. Suppose $x = (x_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}] \cap Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}']$. This implies that

$$\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)_{i,j}$$

and

$$\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M'_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)_{i,j}$$

both are in Q. Now by using part (3) of Remark 1.3, we have

$$
\begin{aligned}
&\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[(M_{k,l} + M'_{k,l})\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}\\
&\qquad\leq D\Big\{\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}\\
&\qquad+ \sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M'_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big\}.
\end{aligned}
$$

Since Q is normal, $\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[(M_{k,l} + M'_{k,l})\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)_{i,j} \in Q.$

Then $x = (x_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M} + \mathcal{M}']$. Hence the proof. □

Theorem 2.4. *Suppose that $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function, $\nu = (\nu_{k,l})$ and $\nu' = (\nu'_{k,l})$ be two double sequences of seminorms. Then*

$$Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}] \cap Q[\Delta_2^{(\gamma)}, p, \nu', u, \mathcal{A}, \mathcal{M}] \subseteq Q[\Delta_2^{(\gamma)}, p, \nu + \nu', u, \mathcal{A}, \mathcal{M}].$$

Proof. One can easily obtain the proof by using part (4) of Remark 1.3. So, we omit it. □

Theorem 2.5. *Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function. If $\nu = (\nu_{k,l})$ and $\nu' = (\nu'_{k,l})$ be two double sequences of seminorms such that $(\nu_{k,l})$ is stronger than $(\nu'_{k,l})$ for each k and l, then $Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}] \subseteq Q[\Delta_2^{(\gamma)}, p, \nu', u, \mathcal{A}, \mathcal{M}]$.*

Proof. Consider a double sequence $x = (x_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}]$. Then

$$\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)_{i,j} \in Q.$$

Since each $\nu_{k,l}$ is stronger than corresponding $\nu'_{k,l}$, we have a natural number $N_{k,l}$ corresponding to each pair of non-negative integer k and l such that $\nu'_{k,l}(w) \leq N_{k,l}\nu_{k,l}(w)$. Let $N = \max\{N_{k,l}\}$. Then $\nu'_{k,l}(w) \leq N\nu_{k,l}(w)$ for all non-negative integers k and l. Thus,

$$\nu'_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big) \leq N\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big).$$

From Remark 1.1, we have

$$\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu'_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}$$
$$\leq \max\{1, N^L\}\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}.$$

Since Q is normal, $\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu'_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)_{i,j} \in Q.$

This implies $x = (x_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu', u, \mathcal{A}, \mathcal{M}]$. □

Corollary 2.6. *Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function. If $\nu = (\nu_{k,l})$ and $\nu' = (\nu'_{k,l})$ be two double sequences of seminorms such that $(\nu_{k,l})$ is equivalent to $(\nu'_{k,l})$ for each k and l. Then*

$$Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}] = Q[\Delta_2^{(\gamma)}, p, \nu', u, \mathcal{A}, \mathcal{M}].$$

Theorem 2.7. *Suppose $\mathcal{M} = (M_{k,l})$ and $\mathcal{M}' = (M'_{k,l})$ be two Musielak Orlicz function such that $M_{k,l}(1)$ is finite for each k and l. Let $\mathcal{A} = (a_{ijkl})$ be a nonnegative four-dimensional bounded-regular matrix. Then*

$$Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}'] \subseteq Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M} \circ \mathcal{M}'].$$

Proof. Consider $x = (x_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}']$. So,

$$\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M'_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}}\Big)_{i,j} \in Q.$$

Since each $(M_{k,l})$ is continuous and $M_{k,l}(0) = 0$ for each k and l, we choose $\varsigma \in (0,1)$ corresponding to an arbitrary $\epsilon > 0$ such that $M_{k,l}(s) < \epsilon$ for $0 \leq s \leq \varsigma$. Let us take

$$s_{k,l} = M'_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)$$

and

$$\sum_{k,l=1,1}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\big(s_{k,l}\big)\Big]^{p_{k,l}} = \sum_{1} a_{ijkl}\Big[M_{k,l}\big(s_{k,l}\big)\Big]^{p_{k,l}} + \sum_{2} a_{ijkl}\Big[M_{k,l}\big(s_{k,l}\big)\Big]^{p_{k,l}}, \tag{2.1}$$

where the first summation is over $s_{k,l} \leq \varsigma$ and the second is taken over $s_{k,l} > \varsigma$. For $s_{k,l} \leq \varsigma$, we have $M_{k,l}(s_{k,l}) < \epsilon$ and hence

$$\sum_{1} a_{ijkl}\Big[M_{k,l}\big(s_{k,l}\big)\Big]^{p_{k,l}} < \sum_{1} a_{ijkl}[\epsilon]^{p_{k,l}}.$$

Now by using Part (2) of Remark 1.1, we have

$$\sum_1 a_{ijkl}\Big[M_{k,l}\big(s_{k,l}\big)\Big]^{p_{k,l}} < \max\{1,\epsilon^L\}\sum_1 a_{ijkl} \le \max\{1,\epsilon^L\}\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}. \tag{2.2}$$

For $s_{k,l} > \varsigma$, we have $s_{k,l} < \left(\frac{s_{k,l}}{\varsigma}\right)$. So, from Part (1) of Remark 1.1, we have

$$M_{k,l}(s_{k,l}) < M_{k,l}\left(\frac{s_{k,l}}{\varsigma}\right) \le \left(1+\left\lfloor\frac{s_{k,l}}{\varsigma}\right\rfloor\right)M_{k,l}(1) \le 2M_{k,l}(1)\frac{s_{k,l}}{\varsigma}.$$

Let $\xi = \max_{k,l}(M_{k,l}(1))$, then $M_{k,l}(s_{k,l}) < 2\xi\frac{s_{k,l}}{\varsigma}$. By using Part (2) of Remark 1.1, we get

$$\sum_2 a_{ijkl}\Big[M_{k,l}\big(s_{k,l}\big)\Big]^{p_{k,l}} \le \max\left(1,\left(\frac{2\xi}{\varsigma}\right)^L\right)\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}[s_{k,l}]^{p_{k,l}}. \tag{2.3}$$

From (2.1), (2.2) and (2.3), we have

$$\begin{aligned}\sum_{k,l=1,1}^{\infty,\infty} a_{ijkl}[M_{k,l}(s_{k,l})]^{p_{k,l}} &\le \max(1,\epsilon^L)\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\\ &+ \max\left(1,\left(\frac{2\xi}{\varsigma}\right)^L\right)\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}[s_{k,l}]^{p_{k,l}}.\end{aligned}$$

Since Q is normal,

$$\begin{aligned}&\left(\sum_{k,l=1,1}^{\infty,\infty} a_{ijkl}[M_{k,l}(s_{k,l})]^{p_{k,l}}\right)_{i,j}\\ &= \left(\sum_{k,l=1,1}^{\infty,\infty} a_{ijkl}\left[(M_{k,l}\circ M'_{k,l})\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\right)\right)\right]^{p_{k,l}}\right)_{i,j} \in Q.\end{aligned}$$

Thus, $x = (x_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}\circ\mathcal{M}']$. Hence the proof. □

Theorem 2.8. *Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function such that $M_{k,l}(s) \le M_{k-1,l-1}(s)$ for all $s \in [0,\infty)$, $\nu = (\nu_{k,l})$ be a double sequence of seminorm such that $\nu_{k,l}(s) \le \nu_{k-1,l-1}(s)$ for all s. Let $\mathcal{A} = (a_{ijkl})$ be a nonnegative four-dimensional bounded-regular matrix such that $a_{ijkl} \le a_{ij(k-1)(l-1)}$ for all non-negative integers i, j, k and l and suppose $p = (p_{k,l} \equiv p)$ is a constant sequence of positive real number. Then*

$$Q[\Delta_2^{(\gamma-1)}, p, \nu, u, \mathcal{A}, \mathcal{M}] \subset Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}].$$

Proof. Suppose $x = (x_{k,l})$ is a double sequence in $Q[\Delta_2^{(\gamma-1)}, p, \nu, u, \mathcal{A}, \mathcal{M}]$. Then

$$\left(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma-1)}x_{k,l}}{\rho}\right)\right)\right]^p\right)_{i,j} \in Q.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Since $M_{k,l}(s) \le M_{k-1,l-1}(s), \nu_{k,l}(s) \le \nu_{k-1,l-1}(s)$ and $a_{ijkl} \le a_{ij(k-1)(l-1)}$, we have

$$\begin{aligned}
&\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma-1)}x_{k-1,l-1}}{\rho}\Big)\Big)\Big]^p \\
&\le \sum_{k,l=0,0}^{\infty,\infty} a_{ij(k-1)(l-1)}\Big[M_{k-1,l-1}\Big(\nu_{k-1,l-1}\Big(\frac{u_{k-1,l-1}\Delta_2^{(\gamma-1)}x_{k-1,l-1}}{\rho}\Big)\Big)\Big]^p \\
&= \sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma-1)}x_{k,l}}{\rho}\Big)\Big)\Big]^p.
\end{aligned}$$

Since Q is normal ,$\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma-1)}x_{k-1,l-1}}{\rho}\Big)\Big)\Big]^p\Big)_{i,j} \in Q$. Now,

$$\begin{aligned}
&\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^p \\
&= \sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma-1)}(x_{k,l}-x_{k-1,l-1})}{\rho}\Big)\Big)\Big]^p \\
&\le D\Big\{\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma-1)}x_{k,l}}{\rho}\Big)\Big)\Big]^p \\
&+ \sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma-1)}x_{k-1,l-1}}{\rho}\Big)\Big)\Big]^p\Big\}.
\end{aligned}$$

Again Q is normal. So, $\Big(\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^p\Big)_{i,j} \in Q$.

Thus, $x = (x_{k,l}) \in Q[\Delta_2^{(\gamma)}, p, \nu, u, \mathcal{A}, \mathcal{M}]$. □

In order to show the strictness of the above inclusion let us consider the following example.

Example 2.9. *Consider $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function with $M_{k,l}(x) = x, (\nu_{k,l})$ be a sequence of seminorm with $\nu_{k,l}(x) = |x|, p_{k,l} = 1, \mathcal{A} = I$ be an identity matrix of infinite order, $\rho = 1, u_{k,l}(x) = x$ and $(x_{k,l}) = 1$ for all non-negative integers k and l. Then*

$$\sup_{i,j}\sum_{k,l=0,0}^{\infty,\infty} a_{ijkl}\Big[M_{k,l}\Big(\nu_{k,l}\Big(\frac{u_{k,l}\Delta_2^{(\gamma)}x_{k,l}}{\rho}\Big)\Big)\Big]^{p_{k,l}} = \sup_{i,j}|\mathcal{J}_{i,j}(\gamma)| \text{ (say)},$$

where $\mathcal{J}_{i,j}(\gamma)$ is the expansion of the series (1.1) *for $x_{k,l} = 1$. For $\gamma = \frac{1}{2}$, $\sup_{i,j}|\mathcal{J}_{i,j}(\gamma)| < \infty$ whereas for $\gamma = -\frac{1}{2}$, $\sup_{i,j}|\mathcal{J}_{i,j}(\gamma)| = \infty$. Thus, $x = (x_{k,l}) \in \ell_\infty[\Delta_2^{(1/2)}, p, \nu, u, I, \mathcal{M}]$ but $x = (x_{k,l}) \notin \ell_\infty[\Delta_2^{(-1/2)}, p, \nu, u, I, \mathcal{M}]$. Therefore, the inclusion relation is strict in general.*

Theorem 2.10. *Let $\mathcal{M} = (M_{k,l})$ be a Musielak Orlicz function and P be a nonnegative four-dimensional bounded-regular matrix whose all entries are 1. If $0 < \inf_{k,l} h_{k,l} < h_{k,l} \le c_{k,l} < \sup_{k,l} c_{k,l} < \infty$ for all non-negative integers k and l. Then*

$$\ell_\infty[\Delta_2^{(\gamma)}, h, \nu, u, P, \mathcal{M}] \subseteq \ell_\infty[\Delta_2^{(\gamma)}, c, \nu, u, P, \mathcal{M}].$$

Proof. Consider that $x = (x_{k,l}) \in \ell_\infty[\Delta_2^{(\gamma)}, h, \nu, u, P, \mathcal{M}]$. This implies

$$\sup_{i,j} \sum_{k,l=0,0}^{\infty,\infty} \left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho}\right)\right)\right]^{h_{k,l}} = \sum_{k,l=0,0}^{\infty,\infty} \left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho}\right)\right)\right]^{h_{k,l}} < \infty.$$

Then for sufficiently large k say k_0 and sufficiently large l say l_0, we have

$$\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho}\right)\right)\right]^{h_{k,l}} \leq 1$$

for all $k \geq k_0$ and $l \geq l_0$. Hence,

$$\left\{\left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho}\right)\right)\right]^{h_{k,l}}\right\}^{\frac{c_{k,l}}{h_{k,l}}} \leq \left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho}\right)\right)\right]^{h_{k,l}}$$

for all $k \geq k_0$ and $l \geq l_0$. Now by taking summation from k_0 to ∞ and l_0 to ∞ on both sides, we have

$$\sum_{k,l=k_0,l_0}^{\infty,\infty} \left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho}\right)\right)\right]^{c_{k,l}} \leq \sum_{k,l=k_0,l_0}^{\infty,\infty} \left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho}\right)\right)\right]^{h_{k,l}} < \infty.$$

Thus, $\sup_{i,j} \sum_{k,l=0,0}^{\infty,\infty} \left[M_{k,l}\left(\nu_{k,l}\left(\frac{u_{k,l}\Delta_2^{(\gamma)} x_{k,l}}{\rho}\right)\right)\right]^{c_{k,l}} < \infty.$

Therefore, $x = (x_{k,l}) \in \ell_\infty[\Delta_2^{(\gamma)}, c, \nu, u, P, \mathcal{M}]$. Hence the proof. □

References

[1] A. Alotaibi, M. Mursaleen and S. A. Mohiuddine, *Some fixed point theorems for Meir-keeler condensing operators with applications to integral equations*, Bull. Belg. Math. Soc. Simon Stevin, **22** (2015), 529-541.

[2] P. Baliarsingh and S. Dutta, *On the classes of fractional order difference sequence spaces and their matrix transformations*, Appl. Math. Comput., **250** (2015), 665-674.

[3] P. Baliarsingh, *Some new difference sequence spaces of fractional order and their dual spaces*, Appl. Math. Comput., **219** (2013), 9737-9742.

[4] P. Baliarsingh, *On difference double sequence spaces of fractional order*, Indian J. Math., **58** (2016), 287-310.

[5] T. J. Bromwich, *An introduction to the theory of infinite series*, Macmillan and Co. Ltd., New York (1965).

[6] H. J. Hamilton, *Transformation of multiple sequences*, Duke Math. J., **2** (1936), 29-60.

[7] J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces*, Israel J. Math., **10** (1971), 379-390.

[8] L. Maligranda, *Orlicz spaces and interpolation*, Seminars in Mathematics **5**, Polish Academy of Science, (1989).

[9] M. Mursaleen and K. J. Ansari, *Approximation of q-Stancu-beta operators which preserve x^2*, Bull. Malays. Math. Sci. Soc., **40** (2017), 1479-1491.

[10] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, **1034** (1983).

[11] A. Pringsheim, *Zur Theori der zweifach unendlichen Zahlenfolgen*, Math. Ann., **53** (1900), 289-321.

[12] K. Raj, A. Choudhary and C. Sharma, *Almost strongly Orlicz double sequence spaces of regular matrices and their applications to statistical convergence*, Asian-Eur. J. Math., doi.org/10.1142/S1793557118500730 (2017).

[13] K. Raj and R. Anand, *Double difference spaces of almost null and almost convergent sequences for Orlicz function*, J. Comput. Anal. Appl., **24** (2018), 773-783.

[14] G. M. Robison, *Divergent double sequences and series*, Trans. Amer. Math. Soc., **28** (1926), 50-73.

School of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J & K (India)
E-mail address: `anuchoudhary407@gmail.com`
E-mail address: `kuldipraj68@gmail.com`

On the dynamics of higher-order anti-competitive system: $x_{n+1} = A + \frac{y_n}{\sum_{i=1}^{k} x_{n-i}}$, $y_{n+1} = B + \frac{x_n}{\sum_{i=1}^{k} y_{n-i}}$

A. Q. Khan* M. A. El-Moneam† E. S. Aly‡ M. A. Aiyashi§

Abstract

We study the boundedness and persistence, asymptotic stability, existence and uniqueness of positive equilibrium point, and rate of convergence of an anti-competitive system of higher-order difference equations. The proposed work is considerably extended and improve some existing results in the literature.

Keywords: difference equations; boundedness; persistence; asymptotic stability; rate of convergence

2010 AMS Mathematics subject classifications: 39A10, 40A05.

1 Introduction

Difference equations or systems of difference equations play a vital role in the development of different sciences ranging from life to decision sciences (see [1–7] and references cited therein). DeVault *et al*. [5] have investigated that every positive solution of the difference equation: $x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}},\ n = 0, 1, \cdots$, converges to a period two solution. Abu-Saris and DeVault [6] have investigated the global stability of the positive equilibrium of the difference equation: $x_{n+1} = A + \frac{x_n}{x_{n-k}},\ n = 0, 1, \cdots$. Zhang *et al*. [7] have studied the global dynamics of the difference equation: $x_{n+1} = A + \frac{x_n}{\sum_{i=1}^{2} y_{n-i}},\ y_{n+1} = B + \frac{y_n}{\sum_{i=1}^{2} x_{n-i}},\ n = 0, 1, \cdots$. In this paper, our goal is to investigate the dynamics of following higher-order anti-competitive system of difference equations:

$$x_{n+1} = A + \frac{y_n}{\sum_{i=1}^{k} x_{n-i}},\ y_{n+1} = B + \frac{x_n}{\sum_{i=1}^{k} y_{n-i}},\ n = 0, 1, \cdots, \tag{1}$$

where initial conditions $x_{-p},\ y_{-p},\ p = k,\ k-1,\ k-2, \cdots, 1, 0$ and $A,\ B$ are positive.

2 Main results

Hereafter we will prove main results for under consideration system.

Theorem 1. *If $ABk^2 > 1$, then the following statements holds:*

(i) Every positive solution $\{(x_n, y_n)\}$ of (1) is bounded and persists.

(ii) The interval $\left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right] \times \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$ is invariant set for (1).

*Department of Mathematics, University of Azad Jammu and Kashmir, Muzaffarabad 13100, Pakistan, e-mail: abdulqadeerkhan1@gmail.com

†Department of Mathematics, Faculty of Science, Jazan University, Jazan, Kingdom of Saudi Arabia, e-mail: maliahmedibrahim@jazanu.edu.sa

‡Department of Mathematics, Faculty of Science, Jazan University, Jazan, Kingdom of Saudi Arabia, e-mail: elkhateeb@jazanu.edu.sa

§Department of Mathematics, Faculty of Science, Jazan University, Jazan, Kingdom of Saudi Arabia, e-mail: maiyashi@jazanu.edu.sa

Proof. (i) If $\{(x_n, y_n)\}$ be a positive solution of (1) then

$$x_n \geq A,\ y_n \geq B,\ n = 0, 1, \cdots . \tag{2}$$

From (1) and (2), one gets

$$x_{n+1} \leq A + \frac{1}{kA} y_n,\ y_{n+1} \leq B + \frac{1}{kB} x_n. \tag{3}$$

Moreover, from (3), one gets

$$x_{n+1} \leq A + \frac{B}{kA} + \frac{1}{k^2AB} x_{n-1},\ y_{n+1} \leq B + \frac{A}{kB} + \frac{1}{k^2AB} y_{n-1}. \tag{4}$$

Now consider

$$\varsigma_{n+1} = A + \frac{B}{kA} + \frac{1}{k^2AB} \varsigma_{n-1},\ \varrho_{n+1} = B + \frac{A}{kB} + \frac{1}{k^2AB} \varrho_{n-1}. \tag{5}$$

Therefore, solution $\{(\varsigma_n, \varrho_n)\}$ of (5) is given by

$$\begin{aligned} \varsigma_n &= c_1 \left(\sqrt{\frac{1}{k^2AB}}\right)^n + c_2 \left(-\sqrt{\frac{1}{k^2AB}}\right)^n + \frac{kB\left(kA^2+B\right)}{k^2AB-1}, \\ \varrho_n &= d_1 \left(\sqrt{\frac{1}{k^2AB}}\right)^n + d_2 \left(-\sqrt{\frac{1}{k^2AB}}\right)^n + \frac{kA\left(kB^2+A\right)}{k^2AB-1}, \end{aligned} \tag{6}$$

where $c_1,\ c_2,\ d_1,\ d_2$ depend upon $\varsigma_{-1},\ \varsigma_0,\ \varrho_{-1},\ \varrho_0$. Assuming $ABk^2 > 1$, then (6) implies that $\{\varsigma_n\}$ and $\{\varrho_n\}$ are bounded. Now considering solution $\{(\varsigma_n, \varrho_n)\}$ of (6) for which

$$\varsigma_{-1} = x_{-1},\ \varsigma_0 = x_0,\ \varrho_{-1} = y_{-1},\ \varrho_0 = y_0, \tag{7}$$

where $x_{-1}, x_0 \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right]$ and $y_{-1}, y_0 \in \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$. From (4) and (7) one gets

$$x_n \leq \frac{kB\left(kA^2+B\right)}{k^2AB-1},\ y_n \leq \frac{kA\left(kB^2+A\right)}{k^2AB-1}. \tag{8}$$

From (2) and (8), we get

$$A \leq x_n \leq \frac{kB\left(kA^2+B\right)}{k^2AB-1},\ B \leq y_n \leq \frac{kA\left(kB^2+A\right)}{k^2AB-1},\ n = 0, 1, \cdots .$$

□

Proof. (ii) Follows from induction. □

Theorem 2. *System (1) has a unique positive equilibrium point* $(\bar{x}, \bar{y}) \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right] \times \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$ *if*

$$k^2 \left(\frac{2k^2B(kA^2+B)}{k^2AB-1}\left(\frac{kB(kA^2+B)}{k^2AB-1} - A\right) - B\right)\left(\frac{2kB(kA^2+B)}{k^2AB-1} - A\right) < 1. \tag{9}$$

Proof. Consider

$$x = A + \frac{y}{kx},\ y = B + \frac{x}{ky}. \tag{10}$$

From (10),

$$y = kx(x-A),\ x = ky(y-B).$$

Defining

$$S(x) = ks(x)(s(x) - B) - x, \tag{11}$$

where

$$s(x) = kx(x-A), \tag{12}$$

and $x \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right]$. We claim that $S(x)=0$ has a unique solution $x \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right]$. From (11) and (12) one gets

$$S'(x) = 2ks(x)s'(x) - kBs'(x) - 1, \tag{13}$$

and

$$s'(x) = 2kx - kA. \tag{14}$$

Now if $\bar{x} \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right]$ be a solution of $S(x)=0$ then from (11) and (12) one gets

$$ks(\bar{x})(s(\bar{x}) - B) = \bar{x}, \tag{15}$$

where

$$s(\bar{x}) = k\bar{x}(\bar{x} - A). \tag{16}$$

In view of (14), (15) and (16), equation (13) becomes

$$\begin{aligned} S'(x) &= k^2\left(2kx(x-A) - B\right)(2x-A) - 1, \\ &\leq k^2\left(\frac{2k^2B(kA^2+B)}{k^2AB-1}\left(\frac{kB(kA^2+B)}{k^2AB-1} - A\right) - B\right)\left(\frac{2kB(kA^2+B)}{k^2AB-1} - A\right) - 1. \end{aligned} \tag{17}$$

Now assume that (9) hold then from (17) one gets $S'(x) < 0$. Hence $S(x) = 0$ has a unique positive solution $\bar{x} \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right]$. □

Theorem 3. *If*

$$\frac{1}{kA} + \frac{kB^2+A}{A(k^2AB-1)} < 1, \ \frac{1}{kB} + \frac{kA^2+B}{B(k^2AB-1)} < 1, \tag{18}$$

then equilibrium $(\bar{x},\bar{y}) \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right] \times \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$ *of the system (1) is locally asymptotically stable.*

Proof. The linearized system of (1) about $(\bar{x},\bar{y})$ is

$$\Phi_{n+1} = E\Phi_n,$$

where

$$\Phi_n = \begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_{n-k} \\ y_n \\ y_{n-1} \\ \vdots \\ y_{n-k} \end{pmatrix}, \ E = \begin{pmatrix} 0 & -\frac{\bar{y}}{k^2\bar{x}^2} & \dots & -\frac{\bar{y}}{k^2\bar{x}^2} & -\frac{\bar{y}}{k^2\bar{x}^2} & \frac{1}{k\bar{x}} & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{k\bar{y}} & 0 & \dots & 0 & 0 & 0 & -\frac{\bar{x}}{k^2\bar{y}^2} & \dots & -\frac{\bar{x}}{k^2\bar{y}^2} & -\frac{\bar{x}}{k^2\bar{y}^2} \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Let us denote $2k+2$ eigenvalues of E as $\kappa_1, \ \kappa_2, \ \dots, \ \kappa_{2k+2}$ and $D = diag(m_1, \ m_2, \ \dots, \ m_{2k+2})$ be a diagonal matrix, where $m_1 = m_{k+2} = 1$, $m_i = m_{k+1+i} = 1 - i\epsilon, \ i = 2, \ 3, \ \cdots, \ k+1$, and

$$0 < \epsilon < \min\left\{\frac{1}{k+1}\left(1 - \frac{1}{kA} - \frac{kB^2+A}{A(K^2AB-1)}\right), \ \ \frac{1}{k+1}\left(1 - \frac{1}{kB} - \frac{kA^2+B}{B(K^2AB-1)}\right)\right\} < 1.$$

Since D is invertible and by computing DED^{-1}, one gets

$$DED^{-1}=\left(\begin{array}{cccccccccc}
0 & -\frac{\bar y}{k^2\bar x^2}m_1m_2^{-1} & \dots & -\frac{\bar y}{k^2\bar x^2}m_1m_k^{-1} & -\frac{\bar y}{k^2\bar x^2}m_1m_{k+1}^{-1} & \frac{1}{k\bar x}m_1m_{k+2}^{-1} & 0 & \dots & 0 & 0\\
m_2m_1^{-1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & \dots & m_{k+1}m_k^{-1} & 0 & 0 & 0 & \dots & 0 & 0\\
\frac{1}{k\bar y}m_{k+2}m_1^{-1} & 0 & \dots & 0 & 0 & 0 & -\frac{\bar x}{k^2\bar y^2}m_{k+2}m_{k+3}^{-1} & \dots & -\frac{\bar x}{k^2\bar y^2}m_{k+2}m_{2k+1}^{-1} & -\frac{\bar x}{k^2\bar y^2}m_{k+2}m_{2k+2}^{-1}\\
0 & 0 & \dots & 0 & 0 & m_{k+3}m_{k+2}^{-1} & 0 & \dots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & m_{2k+2}m_{2k+1}^{-1} & 0
\end{array}\right). \tag{19}$$

From $m_1>m_2>\cdots>m_{k+1}>0$ and $m_{k+2}>m_{k+3}>\cdots>m_{2k+2}>0$, one has

$$m_2m_1^{-1}<1,\ m_3m_2^{-1}<1,\ \cdots,\ m_{k+1}m_k^{-1}<1, m_{k+3}m_{k+2}^{-1}<1,\ m_{k+4}m_{k+3}^{-1}<1,\ \cdots,\ m_{2k+2}m_{2k+1}^{-1}<1.$$

Also,

$$\begin{aligned}
\frac{\bar y}{k^2\bar x^2}m_1m_2^{-1}+\cdots+\frac{\bar y}{k^2\bar x^2}m_1m_{k+1}^{-1}+\frac{1}{k\bar x}m_1m_{k+2}^{-1} &= \frac{1}{k\bar x}+\frac{\bar y}{k^2\bar x^2}\left(\frac{1}{1-2\epsilon}+\cdots+\frac{1}{1-(k+1)\epsilon}\right),\\
&< \left(\frac{1}{k\bar x}+\frac{\bar y}{k\bar x^2}\right)\frac{1}{1-(k+1)\epsilon},\\
&< \left(\frac{1}{kA}+\frac{kB^2+A}{A\left(k^2AB-1\right)}\right)\frac{1}{1-(k+1)\epsilon}<1.
\end{aligned}$$

And

$$\begin{aligned}
\frac{1}{k\bar y}m_{k+2}m_1^{-1}+\frac{\bar x}{k^2\bar y^2}m_{k+2}m_{k+3}^{-1}+\cdots+\frac{\bar x}{k^2\bar y^2}m_{k+2}m_{2k+2}^{-1} &= \frac{1}{k\bar y}+\frac{\bar x}{k^2\bar y^2}\left(\frac{1}{1-2\epsilon}+\cdots+\frac{1}{1-(k+1)\epsilon}\right),\\
&< \left(\frac{1}{k\bar y}+\frac{\bar x}{k\bar y^2}\right)\frac{1}{1-(k+1)\epsilon},\\
&< \left(\frac{1}{kB}+\frac{kA^2+B}{B\left(k^2AB-1\right)}\right)\frac{1}{1-(k+1)\epsilon}<1.
\end{aligned}$$

Since E has the same eigenvalues as DED^{-1} and hence

$$\begin{aligned}
\max_{1\le n\le 2k+2}|\kappa_n|\le\|DED^{-1}\|_\infty &= \max\{m_2m_1^{-1},\cdots,m_{k+1}m_k^{-1},m_{k+3}m_{k+2}^{-1},\cdots,m_{2k+2}m_{2k+1}^{-1},\\
&\quad \frac{1}{k\bar x}+\frac{\bar y}{k^2\bar x^2}\left(\frac{1}{1-2\epsilon}+\cdots+\frac{1}{1-(k+1)\epsilon}\right),\frac{1}{k\bar y}+\\
&\quad \frac{\bar x}{k^2\bar y^2}\left(\frac{1}{1-2\epsilon}+\cdots+\frac{1}{1-(k+1)\epsilon}\right)\}<1.
\end{aligned} \tag{20}$$

Thus equation (20) implies that $(\bar x,\bar y)\in\left[A,\frac{kB\left(kA^2+B\right)}{k^2AB-1}\right]\times\left[B,\frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$ of (1) is locally asymptotically stable. $\square$

Theorem 4. *Equilibrium* $(\bar x,\bar y)\in\left[A,\frac{kB\left(kA^2+B\right)}{k^2AB-1}\right]\times\left[B,\frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$ *of (1) is globally asymptotically stable.*

Proof. Let $\{(x_n,y_n)\}$ be arbitrary solution of (1). Also let $\lim_{n\to\infty}\sup x_n=L_1,\ \lim_{n\to\infty}\inf x_n=l_1,\ \lim_{n\to\infty}\sup y_n=L_2,\ \lim_{n\to\infty}\inf y_n=l_2$ where $l_i,L_i\in(0,\infty),\ i=1,2$. Then from (1) one gets

$$L_1\le A+\frac{L_2}{kl_1},\ l_1\ge A+\frac{l_2}{kL_1}. \tag{21}$$

And

$$L_2 \leq B + \frac{L_1}{kl_2},\ l_2 \geq B + \frac{l_1}{kL_2}. \tag{22}$$

From (21), we have

$$Ak(L_1 - l_1) \leq L_2 - l_2. \tag{23}$$

From (22), we get

$$Bk(L_2 - l_2) \leq L_1 - l_1. \tag{24}$$

From (23) and (24), we get

$$(ABk^2 - 1)(L_1 - l_1) \leq 0,$$

which implies that $l_1 = L_1$. Similarly it is easy to prove that $l_2 = L_2$. □

Theorem 5. *Assuming $\{(x_n, y_n)\}$ be a positive solution of (1) such that $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$, where $(\bar{x}, \bar{y}) \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right] \times \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$. Then, the error vector ξ_n satisfying*

$$\lim_{n\to\infty} \sqrt[n]{\|\xi_n\|} = |\kappa E|,\ \lim_{n\to\infty} \frac{\|\xi_{n+1}\|}{\|\xi_n\|} = |\kappa E|,$$

where κE are the characteristic roots of E.

Proof. If $\{(x_n, y_n)\}$ be any solution of (1) such that $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$. To find error term one has

$$x_{n+1} - \bar{x} = \frac{y_n}{\sum_{i=1}^{k} x_{n-i}} - \frac{\bar{y}}{k\bar{x}} = -\sum_{i=1}^{k} \frac{\bar{y}}{k\bar{x}\left(\sum_{i=1}^{k} x_{n-i}\right)} (x_{n-i} - \bar{x}) + \frac{1}{\sum_{i=1}^{k} x_{n-i}} (y_n - \bar{y}),$$

$$y_{n+1} - \bar{y} = \frac{x_n}{\sum_{i=1}^{k} y_{n-i}} - \frac{\bar{x}}{k\bar{y}} = \frac{1}{\sum_{i=1}^{k} y_{n-i}} (x_n - \bar{x}) - \sum_{i=1}^{k} \frac{\bar{x}}{k\bar{y}\left(\sum_{i=1}^{k} y_{n-i}\right)} (y_{n-i} - \bar{y}).$$

Denote $\epsilon_n^1 = x_n - \bar{x}$ and $\epsilon_n^2 = y_n - \bar{y}$, one has

$$\epsilon_{n+1}^1 = \sum_{i=1}^{k} A_{ni}\epsilon_{n-i}^1 + B_n\epsilon_n^2,\ \epsilon_{n+1}^2 = C_n\epsilon_n^1 + \sum_{i=1}^{k} D_{ni}\epsilon_{n-i}^2,$$

where

$$A_{n1} = A_{n2} = \cdots = A_{nk} = -\frac{\bar{y}}{k\bar{x}\left(\sum_{i=1}^{k} x_{n-i}\right)},\ B_n = \frac{1}{\sum_{i=1}^{k} x_{n-i}},$$

$$C_n = \frac{1}{\sum_{i=1}^{k} y_{n-i}},\ D_{n1} = D_{n2} = \cdots = D_{nk} = -\sum_{i=1}^{k} \frac{\bar{x}}{k\bar{y}\left(\sum_{i=1}^{k} y_{n-i}\right)}.$$

Taking the limits, we obtain

$$\lim_{n\to\infty} A_{n1} = \lim_{n\to\infty} A_{n2} = \cdots = \lim_{n\to\infty} A_{nk} = -\frac{\bar{y}}{k^2\bar{x}^2},\ \lim_{n\to\infty} B_n = \frac{1}{k\bar{x}},$$

$$\lim_{n\to\infty} C_n = \frac{1}{k\bar{y}},\ \lim_{n\to\infty} D_{n1} = \lim_{n\to\infty} D_{n2} = \cdots = \lim_{n\to\infty} D_{nk} = -\frac{\bar{x}}{k^2\bar{y}^2}.$$

Hence we have system (1.10) of [8] where

$$\xi_{n+1} = E\xi_n, \tag{25}$$

where $\xi_n = \begin{pmatrix} \epsilon^1_n \\ \epsilon^1_{n-1} \\ \vdots \\ \epsilon^1_{n-k} \\ \epsilon^2_n \\ \epsilon^2_{n-1} \\ \vdots \\ \epsilon^2_{n-k} \end{pmatrix}$, $E = \begin{pmatrix} 0 & -\frac{\bar{y}}{k^2\bar{x}^2} & \dots & -\frac{\bar{y}}{k^2\bar{x}^2} & -\frac{\bar{y}}{k^2\bar{x}^2} & \frac{1}{k\bar{x}} & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{k\bar{y}} & 0 & \dots & 0 & 0 & 0 & -\frac{\bar{x}}{k^2\bar{y}^2} & \dots & -\frac{\bar{x}}{k^2\bar{y}^2} & -\frac{\bar{x}}{k^2\bar{y}^2} \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$. This is similar to linearized system of (1) about $(\bar{x}, \bar{y})$. $\square$

3 Conclusion

In the present work, dynamics of following higher-order anti-competitive system is studied:

$$x_{n+1} = A + \frac{y_n}{\sum_{i=1}^{k} x_{n-i}},\ y_{n+1} = B + \frac{x_n}{\sum_{i=1}^{k} y_{n-i}}.$$

Our investigations reveal that if $ABk^2 > 1$, then $\{(x_n, y_n)\}$ of this system is bounded and persists and the region $\left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right] \times \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$ is invariant set. It is proved that if $\frac{1}{kA} + \frac{kB^2+A}{A(k^2AB-1)} < 1$ and $\frac{1}{kB} + \frac{kA^2+B}{B(k^2AB-1)} < 1$ then equilibrium $(\bar{x}, \bar{y}) \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right] \times \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$ of the system is locally asymptotically stable. Finally global dynamics and rate of convergence that converges to $(\bar{x}, \bar{y}) \in \left[A, \frac{kB\left(kA^2+B\right)}{k^2AB-1}\right] \times \left[B, \frac{kA\left(kB^2+A\right)}{k^2AB-1}\right]$ of (1) are also demonstrated.

Acknowledgements

A. Q. Khan research is supported by the Higher Education Commission(HEC) of Pakistan.

References

[1] E. A. Grove, G. Ladas, Periodicities in nonlinear difference equations, Chapman and Hall/CRC Press, Boca Raton, (2004).

[2] V. L. Kocic, G. Ladas, Global behavior of nonlinear difference equations of higher order with applications, Kluwer Academic Publishers, Dordrecht, (1993).

[3] A. S, Kurbanli, On the behavior of positive solutions of the system of rational difference equations $x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1}-1}$, $y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1}-1}$, $z_{n+1} = \frac{1}{y_n z_n}$, *Advances in Difference Equations*, 2011, 2011:40.

[4] E. M. Elsayed, T. F. Ibrahim, Periodicity and solutions for some systems of nonlinear rational difference equations, *Hacettepe Journal of Mathematics and Statistics*, 44(6)(2015):1361-1390.

[5] R. DeVault, G. Ladas, S. W. Schultz, On the recursive sequence $x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}$, *Proceedings of the American Mathematical Society*, 126(11)(1998):3257-3261.

[6] R. M. Abu-Saris, R. DeVault, Global stability of $x_{n+1} = A + \frac{x_n}{x_{n-k}}$, *Applied Mathematics Letters*, 16(2003):173-178.

[7] Q. Zhang, L. Yang, D. Liao, Global asymptotic behavior of positive solutions to the system of rational difference equations, *Journal of Southwest University*, 7(2012):12-15.

[8] M. Pituk, More on Poincare's and Perron's theorems for difference equations, *Journal of Difference Equations and Applications*, 8(2002):201-216.

Stability of a modified within-host HIV dynamics model with antibodies

Ali Al-Qahtani[a], Shaban Aly[a], Ahmed Elaiw[b] and E. Kh. Elnahary[c]
[a]Department of Mathematics, Faculty of Science, King Khalid University, Saudi Arabia.
[b]Department of Mathematics, Faculty of Science, King Abdulaziz University, Saudi Arabia.
[c]Department of Mathematics, Faculty of Science, Sohag University, Sohag, Egypt.
Email: a_m_elaiw@yahoo.com (A. Elaiw)

Abstract

We investigate a modified HIV infection model with antibodies and latency. The model consider saturated HIV-CD4$^+$ T cells and HIV-macrophages incidence rates. We show that the solutions of the proposed model are nonnegative and bounded. We established that the global stability of the three steady states of the model depend on threshold parameters R_0 and R_1. Using Lyapunov function, we established the global stability of the steady states of the model. The theoretical results are confirmed by numerical simulations. The results show that antibodies can reduce the HIV infection.

1 Introduction

Constructing and analyzing of within-host human immunodeficiency virus (HIV) dynamics models have become one of the hot topics during the last decades [1]-[18]. These works can help researchers for better understanding the HIV dynamical behavior and providing new suggestions for clinical treatment. A vast of the mathematical models presented in the literature have focused on modeling the interaction between three main compartments, uninfected CD4$^+$ T cells (s), infected cells (u) and free HIV particles (p). Other models have differentiated between latent and active infected cells [19]-[23], an HIV mathematical model has been presented by inroducing a new variable (w) for the latently infected cells as:

$$\dot{s} = \rho - \delta s - \lambda sp, \tag{1}$$

$$\dot{w} = \lambda sp - (\alpha + \beta) w, \tag{2}$$

$$\dot{u} = \beta w - au, \tag{3}$$

$$\dot{p} = ku - gp, \tag{4}$$

where, ρ is the creation rate of the uninfected CD4$^+$ T cells, δ, α, a and g are the death rate constants of the four compartments s, w, u and p, respectively. The term βw represents the activation rate of the latently infected cells. The HIV-CD4$^+$T cell incidence rate is given by λsp. Parameter k represents the rate constant of free virus production. Sun et. al. [24] have modified the above model by considering the saturated infection rate

$\frac{\lambda sp}{s+p}$ as:

$$\dot{s} = \rho - \delta s - \frac{\lambda sp}{s+p}, \tag{5}$$

$$\dot{w} = \frac{\lambda sp}{s+p} - (\alpha + \beta) w, \tag{6}$$

$$\dot{u} = \beta w - au, \tag{7}$$

$$\dot{p} = ku - gp, \tag{8}$$

Model (5)-(8) consider one type of target cells (CD4$^+$ T cells). Moreover, the model does not acount the presence of the antibodies which are important in reducing the HIV infection. To have more accurate HIV model we improve model (5)-(8) by taking into account the dynamics of HIV with two target cells, CD4$^+$T cells and macrophages and antibodies. The global stability of the model is proven by using Lyapunov method.

2 The modified HIV

We propose the following model:

$$\dot{s}_i = \rho_i - \delta_i s_i - \frac{\lambda_i s_i p}{s_i + p}, \qquad i = 1, 2, \tag{9}$$

$$\dot{w}_i = \frac{\lambda_i s_i p}{s_i + p} - (\alpha_i + \beta_i) w_i, \qquad i = 1, 2, \tag{10}$$

$$\dot{u}_i = \beta_i w_i - a_i u_i, \qquad i = 1, 2, \tag{11}$$

$$\dot{p} = \sum_{i=1}^{2} k_i u_i - gp - \mu pz, \tag{12}$$

$$\dot{z} = rpz - \zeta z. \tag{13}$$

where, $z(t)$ represents the populations of the antibody immune cells. The antibodies are proliferated and die at rates rpz and ζz, respectively. The HIV particles are killed by antibodies at rate μpz.

2.1 Preliminaries.

Lemma 1. The solutions of model (9)-(13) with the initial conditions $s_i(0), w_i(0), u_i(0), p(0)$ and $z(0)$ are nonnegativite and bounded for $t \geq 0$.

Proof. We have

$$\dot{s}_i \mid_{s_i=0} = \rho_i > 0, \qquad \dot{w}_i \mid_{w_i=0} = \frac{\lambda_i s_i p}{s_i + p} \geq 0 \ \ \forall \ s_i \geq 0, p \geq 0, \qquad \dot{u}_i \mid_{u_i=0} = \beta_i w_i \geq 0 \quad \forall \ w_i \geq 0, \ i = 1, 2$$

$$\dot{p} \mid_{p=0} = \sum_{i=1}^{2} k_i u_i \geq 0 \ \ \forall \ \ u_i \geq 0, \qquad \dot{z} \mid_{z=0} = 0.$$

This shows the nonnegativity of the model's solutions. Now we let $G_i(t) = s_i(t) + w_i(t) + u_i(t)$, then

$$\dot{G}_i = \rho_i - \delta_i s_i - \alpha_i w_i - a_i u_i \leq \rho_i - \kappa_i (s_i + w_i + u_i) = \rho_i - \kappa_i G_i,$$

where $\kappa_i = \min\{\delta_i, \alpha_i, a_i\}, i = 1, 2$. Hence $0 \leq G_i(t) \leq M_i$ where, $M_i = \frac{\rho_i}{\kappa_i}$. therefore $s_i(t), w_i(t)$ and $u_i(t)$ are all bonded. Let $G_3(t) = p(t) + \frac{\mu}{r} z(t)$, then

$$\dot{G}_3(t) = \sum_{i=1}^{2} k_i u_i - gp - \frac{\mu\zeta}{r} z \leq \sum_{i=1}^{2} k_i M_i - \kappa_3 \left(p + \frac{\mu}{r} z\right) = \sum_{i=1}^{2} k_i M_i - \kappa_3 G_3(t),$$

where $\kappa_3 = \min\{g, \zeta\}$. Hence $p(t) \leq M_3$ and $z(t) \leq M_4$ for $t \geq 0$ where, $M_3 = \frac{1}{\kappa_3}\sum_{i=1}^{2} k_i M_i$ and $M_4 = \frac{rM_3}{\mu}$. So that, there is a bounded subset of D

$$\Gamma = \{(s_1, s_2, w_1, w_2, u_1, u_2, p, z) \in H : 0 \leq s_i + w_i + u_i \leq M_i, 0 \leq p \leq M_3, 0 \leq w \leq M_4\}.$$

is positively invariant with respect to system (9)-(13).

Lemma 2. For system (9)-(13) there exist two bifurcation parameters R_0 and R_1 with $R_0 > R_1$ such that
(i) if $R_0 \leq 1$, then the system has only one steady state Π_0,
(ii) if $R_1 \leq 1 < R_0$, then the system has only two steady states Π_0 and Π_1,
(iii) if $R_1 > 1$, then the system has three steady states Π_0,$\Pi_1$and Π_2.
Proof. Let

$$\rho_i - \delta_i s_i - \frac{\lambda_i s_i p}{s_i + p} = 0, \tag{14}$$

$$\frac{\lambda_i s_i p}{s_i + p} - (\alpha_i + \beta_i) w_i = 0, \tag{15}$$

$$\beta_i w_i - a_i u_i = 0, \tag{16}$$

$$\sum_{i=1}^{2} k_i u_i - gp - \mu pz = 0, \tag{17}$$

$$rpz - \zeta z = 0. \tag{18}$$

Eq. (18) we obtain two possible solutions, $z = 0$ or $p = \frac{\zeta}{r}$. First, we consider the case $z = 0$, then from Eqs. (15)-(16) we can get:

$$w_i = \frac{\lambda_i s_i p}{(\alpha_i + \beta_i)(s_i + p)}, \quad u_i = \frac{\lambda_i \beta_i s_i p}{a_i (\alpha_i + \beta_i)(s_i + p)}, \tag{19}$$

where $s_i^0 = \frac{\rho_i}{\delta_i}$. From Eq. (17) we obtain

$$\left(\sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i s_i}{a_i g (\alpha_i + \beta_i)(s_i + p)} - 1\right) gp = 0. \tag{20}$$

Eq. (20) has two possible solutions $p = 0$ or $\sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i s_i}{a_i g(\alpha_i+\beta_i)(s_i+p)} = 1$.

If $p = 0$, then substituting it in Eq. (19) leads to the uninfected steady state $\Pi_0 = (s_1^0, s_2^0, 0, 0, 0, 0, 0, 0)$. If $p \neq 0$, we have

$$\sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i s_i}{a_i g (\alpha_i + \beta_i)(s_i + p)} - 1 = 0. \tag{21}$$

Eq. (14) implies that

$$s_i^{\pm} = \frac{1}{2}\left(\left(s_i^0 - \varphi_i p\right) \pm \sqrt{\left(\varphi_i p - s_i^0\right)^2 + 4 s_i^0 p}\right),$$

where, $s_i^0 = \frac{\rho_i}{\delta_i}, \varphi_i = \frac{\lambda_i}{\delta_i} + 1, i = 1, 2$. Clearly if $p > 0$ then $s_i^- < 0$ and $s_i^+ > 0$, then we choose $s_i = s_i^+$

$$s_i = \frac{1}{2}\left(\left(s_i^0 - \varphi_i p\right) + \sqrt{\left(\varphi_i p - s_i^0\right)^2 + 4 s_i^0 p}\right). \tag{22}$$

Substituting from Eqs. (14) and (19) into Eq. (17) we get

$$\sum_{i=1}^{2} \frac{k_i \beta_i}{d_{3i}(\alpha_i + \beta_i)} (\rho_i - \delta_i s_i) - gp = 0. \tag{23}$$

Since s_i is a function of p then from Eq. (23) we can define a function $H_1(p)$ as:

$$H_1(p) = \sum_{i=1}^{2} \frac{k_i \beta_i}{a_i (\alpha_i + \beta_i)} (\rho_i - \delta_i s_i(p)) - gp = 0. \tag{24}$$

We need to show that there exists a $p > 0$ such that $H_1(p) = 0$. It is clear that, if $p = 0$, then $s_i = s_i^0$ and $H_1(0) = 0$ and when $p = \hat{p} = \sum_{i=1}^{2} \frac{k_i \rho_i \beta_i}{a_i g(\alpha_i + \beta_i)} > 0$, we have $\hat{s}_i = s_i(\hat{p}) > 0$ and

$$H_1(\hat{p}) = -\sum_{i=1}^{2} \frac{k_i \beta_i \delta_i \hat{s}_i}{a_i g(\alpha_i + \beta_i)} < 0.$$

Since $H_1(p)$ is continuous for all $p \geq 0$, we obtain

$$H_1^{'}(0) = g\left(\sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i}{a_i g(\alpha_i + \beta_i)} - 1\right).$$

Therefore, $H_1^{'}(0) > 0$, if

$$\sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i}{a_i g(\alpha_i + \beta_i)} > 1 \tag{25}$$

It means that if condition (25) is satisfied, then there exists $\tilde{p} \in (0, \hat{p})$ such that $H_1(\tilde{p}) = 0$. From Eqs. (19) and (22), we have $\tilde{s}_i, \tilde{w}_i, \tilde{u}_i, \tilde{p} > 0$. Thus, an infection steady state without antibodies $\Pi_1 = (\tilde{s}_1, \tilde{s}_2, \tilde{w}_1, \tilde{w}_2, \tilde{u}_1, \tilde{u}_2, \tilde{p}, 0)$ exists when $\sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i}{a_i g(\alpha_i + \beta_i)} > 1$.Now we can define

$$R_0 = \sum_{i=1}^{2} R_{0i} = \sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i}{a_i g(\alpha_i + \beta_i)},$$

Now if $z \neq 0$, then from Eqs. Eqs. (14)-(16),

$$\bar{s}_i = \frac{1}{2}\left[\left(s_i^0 - \varphi_i \frac{\zeta}{r}\right) + \sqrt{\left(\varphi_i \frac{\zeta}{r} - s_i^0\right)^2 + \frac{4\zeta s_i^0}{r}}\right], \qquad \bar{w}_i = \frac{\lambda_i \bar{s}_i \bar{p}}{(\alpha_i + \beta_i)(\bar{s}_i + \bar{p})},$$
$$\bar{u}_i = \frac{\lambda_i \beta_i \bar{s}_i \bar{p}}{a_i (\alpha_i + \beta_i)(\bar{s}_i + \bar{p})}, \qquad \bar{z} = \frac{g}{\mu}\left(\sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i \bar{s}_i}{a_i g(\alpha_i + \beta_i)(\bar{s}_i + \bar{p})} - 1\right),$$

Thus, $\bar{z} > 0$ when $\sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i \bar{s}_i}{a_i g(\alpha_i + \beta_i)(\bar{s}_i + \bar{p})} > 1$. Let us define the parameter R_1 as:

$$R_1 = \sum_{i=1}^{2} \frac{k_i \lambda_i \beta_i \bar{s}_i}{a_i g(\alpha_i + \beta_i)(\bar{s}_i + \bar{p})},$$

If $R_1 > 1$, then $\bar{z} = \frac{g}{\mu}(R_1 - 1) > 0$ and exists an infection steady state with antibodies $\Pi_2 = (\bar{s}_1, \bar{s}_2, \bar{w}_1, \bar{w}_2, \bar{u}_1, \bar{u}_2, \bar{p}, \bar{z})$ if $R_1 > 1$.

2.2 Global properties

We will use the following function throughout the paper, $F : (0, \infty) \longrightarrow [0, \infty)$ as $F(q) = q - 1 - \ln q$.

Theorem 1. The steady state Π_0 is globally asymptotically stable when $R_0 \leq 1$.

Proof. Define

$$W_{01}=\sum_{i=1}^{2}\gamma_i\left[w_i+\frac{(\alpha_i+\beta_i)}{\beta_i}u_i\right]+p+\frac{\zeta}{r}z,$$

where, $\gamma_i=\frac{k_i\beta_i}{a_i(\alpha_i+\beta_i)}$. We evaluate $\frac{dW_{01}}{dt}$ along the solutions of system (9)-(13) as:

$$\begin{aligned}\frac{dW_{01}}{dt}&=\sum_{i=1}^{2}\gamma_i\left[\dot{w}_i+\frac{(\alpha_i+\beta_i)}{\beta_i}\dot{u}_i\right]+\dot{p}+\frac{\mu}{r}\dot{z}\\&=\sum_{i=1}^{2}\gamma_i\left[\frac{\lambda_i s_i p}{s_i+p}-(\alpha_i+\beta_i)\,w_i+\frac{(\alpha_i+\beta_i)}{\beta_i}(\beta_i w_i-a_i u_i)\right]+\sum_{i=1}^{2}k_i u_i-gp-\mu pz+\frac{\mu}{r}(rpz-\zeta z).\end{aligned}\tag{26}$$

Eq. (26) can be simplified as

$$\begin{aligned}\frac{dW_{01}}{dt}&=\sum_{i=1}^{2}\gamma_i\frac{\lambda_i s_i p}{s_i+p}-gp-\frac{\mu\zeta}{r}z\le\sum_{i=1}^{2}\gamma_i\lambda_i p-gp-\frac{\mu\zeta}{r}z\\&=g\left(\sum_{i=1}^{2}\frac{k_i\lambda_i\beta_i}{a_i g(\alpha_i+\beta_i)}-1\right)p-\frac{\mu\zeta}{r}z=g(R_0-1)\,p-\frac{\mu\zeta}{r}z.\end{aligned}$$

If $R_0\le 1$, then $\frac{dW_{01}}{dt}\le 0$ holds in Γ. Moreover, $\frac{dW_{01}}{dt}=0$ when $p=0$ and $z=0$. Hence the largest compact invariant set in Γ is

$$\begin{aligned}Q_1&=\left\{(s_1,s_2,w_1,w_2,u_1,u_2,p,z)\in\Gamma\mid\frac{dW_{01}}{dt}=0\right\}\\&=\{(s_1,s_2,w_1,w_2,u_1,u_2,p,z)\in\Gamma\mid p=0,z=0\}.\end{aligned}$$

LaSalle's invariance principle yields $\lim_{t\to+\infty}p(t)=0$ and $\lim_{t\to+\infty}z(t)=0$. One can get limit equations:

$$\dot{s}_i=\rho_i-\delta_i s_i,\tag{27}$$
$$\dot{w}_i=-(\alpha_i+\beta_i)\,w_i,\tag{28}$$
$$\dot{u}_i=\beta_i w_i-a_i u_i.\tag{29}$$

Define a function W_{02} by

$$W_{02}=\sum_{i=1}^{2}\gamma_i\left[s_i^0F\left(\frac{s_i}{s_i^0}\right)+w_i+\frac{(\alpha_i+\beta_i)}{\beta_i}u_i\right].$$

Then

$$\begin{aligned}\frac{dW_{02}}{dt}&=\sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{s_i^0}{s_i}\right)\dot{s}_i+\dot{w}_i+\frac{(\alpha_i+\beta_i)}{\beta_i}\dot{u}_i\right]\\&=\sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{s_i^0}{s_i}\right)(\rho_i-\delta_i s_i)-(\alpha_i+\beta_i)\,w_i+\frac{(\alpha_i+\beta_i)}{\beta_i}(\beta_i w_i-a_i u_i)\right]\\&=-\sum_{i=1}^{2}\gamma_i\delta_i\frac{(s_i-s_i^0)^2}{s_i}-\sum_{i=1}^{2}k_i u_i.\end{aligned}$$

Therefore, $\frac{dW_{02}}{dt}\le 0$ holds in Q_1 and $\frac{dW_{02}}{dt}=0$ if and only if $s_i=s_i^0$ and $u_i=0$. There is the largest compact invariant set in Q_1:

$$\begin{aligned}Q_2&=\left\{(s_1,s_2,w_1,w_2,u_1,u_2,p,z)\in Q_1\mid\frac{dW_{02}}{dt}=0\right\}\\&=\left\{(s_1,s_2,w_1,w_2,u_1,u_2,p,z)\in Q_1\mid s_i=s_i^0,w_i\ge 0,u_i=0\right\}.\end{aligned}$$

In Q_2, from Eq. (29) we get $\beta_i w_i - a_i(0) = 0$, and then $w_i = 0$. So

$$\begin{aligned}Q_2 &= \left\{(s_1, s_2, w_1, w_2, u_1, u_2, p, z) \in Q_1 \mid \frac{dW_{02}}{dt} = 0\right\} \\ &= \left\{(s_1, s_2, w_1, w_2, u_1, u_2, p, z) \in Q_1 \mid s_i = s_i^0, w_i = 0, u_i = 0\right\} \\ &= \{\Pi_0\}.\end{aligned}$$

Hence, if $R_0 \leq 1$, all solution trajectories in Γ approach the uninfected steady state Π_0.

Theorem 2. The steady state Π_1 is globally asymptotically stable when $R_1 \leq 1 < R_0$.

Proof. We introduce

$$W_1 = \sum_{i=1}^{2} \gamma_i \left[s_i - \tilde{s}_i - \int_{\tilde{s}_i}^{s_i} \frac{(\alpha_i + \beta_i)\, \tilde{w}_i (\tau + \tilde{p})}{\lambda_i \tau \tilde{p}} d\tau + \tilde{w}_i F\left(\frac{w_i}{\tilde{w}_i}\right) + \frac{(\alpha_i + \beta_i)}{\beta_i} \tilde{u}_i F\left(\frac{u_i}{\tilde{u}_i}\right) \right] + \tilde{p} F\left(\frac{p}{\tilde{p}}\right) + \frac{\mu}{r} z.$$

Evaluating $\frac{dW_1}{dt}$ along the trajectories of system (9)-(13):

$$\begin{aligned}\frac{dW_1}{dt} &= \sum_{i=1}^{2} \gamma_i \left[\left(1 - \frac{(\alpha_i + \beta_i)\, \tilde{w}_i (s_i + \tilde{p})}{\lambda_i s_i \tilde{p}}\right) \dot{s}_i + \left(1 - \frac{\tilde{w}_i}{w_i}\right) \dot{w}_i + \frac{(\alpha_i + \beta_i)}{\beta_i} \left(1 - \frac{\tilde{u}_i}{u_i}\right) \dot{u}_i \right] + \left(1 - \frac{\tilde{p}}{p}\right) \dot{p} + \frac{\mu}{r} \dot{z} \\ &= \sum_{i=1}^{2} \gamma_i \left[\left(1 - \frac{(\alpha_i + \beta_i)\, \tilde{w}_i (s_i + \tilde{p})}{\lambda_i s_i \tilde{p}}\right) \left(\rho_i - \delta_i s_i - \frac{\lambda_i s_i p}{s_i + p}\right) + \left(1 - \frac{\tilde{w}_i}{w_i}\right) \left(\frac{\lambda_i s_i p}{s_i + p} - (\alpha_i + \beta_i)\, w_i\right) \right. \\ &\quad \left. + \frac{(\alpha_i + \beta_i)}{\beta_i} \left(1 - \frac{\tilde{u}_i}{u_i}\right) (\beta_i w_i - a_i u_i) \right] + \left(1 - \frac{\tilde{p}}{p}\right) \left(\sum_{i=1}^{2} k_i u_i - gp - \mu p z\right) + \frac{\mu}{r} (rpz - \zeta z).\end{aligned} \tag{30}$$

Simplify Eq. (30) as:

$$\begin{aligned}\frac{dW_1}{dt} &= \sum_{i=1}^{2} \gamma_i \left[\rho_i - \delta_i s_i - \frac{(\alpha_i + \beta_i)\, \tilde{w}_i (s_i + \tilde{p})}{\lambda_i s_i \tilde{p}} \left(\rho_i - \delta_i s_i - \frac{\lambda_i s_i p}{s_i + p}\right) - \frac{\lambda_i s_i p}{s_i + p} \frac{\tilde{w}_i}{w_i} + (\alpha_i + \beta_i)\, \tilde{w}_i \right. \\ &\quad \left. - (\alpha_i + \beta_i)\, w_i \frac{\tilde{u}_i}{u_i} + \frac{a_i (\alpha_i + \beta_i)}{\beta_i} \tilde{u}_i - \frac{a_i (\alpha_i + \beta_i)}{\beta_i} u_i \frac{\tilde{p}}{p} \right] - gp + g\tilde{p} + \mu \left(\tilde{p} - \frac{\zeta}{r}\right) z.\end{aligned} \tag{31}$$

From the conditions of Π_1, we obtain

$$\begin{aligned}&\rho_i = \delta_i \tilde{s}_i + (\alpha_i + \beta_i)\, \tilde{w}_i, \quad \frac{\lambda_i \tilde{s}_i \tilde{p}}{\tilde{s}_i + \tilde{p}} = (\alpha_i + \beta_i)\, \tilde{w}_i, \quad \frac{a_i (\alpha_i + \beta_i)}{\beta_i} \tilde{u}_i = (\alpha_i + \beta_i)\, \tilde{w}_i, \\ &g\tilde{p} = \sum_{i=1}^{2} k_i \tilde{u}_i, \quad \lambda_i = \frac{(\alpha_i + \beta_i)\, \tilde{w}_i (\tilde{s}_i + \tilde{p})}{\tilde{s}_i \tilde{p}}, \quad \frac{(\alpha_i + \beta_i)\, \tilde{w}_i (s_i + \tilde{p})}{\lambda_i s_i \tilde{p}} = \frac{\tilde{s}_i (s_i + \tilde{p})}{s_i (\tilde{s}_i + \tilde{p})},\end{aligned}$$

then, we have

$$\begin{aligned}\frac{dW_1}{dt} &= \sum_{i=1}^{2} \gamma_i \left[\delta_i \tilde{s}_i \left(1 - \frac{s_i}{\tilde{s}_i} - \frac{\tilde{s}_i (s_i + \tilde{p})}{s_i (\tilde{s}_i + \tilde{p})} + \frac{s_i + \tilde{p}}{\tilde{s}_i + \tilde{p}}\right) + (\alpha_i + \beta_i)\, \tilde{w}_i \left(-1 - \frac{p}{\tilde{p}} + \frac{p (s_i + \tilde{p})}{\tilde{p} (s_i + p)} + \frac{s_i + p}{s_i + \tilde{p}}\right) \right. \\ &\quad \left. + (\alpha_i + \beta_i)\, \tilde{w}_i \left(5 - \frac{\tilde{s}_i (s_i + \tilde{p})}{s_i (\tilde{s}_i + \tilde{p})} - \frac{s_i \tilde{w}_i p (\tilde{s}_i + \tilde{p})}{\tilde{s}_i w_i \tilde{p} (s_i + p)} - \frac{w_i \tilde{u}_i}{\tilde{w}_i u_i} - \frac{u_i \tilde{p}}{\tilde{u}_i p} - \frac{s_i + p}{s_i + \tilde{p}}\right) \right] + \mu (\tilde{p} - \bar{p})\, z.\end{aligned} \tag{32}$$

Eq. (32) becomes

$$\begin{aligned}\frac{dW_1}{dt} &= \sum_{i=1}^{2} \gamma_i \left[-\frac{\delta_i \tilde{p} (s_i - \tilde{s}_i)^2}{s_i (s_i + p)} - (\alpha_i + \beta_i)\, \tilde{w}_i \frac{s_i (p - \tilde{p})^2}{\tilde{p} (s_i + p)(s_i + \tilde{p})} \right. \\ &\quad \left. + (\alpha_i + \beta_i)\, \tilde{w}_i \left(5 - \frac{\tilde{s}_i (s_i + \tilde{p})}{s_i (\tilde{s}_i + \tilde{p})} - \frac{s_i \tilde{w}_i p (\tilde{s}_i + \tilde{p})}{\tilde{s}_i w_i \tilde{p} (s_i + p)} - \frac{w_i \tilde{u}_i}{\tilde{w}_i u_i} - \frac{u_i \tilde{p}}{\tilde{u}_i p} - \frac{s_i + p}{s_i + \tilde{p}}\right) \right] + \mu (\tilde{p} - \bar{p})\, z.\end{aligned}$$

Using the rule

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \sqrt[n]{\prod_{i=1}^{n} a_i},$$

we obtain

$$\frac{\tilde{s}_i\left(s_i+\tilde{p}\right)}{s_i\left(\tilde{s}_i+\tilde{p}\right)}+\frac{s_i\tilde{w}_i p\left(\tilde{s}_i+\tilde{p}\right)}{\tilde{s}_i w_i\tilde{p}\left(s_i+p\right)}+\frac{w_i\tilde{u}_i}{\tilde{w}_i u_i}+\frac{u_i\tilde{p}}{\tilde{u}_i p}+\frac{s_i+p}{s_i+\tilde{p}}-5\geq 0.$$

Now we show that if $R_1 \leq 1$ then $\tilde{p} \leq \frac{\zeta}{r} = \bar{p}$. This can be shown if we prove that

$$sgn\left(\bar{s}_i-\tilde{s}_i\right)=sgn\left(\tilde{p}-\bar{p}\right)=sgn\left(R_1-1\right).$$

Suppose that, $sgn\left(\bar{p}-\tilde{p}\right)=sgn\left(\bar{s}_i-\tilde{s}_i\right)$.

$$\left(\rho_i-\delta_i\bar{s}_i\right)-\left(\rho_i-\delta_i\tilde{s}_i\right)=\frac{\lambda_i\bar{s}_i\bar{p}}{\bar{s}_i+\bar{p}}-\frac{\lambda_i\tilde{s}_i\tilde{p}}{\tilde{s}_i+\tilde{p}}=\lambda_i\left[\frac{\left(\bar{p}-\tilde{p}\right)s_i^2}{\left(\bar{s}_i+\bar{p}\right)\left(\bar{s}_i+\tilde{p}\right)}+\frac{\left(\bar{s}_i-\tilde{s}_i\right)\bar{p}^2}{\left(\bar{s}_i+\bar{p}\right)\left(\tilde{s}_i+\tilde{p}\right)}\right].$$

This yields, $sgn\left(\tilde{s}_i-\bar{s}_i\right)=sgn\left(\bar{s}_i-\tilde{s}_i\right)$, which leads to contradiction and then $sgn\left(\tilde{p}-\bar{p}\right)=sgn\left(\bar{s}_i-\tilde{s}_i\right)$. Using the condition for the steady state Π_1 we have $\sum_{i=1}^{2}\frac{k_i\lambda_i\beta_i\tilde{s}_i}{a_i g(\alpha_i+\beta_i)(\tilde{s}_i+\tilde{p})}=1$, then

$$\begin{aligned}
R_1-1&=\sum_{i=1}^{2}\frac{k_i\lambda_i\beta_i\bar{s}_i}{a_i g\left(\alpha_i+\beta_i\right)\left(\bar{s}_i+\bar{p}\right)}-\sum_{i=1}^{2}\frac{k_i\lambda_i\beta_i\tilde{s}_i}{a_i g\left(\alpha_i+\beta_i\right)\left(\tilde{s}_i+\tilde{p}\right)}\\
&=\sum_{i=1}^{2}\frac{k_i\lambda_i\beta_i}{a_i g\left(\alpha_i+\beta_i\right)}\left(\frac{\left(\bar{s}_i-\tilde{s}_i\right)\tilde{p}+\left(\tilde{p}-\bar{p}\right)\tilde{s}_i}{\left(\bar{s}_i+\bar{p}\right)\left(\tilde{s}_i+\tilde{p}\right)}\right).
\end{aligned}\tag{33}$$

From (33) we get $sgn\left(R_1-1\right)=sgn\left(\tilde{p}-\bar{p}\right)$. So that, if $R_1\leq 1$ then $\tilde{p}\leq\frac{\zeta}{r}=\bar{p}$. So that, if $R_1\leq 1$ then $\frac{dW_1}{dt}\leq 0$ holds in Γ and $\frac{dW_1}{dt}=0$ when $s_i=\tilde{s}_i, w_i=\tilde{w}_i, u_i=\tilde{u}_i, p=\tilde{p}, z=0$. Hence the largest compact invariant subset in Γ is

$$\begin{aligned}
Q_3&=\left\{\left(s_1,s_2,w_1,w_2,u_1,u_2,p,z\right)\in\Gamma \mid \frac{dW_1}{dt}=0\right\}\\
&=\left\{\left(s_1,s_2,w_1,w_2,u_1,u_2,p,z\right)\in\Gamma \mid s_i=\tilde{s}_i, w_i=\tilde{w}_i, u_i=\tilde{u}_i, p=\tilde{p}, z=0\right\}\\
&=\left\{\Pi_1\right\}.
\end{aligned}$$

It follows that, if $R_1\leq 1$ then Π_1 is GAS in Γ by LIP.

Theorem 3. The steady state Π_2 is globally asymptotically stable when $R_1>1$.

Proof. Define

$$W_2=\sum_{i=1}^{2}\gamma_i\left[s_i-\bar{s}_i-\int_{\bar{s}_i}^{s_i}\frac{\left(\alpha_i+\beta_i\right)\bar{w}_i\left(\tau+\bar{p}\right)}{\lambda_i\tau\bar{p}}d\tau+\bar{w}_iF\left(\frac{w_i}{\bar{w}_i}\right)+\frac{\left(\alpha_i+\beta_i\right)}{\beta_i}\bar{u}_iF\left(\frac{u_i}{\bar{u}_i}\right)\right]+\bar{p}F\left(\frac{p}{\bar{p}}\right)+\frac{\mu}{r}\bar{z}F\left(\frac{z}{\bar{z}}\right).$$

Then $\frac{dW_2}{dt}$ is given as:

$$\begin{aligned}
\frac{dW_2}{dt}&=\sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{\left(\alpha_i+\beta_i\right)\bar{w}_i\left(s_i+\bar{p}\right)}{\lambda_i s_i\bar{p}}\right)\dot{s}_i+\left(1-\frac{\bar{w}_i}{w_i}\right)\dot{w}_i+\frac{\left(\alpha_i+\beta_i\right)}{\beta_i}\left(1-\frac{\bar{u}_i}{u_i}\right)\dot{u}_i\right]+\left(1-\frac{\bar{p}}{p}\right)\dot{p}+\frac{\mu}{r}\left(1-\frac{\bar{z}}{z}\right)\dot{z}\\
&=\sum_{i=1}^{2}\gamma_i\left[\left(1-\frac{\left(\alpha_i+\beta_i\right)\bar{w}_i\left(s_i+\bar{p}\right)}{\lambda_i s_i\bar{p}}\right)\left(\rho_i-\delta_i s_i-\frac{\lambda_i s_i p}{s_i+p}\right)+\left(1-\frac{\bar{w}_i}{w_i}\right)\left(\frac{\lambda_i s_i p}{s_i+p}-\left(\alpha_i+\beta_i\right)w_i\right)\right.\\
&\quad\left.+\frac{\left(\alpha_i+\beta_i\right)}{\beta_i}\left(1-\frac{\bar{u}_i}{u_i}\right)\left(\beta_i w_i-a_i u_i\right)\right]+\left(1-\frac{\bar{p}}{p}\right)\left(\sum_{i=1}^{2}k_i u_i-gp-\mu pz\right)+\frac{\mu}{r}\left(1-\frac{\bar{z}}{z}\right)\left(rpz-\zeta z\right).
\end{aligned}\tag{34}$$

Eq. (34) can be simplified as:

$$\frac{dW_2}{dt} = \sum_{i=1}^{2}\gamma_i \left[\rho_i - \delta_i s_i - \frac{(\alpha_i+\beta_i)\,\bar{w}_i\,(s_i+\bar{p})}{\lambda_i s_i \bar{p}}\left(\rho_i - \delta_i s_i - \frac{\lambda_i s_i p}{s_i+p}\right) - \frac{\lambda_i s_i p}{s_i+p}\frac{\bar{w}_i}{w_i} + (\alpha_i+\beta_i)\,\bar{w}_i \right.$$
$$\left. - (\alpha_i+\beta_i)\,w_i\frac{\bar{u}_i}{u_i} + \frac{a_i(\alpha_i+\beta_i)}{\beta_i}\bar{u}_i - \frac{a_i(\alpha_i+\beta_i)}{\beta_i}u_i\frac{\bar{p}}{p}\right] - gp + g\bar{p} - \mu p\bar{z} + \frac{\mu\zeta}{r}\bar{z}.$$

Using conditions of Π_2 we get

$$\rho_i = \delta_i\bar{s}_i + (\alpha_i+\beta_i)\,\bar{w}_i, \quad \frac{a_i(\alpha_i+\beta_i)}{\beta_i}\bar{u}_i = (\alpha_i+\beta_i)\,\bar{w}_i, \quad \frac{(\alpha_i+\beta_i)\,\bar{w}_i\,(s_i+\bar{p})}{\lambda_i s_i\bar{p}} = \frac{\bar{s}_i\,(s_i+\bar{p})}{s_i\,(\bar{s}_i+\bar{p})},$$
$$\lambda_i = \frac{(\alpha_i+\beta_i)\,\bar{w}_i\,(\bar{s}_i+\bar{p})}{\bar{s}_i\bar{p}}, \quad g\bar{p} = \sum_{i=1}^{2}k_i\bar{u}_i - \mu\bar{p}\bar{z}, \quad gp = \frac{p}{\bar{p}}\sum_{i=1}^{2}k_i\bar{u}_i - \mu p\bar{z}$$

Then, we have

$$\frac{dW_2}{dt} = \sum_{i=1}^{2}\gamma_i \left[\delta_i\bar{s}_i\left(1 - \frac{s_i}{\bar{s}_i} - \frac{\bar{s}_i\,(s_i+\bar{p})}{s_i\,(\bar{s}_i+\bar{p})} + \frac{s_i+\bar{p}}{\bar{s}_i+\bar{p}}\right) + (\alpha_i+\beta_i)\,\bar{w}_i\left(-1 - \frac{p}{\bar{p}} + \frac{p\,(s_i+\bar{p})}{\bar{p}\,(s_i+p)} + \frac{s_i+p}{s_i+\bar{p}}\right)\right.$$
$$\left. + (\alpha_i+\beta_i)\,\bar{w}_i\left(5 - \frac{\bar{s}_i\,(s_i+\bar{p})}{s_i\,(\bar{s}_i+\bar{p})} - \frac{\bar{w}_i s_i p\,(\bar{s}_i+\bar{p})}{w_i\bar{s}_i\bar{p}\,(s_i+p)} - \frac{w_i\bar{u}_i}{\bar{w}_i u_i} - \frac{\bar{p}u_i}{p\bar{u}_i} - \frac{s_i+p}{s_i+\bar{p}}\right)\right]. \tag{35}$$

Eq. (35) becomes

$$\frac{dW_2}{dt} = \sum_{i=1}^{2}\gamma_i \left[-\frac{\delta_i\bar{p}\,(s_i-\bar{s}_i)^2}{s_i\,(\bar{s}_i+\bar{p})} - (\alpha_i+\beta_i)\,\bar{w}_i\frac{s_i\,(p-\bar{p})^2}{\bar{p}\,(s_i+p)\,(s_i+\bar{p})}\right.$$
$$\left. + (\alpha_i+\beta_i)\,\bar{w}_i\left(5 - \frac{\bar{s}_i\,(s_i+\bar{p})}{s_i\,(\bar{s}_i+\bar{p})} - \frac{s_i\bar{w}_i p\,(\bar{s}_i+\bar{p})}{\bar{s}_i w_i\bar{p}\,(s_i+p)} - \frac{w_i\bar{u}_i}{\bar{w}_i u_i} - \frac{u_i\bar{p}}{\bar{u}_i p} - \frac{s_i+p}{s_i+\bar{p}}\right)\right].$$

It follows that, $\frac{dW_2}{dt} \leq 0$ for all $s_i, w_i, u_i, p, z > 0$ and $\frac{dW_2}{dt} = 0$ when $s_i = \bar{s}_i, w_i = \bar{w}_i, u_i = \bar{u}_i, p = \bar{p}, z = \bar{z}$. Hence

$$Q_4 = \left\{(s_1, s_2, w_1, w_2, u_1, u_2, p, z) \in \Gamma \mid \frac{dW_2}{dt} = 0\right\}$$
$$= \{(s_1, s_2, w_1, w_2, u_1, u_2, p, z) \in \Gamma \mid s_i = \bar{s}_i, w_i = \bar{w}_i, u_i = \bar{u}_i, p = \bar{p}, z = \bar{z}\}$$
$$= \{\Pi_2\}.$$

It follows that, if $R_1 > 1$ then Π_2 is GAS in Γ by LIP.

3 Simulations

We support our results by numerical simulations using the values of the parameters given in Table 1.

For the parameters $\bar{\lambda}_1$, $\bar{\lambda}_2$ and r we have three cases to show its effect on the stability of the system. We assume that $\varepsilon_1 = \varepsilon_2 = 0$ (there is no treatment). The initial condtions are considered to be: $s_1(0) = 500, s_2(0) = 20, w_1(0) = 1, w_2(0) = 0.3, u_1(0) = 20, u_2(0) = 0.2, p(0) = 90, z(0) = 40$.

Case (I) $R_0 \leq 1$. We consider $\bar{\lambda}_1 = 0.002, \bar{\lambda}_2 = 0.00001$ and $r = 0.0001$. Then, $R_0 = 0.2469 < 1$ and $R_1 = 0.1062 < 1$. This means that Π_0 is GAS. From Figures 1-8 we can see that the trajectory of the system converges the steady state $\Pi_0(830, 24.6, 0, 0, 0, 0, 0, 0)$.

Case (II) $R_1 \leq 1 < R_0$. Choosing $\bar{\lambda}_1 = 0.02, \bar{\lambda}_2 = 0.0005$ and $r = 0.0001$. In this case, $R_0 = 2.5694$ and $R_1 = 0.7141 < 1$ and Π_1 exists with $\Pi_1 = (448.116, 17.9, 2.949, 0.436, 32.439, 0.218, 650.956, 0)$. According to Theorem 2, Π_1 is GAS. Figures 1-8 show the validity of the theoretical results of Theorem 2.

Table 1: The values of parameters of the models.

Parameter	Value	Parameter	Value	Parameter	Value	Parameter	Value
ρ_1	11.537	ρ_2	0.03198	k_1	10	k_2	5
δ_1	0.0139	δ_2	0.001	g	0.5	μ	0.01
α_1	0.57	α_2	0.5	ζ	0.05	f	0.5
a_1	0.1	a_2	0.02	h	0.5	λ_1	varied
β_1	1.1	β_2	0.01	λ_2	varied	r	varied

Case (III) $R_1 > 1$. We take $\bar{\lambda}_1 = 0.02, \bar{\lambda}_2 = 0.0005$ and $r = 0.002$. Then we get $R_0 = 2.5694 > 1$ and $R_1 = 2.3288 > 1$. Figures 1-8 show that, the steady state $\Pi_2(762.485, 19.254, 0.521, 0.348, 5.735, 0.174, 50, 66.438)$ is GAS which confirm the results of Theorem 3.

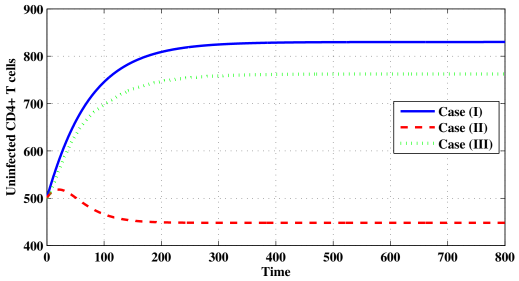


Figure 1: The concentration of uninfected CD4$^+$T cells.

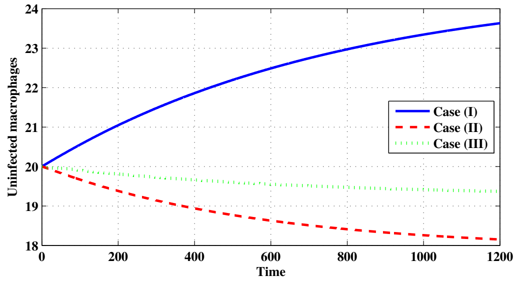


Figure 2: The concentration of uninfected macrophages.

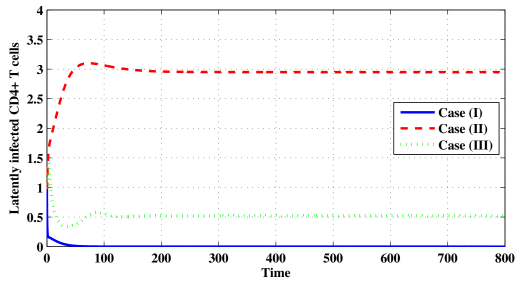


Figure 3: The concentration of latently infected CD4$^+$T cells.

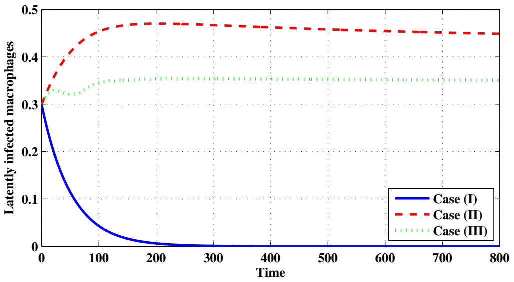


Figure 4: The concentration of latently infected macrophages.

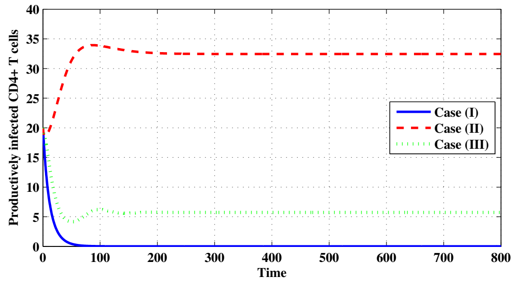


Figure 5: The concentration of productively infected CD4$^+$T cells.

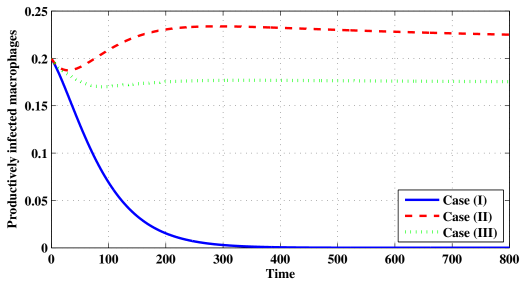


Figure 6: The concentration of productively infected macrophages.

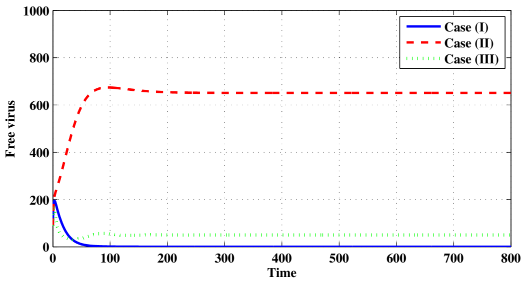


Figure 7: The concentration of HIV.

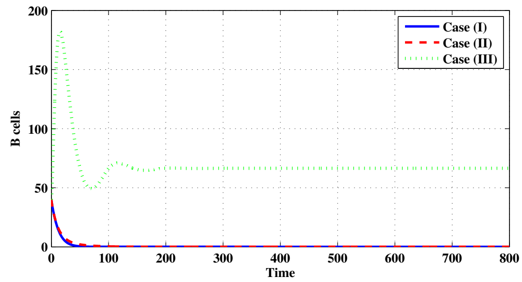


Figure 8: The concentration of B cells.

References

[1] D. Wodarz and M. A. Nowak, Mathematical models of HIV pathogenesis and treatment, BioEssays, **24**, (2002), 1178-1187.

[2] L. Rong and A. S. Perelson, Modeling HIV persistence, the latent reservoir, and viral blips, Journal of Theoretical Biology, **260** (2009), 308-331.

[3] M. A. Nowak and C. R. M. Bangham, *Population dynamics of immune responses to persistent viruses*, Science, **272** (1996) 74-79.

[4] M. A. Nowak, and R. M. May, *Virus dynamics: Mathematical Principles of Immunology and Virology*, Oxford Uni., Oxford, 2000.

[5] A. M. Elaiw, A. M. Althiabi, M. A. Alghamdi and N. Bellomo, *Dynamical behavior of a general HIV-1 infection model with HAART and cellular reservoirs*, Journal of Computational Analysis and Applications, **24**(4) (2018), 728-743.

[6] A. M. Elaiw, A. A. Raezah and A. S. Alofi, *Stability of delay-distributed virus dynamics model with cell-to-cell transmission and CTL immune response*, Journal of Computational Analysis and Applications, **25(**8) (2018), 1518-1531.

[7] A. S. Perelson, and P. W. Nelson, *Mathematical analysis of HIV-1 dynamics in vivo*, SIAM Rev., 41 (1999), 3-44.

[8] P. K. Roy, A. N. Chatterjee, D. Greenhalgh, and Q. J.A. Khan, *Long term dynamics in a mathematical model of HIV-1 infection with delay in different variants of the basic drug therapy model*, Nonlinear Anal. Real World Appl., **14** (2013), 1621-1633.

[9] Huang, Y., S. L. Rosenkranz, H.Wu, *Modeling HIV dynamic and antiviral response with consideration of time-varying drug exposures, adherence and phenotypic sensitivity*, Mathematical Biosciences, 184(2), pp. 165-186, 2003.

[10] G. Huang, Y. Takeuchi and W. Ma, *Lyapunov functionals for delay differential equations model of viral infections*, SIAM J. Appl. Math., **70**(7) (2010), 2693-2708.

[11] A.M. Elaiw and S.A. Azoz, *Global properties of a class of HIV infection models with Beddington-DeAngelis functional response,* Mathematical Methods in the Applied Sciences, **36** (2013), 383-394.

[12] A.M. Elaiw, *Global properties of a class of HIV models*, Nonlinear Analysis: Real World Applications, **11** (2010), 2253-2263.

[13] C. Monica and M. Pitchaimani, *Analysis of stability and Hopf bifurcation for HIV-1 dynamics with PI and three intracellular delays*, Nonlinear Analysis: Real World Applications, **27** (2016), 55-69.

[14] M. Y. Li and L. Wang, *Backward bifurcation in a mathematical model for HIV infection in vivo with anti-retroviral treatment*, Nonlinear Analysis: Real World Applications, **17** (2014), 147-160.

[15] S. Liu, and L. Wang, *Global stability of an HIV-1 model with distributed intracellular delays and a combination therapy*, Mathematical Biosciences and Engineering, **7**(3) (2010), 675-685.

[16] A. M. Elaiw and N. H. AlShamrani, *Stability of a general delay-distributed virus dynamics model with multi-staged infected progression and immune response*, Mathematical Methods in the Applied Sciences, **40(3)** (2017), 699-719.

[17] A.M. Elaiw and N.A. Almuallem, *Global dynamics of delay-distributed HIV infection models with differential drug efficacy in cocirculating target cells*. Mathematical Methods in the Applied Sciences 2016; **39**:4-31.

[18] X. Wang, A. M. Elaiw, X. Song, *Global properties of a delayed HIV infection model with CTL immune response*, Applied Mathematics and Computation, **218** (2012), 9405-9414.

[19] D.S. Callaway, and A.S. Perelson, *HIV-1 infection and low steady state viral loads*, Bull. Math. Biol., **64** (2002), 29-64.

[20] B. Buonomo, and C. Vargas-De-Le, *Global stability for an HIV-1 infection model including an eclipse stage of infected cells*, Journal of Mathematical Analysis and Applications, **385** (2012), 709-720.

[21] H. Wang, R. Xu, Z. Wang, H. Chen, *Global dynamics of a class of HIV-1 infection models with latently infected cells*, Nonlinear Analysis: Modelling and Control, **20** (1) (2012), 21-37.

[22] A. Korobeinikov, *Global properties of basic virus dynamics models*, Bull. Math. Biol. 66, (2004), 879-883.

[23] A. M. Elaiw and N. H. AlShamrani, *Global stability of humoral immunity virus dynamics models with nonlinear infection rate and removal*, Nonlinear Analysis: Real World Applications, **26**, (2015), 161-190.

[24] Q. Sun, L. Min and Y. Kuang," Global stability of infection-free state and endemic infection state of a modified human immunodeficiency virus infection model," *IET Systems Biology*, vol. 9, pp. 95 -103, 2015.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

FOURIER SERIES OF TWO VARIABLE HIGHER-ORDER FUBINI FUNCTIONS

LEE CHAE JANG, GWAN-WOO JANG, DAE SAN KIM, AND TAEKYUN KIM

Abstract. In this paper, we consider the two variable higher-order Fubini functions and investigate their Fourier series expansions. In addition, we will express those functions in terms of Bernoulli functions and obtain as a consequence the corresponding polynomial identities for the two variable higher-order Fubini polynomials.

1. Introduction

For each nonnegative integer r, the two variable Fubini polynomials $F_m^{(r)}(x;y)$ of order r are defined by

$$\frac{e^{xt}}{(1-y(e^t-1))^r}=\sum_{m=0}^{\infty}F_m^{(r)}(x;y)\frac{t^m}{m!},\quad \text{(see [4, 7])}. \tag{1.1}$$

However, in this paper y will be an arbitrary but fixed nonzero real number, and hence $F_m^{(r)}(x;y)$ are polynomials in x, for each $0\neq y\in\mathbb{R}$.

In the case of $r=1$, $F_m(x;y)=F_m^{(1)}(x;y)$ are called two variable Fubini polynomials and they were introduced by Kilar and Simsek in [4]. For $x=0$, $F_m^{(r)}(y)=F_m^{(r)}(0;y)$ are called Fubini polynomials of order r, and $F_m^{(r)}=F_m^{(r)}(1)=F_m^{(r)}(0;1)$ Fubini numbers of order r. Further, $F_m^{(r)}(x;1)$ are called ordered Bell polynomials of order r and they are denoted by $Ob_m^{(r)}(x)$; $F_m^{(r)}(1)=F_m^{(r)}(0;1)$ are also called ordered Bell numbers of order r and they are also denoted by $Ob_m^{(r)}$. Thus $Ob_m^{(r)}(x)$ and $Ob_m^{(r)}$ are respectively given by

$$\frac{e^{xt}}{(2-e^t)^r}=\sum_{m=0}^{\infty}Ob_m^{(r)}(x)\frac{t^m}{m!}, \tag{1.2}$$

$$\frac{1}{(2-e^t)^r}=\sum_{m=0}^{\infty}Ob_m^{(r)}\frac{t^m}{m!}, \tag{1.3}$$

(see [1, 3, 5]).

As we see from (1.1), $F_m^{(r)}(x;y)$ are Appell polynomials and hence

$$\frac{d}{dx}F_m^{(r)}(x;y)=mF_{m-1}^{(r)}(x;y),\qquad (m\geq 1). \tag{1.4}$$

Also, we have

$$yF_m^{(r)}(x+1;y)=(y+1)F_m^{(r)}(x;y)-F_m^{(r-1)}(x;y),\qquad (m\geq 0). \tag{1.5}$$

2010 *Mathematics Subject Classification.* 11B83, 42A16.

Key words and phrases. Fourier series, two variable higher-order Fubini function, two variable higher-order Fubini polynomial, Bernoulli function.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Indeed,

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left(F_m^{(r)}(x+1;y)-F_m^{(r)}(x;y)\right)\frac{t^m}{m!}\\
&=\frac{e^{xt}(e^t-1)}{(1-y(e^t-1))^r}\\
&=\frac{1}{y}\left(\frac{e^{xt}}{(1-y(e^t-1))^r}-\frac{e^{xt}}{(1-y(e^t-1))^{r-1}}\right)\\
&=\frac{1}{y}\sum_{m=0}^{\infty}\left(F_m^{(r)}(x;y)-F_m^{(r-1)}(x;y)\right)\frac{t^m}{m!}.
\end{aligned}\tag{1.6}$$

The identity (1.5) follows from this. In turn, from (1.4) and (1.5), we obtain

$$F_m^{(r)}(1;y)-F_m^{(r)}(y)=\frac{1}{y}\left(F_m^{(r)}(y)-F_m^{(r-1)}(y)\right),\tag{1.7}$$

$$\begin{aligned}
\int_0^1 F_m^{(r)}(x;y)dx&=\frac{1}{m+1}\left(F_{m+1}^{(r)}(1;y)-F_{m+1}^{(r)}(y)\right)\\
&=\frac{1}{(m+1)y}\left(F_{m+1}^{(r)}(y)-F_{m+1}^{(r-1)}(y)\right).
\end{aligned}\tag{1.8}$$

As is well-known, the Bernoulli polynomials $B_m(x)$ are given by

$$\frac{t}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!},\quad \text{(see [2])}.\tag{1.9}$$

For any real number x, the fractional part of x is denoted by $< x >= x-[x]\in[0,1)$. We also need the following facts about Bernoulli functions $B_m(< x >)$:

(a) for $m\geq 2$,

$$B_m(< x >)=-m!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^m},\tag{1.10}$$

(b) for $m=1$,

$$-\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{2\pi in}=\begin{cases}B_1(< x >), & \text{for } x\in\mathbb{R}-\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}\tag{1.11}$$

In this paper, we will consider the two variable higher-order Fubini functions $F_m^{(r)}(< x >;y)$, for each $0\neq y\in\mathbb{R}$, and derive their Fourier series expansions. In addition, we will express those functions in terms of Bernoulli functions and obtain as a consequence the corresponding polynomial identities for the two variable higher-order Fubini polynomials. For some related to Fourier series, we refer the reader to [5,6,8].

2. Main results

In this section, we assume that $m\geq 1$, $r\geq 1$, and $0\neq y\in\mathbb{R}$. For convenience, we set

$$\Delta_m^{(r)}(y)=F_m^{(r)}(1;y)-F_m^{(r)}(y)=\frac{1}{y}\left(F_m^{(r)}(y)-F_m^{(r-1)}(y)\right).\tag{2.1}$$

We note here that

$$F_m^{(r)}(1;y)=F_m^{(r)}(y) \Leftrightarrow \Delta_m^{(r)}(y)=0 \\ \Leftrightarrow F_m^{(r)}(y)=F_m^{(r-1)}(y), \tag{2.2}$$

and

$$\int_0^1 F_m^{(r)}(x;y)dx=\frac{1}{m+1}\Delta_{m+1}^{(r)}(y). \tag{2.3}$$

Before we move on our discussion for Fourier series expansions of $F_m^{(r)}(<x>;y)$, in passing we note the following:

$$\frac{1}{(1-y)^r}F_m^{(r)}(\frac{y}{1-y})=\sum_{k=0}^{\infty}\binom{r+k-1}{k}k^m y^k, \tag{2.4}$$

from which, by letting $y=\frac{1}{2}$, we get

$$Ob_m^{(r)}=F_m^{(r)}(1)=\frac{1}{2^r}\sum_{k=0}^{\infty}\binom{r+k-1}{k}\frac{k^m}{2^k}. \tag{2.5}$$

Indeed, we may see (2.4) from

$$\begin{aligned}\frac{1}{(1-y)^r}\sum_{m=0}^{\infty}F_m^{(r)}\left(\frac{y}{1-y}\right)\frac{t^m}{m!}&=(1-ye^t)^{-r}\\&=\sum_{k=0}^{\infty}\binom{r+k-1}{k}y^k e^{kt}\\&=\sum_{k=0}^{\infty}\binom{r+k-1}{k}y^k\sum_{m=0}^{\infty}k^m\frac{t^m}{m!}\\&=\sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty}\binom{r+k-1}{k}k^m y^k\right)\frac{t^m}{m!}.\end{aligned} \tag{2.6}$$

$F_m^{(r)}(<x>;y)$ is a periodic function on $\mathbb{R}$ with period 1 and piecewise C^∞. Further, in view of (2.2), $F_m^{(r)}(<x>;y)$ is continuous from those (r,m) with $\Delta_m^{(r)}(y)=0$ (or equivalently $F_m^{(r)}(y)=F_m^{(r-1)}(y)$), and is discontinuous with jump disconitinuities at integers for those (r,m) with $\Delta_m^{(r)}(y)\neq 0$ (or equivalently $F_m^{(r)}(y)\neq F_m^{(r-1)}(y)$).

The Fourier series of $F_m^{(r)}(<x>;y)$ is

$$\sum_{n=-\infty}^{\infty}C_n^{(m,r,y)}e^{2\pi inx} \tag{2.7}$$

where

$$\begin{aligned}C_n^{(m)}=C_n^{(m,r,y)}&=\int_0^1 F_m^{(r)}(<x>;y)e^{-2\pi inx}dx\\&=\int_0^1 F_m^{(r)}(x;y)e^{-2\pi inx}dx.\end{aligned} \tag{2.8}$$

Now, we would like to determine the Fourier coefficients $C_n^{(m)}$.

Case 1 : $n \neq 0$.

$$\begin{aligned} C_n^{(m)} &= \int_0^1 F_m^{(r)}(x;y)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}\left[F_m^{(r)}(x;y)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \left(\frac{\partial}{\partial x}F_m^{(r)}(x;y)\right)e^{-2\pi inx}dx \\ &= -\frac{1}{2\pi in}(F_m^{(r)}(1;y) - F_m^{(r)}(y)) + \frac{m}{2\pi in}\int_0^1 F_{m-1}^{(r)}(x;y)e^{-2\pi inx}dx \\ &= \frac{m}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m^{(r)}(y). \end{aligned} \tag{2.9}$$

Thus we have shown that

$$C_n^{(m)} = \frac{m}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m^{(r)}(y), \tag{2.10}$$

from which by induction on m we get

$$C_n^{(m)} = -\frac{1}{m+1}\sum_{j=1}^{m}\frac{(m+1)_j}{(2\pi in)^j}\Delta_{m-j+1}^{(r)}(y). \tag{2.11}$$

Case 2: $n = 0$.

$$C_0^{(m)} = \int_0^1 F_m^{(r)}(x;y)dx = \frac{1}{m+1}\Delta_{m+1}^{(r)}(y). \tag{2.12}$$

Assume first that $\Delta_m^{(r)}(y) = 0$. Then $F_m^{(r)}(1;y) = F_m^{(r)}(y)$. As $F_m^{(r)}(< x >;y)$ is piecewise C^∞ and continuous, the Fourier series of $F_m^{(r)}(< x >;y)$ converges uniformly to $F_m^{(r)}(< x >;y)$, and

$$\begin{aligned} & F_m^{(r)}(< x >;y) \\ &= \frac{1}{m+1}\Delta_{m+1}^{(r)}(y) + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+1}\sum_{j=1}^{m}\frac{(m+1)_j}{(2\pi in)^j}\Delta_{m-j+1}^{(r)}(y)\right)e^{2\pi inx} \\ &= \frac{1}{m+1}\Delta_{m+1}^{(r)}(y) + \frac{1}{m+1}\sum_{j=1}^{m}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}(y)\left(-j!\sum_{n=-\infty,n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right) \\ &= \frac{1}{m+1}\sum_{j=0,j\neq 1}^{m}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}(y)B_j(< x >) \\ &\quad + \Delta_m^{(r)}(y) \times \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{R} - \mathbb{Z}, \\ 0, & \text{for} \quad x \in \mathbb{Z}. \end{cases} \end{aligned} \tag{2.13}$$

We are ready to state our first result.

Theorem 2.1. *For positive integers r, l, and $0 \neq y \in \mathbb{R}$, we let*

$$\Delta_l^{(r)}(y) = F_l^{(r)}(1;y) - F_l^{(r)}(y) = \frac{1}{y}\left(F_l^{(r)}(y) - F_l^{(r-1)}(y)\right). \tag{2.14}$$

Assume that $\Delta_m^{(r)}(y) = 0$. Then we have the following.

(a) $F_m^{(r)}(<x>;y)$ *has the Fourier series expansion*

$$F_m^{(r)}(<x>;y) = \frac{1}{m+1}\Delta_{m+1}^{(r)}(y) + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+1}\sum_{j=1}^{m}\frac{(m+1)_j}{(2\pi in)^j}\Delta_{m-j+1}^{(r)}(y)\right)e^{2\pi inx}, \tag{2.15}$$

for all $x \in \mathbb{R}$, *where the convergence is uniform.*

(b)

$$F_m^{(r)}(<x>;y) = \frac{1}{m+1}\sum_{j=0,j\neq 1}^{m}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}(y)B_j(<x>), \tag{2.16}$$

for all $x \in \mathbb{R}$.

Assume next that $\Delta_m^{(r)}(y) \neq 0$. Then $F_m^{(r)}(1;y) \neq F_m^{(r)}(y)$. Hence $F_m^{(r)}(<x>;y)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $F_m^{(r)}(<x>;y)$ converges pointwise to $F_m^{(r)}(<x>;y)$, for $x \in \mathbb{R}-\mathbb{Z}$, and converges to

$$\frac{1}{2}(F_m^{(r)}(y) + F_m^{(r)}(1;y)) = F_m^{(r)}(y) + \frac{1}{2}\Delta_m^{(r)}(y), \tag{2.17}$$

for $x \in \mathbb{Z}$. We are now ready to state our second result.

Theorem 2.2. *For positive integers* r, l, *and* $0 \neq y \in \mathbb{R}$, *we let*

$$\Delta_l^{(r)}(y) = F_l^{(r)}(1;y) - F_l^{(r)}(y) = \frac{1}{y}\left(F_l^{(r)}(y) - F_l^{(r-1)}(y)\right). \tag{2.18}$$

Assume that $\Delta_m^{(r)}(y) \neq 0$. *Then we have the following.*

(a)

$$\begin{aligned}&\frac{1}{m+1}\Delta_{m+1}^{(r)}(y) + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+1}\sum_{j=1}^{m}\frac{(m+1)_j}{(2\pi in)^j}\Delta_{m-j+1}^{(r)}(y)\right)e^{2\pi inx}\\ &= \begin{cases} F_m^{(r)}(<x>;y), & \text{for} \quad x \in \mathbb{R}-\mathbb{Z},\\ F_m^{(r)}(y) + \frac{1}{2}\Delta_m^{(r)}(y), & \text{for} \quad x \in \mathbb{Z}.\end{cases}\end{aligned} \tag{2.19}$$

(b)

$$\frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}(y)B_j(<x>) = F_m^{(r)}(<x>;y) \tag{2.20}$$

for all $x \in \mathbb{R}-\mathbb{Z}$;

$$\frac{1}{m+1}\sum_{j=0,j\neq 1}^{m}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}(y)B_j(<x>) = F_m^{(r)}(y) + \frac{1}{2}\Delta_m^{(r)}(y) \tag{2.21}$$

for all $x \in \mathbb{Z}$.

We remark that the case of $y = 1$ had been treated in the previous paper [?]. From Theorems 2.1 and 2.2, we have

$$F_m^{(r)}(<x>;y) = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}\Delta_{m-j+1}^{(r)}(y)B_j(<x>), \tag{2.22}$$

for all $x \in \mathbb{R} - \mathbb{Z}$ and $0 \neq y \in \mathbb{R}$. We immediately obtain the following polynomial identities from this observation.

Corollary 2.3. *We have the following polynomial identities for two variable higher-order Fubini polynomials*

(a) $F_m^{(r)}(x;y) = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}\left(F_{m-j+1}^{(r)}(1;y) - F_{m-j+1}^{(r)}(0;y)\right)B_j(x)$,

(b) $yF_m^{(r)}(x;y) = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}\left(F_{m-j+1}^{(r)}(y) - F_{m-j+1}^{(r-1)}(y)\right)B_j(x)$.

For $x = 0$, we have the following identities for higher-order Fubini polynomials.

Corollary 2.4. *We have the following polynomial identities for higher-order Fubini polynomials*

(a) $F_m^{(r)}(y) = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}B_j\left(F_{m-j+1}^{(r)}(1;y) - F_{m-j+1}^{(r)}(0;y)\right)$,

(b) $yF_m^{(r)}(y) = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}B_j\left(F_{m-j+1}^{(r)}(y) - F_{m-j+1}^{(r-1)}(y)\right)$.

Finally, for $y = 1$, we get the following identities for higher-order ordered Bell polynomials.

Corollary 2.5. *We have the following polynomial identities for higher-order ordered Bell polynomials*

(a) $Ob_m^{(r)}(x) = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}\left(Ob_{m-j+1}^{(r)}(1) - Ob_{m-j+1}^{(r)}\right)B_j(x)$,

(b) $Ob_m^{(r)}(x) = \frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}\left(Ob_{m-j+1}^{(r)} - Ob_{m-j+1}^{(r-1)}\right)B_j(x)$,

References

1. R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete mathematics. A foundation for computer science*,Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, 1989. xiv+625 pp. ISBN: 0-201-14236-8 1
2. D. S. Kim, T. Kim, H.-I. Kwon, *Fourier series r-derangement and higher-order derangement functions*, Adv. Stud. Contemp. Math., **28(1)**(2018), 1–11.
3. G.-W.Jang, T. Kim, *Some identities of oredered Bell numbers arising from differential equations*, Adv. Stud. Contemp. Math., **27(3)**(2017), 385–397. 1
4. D. S. Kim, T. Kim, H.-I. Kwon, *A note on degenerate Fubini polynomials*,Proc. Jangjeon Math. Soc., **20(4)**(2017),521–717. 1
5. T. Kim, D. S. Kim, D. V. Dolgy, J.-W. Park, *Fourier series of sums of products of ordered Bell and poly-Bernoulli functions,* J. Inequal. Appl., **(2017)** 2017:84, 17 pp. 1, 1
6. E. Kreyszig *Advanced engineering mathematics. Tenth edition*, John Wiley and Sons, Inc., New York, 2011. ISBN:978-0-470-64613-7 1
7. M. Mursan, G. Toader, *A generalization of Fubini's numbers*, Studia Univ. Babes-Bolyai Math., **31**(1986), 60-65.
8. D. G. Zill, M. R. Cullen *Advanced Engineering Mathematics. Third edition*, Jones and Bartlett Publishers 40 Tall Pine Drive Sudbury, MA 01776, 2006. ISBN: 978-0-7637-3914-0 1

Graduate School of Education, Konkuk University, Seoul 143-701, republic of Korea
E-mail address: lcjang@konkuk.ac.kr

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail address: jgw5687@naver.com

Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
E-mail address: dskim@sogang.ac.kr

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail address: tkkim@kw.ac.kr

WEIGHTED COMPOSITION OPERATORS FROM DIRICHLET TYPE SPACES TO SOME WEIGHTED-TYPE SPACES

MANISHA DEVI, AJAY K. SHARMA AND KULDIP RAJ

ABSTRACT. Let $K : [0,\infty) \to [0,\infty)$ be a right continuous increasing function and $\nu : \mathbb{D} \to (0,\infty)$ be any continuous function. In this paper by considering K and ν as weight functions, we characterize the boundedness and compactness of weighted composition operators from Dirichlet type spaces to some weighted-type spaces.

1. Introduction and Preliminaries

Let $\mathbb{D}$ be the open unit disk and $\partial\mathbb{D}$ be its boundary in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ denotes the class of all holomorphic functions on $\mathbb{D}$, $S(\mathbb{D})$ be the class of all holomorphic self-maps of $\mathbb{D}$ and H^∞ be the space of all bounded analytic functions on $\mathbb{D}$. Let $dA(z) = \frac{dxdy}{\pi} = r\frac{drd\theta}{\pi}$ be the normalized area measure on $\mathbb{D}$.
A continuous function $\nu : \mathbb{D} \longrightarrow (0,\infty)$ is called weight. For $\nu(z) = \nu(|z|)$, $z \in \mathbb{D}$, weight is radial and weight is a standard weight if $\lim_{|z|\to 1^-} \nu(z) = 0$.
For weight ν, the *Bers-type space* $\mathcal{A}_\nu$ is the collection of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} \nu(z)|f(z)| < \infty$$

and with the norm

$$\|f\|_{\mathcal{A}_\nu} = \sup_{z\in\mathbb{D}} \nu(z)|f(z)|,$$

it is a non-separable Banach space. The closure of the set of polynomials in $\mathcal{A}_\nu$ forms a separable Banach space. This set is denoted by $\mathcal{A}_{\nu,0}$ and contains exactly of those $f \in \mathcal{A}_\nu$ such that

$$\lim_{|z|\to 1^-} \nu(z)|f(z)| = 0.$$

The *Bloch-type space* $\mathcal{B}_\nu$ on $\mathbb{D}$ with the weight ν is the space of all holomorphic functions f on $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}} \nu(z)|f'(z)| < \infty.$$

2010 *Mathematics Subject Classification.* 47B33, 30D55, 30H05, 30E05.
Key words and phrases. Weighted composition operator, Dirichlet type space, Bloch-type space, Bers-type spaces, boundedness, compactness.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

The *little Bloch-type space* $\mathcal{B}_{\nu,0}$ is the closure of the set of polynomials in $\mathcal{B}_\nu$ and contains all those $f \in \mathcal{B}_\nu$ such that

$$\lim_{|z|\to 1} \nu(z)|f'(z)| = 0$$

and with the norm

$$\|f\|_{\mathcal{B}_\nu} = |f(0)| + \sup_{z\in\mathbb{D}} \nu(z)|f'(z)| < \infty,$$

both $\mathcal{B}_\nu$ and $\mathcal{B}_{\nu,0}$ form Banach spaces.
For more information about these spaces one may refer [20] and references therein.

Let $\varphi \in S(\mathbb{D})$ and ψ be an analytic map on $\mathbb{D}$. The operator C_φ so called as the *composition operator* and is defined as $C_\varphi f = f \circ \varphi$, $f \in H(\mathbb{D})$. The operator M_ψ which is called as the *multiplication operator* is defined by $M_\psi f = \psi \cdot f$, $f \in H(\mathbb{D})$. For $f \in H(\mathbb{D})$, the *weighted composition operator* on $H(\mathbb{D})$ is defined by

$$(W_{\psi,\varphi} f)(z) = \psi(z) f(\varphi(z)),$$

where $\psi \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$ and $z \in \mathbb{D}$.

It can be easily seen that for $\psi \equiv 1$, the operator reduced to C_φ. If $\varphi(z) = z$, operator get reduced to M_ψ. This operator is basically a linear transformation of $H(\mathbb{D})$ defined by $(W_{\psi,\varphi} f)(z) = \psi(z) f(\varphi(z)) = (M_\psi C_\varphi f)(z)$, for f in $H(\mathbb{D})$ and z in $\mathbb{D}$. The basic problem is to give the function-theoretic characterization when between various function spaces ψ and φ induce bounded or compact weighted composition operator. Various holomorphic functions spaces on various domains have been studied for the the boundedness and compactness of weighted composition operators acting on them. Moreover, a number of papers have been studied on these operators acting on different spaces of holomorphic functions on various domains for more detail (see [1], [5], [7]-[11], [13], [15], [19]).

Consider a function $K : [0,\infty) \to [0,\infty)$ which is right continuous and increasing. The *Dirichlet type space* $\mathcal{D}_K$ consists of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|^2_{\mathcal{D}_K} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 K\big(1 - |z|^2\big) dA(z) < \infty.$$

For more about the Dirichlet type spaces we refer ([2], [3], [4], [12], [14], [16]). In this paper we consider function K as a weight function satisfying the following two conditions:

(a) $K_1(t) = \int_0^t K(s)\frac{ds}{s} \approx K(t)$, $0 < t < 1$;

(b) $K_2(t) = t\int_t^\infty K(s)\frac{ds}{s^2} \approx K(t)$, $t > 0$.

From condition (b), we get that $K(2t) \approx K(t)$ for $0 < t < 1$. Also there exist $C > 0$ sufficiently small for which $t^{-C}K_1(t)$ is increasing and $K_2(t)t^{C-1}$ is decreasing (see [4],

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

[17], [18]).

This paper is entirely devoted to characterize the boundedness and compactness of operator $W_{\psi,\varphi}$ from Dirichlet type spaces to the Bers-type space and Bloch-type space.

Throughout this paper, C will represents a constant which may differ from one occurrence to another. The notation $A \lesssim B$ means that there exist $C > 0$ such that $A \leq CB$. We write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

The paper is organized in a systematic manner. Section 1 covers the introduction and literature part. Lemmas that are used to formulate our main theorems are kept in Section 2. Section 3 contains the boundedness and compactness of the oparator $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$. Section 4 considers the boundedness and compactness of the oparator $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{A}_\nu$.

2. **Auxiliary Results**

To arrive at the main results we use some lemmas, as given below

Lemma 2.1. [4] *Let K be a weight function. Then for any $w \in \mathbb{D}$ and $\varepsilon > 0$, we have*

$$f_z(w) = \frac{(1-|z|^2)^{\varepsilon/2}}{\sqrt{K(1-|z|^2)}(1-w\bar{z})^{1+\varepsilon/2}}$$

is in $\mathcal{D}_K$. Moreover,

$$\sup_{z\in\mathbb{D}} \|f_z\|_{\mathcal{D}_K} \approx 1,$$

and f_z converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|z| \to 1^-$.

The following two lemmas can be proved easily by following the Lemma 2.1 and [4].

Lemma 2.2. [4] *Let K be a weight function. Then for every $f \in \mathcal{D}_K$ we have*

$$|f(z)| \leq C\frac{\|f\|_{\mathcal{D}_K}}{\sqrt{K(1-|z|^2)}(1-|z|^2)}, \quad z \in \mathbb{D}.$$

Lemma 2.3. [4] *Let K be a weight function and n be a positive integer. Then for every $f \in \mathcal{D}_K$ we have*

$$|f^{(n)}(z)| \leq C\frac{\|f\|_{\mathcal{D}_K}}{\sqrt{K(1-|z|^2)}(1-|z|^2)^{n+1}}, \quad z \in \mathbb{D}.$$

The following criterion characterize the compactness. It was given for the first time in [6]. Since the proof is standard, so we omit it.

Lemma 2.4. *Let ν be the standard weight and the operator $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is bounded. Then $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is compact if and only if for any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{D}_K$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have*

$$\lim_{n\to\infty} \|W_{\psi,\varphi} f_n\|_{\mathcal{B}_\nu} = 0.$$

3. Boundedness and compactness of weighted composition operator from Dirichlet type space to Bloch-type space

Theorem 3.1. *Let ν and K be two weight functions, $\psi \in H(\mathbb{D})$ and φ be a self analytic map on $\mathbb{D}$. Then the operator $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is bounded if and only if the following conditions are satisfied:*

$$(i)\ M_1 = \sup_{z\in\mathbb{D}} \frac{\nu(z)|\psi'(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)} < \infty;$$
$$(ii)\ M_2 = \sup_{z\in\mathbb{D}} \frac{\nu(z)|\psi(z)\varphi'(z)|}{\sqrt{K(1-|\varphi(z)|^2)^2}(1-|\varphi(z)|^2)^2} < \infty.$$

Furthermore, if the operator $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is bounded, then

$$M_1 + M_2 \lesssim \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu} \lesssim 1 + M_1 + M_2.$$

Proof. First suppose that condition (i) and (ii) hold. Using Lemma 2.2 we have,

$$\begin{aligned} \nu(z)|(W_{\psi,\varphi}f)'(z))| &\leq \nu(z)|\psi'(z)||f(\varphi(z))| + \nu(z)|\psi(z)\varphi'(z)||f'(\varphi(z))| \\ &\lesssim \Bigg(\frac{\nu(z)|\psi'(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)} \\ &\quad + \frac{\nu(z)|\psi(z)\varphi'(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^2} \Bigg) \|f\|_{\mathcal{D}_K}. \end{aligned} \tag{3.1}$$

Also,

$$\begin{aligned} |(W_{\psi,\varphi}f)(0)| &= |\psi(0)||f(\varphi(0))| \\ &\lesssim \frac{|\psi(0)|}{\sqrt{K(1-|\varphi(0)|^2)}(1-|\varphi(0)|^2)} \|f\|_{\mathcal{D}_K}. \end{aligned} \tag{3.2}$$

From conditions $(i), (ii)$ and equations (3.1) and (3.2), we get

$$\begin{aligned} \|W_{\psi,\varphi}f\|_{\mathcal{B}_\nu} &= |\psi(0)||f(\varphi(0))| + \sup_{z\in\mathbb{D}} \nu(z)|(W_{\psi,\varphi}f)'(z)| \\ &\lesssim \left(\frac{|\psi(0)|}{\sqrt{K(1-|\varphi(0)|^2)}(1-|\varphi(0)|^2)} + M_1 + M_2 \right) \|f\|_{\mathcal{D}_K} \\ &\lesssim \big(1 + M_1 + M_2\big) \|f\|_{\mathcal{D}_K}. \end{aligned}$$

Therefore, $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is bounded and

$$\|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu} \lesssim 1 + M_1 + M_2. \tag{3.3}$$

Conversely, suppose that $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is bounded. Let $z = \varphi(\zeta)$, $\zeta \in \mathbb{D}$ and

$$g_z(w) = \tau_z(w) f_z(w), \tag{3.4}$$

where $f_z(w)$ is defined in Lemma 2.1 and $\tau_z(w)$ is defined as

$$\tau_z(w) = 1 - \frac{1-|z|^2}{1-\bar{z}w}. \tag{3.5}$$

Then $\tau_z \in H^\infty$ as

$$\sup_{w\in\mathbb{D}} |\tau_z(w)| \leq \sup_{w\in\mathbb{D}} \left(1 + \frac{1-|z|^2}{1-|z||w|}\right) \leq 3.$$

Therefore, $g_z \in \mathcal{D}_K$ and $\sup_{w\in\mathbb{D}} \|g_z\|_{\mathcal{D}_K} \lesssim 1$. From equation (3.5) we have,

$$\tau_z(z) = 0 \tag{3.6}$$

and

$$\tau_z'(w) = \frac{-\bar{z}(1-|z|^2)}{(1-\bar{z}w)^2}.$$

Thus,

$$\tau_z'(z) = \frac{-\bar{z}}{(1-|z|^2)}. \tag{3.7}$$

Therefore, $g_z(z) = 0$, using the value of $f_z(z)$ and from (3.7), we obtain

$$\begin{aligned} g_z'(z) &= \tau_z'(z) f_z(z) + \tau_z(z) f_z'(z) \\ &= \frac{-\bar{z}}{\sqrt{K(1-|z|^2)}(1-|z|^2)^2}. \end{aligned}$$

Using the above fact, we get

$$\begin{aligned} \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu} &\gtrsim \|W_{\psi,\varphi} g_{\varphi(\zeta)}\|_{\mathcal{B}_\nu} \\ &\geq \nu(\zeta)|\psi'(\zeta) g_{\varphi(\zeta)}(\varphi(\zeta)) + \psi(\zeta)\varphi'(\zeta) g'_{\varphi(\zeta)}(\varphi(\zeta))| \\ &\geq \nu(\zeta)|\psi(\zeta)\varphi'(\zeta) g'_{\varphi(\zeta)}(\varphi(\zeta))| \\ &\geq \frac{\nu(\zeta)|\psi(\zeta)\varphi'(\zeta)||\varphi(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)^2}. \end{aligned}$$

When $\delta \in (0,1)$ is fixed, we have

$$\sup_{|\varphi(\zeta)|>\delta} \frac{\nu(\zeta)|\psi(\zeta)\varphi'(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)^2} \lesssim \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu}. \tag{3.8}$$

Taking $f_z(w) \equiv 1 \in \mathcal{D}_K$, implies that

$$\sup_{w\in\mathbb{D}} \nu(w)|\psi'(w)| = \|W_{\psi,\varphi}(1)\|_{\mathcal{B}_\nu} \lesssim \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu}. \tag{3.9}$$

Again taking $f(w) = w \in \mathcal{D}_K$, using the asymptotic estimate (3.9) and boundedness of φ, we get

$$\sup_{w\in\mathbb{D}} \nu(w)|\psi(w)\varphi'(w)| \lesssim \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu}. \tag{3.10}$$

Using (3.10) and the compactness of φ, we easily get

$$\begin{aligned} &\sup_{|\varphi(\zeta)|\leq\delta} \frac{\nu(\zeta)|\psi(\zeta)\varphi'(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)^2} \\ &\lesssim \left(\frac{1}{\sqrt{K(1-\delta^2)}(1-\delta^2)^2}\right)\|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu}. \end{aligned} \tag{3.11}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Further, from (3.8) and (3.11), we obtain

$$\sup_{\zeta\in\mathbb{D}} \frac{\nu(\zeta)|\psi(\zeta)\varphi'(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)^2} \lesssim \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu}. \tag{3.12}$$

Again, for f_z as defined in Lemma 2.1, we have

$$\begin{aligned}\|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu} &\gtrsim \|W_{\psi,\varphi}f_{\varphi(\zeta)}\|_{\mathcal{B}_\nu}\\ &\geq \nu(\zeta)\big|\psi'(\zeta)f_{\varphi(\zeta)}(\varphi(\zeta))+\psi(\zeta)\varphi'(\zeta)f'_{\varphi(\zeta)}(\varphi(\zeta))\big|\\ &\geq \frac{\nu(\zeta)|\psi'(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)}\\ &\quad+\frac{(1+\epsilon/2)\nu(\zeta)|\psi(\zeta)||\varphi'(\zeta)||\varphi(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)^2}.\end{aligned}$$

By using the boundedness of φ, we get

$$\begin{aligned}&\frac{\nu(\zeta)|\psi'(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)}\\ &\leq \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu}+C\frac{\nu(\zeta)|\psi(\zeta)||\varphi'(\zeta)||\varphi(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)^2}.\end{aligned} \tag{3.13}$$

Taking the supremum over $\zeta\in\mathbb{D}$ in (3.13) and using (3.12), we get

$$\sup_{\zeta\in\mathbb{D}} \frac{\nu(\zeta)|\psi'(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)} \lesssim \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu}. \tag{3.14}$$

From (3.12) and (3.14), we obtain

$$M_1+M_2 \lesssim \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu}. \tag{3.15}$$

Hence, from (3.3) and (3.15), we get

$$M_1+M_2 \lesssim \|W_{\psi,\varphi}\|_{\mathcal{D}_K\to\mathcal{B}_\nu} \lesssim 1+M_1+M_2.$$

□

Theorem 3.2. *Let ν be a standard weight, $\psi\in H(\mathbb{D})$ and φ be a self analytic map on $\mathbb{D}$. Let K be a weight function. Assume that $W_{\psi,\varphi}:\mathcal{D}_K\to\mathcal{B}_\nu$ is bounded. Then the operator $W_{\psi,\varphi}:\mathcal{D}_K\to\mathcal{B}_\nu$ is compact if and only if the following conditions are satisfied:*

$$(i)\ \lim_{|\varphi(z)|\to1} \frac{\nu(z)|\psi'(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)}=0;$$

$$(ii)\ \lim_{|\varphi(z)|\to1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^2}=0.$$

Proof. First suppose that (i) and (ii) hold. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of functions in $\mathcal{D}_K$ that converges to zero uniformly on compact subset of $\mathbb{D}$. To prove the compactness of $W_{\psi,\varphi}$, we have to show that $\|W_{\psi,\varphi}f_n\|_{\mathcal{B}_\nu}\to0$ as $n\to\infty$.
Condition (i) and (ii) implies that for any $\varepsilon>0$, there exists $\delta\in(0,1)$ such that

$$\frac{\nu(z)|\psi'(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)}<\varepsilon \tag{3.16}$$

and

$$\frac{\nu(z)|\psi(z))\varphi'(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)^2} < \varepsilon, \tag{3.17}$$

whenever $\delta < |\varphi(z)| < 1$.
Let $A = \{z \in \mathbb{D} : |z| \leq \delta\}$ be a compact subset of $\mathbb{D}$. We have

$$\begin{aligned}
\|W_{\psi,\varphi} f_n\|_{\mathcal{B}_\nu} &= |\psi(0)||f_n(\varphi(0))| + \sup_{\zeta\in\mathbb{D}} \nu(\zeta)|(W_{\psi,\varphi} f_n)'(\zeta)| \\
&\leq |\psi(0)||f_n(\varphi(0))| + \sup_{\zeta\in\mathbb{D}} \nu(\zeta)|\psi'(\zeta)||f_n(\varphi(\zeta))| \\
&\quad + \sup_{\zeta\in\mathbb{D}} \nu(\zeta)|\psi(\zeta)\varphi'(\zeta)||f_n'(\varphi(\zeta))| \\
&\leq |\psi(0)||f_n(\varphi(0))| + \sup_{\{\zeta\in\mathbb{D}:\varphi(\zeta)\in A\}} \nu(\zeta)|\psi'(\zeta)||f_n(\varphi(\zeta))| \\
&\quad + \sup_{\{\zeta\in\mathbb{D}:\delta<|\varphi(\zeta)|<1\}} \nu(\zeta)|\psi'(\zeta)||f_n(\varphi(\zeta))| \\
&\quad + \sup_{\{\zeta\in\mathbb{D}:\varphi(\zeta)\in A\}} \nu(\zeta)|\psi(\zeta)\varphi'(\zeta)||f_n'(\varphi(\zeta))| \\
&\quad + \sup_{\{\zeta\in\mathbb{D}:\delta<|\varphi(\zeta)|<1\}} \nu(\zeta)|\psi(\zeta)\varphi'(\zeta)||f_n'(\varphi(\zeta))| \\
&\leq |\psi(0)||f_n(\varphi(0))| + \|\psi\|_{\mathcal{B}_\nu} \sup_{z\in A}|f_n(z)| + N \sup_{z\in A}|f_n'(z)| \\
&\quad + C \sup_{\{\zeta\in\mathbb{D}:\delta<|\varphi(\zeta)|<1\}} \frac{\nu(\zeta)|\psi'(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)} \|f_n\|_{\mathcal{D}_K} \\
&\quad + C \sup_{\{\zeta\in\mathbb{D}:\delta<|\varphi(\zeta)|<1\}} \frac{\nu(\zeta)|\psi(\zeta)\varphi'(\zeta)|}{\sqrt{K(1-|\varphi(\zeta)|^2)}(1-|\varphi(\zeta)|^2)^2} \|f_n\|_{\mathcal{D}_K},
\end{aligned} \tag{3.18}$$

where we have $|f_n(\varphi(0))| < \varepsilon$, $\sup\limits_{z\in A}|f_n(z)| < \varepsilon$ and $\sup\limits_{z\in A}|f_n'(z)| < \varepsilon$, for some $N_0 \in \mathbb{N}$ and for all $n \geq N_0$. Also we have used the fact that $\psi \in \mathcal{B}_\nu$ and $N = \sup\limits_{\zeta\in\mathbb{D}} \nu(\zeta)|\psi(\zeta)\varphi'(\zeta)| < \infty$. Using the above fact in (3.18) and along with (3.16) and (3.17), we get $\|W_{\psi,\varphi} f_n\|_{\mathcal{B}_\nu} < C\varepsilon$, for $n \geq N_0$. Since $\varepsilon > 0$ is arbitrary, so we have $\|W_{\psi,\varphi} f_n\|_{\mathcal{B}_\nu} \to 0$ as $n \to \infty$. Hence, $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is compact.
Conversely, suppose that $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is compact. Let $(\zeta_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(\zeta_n)| \to 1$ as $n \to \infty$. Suppose such a sequence does not exist, then (i) & (ii) are vacuously satisfied. Let $g_n(w) = \tau_{\varphi(\zeta_n)}(w) f_{\varphi(\zeta_n)}(w)$, where f_z and τ_z are defined earlier in Lemma 2.1 and Theorem 3.1. Then, $\|\tau_{\varphi(\zeta_n)}\|_{\mathcal{D}_K} \lesssim 1$, $\|f_{\varphi(\zeta_n)}\|_{\mathcal{D}_K} \lesssim 1$ and $(f_{\varphi(\zeta_n)})_{n\in\mathbb{N}}$ converges to zero uniformly on compact subset of $\mathbb{D}$ as $n \to \infty$. So, $\|g_n\|_{\mathcal{D}_K} \lesssim 1$ and $(g_n)_{n\in\mathbb{N}}$ converges to zero uniformly on compact subset of $\mathbb{D}$ as $n \to \infty$.
Since $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is compact, so we have

$$\|W_{\psi,\varphi} g_n\|_{\mathcal{B}_\nu} \to 0 \text{ as } n \to \infty.$$

Also, we have (as in Theorem 3.1),

$$\|W_{\psi,\varphi} g_n\|_{\mathcal{B}_\nu} \geq \frac{\nu(\zeta_n)|\psi(\zeta_n)\varphi'(\zeta_n)|\varphi(\zeta_n)|}{\sqrt{K(1-|\varphi(\zeta_n)|^2)}(1-|\varphi(\zeta_n)|^2)^2}.$$

Using the above two facts, we get

$$\lim_{|\varphi(\zeta_n)|\to 1} \frac{\nu(\zeta_n)|\psi(\zeta_n)\varphi'(\zeta_n)|}{\sqrt{K(1-|\varphi(\zeta_n)|^2)}(1-|\varphi(\zeta_n)|^2)^2} = 0. \tag{3.19}$$

Using Lemma 2.1, we have $\sup_{n\in\mathbb{N}} \|f_{\varphi(\zeta_n)}\|_{\mathcal{D}_K} \lesssim 1$ and $f_{\varphi(\zeta_n)}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. Since $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{B}_\nu$ is compact. Therefore,

$$\lim_{n\to\infty} \|W_{\psi,\varphi} f_{\varphi(\zeta_n)}\|_{\mathcal{B}_\nu} = 0. \tag{3.20}$$

From (3.13), we obtain

$$\begin{aligned}\frac{\nu(\zeta_n)|\psi'(\zeta_n)|}{\sqrt{K(1-|\varphi(\zeta_n)|^2)}(1-|\varphi(\zeta_n)|^2)} &\leq \|W_{\psi,\varphi} f_{\varphi(\zeta_n)}\|_{\mathcal{D}_K\to\mathcal{B}_\nu} \\ &\quad + C\frac{\nu(\zeta_n)|\psi(\zeta_n)\varphi'(\zeta_n)|}{\sqrt{K(1-|\varphi(\zeta_n)|^2)}(1-|\varphi(\zeta_n)|^2)^2},\end{aligned}$$

which on combining with (3.19) and (3.20) gives

$$\lim_{|\varphi(\zeta_n)|\to 1} \frac{\nu(\zeta_n)|\psi'(\zeta_n)|}{\sqrt{K(1-|\varphi(\zeta_n)|^2)}(1-|\varphi(\zeta_n)|^2)} = 0. \tag{3.21}$$

Hence, the result follows from (3.19) and (3.21). ☐

4. Boundedness and compactness of weighted composition operator from Dirichlet type space to Bers-type space

In this section, we consider the Bers-type spaces and characterize the boundedness and compactness of operator $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{A}_\nu$. We omit the proofs as these are similar to Theorem 3.1 and 3.2 of Section 3.

Theorem 4.1. *Let ν be a weight and K be a weight function, $\psi \in H(\mathbb{D})$ and φ be a self analytic map on $\mathbb{D}$. Then the operator $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{A}_\nu$ is bounded if and only if the following condition is satisfied:*

$$l_1 = \sup_{z\in\mathbb{D}} \frac{\nu(z)|\psi(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)} < \infty.$$

Theorem 4.2. *Let ν be a standard weight, $\psi \in H(\mathbb{D})$ and φ be a self analytic map on $\mathbb{D}$. Let K be a weight function. Assume that the operator $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{A}_\nu$ is bounded. Then $W_{\psi,\varphi} : \mathcal{D}_K \to \mathcal{A}_\nu$ is compact if and only if the following condition is satisfied:*

$$\lim_{|\varphi(z)|\to 1^-} \frac{\nu(z)|\psi(z)|}{\sqrt{K(1-|\varphi(z)|^2)}(1-|\varphi(z)|^2)} = 0.$$

Acknowledgements.
The first author is thankful to DST (India) for Inspire fellowship [DST/INSPIRE Fellowship/2016/IF160486].

References

[1] R.F. Allen and F. Colonna, Weighted composition operators on the Bloch space of a bounded homogeneous domain, *Oper. Theory: Adv. Appl.*, **202** (2010), 11–37.

[2] H.A. Alsaker, Multipliers of the Dirichlet Space, *Masters thesis*, Bergen (2009).

[3] N. Arcozzi, R. Rochberg, E.T. Sawyer and B.D. Wick, The Dirichlet space: a survey, *New York J. Math.*, **17A** (2011), 45–86.

[4] G. Bao, Z. Lou, R. Qian and H. Wulan, On multipliers of Dirichlet type spaces, *Complex Anal. Oper. Theory*, **9** (2015), 1701–1732.

[5] F. Colonna and S. Li, Weighted composition operators from the minimal Möbius invariant space into the Bloch space, *Mediterr. J. Math.*, **10** (2013), 39–409.

[6] C.C. Cowen and B.D. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press (1995).

[7] Z.J. Jiang, Weighted composition operators from weighted Bergman spaces to some spaces of analytic functions on the upper half plane, *Util. Math.*, **93** (2014), 205–212.

[8] S. Li and S. Stević, Weighted composition operators from Bergman-type spaces into Bloch spaces, *Proc. Indian Acad. Sci. Math. Sci.*, **117** (2007), 371–385.

[9] S. Li and S. Stević, Weighted composition operators from H^∞ to the Bloch space on the polydisc, *Abstr. Appl. Anal.*, Volume **2007**, Article ID 48478, 13 pp.

[10] S. Li and S. Stević, Weighted composition operators from Zygmund spaces into Bloch spaces, *Appl. Math. Comput.*, **206** (2008), 825–831.

[11] S. Li and S. Stević, Weighted composition operators between H^∞ and α-Bloch spaces in the unit ball, *Taiwan J. Math. Soc.*, **12** (2008), 1625–1639.

[12] R. Rochberg and Z. Wu, A new characterization of Dirichlet type spaces and applications, *Illinois J. Math.*, **37** (1993), 101–122.

[13] A.K. Sharma and S.I. Ueki, Compact composition operator on the Bloch space and the growth space of the upper half-plane, *Mediterr. J. Math.*, **14** (2017), 9pp.

[14] D. Stegenga, Multipliers of the Dirichlet space, *Illinois J. Math.*, **24** (1980), 113–139.

[15] S. Stević, A.K. Sharma and S.D. Sharma, Generalized integration operators from the space of integral transforms into Bloch type spaces, *J. Comput. Anal. Appl.*, **14** (2012), 1139–1147.

[16] G. Taylor, Multipliers on $\mathcal{D}_\alpha$, *Trans. Amer. Math. Soc.*, **123** (1966), 229–240.

[17] H. Wulan and J. Zhou, $\mathcal{Q}_K$ and Morrey type spaces, *Ann. Acad. Sci. Fenn. Math.*, **38** (2013), 193–207.

[18] H. Wulan and K. Zhu, Lacunary Series in $\mathcal{Q}_K$ Spaces, *Stud. Math.*, **173** (2007), 217–230.

[19] W. Yang, Weighted composition operators from Bloch-type spaces to weighted-type spaces, *Ars Comb.*, **93** (2009), 265–274.

[20] K. Zhu, Spaces of Holomorphic Functions in the Unit Ball, *Springer*, New York (2005).

School of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J&K, India

E-mail address: manishadevi192@gmail.com

E-mail address: aksju76@gmail.com

E-mail address: kuldipraj68@gmail.com

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

OSCILLATION CRITERIA FOR DIFFERENTIAL EQUATIONS WITH SEVERAL NON-MONOTONE DEVIATING ARGUMENTS

G. M. MOREMEDI[1], H. JAFARI[1], AND I. P. STAVROULAKIS[*,1]

Abstract. Consider the first-order linear differential equation with several retarded arguments $x'(t)+\sum_{i=1}^{m} p_i(t)\, x(\tau_i(t)) = 0,\ t \geq t_0$, where the functions $p_i,\ \tau_i \in C\left([t_0,\infty),\ \mathbb{R}^+\right)$, for every $i = 1,2,\ldots,m,\ \tau_i(t) \leq t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau_i(t) = \infty$. New oscillation criteria which essentially improve known results in the literature are established.An example illustrating the results is given.

1. Introduction

Consider the first-order linear differential equation with several non-monotone retarded arguments

$$x'(t) + \sum_{i=1}^{m} p_i(t)\, x(\tau_i(t)) = 0,\ t \geq t_0, \tag{1.1}$$

where the functions $p_i,\ \tau_i \in C([t_0,\infty), \mathbb{R}^+)$, for every $i = 1,2,\ldots,m$, (here $\mathbb{R}^+ = [0,\infty)$), $\tau_i(t) \leq t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau_i(t) = \infty$.

Let $T_0 \in [t_0,+\infty)$, $\tau(t) = \min\{\tau_i(t) : i = 1,\ldots,m\}$ and $\tau_{-1}(t) = \sup\{s : \tau(s) \leq t\}$. By a solution of the equation (1.1) we understand a function $x \in C([T_0,+\infty), \mathbb{R})$, continuously differentiable on $[\tau_{-1}(T_0), +\infty]$ and that satisfies (1.1) for $t \geq \tau_{-1}(T_0)$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *non-oscillatory.*

In the special case where $m = 1$ equation (1.1) reduces to the equation

$$x'(t) + p(t)\, x(\tau(t)) = 0,\ t \geq t_0, \tag{1.2}$$

where the functions $p,\ \tau \in C([t_0,\infty), \mathbb{R}^+)$, $\tau(t) \leq t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$.

For the general theory of these equations the reader is referred to [13,16,18,19,32].

The problem of establishing sufficient conditions for the oscillation of all solutions to the differential equations (1.1) and (1.2) has been the subject of many investigations. See, for example, [1-40] and the references cited therein.

In the case of monotone argumetns, a survey of the most interesting oscillation conditions for Eq.(1.2) can be found in [36]. While in the general case of non-monotone arguments we mention the following interesting sufficient oscillation conditions.

2010 *Mathematics Subject Classification.* Primary 34K11; Secondary 34K06.

Key words and phrases. Oscillation, Retarded, Differential equations, Non-monotone arguments.

1

In 1994, Koplatadze and Kvinikadze [26] established the following: Assume

$$\sigma(t) := \sup_{s \leq t} \tau(s), \qquad t \geq 0. \tag{1.3}$$

Clearly $\sigma(t)$ is non-decreasing and $\tau(t) \leq \sigma(t)$ for all $t \geq 0$. *Let* $k \in \mathbb{N}$ *exist such that*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left\{ \int_{\sigma(s)}^{\sigma(t)} p(\xi)\psi_k(\xi)d\xi \right\} ds > 1 - c(\mathfrak{a}), \tag{1.4}$$

where $\mathfrak{a} := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \leq \frac{1}{e}$,

$$\psi_1(t) = 0,\ \psi_k(t) = \exp\left\{ \int_{\tau(t)}^{t} p(\xi)\psi_{k-1}(\xi)d\xi \right\},\ k = 2, 3, ... \text{ for } t \in \mathbb{R}^+. \tag{1.5}$$

and

$$c(\mathfrak{a}) = \begin{cases} 0 & \text{if } \mathfrak{a} > \frac{1}{e}, \\ \frac{1}{2}\left(1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}\right) & \text{if } 0 < \mathfrak{a} \leq \frac{1}{e}\ . \end{cases} \tag{1.6}$$

Then all solutions of equation (1.2) oscillate.

In 2011 Braverman and Karpuz [6] derived the following sufficient oscillation condition for Eq.(1.2)

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left\{ \int_{\tau(s)}^{\sigma(t)} p(\xi)d\xi \right\} ds > 1, \tag{1.7}$$

while in 2014 Stavroulakis [37] improved the above condition as follows:

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left\{ \int_{\tau(s)}^{\sigma(t)} p(\xi)d\xi \right\} ds > 1 - \frac{1}{2}\left(1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}\right) \tag{1.8}$$

In 2018 Chatzarakis, Purnaras and Stavroulakis [9] improved further these conditions as follows: Assume that for some $k \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{\sigma(t)} P_k(u)\, du \right) ds > 1, \tag{1.9}$$

or

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{\sigma(t)} P_k(u)\, du \right) ds > 1 - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2}, \tag{1.10}$$

where $0 < \mathfrak{a} \leq \frac{1}{e}$, and

$$P_k(t) = p(t) \left[1 + \int_{\tau(t)}^{t} p(s) \exp\left(\int_{\tau(s)}^{t} p(u) \exp\left(\int_{\tau(u)}^{u} P_{k-1}(\xi)\, d\xi \right) du \right) ds \right]$$

with $P_0(t) = p(t)$. Then all solutions of Eq. (1.2) oscillate.

Concerning the differential equation (1.1) with several non-monotone arguments the following related oscillation results have been recently published.

Assume that there exist non-decreasing functions $\sigma_i \in C([t_0, \infty), \mathbb{R}^+)$ such that

$$\tau_i(t) \le \sigma_i(t) \le t, \; i = 1, 2, \ldots, m. \tag{1.11}$$

In 2015 Infante, Kopladatze and Stavroulakis [21] proved that if

$$\limsup_{t\to\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} \sum_{i=1}^{m} p_i(\xi) \exp\left(\int_{\tau_i(\xi)}^{\xi} \sum_{i=1}^{m} p_i(u)\,du \right) d\xi \right) ds \right]^{1/m} > \frac{1}{m^m}, \tag{1.12}$$

then all solutions of Eq. (1.1) oscillate.

Also in 2015 Kopladatze [27] improved the above condition as follows: Let there exist some $k \in \mathbb{N}$ such that

$$\limsup_{t\to\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left(m \int_{\tau_i(s)}^{\sigma_i(t)} \left(\prod_{\ell=1}^{m} p_\ell(\xi) \right)^{\frac{1}{m}} \psi_k(\xi)\,d\xi \right) ds \right]^{\frac{1}{m}}$$
$$> \frac{1}{m^m} \left[1 - \prod_{i=1}^{m} c_i(\alpha_i) \right], \tag{1.13}$$

where

$$\psi_1(t) = 0, \; \psi_i(t) = \exp\left(\sum_{j=1}^{m} \int_{\tau_j(t)}^{t} \left(\prod_{\ell=1}^{m} p_\ell(s) \right)^{\frac{1}{m}} \psi_{i-1}(s)\,ds \right), \; i = 2, 3, \ldots,$$

$$0 < \alpha_i := \liminf_{t\to\infty} \int_{\sigma_i(t)}^{t} p_i(s)\,ds < \frac{1}{e}, \; i = 1, 2, \ldots, m, \tag{1.14}$$

and

$$c_i(\alpha_i) = \frac{1 - \alpha_i - \sqrt{1 - 2\alpha_i - \alpha_i^2}}{2}, \; i = 1, 2, \ldots, m, \tag{1.15}$$

then all solutions of Eq. (1.1) oscillate.

In 2016 Bravermen, Chatzarakis and Stavroulakis [7] obtained the following iterative sufficient oscillation conditions

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(u)\, a_r(h(t), \tau_i(u))\,du > 1, \tag{1.16}$$

or

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(u)\, a_r(h(t), \tau_i(u))\,du > 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}, \tag{1.17}$$

or

$$\liminf_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(u)\, a_r(h(t), \tau_i(u))\,du > \frac{1}{e}, \tag{1.18}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

where

$$h(t)=\max_{1\le i\le m} h_i(t) \text{ and } h_i(t)=\sup_{t_0\le s\le t}\tau_i(s),\ i=1,2,\dots,m,$$

$$0<\alpha:=\liminf_{t\to\infty}\int_{h(t)}^{t}\sum_{i=1}^{m}p_i(s)\,ds\le\frac{1}{e} \tag{1.19}$$

and

$$a_1(t,s)=\exp\left(\int_s^t\sum_{i=1}^{m}p_i(u)\,du\right),$$

$$a_{r+1}(t,s)=\exp\left(\int_s^t\sum_{i=1}^{m}p_i(u)\,a_r(u,\tau_i(u))\,du\right),\ r\in\mathbb{N}.$$

Also, in 2016 Akca, Chatzarakis and Stavroulakis [1] improved that result replacing condition (1.8) by the iterative condition

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\sum_{i=1}^{m}p_i(u)\,a_r(h(u),\tau_i(u))\,du>\frac{1+\ln\lambda_0}{\lambda_0} \tag{1.20}$$

where λ_0 is the smaller root of the equation $\lambda=e^{\alpha\lambda}$,

$$0<\alpha:=\liminf_{t\to\infty}\int_{\tau(t)}^{t}\sum_{i=1}^{m}p_i(s)\,ds\le\frac{1}{e}$$

and $\tau(t)=\max_{1\le i\le m}\{\tau_i(t)\}$.

In 2017 Chatzarakis [8] derived the following results: Assume that for some $k\in\mathbb{N}$

$$\limsup_{t\to\infty}\int_{h(t)}^{t}P(s)\exp\left(\int_{\tau(s)}^{h(t)}P_k(u)\,du\right)ds>1, \tag{1.21}$$

or

$$\limsup_{t\to\infty}\int_{h(t)}^{t}P(s)\exp\left(\int_{\tau(s)}^{h(t)}P_k(u)\,du\right)ds>1-\frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{1.22}$$

or

$$\limsup_{t\to\infty}\int_{h(t)}^{t}p(s)\exp\left(\int_{\tau(s)}^{t}P_k(u)du\right)ds>\frac{2}{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}, \tag{1.23}$$

or

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t}p(s)\exp\left(\int_{\tau(s)}^{\sigma(s)}P_k(u)du\right)ds>\frac{1+\ln\lambda_1}{\lambda_1}-\frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{1.24}$$

or

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t}p(s)\exp\left(\int_{\tau(s)}^{\sigma(s)}P_k(u)du\right)ds>\frac{1}{e}, \tag{1.25}$$

where $h(t), \tau(t), \alpha$ are defined as above, λ_1 is the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$, and

$$P_k(t) = P(t)\left[1 + \int_{\tau(t)}^{t} P(s)\exp\left(\int_{\tau(s)}^{t} P(u)\exp\left(\int_{\tau(u)}^{u} P_{k-1}(\xi)\,d\xi\right)du\right)ds\right]$$

with $P_0(t) = P(t) = \sum_{i=1}^{m} p_i(t)$. Then all solutions of Eq. (1.1) oscillate.

Recently Bereketoglu et al [4] improved the above conditions as follows:
Assume that there exist non-decreasing functions $\sigma_i \in C([t_0,\infty), \mathbb{R}^+)$ such that (1.11) is satisfied and for some $k \in \mathbb{N}$

$$\limsup_{t\to\infty} \prod_{j=1}^{m}\left[\prod_{i=1}^{m}\left(\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du\right)ds\right)\right]^{1/m} > \frac{1}{m^m}, \tag{1.26}$$

or

$$\limsup_{t\to\infty} \prod_{j=1}^{m}\left[\prod_{i=1}^{m}\left(\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du\right)ds\right)\right]^{1/m} > \frac{1}{m^m}\left[1 - \prod_{i=1}^{m} c_i(\alpha_i)\right], \tag{1.27}$$

where

$$P_k(t) = \sum_{j=1}^{m} p_j(t)\left\{1 + m\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{t} P_{k-1}(u)\,du\right)ds\right]^{1/m}\right\},$$

with

$$P_0(t) = m\left[\prod_{\ell=1}^{m} p_\ell(t)\right]^{1/m},$$

α_i is given by (1.14) and $c_i(\alpha_i)$ by (1.15). Then all solutions of Eq.(1.1) oscillate.

In 2018 Attia et al [3] established the following oscillation conditions.
Assume that

$$0 < \rho := \liminf_{t\to\infty} \int_{g(t)}^{t} \sum_{k=1}^{n} p_k(s)ds \leq \frac{1}{\mathrm{e}},$$

and

$$\limsup_{t\to\infty} \left(\int_{g(t)}^{t} Q(v)dv + c(\rho)\mathrm{e}^{\int_{g(t)}^{t}\sum_{i=1}^{n} p_i(s)ds}\right) > 1,$$

where

$$Q(t) = \sum_{k=1}^{n}\sum_{i=1}^{n} p_i(t)\int_{\tau_i(t)}^{t} p_k(s)\mathrm{e}^{\int_{g_k(t)}^{t}\sum_{i=1}^{n} p_i(s)ds + (\lambda(\rho)-\epsilon)\int_{\tau_k(s)}^{g_k(t)}\sum_{\ell=1}^{n} p_\ell(u)du}ds, \quad \epsilon \in (0,\, \lambda(\rho)),$$

or

$$\limsup_{t\to\infty} \left(\int_{g(t)}^{t} Q_1(v)dv + c(\rho)\mathrm{e}^{\int_{g(t)}^{t}\sum_{i=1}^{n} p_i(s)ds}\right) > 1,$$

where

$$Q_1(t) = \sum_{k=1}^{n}\sum_{i=1}^{n} p_i(t) \int_{\tau_i(t)}^{t} p_k(s) \mathrm{e}^{\int_{g_k(t)}^{t} \sum_{i=1}^{n} p_i(s)ds + \int_{\tau_k(s)}^{g_k(t)} \sum_{\ell=1}^{n} (\lambda(q_\ell)-\epsilon_\ell) p_\ell(u)du} ds, \quad \epsilon_\ell \in (0,\, \lambda(q_\ell)),$$

and

$$q_\ell = \liminf_{t\to\infty} \int_{\tau_\ell(t)}^{t} p_\ell(s)ds, \quad \ell = 1, 2, ..., m$$

or

$$\limsup_{t\to\infty} \left(\prod_{j=1}^{n} \left(\prod_{k=1}^{n} \int_{g_j(t)}^{t} R_k(s)ds \right)^{\frac{1}{n}} + \frac{\prod_{k=1}^{n} c(\beta_k)}{n^n} \mathrm{e}^{\sum_{k=1}^{n} \int_{g_k(t)}^{t} \sum_{\ell=1}^{n} p_\ell(s)ds} \right) > \frac{1}{n^n},$$

where

$$R_k(s) = \mathrm{e}^{\int_{g_k(s)}^{s} \sum_{i=1}^{n} p_i(u)du} \sum_{i=1}^{n} p_i(s) \int_{\tau_i(s)}^{s} p_k(u) \mathrm{e}^{(\lambda(\rho)-\epsilon) \int_{\tau_k(u)}^{g_k(s)} \sum_{\ell=1}^{n} p_\ell(v)dv} du, \quad \epsilon \in (0,\, \lambda(\rho)),$$

and

$$0 < \beta_k := \liminf_{t\to\infty} \int_{\sigma_i(t)}^{t} p_i(s)ds \le \frac{1}{\mathrm{e}}.$$

Then Eq. (1.1) is oscillatory.

In this paper we further investigate the problem and derive oscillation conditions which essentially improve all the above mentioned conditions.

2. Main Results

Our main results are the following two theorems

Theorem 1. *Assume that there exist non-decreasing functions $\sigma_i \in C\left([t_0, \infty), \mathbb{R}^+\right)$ such that (1.11) is satisfied and for some $k \in \mathbb{N}$*

$$\limsup_{t\to\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\, du \right) ds \right]^{1/m} > \frac{1}{m^m}, \tag{2.1}$$

where

$$P_k(t) = P(t) \left[1 + \int_{\sigma_i(t)}^{t} P(s) \exp\left(\int_{\tau_i(s)}^{t} P(u) \exp\left(\int_{\tau_i(u)}^{u} P_{k-1}(\xi)\, d\xi \right) du \right) ds \right] \tag{2.2}$$

with

$$P_0(t) = P(t) = \sum_{i=1}^{m} p_i(t).$$

Then all solutions of Eq.(1.1) oscillate.

Proof. Suppose for the sake of contradiction that Eq.(1.1) has a non-oscillatory solution $x(t)$. Since $-x(t)$ is also a solution to (1.1), we confine ourselves only to the case that $x(t)$ is an eventually positive solution of Eq.(1.1). Then there exists $t_1 > t_0$ such that $x(t) > 0,\ x(\tau_i(t)) > 0,\ x(\sigma_i(t)) > 0$. Thus, from Eq.(1.1) it

follows that $x'(t) \leq 0$ for all $t \geq t_1$ and therefore $x(t)$ is non-increasing and taking into account that $\tau_i(t) \leq t$, it follows

$$x'(t) + \sum_{i=1}^{m} p_i(t)\, x(t) \leq 0,\ \ t \geq t_1. \tag{2.3}$$

Dividing the last inequality by $x(t)$ and integrating from $\tau_i(t)$ to t for sufficiently large t, we have

$$x(\tau_i(t)) \geq x(t) \exp\left(\int_{\tau_i(t)}^{t} \sum_{\ell=1}^{m} p_\ell(\xi)\, d\xi\right),\ \ i = 1, 2, ..., m. \tag{2.4}$$

Dividing (1.1) by $x(t)$ and integrating from $\tau_i(s)$ to t, $s \leq t$, we obtain

$$x(\tau_i(s)) = x(t) \exp\left(\int_{\tau_i(s)}^{t} \sum_{\ell=1}^{m} p_\ell(u)\, \frac{x(\tau_\ell(u))}{x(u)} du\right),\ \ i = 1, 2, ..., m. \tag{2.5}$$

Combining the last two relations, we obtain

$$x(\tau_i(s)) \geq x(t) \exp\left(\int_{\tau_i(s)}^{t} \sum_{\ell=1}^{m} p_\ell(u) \exp\left(\int_{\tau_i(u)}^{u} \sum_{\ell=1}^{m} p_\ell(\xi)\, d\xi\right) du\right). \tag{2.6}$$

Now, integrating (1.1) from $\tau_i(t)$ to t and using (2.6) for sufficiently large t, we have

$$x(\tau_i(t)) \geq x(t) \left[1 + \int_{\tau_i(t)}^{t} \sum_{\ell=1}^{m} p_\ell(s) \exp\left(\int_{\tau_i(s)}^{t} \sum_{\ell=1}^{m} p_\ell(u) \exp\left(\int_{\tau_i(u)}^{u} \sum_{\ell=1}^{m} p_\ell(\xi)\, d\xi\right) du\right) ds\right]. \tag{2.7}$$

Multiplying the last inequality by $p_i(t)$ [cf.10,3,4] and taking the sum over i $(\ i = 1, 2, ..., m)$, we have

$$x'(t) + P_1(t)\, x(t) \leq 0,\ \ t \geq t_1, \tag{2.8}$$

where

$$P_1(t) = P(t) \left[1 + \int_{\tau_i(t)}^{t} P(s) \exp\left(\int_{\tau_i(s)}^{t} P(u) \exp\left(\int_{\tau_i(u)}^{u} P_0(\xi) d\xi\right) du\right) ds\right].$$

Observe that (2.8) resembles with (2.3), where $\sum_{i=1}^{m} p_i(t)$ $[= P(t) = P_0(t)]$ is replaced by $P_1(t)$, and following the same steps as from (2.3) to (2.8), for sufficiently large t we find

$$x'(t) + P_2(t)\, x(t) \leq 0, \tag{2.9}$$

where

$$P_2(t) = P(t) \left[1 + \int_{\tau_i(t)}^{t} P(s) \exp\left(\int_{\tau_i(s)}^{t} P(u) \exp\left(\int_{\tau_i(u)}^{u} P_1(\xi) d\xi\right) du\right) ds\right].$$

Repeating the above procedure, it follows by induction, that for sufficiently large t

$$x'(t) + P_k(t)\, x(t) \leq 0, \tag{2.10}$$

where $P_k(t)$ is given by

$$P_k(t) = P(t)\left[1 + \int_{\tau_i(t)}^{t} P(s)\exp\left(\int_{\tau_i(s)}^{t} P(u)\exp\left(\int_{\tau_i(u)}^{u} P_{k-1}(\xi)d\xi\right)du\right)ds\right].$$

Dividing (2.10) by $x(t)$ and integrating from $\tau_i(s)$ to $\sigma_i(t)$, $s \leq t$, for sufficiently large t, we get

$$x(\tau_i(s)) \geq x(\sigma_i(t))\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\, du\right). \tag{2.11}$$

On the other hand, integrating (1.1) from $\sigma_j(t)$ to t for sufficiently large t, we have

$$x(\sigma_j(t)) = x(t) + \sum_{i=1}^{m}\int_{\sigma_j(t)}^{t} p_i(s)\, x(\tau_i(s))\, ds. \tag{2.12}$$

Combining (2.12) with (2.11) and using the arithmetic mean-geometric mean inequality, we obtain

$$x(\sigma_j(t)) \geq m\left[\prod_{i=1}^{m} x(\sigma_i(t))\right]^{1/m}\left[\prod_{i=1}^{m}\left(\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\, du\right)ds\right)\right]^{1/m}.$$

Now, taking the product on both sides of the last inequality, we find

$$\prod_{j=1}^{m} x(\sigma_j(t)) \geq m^m\left[\prod_{j=1}^{m} x(\sigma_j(t))\right]\left[\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\, du\right)ds\right]^{1/m}\right].$$

Hence

$$\limsup_{t\to\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\, du\right)ds\right]^{1/m} \leq \frac{1}{m^m}$$

which contradicts (2.1). □

For the next theorem we need the following lemma (See [39,13,26,27,4]).

Lemma 1. *Let there exist non-decreasing functions $\sigma_i \in C([t_0,\infty), \mathbb{R}^+)$ such that condition (1.11) is fulfilled and equation (1.1) has an eventually positive solution $x : [t_0, +\infty) \to (0, +\infty)$. Then*

$$\liminf_{t\to\infty}\frac{x(t)}{x(\sigma_i(t))} \geq c_i(\alpha_i),\ i = 1, 2, \ldots, m,$$

where α_i and $c_i(\alpha_i)$ are given by (1.14) and (1.15).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Theorem 2. *Assume that there exist non-decreasing functions $\sigma_i \in C([t_0,\infty),\mathbb{R}^+)$ such that (1.11) is satisfied and for some $k \in \mathbb{N}$*

$$\limsup_{t\to\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\left(\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du\right)ds\right)\right]^{1/m} > \frac{1}{m^m}\left[1-\prod_{i=1}^{m} c_i(\alpha_i)\right], \tag{2.13}$$

where $P_k(u)$ is given by (2.2), α_i by (1.14) and $c_i(\alpha_i)$ by (1.15). Then all solutions of Eq.(1.1) oscillate.

As in the proof of Theorem 1, we assume, for the sake of contradiction, that Eq.(1.1) has a non-oscillatory solution $x(t)$ and derive (2.11) and (2.12). Combining (2.12) with (2.11) and using the arithmetic mean-geometric mean inequality for sufficiently large t, we get

$$x(\sigma_j(t)) \geq x(t)+m\left[\prod_{i=1}^{m} x(\sigma_i(t))\right]^{1/m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du\right)ds\right]^{1/m}.$$

Taking the product on both sides of the last inequalities and using Lemma 1, as in proof of [4, Theorem 2], we find

$$\begin{aligned}
&\limsup_{t\to\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du\right)ds\right]^{1/m} \leq \\
&\leq \frac{1}{m^m}\left[1-\liminf_{t\to\infty}\frac{x^m(t)}{\prod_{i=1}^{m} x(\sigma_i(t))}\right] \\
&\leq \frac{1}{m^m}\left[1-\prod_{i=1}^{m} c_i(\alpha_i)\right]
\end{aligned}$$

which contradicts (2.13).

Remark 1. It is clear that the left-hand sides of both conditions (2.1) and (2.13) are identically the same and also the right-hand side of (2.13) reduces to (2.1) when $c_i(\alpha_i)=0$.Thus, it seems that Theorem 2 is exactly the same as Theorem 1, when $c_i(\alpha_i)=0$. One may notice, however, that the condition (1.14) is required in Theorem 2 but not in Theorem 1.

In the case of monotone arguments we have the following theorem.

Theorem 3. *Let τ_i be non-decreasing functions and for some $k \in \mathbb{N}$*

$$\limsup_{t\to\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\tau_j(t)}^{t}\left(p_i(s)\exp\left(\int_{\tau_i(s)}^{\tau_i(t)} P_k(u)\,du\right)ds\right)\right]^{1/m} > \begin{cases} \frac{1}{m^m} \\ or \\ \frac{1}{m^m}\left[1-\prod_{i=1}^{m} c_i(\alpha_i)\right] \end{cases}$$

where

$$P_k(t) = P(t)\left[1+\int_{\tau_i(t)}^{t} P(s)\exp\left(\int_{\tau_i(s)}^{t} P(u)\exp\left(\int_{\tau_i(u)}^{u} P_{k-1}(\xi)d\xi\right)du\right)ds\right].$$

with

$$P_0(t) = P(t) = \sum_{j=1}^{m} p_j(t),$$

α_i *is given by (1.14), and* $c_i(\alpha_i)$ *by (1.15). Then all solutions of (1.1) osillate.*

3. Corollaries and Examples

In the case $m = 2$, Eq.(1.1) reduces to the equation

$$x'(t) + p_1(t)\, x(\tau_1(t)) + p_2(t)\, x(\tau_2(t)) = 0. \tag{3.1}$$

From Theorems 1 and 2 the following corollary is immediate

Corollary 1. *Assume that (1.11) holds and for for* $k \in \mathbb{N}$

$$\limsup_{t\to\infty} \prod_{j=1}^{2}\left[\prod_{i=1}^{2}\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\,du\right)ds\right]^{1/2} > \begin{cases} \frac{1}{4} \\ or \\ \frac{1}{4}\left[1-\prod_{i=1}^{2} c_i(\alpha_i)\right] \end{cases},$$

where,

$$P_k(t) = P(t)\left[1+\int_{\sigma_i(t)}^{t} P(s)\exp\left(\int_{\tau_i(s)}^{t} P(u)\exp\left(\int_{\tau_i(u)}^{u} P_{k-1}(\xi)d\xi\right)du\right)ds\right],$$

$$P_0(t) = 2\left(p_1(t)\, p_2(t)\right)^{1/2},$$

and for $i = 1, 2$, α_i *is given by (1.14) and* $c_i(\alpha_i)$ *by (1.15). Then all solutions of Eq.(3.1) oscillate.*

Corollary 2. *Assume that there exist a non-decreasing function* $\sigma(t)$ *such that* $\tau(t) \le \sigma(t) \le t$ *and for some* $k \in \mathbb{N}$

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(t)} P_k(u)\,du\right)ds > \begin{cases} 1 \\ or \\ 1-c(\alpha) \end{cases} \tag{3.2}$$

where

$$P_k(t) = p(t)\left[1+\int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} p(u)\exp\left(\int_{\tau(u)}^{u} P_{k-1}(\xi)\,d\xi\right)du\right)ds\right],\ P_0(t) = p(t),$$

$$0 < \alpha := \liminf_{t\to\infty}\int_{\sigma(t)}^{t} p(s)\,ds \le \frac{1}{e}, \tag{3.3}$$

and

$$c(\alpha) = \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}$$

Then all solutions of Eq.(1.2) oscillate.

The following example (cf.[6],[21],[4]) is given to illustrate our results. It is to be pointed out that in this example it is shown that our conditions essentially improve all the related known conditions in the literature.

Example 1. *(Cf. [6],[21],[4]) Consider the equation*

$$x'(t) + px(\tau(t)) = 0, \; t \geq 0, \; p > 0. \tag{3.4}$$

with the retarded argument

$$\tau(t) = \begin{cases} t-1, & t \in [3n, \; 3n+1], \\ -3t + (12n+3), & t \in [3n+1, \; 3n+2], \\ 5t - (12n+13), & t \in [3n+2, \; 3n+3]. \end{cases}$$

For this equation, as in [6,21,4], one may choose the funtion

$$\sigma(t) = \begin{cases} t-1, & t \in [3n, \; 3n+1], \\ -3n, & t \in [3n+1, \; 3n+2.6], \\ 5t - (12n+13), & t \in [3n+2.6, \; 3n+3]. \end{cases}$$

If we choose $t_n = 3n+3$, (cf. [6, Example 1] and [21, Example 4.2]), then for $k = 1$, the condition (2.1) of Theorem 1 (or the condition (3.2) of Corollary 2) reduces to

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{\sigma(t)} P_1(u)\,du\right) ds \geq \lim_{n\to\infty} \int_{3n+2}^{3n+3} p\exp\left(\int_{5s-(12n+13)}^{3n+2} P_1(u)\,du\right) ds,$$

where

$$\begin{aligned} P_1(t) &= p\left[1 + \int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{t} p\exp\left(\int_{\tau(u)}^{u} pd\xi\right) du\right) ds\right] \\ &\geq p\left[1 + \int_{3n+2}^{3n+3} p\exp\left(\int_{5s-(12n+13)}^{3n+3} p\exp(p)du\right) ds\right] \\ &= p\left[1 + \left(\frac{e^{6pe^p} - e^{pe^p}}{5}\right) e^{-p}\right] \end{aligned}$$

Therefore

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{\sigma(t)} P_1(u)\,du\right) ds \geq \frac{p}{5P_1}\left(e^{5P_1} - 1\right),$$

where $P_1 = p\left[1 + \left(\frac{e^{6pe^p} - e^{pe^p}}{5}\right) e^{-p}\right]$*.For* $p = 0.255, \; P_1 \approx 0.484721,$ *and so*

$$\frac{p}{5P_1}\left(e^{5P_1} - 1\right) \approx 1.082293 > 1.$$

Therefore all solutions of Eq.(3.4) oscillate.

Observe, however, that when we consider the conditions stated in [6], [37] [21], [27], [7], [1] and[4] for the above equation (3.4), we obtain the following:

1. Observe that, for $t_n = 3n+3$,

$$\int_{\sigma(3n+3)}^{3n+3} p\exp\left\{\int_{\tau(s)}^{\sigma(3n+3)} pd\xi\right\} ds = \int_{3n+2}^{3n+3} p\exp\left\{\int_{5s-(12n+13))}^{3n+2} pd\xi\right\} ds = \frac{e^{5p}-1}{5}$$

and condition (1.7) reduces to

$$\frac{e^{5p}-1}{5} > 1.$$

But, for $p = 0.255$

$$\frac{e^{5p}-1}{5} \approx 0.51574 < 1$$

therefore the condition (1.7) is not satisfied.

2. Similarly, in the condition (1.8),

$$\mathfrak{a} = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds = \lim_{n\to\infty} \int_{3n+2}^{3n+3} pds = p$$

and

$$c(\mathfrak{a}) = c(p) = \frac{1-p-\sqrt{1-2p-p^2}}{2}.$$

and, as before, (1.8) reduces to

$$\frac{e^{5p}-1}{5} > 1 - \frac{1-p-\sqrt{1-2p-p^2}}{2}$$

Taking $p = 0.255$ the left-hand side of (1.8) is equal to 0.51574 while the right-hand side is 0.95345. Therefore this condition is not satisfied.

3. The condition (1.12) reduces to

$$\lim\sup_{t\to+\infty} \int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{\sigma(t)} p\exp\left(\int_{\tau(\xi)}^{\xi} pdu\right) d\xi\right) ds > 1, \tag{3.5}$$

and, as in [20,Example 4.2], the choice of $t_n = 3n+3$, leads to the inequality

$$\frac{\left(e^{5pe^p}-1\right)}{5e^p} > 1. \tag{3.6}$$

Observe, however, that for $p = 0.255$,

$$\frac{\left(e^{5pe^p}-1\right)}{5e^p} \approx 0.64849 < 1.$$

Therefore the condition (3.6) is not satisfied.

4. The condition (1.13), for $k = 2$, reduces to

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{\sigma(t)} p\ \psi_2(\xi)\,d\xi\right) ds > 1 - c(\alpha), \tag{3.7}$$

where $\psi_2(\xi) = 1$, and for $t_n = 3n + 3$, as before, it leads to

$$\frac{e^{5p} - 1}{5} > \frac{1 - p - \sqrt{1 - 2p - p^2}}{2}$$

For $p = 0.255$, we have

$$\frac{e^{5p} - 1}{5} \approx 0.51574,$$

while the right-hand side

$$1 - c(p) \approx 0.95345.$$

Therefore the condition (3.7) is not satisfied.

5. The condition (1.16) for $r = 1$ reduces to

$$\limsup_{t\to\infty} \int_{h(t)}^{t} pa_1(h(t), \tau(s))\, ds > 1, \tag{3.8}$$

where

$$h(t) = \sigma(t) \text{ and } a_1(t, s) = \exp\left(\int_{s}^{t} pdu\right).$$

That is, to the condition

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p \exp\left(\int_{\tau(s)}^{\sigma(t)} pd\xi\right) ds > 1, \tag{3.9}$$

and, as before, for $t_n = 3n + 3$ and $p = 0.255$, we have

$$\frac{e^{5p} - 1}{5} \approx 0.51574 < 1. \tag{3.10}$$

Therefore the condition (3.8) is not satisfied.

6. Similarly, condition (1.20) for $r = 1$ reduces to

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p \exp\left(\int_{\tau(s)}^{\sigma(t)} pd\xi\right) ds > \frac{1 + \ln\lambda_0}{\lambda_0}, \tag{3.11}$$

where λ_0 is the smaller root of the equation $\lambda = e^{p\lambda}$. As before, for $t_n = 3n + 3$ and $p = 0.255$, we have

$$\frac{e^{5p} - 1}{5} \approx 0.51574,$$

while

$$\frac{1 + \ln\lambda_0}{\lambda_0} \approx 0.94664$$

Therefore the condition (3.11) is not satisfied.

7. For $k = 1$, condition (1.26) reduces to

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{\sigma(t)} P_1(u)\,du\right)ds>1. \tag{3.12}$$

If we choose $t_n = 3n+3$,

$$\begin{aligned} P_1(t) &= p\left\{1+\int_{\sigma(t)}^{t} p\exp\left(\int_{\tau(s)}^{t} pdu\right)ds\right\}=p\left\{1+\int_{3n+2}^{3n+3} p\exp\left(\int_{5s-(12n+13)}^{3n+3} pdu\right)ds\right\} \\ &= p\left(1+\frac{e^{6p}-e^{p}}{5}\right). \end{aligned}$$

and, as before, (3.12) reduces to

$$\frac{p}{5P_1}\left(e^{5P_1}-1\right)>1.$$

For $p=0.255$ we find $P_1\approx 0.424232$ and so

$$\frac{p}{5P_1}\left(e^{5P_1}-1\right)\approx 0.882491<1.$$

Therefore the condition (3,12) is not satisfied

We conclude, therefore, that for $p=0.255$ no one of the conditions (1.7), (1.8), (1.12), (1.13) for $k=2$, (1.16) and (1.20) for $r=1$, and (1.26) is satisfied.

It should be also pointed out that not only for this value of $p=0.255$ but for all values of $p>0.255$, especially for all values of $p\in[0.255,0.358]$, (cf. [21, Example 4.2]),

$$\frac{p}{5P_1}\left(e^{5P_1}-1\right)>1$$

and therefore all solutions of (3.4) oscillate. Observe, however, that for $p=0.358$

$$\frac{e^{5p}-1}{5}\approx 0.99789<1,$$

also for $p=0.3$

$$\frac{\left(e^{5pe^{p}}-1\right)}{5e^{p}}\approx 0.974101<1,$$

$$\frac{e^{5p}-1}{5}\approx 0.696337<0.912993\approx\frac{1+\ln\lambda_0}{\lambda_0},$$

and for $p=0.263$, $P_1\approx 0.44944$ and so

$$\frac{p}{5P_1}\left(e^{5P_1}-1\right)\approx 0.99024<1.$$

Therefore for all values of $p\in[0.255,0.358]$ the conditions of Corollary 2 are satisfied and so all solutions to Eq.(3.4) oscillate, while no one of the above mentioned conditions is satisfied for these values of $p\in[0.255,0.358]$.

References

[1] Akca, H., Chatzarakis, G. E. and Stavroulakis, I. P., An oscillation criterion for delay differential equations with several non-monotone arguments, Appl. Math. Lett., **59** (2016), 101-108, (arXiv: 1703/04144v1[math.CA]).

[2] Arino, O., Gyori, I. and Jawhari, A., Oscillation criteria in delay equations, J. Differential Equations, **53** (1984), 115-123.

[3] E.R. Attia, V. Benekas, H.A.El-Morshedy and I.P. Stavroulakis, Oscillation of first-order linear differential equations with several non-monotone delays, Open Math. **16** (2018), 83-94.

[4] H. Bereketoglu, F. Karakoc, G.S. Oztepe and I.P Stavroulakis, Oscillations of first-order differential equations with several non-monotone retarded arguments, Georgian Math. J. (to appear).

[5] Berezansky, L. and Braverman, E., On some constants for oscillation and stability of delay equations, Proc. Amer. Math. Soc., **139** (2011), 4017-4026.

[6] Braverman, E. and Karpuz, B., On oscillation of differential and difference equations with non-monotone delays, Appl. Math. Comput., **218** (2011), 3880-3887.

[7] Braverman, E., Chatzarakis, G. E. and Stavroulakis, I. P., Iterative oscillation tests for differential equations with several non-monotone arguments, Adv. Differential Equations (2016), **2016:87**, 18 pp.

[8] G.E.Chatzarakis, Oscillations caused by several non-monotone deviating arguments, Diff. Equ. Appl. **9** (2017), 285-310.

[9] G.E.Chatzarakis, I.K. Purnaras and I.P. Stavroulakis, Oscillation tests for differential equations with deviating arguments, Adv. Math. Sci, Appl. 27 (2018), 1-28.

[10] El-Morshedy, H. A. and Attia, E. R., New oscillation criterion for delay differential equations with non-monotone arguments, Appl. Math. Lett., **54** (2016), 54-59.

[11] Elbert, A. and Stavroulakis, I. P., Oscillations of first order differential equations with deviating arguments, Recent trends in differential equations, World Sci. Ser. Appl. Anal.,1, World Sci. Publ., River Edge, NJ, 1992, pp. 163-178.

[12] Elbert, A. and Stavroulakis, I. P., Oscillation and nonoscillation criteria for delay differential equations, Proc. Amer. Math. Soc., **123** (1995), 1503-1510.

[13] Erbe, L. H., Kong, Q. and Zhang, B. G., *Oscillation Theory for Functional Differential Equations,* Monographs and Textbooks in Pure and Applied Mathematics, 190, Marcel Dekker, Inc., New York, 1995.

[14] Erbe, L. H. and Zhang, B. G., Oscillation for first order linear differential equations with deviating arguments, Differential Integral Equations, **1** (1988), 305-314.

[15] Fukagai, N. and Kusano, T., Oscillation theory of first order functional differential equations with deviating arguments, Ann. Mat. Pura Appl. (4), **136** (1984), 95-117.

[16] Gopalsamy, K., *Stability and Oscillations in Delay Differential Equations of Population Dynamics,* Mathematics and its Applications, 74, Kluwer Academic Publishers Group Dordrecht, 1992.

[17] Grammatikopoulos, M. K., Kopladatze, R. G. and Stavroulakis, I. P., On the oscillation of solutions of first order differential equations with retarded arguments, Georgian Math. J., **10** (2003), 63-76.

[18] Gyori, I. and Ladas, G., *Oscillation Theory of Delay Differential Equations with Applications,* Oxford Mathematical Monographs , Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1991.

[19] Hale, J. K., *Theory of Functional Differential Equations,* Second edition, Applied Mathematical Sciences, Vol. 3, Springer-Verlag, New York-Heidelberg, 1977.

[20] Hunt, B. R. and Yorke, J. A., When all solutions of $x' = \sum q_i(t)\, x(t - T_i(t))$ oscillate, J. Differential Equations, **53** (1984), 139-145.

[21] Infante, G., Kopladatze, R. and Stavroulakis, I. P., Oscillation criteria for differential equations with several retarded arguments, Funkcial. Ekvac. **58** (2015), 347-364.

[22] Jaroš, J. and Stavroulakis, I. P., Oscillation tests for delay equations, Rocky Mountain J. Math., **29** (1999), 197-207.

[23] Jian, C., Oscillation of linear differential equations with deviating argument, Math. Practice Theory, **1** (1991), 32-41 (in Chinese).

[24] Kon, M., Sficas, Y. G. and Stavroulakis, I. P., Oscillation criteria for delay equations, Proc. Amer. Math. Soc., **128** (2000), 2989-2997.

[25] Kopladatze, R. G. and Chanturija T. A., On the oscillatory and monotonic solutions of first order differential equations with deviating arguments, Differentsial'nye Uravneniya, **18** (1982), 1463-1465.
[26] Kopladatze, R. G. and Kvinikadze, G., On the oscillation of solutions of first order delay differential inequalities and equations, Georgian Math. J., **1** (1994), 675-685.
[27] Kopladatze, R. G., Specific properties of solutions of first order differential equations with several delay arguments, J. Contemp. Math. Anal. **50** (2015), 229-235.
[28] Kwong, M. K., Oscillation of first-order delay equations, J. Math. Anal. Appl., **156** (1991), 274-286.
[29] Ladas G., Sharp conditions for oscillations caused by delay, Applicable Anal., **9** (1979), 93-98.
[30] Ladas, G., Lakshmikantham, V. and Papadakis, J. S., Oscillations of higher-order retarded differential equations generated by retarded arguments, Delay and Functional Differential Equations and Their Applications, Academic Press, New York, 1972, pp. 219-231.
[31] Ladas, G. and Stavroulakis, I. P., Oscillations caused by several retarded and advanced arguments, J. Differential Equations, **44** (1982), 134-152.
[32] Ladde, G. S., Lakshmikantham, V. and Zhang, B. G., *Oscillation Theory of Differential Equations with Deviating Arguments,* Monographs and Textbooks in Pure and Applied Mathematics, 110, Marcel Dekker, Inc., New York, 1987.
[33] Li, B., Oscillations of first order delay differential equations, Proc. Amer. Math. Soc., **124** (1996), 3729-3737.
[34] Myshkis, A. D., Linear homogeneous differential equations of first order with deviating arguments, Uspehi Matem. Nauk (N.S.), **5** (1950), 160-162 (in Russian).
[35] G.M.Moremedi and I.P. Stavroulakis, A survey on the oscillation of solutions of differential equations with a monotone or non-monotone argument, Springer Proc. Math. Stat. (to appear).
[36] Sficas, Y.G. and Stavroulakis, I. P., Oscillation criteria for first-order delay equations, Bull. London Math. Soc., **35** (2003), 239-246.
[37] Stavroulakis, I. P., Oscillation criteria for delay and difference equations with non-monotone arguments, Appl. math. Comput. 226 (2014), 661-672.
[38] Wang, Z. C., Stavroulakis, I. P. and Qian, X. Z., A survey on the oscillation of solutions of first order linear differential equations with deviating arguments, Appl. Math. E-Notes, **2** (2002), 171-191.
[39] Yu, J. S., Wang, Z. C., Zhang, B. G. and Qian, X. Z., Oscillations of differential equations with deviating arguments, Panamer. Math. J., **2** (1992), 59-78.
[40] Zhou, Y. and Yu, Y. H., On the oscillation of solutions of first order differential equations with deviating arguments, Acta Math. Appl. Sinica (English Ser.), **15**

[1]Department of Mathematical Sciences, University of South Africa, 0003, South Africa
E-mail address, M. Moremedi: `moremgm@unisa.ac.za`

E-mail address, H. jafari: `hjafari@unisa.ac.za`

∗Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
E-mail address, I. P. Stavroulakis: `ipstav@uoi.gr`

Hyers–Ulam stability of second-order nonhomogeneous linear difference equations with a constant stepsize

Masakazu Onitsuka

Department of Applied Mathematics, Okayama University of Science,
Okayama 700-0005, Japan
e-mail: onitsuka@xmath.ous.ac.jp

August 7, 2018

Abstract

The present paper is concerned with Hyers–Ulam stability of the second-order linear difference equation $\Delta_h^2 x(t) + \alpha\Delta_h x(t) + \beta x(t) = f(t)$ on $h\mathbb{Z}$, where $\Delta_h x(t) = (x(t+h) - x(t))/h$ and $h\mathbb{Z} = \{hk |\, k \in \mathbb{Z}\}$ for the stepsize $h > 0$; α and β are real numbers; $f(t)$ is a real-valued function on $h\mathbb{Z}$. The purpose of this paper is to find an explicit HUS constant for the second-order linear difference equation whose characteristic equation has real roots. It is clarified that an HUS constant changes by the influence of the stepsize.

Keywords: Hyers–Ulam stability; HUS constant; second-order linear difference equation; stepsize.
2010 Mathematics Subject Classiffication: Primary 39B82; Secondary 39A06; 65Q10.

1 Introduction

Hyers–Ulam stability is originated from in the field of functional equations. In 1940, this problem was posed by Ulam [32, 33]. In the next year, it was solved by Hyers [9]. After that, there has been an increasing interest in studying Hyers–Ulam stability of functional equations, differential equations and difference equations (see [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 31, 34, 36]). In this paper, we will deal with Hyers–Ulam stability of the second-order nonhomogeneous linear difference equation

$$\Delta_h^2 x(t) + \alpha\Delta_h x(t) + \beta x(t) = f(t) \tag{1.1}$$

on $h\mathbb{Z}$, where

$$\Delta_h x(t) = \frac{x(t+h) - x(t)}{h} \quad \text{and} \quad h\mathbb{Z} = \{hk |\, k \in \mathbb{Z}\}$$

for the stepsize $h > 0$; α and β are real numbers; $f(t)$ is a real-valued function on $h\mathbb{Z}$. If $1 - \alpha h + \beta h^2 = 0$ holds, then we no longer have a second-order difference equation. For this reason, we assume that

$$1 - \alpha h + \beta h^2 \neq 0. \tag{1.2}$$

It is well-known that the global existence and uniqueness of solutions of (1.1) are guaranteed for the initial-value problem. We say that (1.1) has "*Hyers–Ulam stability*" on $h\mathbb{Z}$ if there exists a constant $K > 0$ with the following property: Let $\varepsilon > 0$ be a given arbitrary constant. If a function $\phi : h\mathbb{Z} \to \mathbb{R}$ satisfies $\left|\Delta_h^2\phi(t) + \alpha\Delta_h\phi(t) + \beta\phi(t) - f(t)\right| \le \varepsilon$ for all $t \in h\mathbb{Z}$, then there exists a solution $x : h\mathbb{Z} \to \mathbb{R}$ of (1.1) such that $|\phi(t) - x(t)| \le K\varepsilon$ for all $t \in h\mathbb{Z}$. We call such K an "*HUS constant*" for (1.1) on $h\mathbb{Z}$. In addition, we call the minimum of HUS constants for (1.1) on $h\mathbb{Z}$ the "*best HUS constant*". Recently, the best HUS constant of various functional equations and linear operators has been discovered by Popa

and Raşa (see [28, 29, 30] and the references cited therein). When $h \to 0$, (1.1) becomes the second-order linear differential equation

$$x'' + \alpha x' + \beta x = f(t), \tag{1.3}$$

that is, (1.1) is an approximation of the ordinary differential equation (1.3). In 2010, Li and Shen [18] proved that (1.3) has HUS on a finite interval I if characteristic equation has two different positive roots. In 2014, Xue [35] extended their results. Since the solution of the difference equation with small stepsize is a good approximate solution of the differential equation, studying Hyers–Ulam stability of difference equation (1.1) will contribute to computer science.

In 2018, the author [22] dealt with Hyers–Ulam stability of the first-order nonhomogeneous linear difference equation

$$\Delta_h x(t) - a x(t) = f(t) \tag{1.4}$$

on $h\mathbb{Z}$, where a is a real number and $f(t)$ is a real-valued function on $h\mathbb{Z}$. We say that (1.4) has "*Hyers–Ulam stability*" on $h\mathbb{Z}$ if there exists a constant $K > 0$ with the following property: Let $\varepsilon > 0$ be a given arbitrary constant. If a function $\phi : h\mathbb{Z} \to \mathbb{R}$ satisfies $|\Delta_h\phi(t) - a\phi(t) - f(t)| \le \varepsilon$ for all $t \in h\mathbb{Z}$, then there exists a solution $x : h\mathbb{Z} \to \mathbb{R}$ of (1.4) such that $|\phi(t) - x(t)| \le K\varepsilon$ for all $t \in h\mathbb{Z}$. Noticing that if $f(t) \equiv 0$ with $a = 0$ or $a = -2/h$, then (1.4) does not have Hyers–Ulam stability on $h\mathbb{Z}$ (see [21]); if $a = -1/h$, then we no longer have a first-order difference equation. For this reason, we assume that

$$a \ne 0,\ -\frac{1}{h} \ \text{ and } \ -\frac{2}{h}.$$

In [22], the author proved that (1.4) has Hyers–Ulam stability on $h\mathbb{Z}$, and the best HUS constant for (1.4) on $h\mathbb{Z}$ is

$$B(a,h) = \begin{cases} \dfrac{1}{|a|}, & \text{if } \ a > 0 \ \text{ or } \ 0 < h < -\dfrac{1}{a}, \\[2mm] \dfrac{1}{|a + 2/h|}, & \text{if } \ -\dfrac{1}{a} < h < -\dfrac{2}{a} \ \text{ or } \ -\dfrac{2}{a} < h. \end{cases}$$

This constant is rewritten as

$$B(a,h) = \frac{1}{||a + 1/h| - 1/h|}. \tag{1.5}$$

Let $\Phi(t)$ be an antidifference of $\phi(t)$ on $h\mathbb{Z}$, that is, $\Delta_h\Phi(t) = \phi(t)$ holds on $h\mathbb{Z}$, and let C be an arbitrary real constant. We denote $\Phi(t) + C$ by $\Delta_h^{-1}\phi(t)$. We can obtain the above fact according to the following results.

Theorem A (see [22, Corollary 2.5])**.** *Suppose that $a > 0$ or $a < -2/h$. Then* (1.4) *has Hyers–Ulam stability with an HUS constant $B(a,h)$ on $h\mathbb{Z}$, where $B(a,h)$ is the constant given by* (1.5)*. Furthermore, if a function $\phi : h\mathbb{Z} \to \mathbb{R}$ satisfies $|\Delta_h\phi(t) - a\phi(t) - f(t)| \le \varepsilon$ for all $t \in h\mathbb{Z}$, then*

$$\lim_{t\to\infty} \left\{ \phi(t)(ah+1)^{-\frac{t}{h}} - \Delta_h^{-1} f(t)(ah+1)^{-\frac{t+h}{h}} \right\}$$

exists, and there exists a unique solution

$$\begin{aligned} x(t) = \Big[&\Delta_h^{-1} f(t)(ah+1)^{-\frac{t+h}{h}} \\ &+ \lim_{t\to\infty} \left\{ \phi(t)(ah+1)^{-\frac{t}{h}} - \Delta_h^{-1} f(t)(ah+1)^{-\frac{t+h}{h}} \right\} \Big] (ah+1)^{\frac{t}{h}} \end{aligned}$$

of (1.4) *such that $|\phi(t) - x(t)| \le B(a,h)\varepsilon$ for all $t \in h\mathbb{Z}$.*

Theorem B (see [22, Corollary 2.6])**.** *Suppose that $-1/h < a < 0$ or $-2/h < a < -1/h$. Then* (1.4) *has Hyers–Ulam stability with an HUS constant $B(a,h)$ on $h\mathbb{Z}$, where $B(a,h)$ is the constant given by* (1.5)*. Furthermore, if a function $\phi : h\mathbb{Z} \to \mathbb{R}$ satisfies $|\Delta_h\phi(t) - a\phi(t) - f(t)| \le \varepsilon$ for all $t \in h\mathbb{Z}$, then*

$$\lim_{t\to-\infty} \left\{ \phi(t)(ah+1)^{-\frac{t}{h}} - \Delta_h^{-1} f(t)(ah+1)^{-\frac{t+h}{h}} \right\}$$

exists, and there exists a unique solution

$$x(t) = \Big[\Delta_h^{-1} f(t)(ah+1)^{-\frac{t+h}{h}} + \lim_{t\to-\infty}\Big\{\phi(t)(ah+1)^{-\frac{t}{h}} - \Delta_h^{-1} f(t)(ah+1)^{-\frac{t+h}{h}}\Big\}\Big](ah+1)^{\frac{t}{h}}$$

of (1.4) *such that* $|\phi(t)-x(t)| \le B(a,h)\varepsilon$ *for all* $t \in h\mathbb{Z}$.

Remark 1.1. We can confirm that the best HUS constant for (1.4) on $h\mathbb{Z}$ is greater than or equal to $B(a,h)$ by the following example. Consider the first-order nonhomogeneous linear difference equation

$$\Delta_h\phi(t) - a\phi(t) - f(t) = \varepsilon(-1)^{\frac{mt}{h}} \tag{1.6}$$

on $h\mathbb{Z}$, where $\varepsilon > 0$ and $m \in \{1,2\}$. Let

$$\phi_0(t) = (ah+1)^{\frac{t}{h}}\Delta_h^{-1} f(t)(ah+1)^{-\frac{t+h}{h}},$$

$$\phi_m(t) = \frac{\varepsilon(-1)^{\frac{mt}{h}}}{\{(-1)^m-1\}/h - a}$$

and $\phi(t) = \phi_0(t) + \phi_m(t)$ for all $t \in h\mathbb{Z}$. Then $\phi(t)$ is a solution of (1.6). Now we will check this fact. Since

$$\begin{aligned} f(t)(ah+1)^{-\frac{t+h}{h}} &= \Delta_h\phi_0(t)(ah+1)^{-\frac{t}{h}} \\ &= \frac{1}{h}\Big\{\phi_0(t+h)(ah+1)^{-\frac{t+h}{h}} - \phi_0(t)(ah+1)^{-\frac{t}{h}}\Big\} \\ &= \frac{\phi_0(t+h)-(ah+1)\phi_0(t)}{h}(ah+1)^{-\frac{t+h}{h}} \\ &= (\Delta_h\phi_0(t) - a\phi_0(t))(ah+1)^{-\frac{t+h}{h}} \end{aligned}$$

holds, $\phi_0(t)$ is a solution of (1.4). From

$$\Delta_h(-1)^{\frac{mt}{h}} = \frac{1}{h}\Big\{(-1)^{\frac{m(t+h)}{h}} - (-1)^{\frac{mt}{h}}\Big\} = \frac{(-1)^m-1}{h}(-1)^{\frac{mt}{h}}, \tag{1.7}$$

we have

$$\Delta_h\phi_m(t) = \frac{\varepsilon\{(-1)^m-1\}(-1)^{\frac{mt}{h}}}{\{(-1)^m-1\}-ah} = \varepsilon(-1)^{\frac{mt}{h}} + a\phi_m(t).$$

That is, $\phi_m(t)$ is a solution of (1.6) with $f(t)\equiv 0$. Using the above facts, we obtain

$$\Delta_h\phi(t) - a\phi(t) = \Delta_h(\phi_0(t)+\phi_m(t)) - a(\phi_0(t)+\phi_m(t)) = f(t) + \varepsilon(-1)^{\frac{mt}{h}}.$$

This means that $\phi(t)$ is a solution of (1.6). Therefore,

$$|\Delta_h\phi(t) - a\phi(t) - f(t)| = \varepsilon$$

holds for all $t \in h\mathbb{Z}$. Since $\phi_0(t)$ is a solution of (1.4), and $(ah+1)^{t/h}$ is a solution of (1.4) with $f(t)\equiv 0$, the general solution of (1.4) is written as

$$x(t) = c(ah+1)^{\frac{t}{h}} + \phi_0(t)$$

for all $t \in h\mathbb{Z}$, where c is an arbitrary constant. Noticing that $c = 0$ holds if and only if $|\phi(t)-x(t)|$ is bounded on $h\mathbb{Z}$. When $c = 0$, we have

$$|\phi(t)-x(t)| = |\phi_m(t)| = \frac{\varepsilon}{|a + \{1-(-1)^m\}/h|}$$

for all $t \in h\mathbb{Z}$ and $m \in \{1,2\}$. This means that the best HUS constant for (1.4) on $h\mathbb{Z}$ is greater than or equal to

$$\max\left\{\frac{1}{|a|}, \frac{1}{|a+2/h|}\right\} = B(a,h).$$

3

Remark 1.2. Theorems A, B and Remark 1.1 imply that the best HUS constant for (1.4) on $h\mathbb{Z}$ is $B(a,h)$ given by (1.5).

The purpose of this paper is to find an HUS constant for (1.1) on $h\mathbb{Z}$. In addition, we will find an explicit solution $x(t)$ of (1.1) such that $|\phi(t)-x(t)|$ is less than or equal to HUS constant multiplied by ε on $h\mathbb{Z}$, where $\phi(t)$ is a function satisfying $\left|\Delta_h^2\phi(t)+\alpha\Delta_h\phi(t)+\beta\phi(t)-f(t)\right|\le\varepsilon$ on $h\mathbb{Z}$. In the next section, we will present main theorems and their proofs, and give an HUS constant for (1.1) on $h\mathbb{Z}$. In Section 3, we will classify HUS constants for (1.1) on $h\mathbb{Z}$ by coefficients α and β. For illustration of the obtained results, we will take an example.

2 HUS constant for the second-order linear difference equations

We can easily see that the quadratic equation

$$\lambda^2+\alpha\lambda+\beta=0 \tag{2.1}$$

is the characteristic equation for the second-order homogeneous linear difference equation

$$\Delta_h^2x(t)+\alpha\Delta_hx(t)+\beta x(t)=0 \tag{2.2}$$

on $h\mathbb{Z}$, where α and β are real numbers with (1.2). In fact, we consider the funtion $x(t)=(\lambda h+1)^{t/h}$ on $h\mathbb{Z}$, where λ is a root of (2.1). Notice that since (1.2), non of λ is equal to $-1/h$. On the other hand, if $\lambda\neq-1/h$ then (1.2) holds. Clearly, $\Delta_hx(t)=\lambda(\lambda h+1)^{t/h}$ and $\Delta_h^2x(t)=\lambda^2(\lambda h+1)^{t/h}$ hold on $h\mathbb{Z}$. Therefore, if (2.1) holds then $x(t)$ is a solution of (2.2). Conversely, (2.1) is satisfied whenever $x(t)$ is a solution of (2.2) on $h\mathbb{Z}$. Thus, $(\lambda h+1)^{t/h}$ is a solution of (2.2) on $h\mathbb{Z}$ if and only if (2.1) holds.

Throughout this paper, we define

$$\Lambda_1=\{\lambda\in\mathbb{R}|\,\lambda>0\},\quad \Lambda_2=\left\{\lambda\in\mathbb{R}\middle|\,-\frac{1}{h}<\lambda<0\right\},$$

and

$$\Lambda_3=\left\{\lambda\in\mathbb{R}\middle|\,-\frac{2}{h}<\lambda<-\frac{1}{h}\right\},\quad \Lambda_4=\left\{\lambda\in\mathbb{R}\middle|\,\lambda<-\frac{2}{h}\right\}.$$

First, the following simple result is obtained by using Theorems A and B.

Theorem 2.1. *Suppose that* (2.1) *has real roots λ_1 and λ_2 with $\lambda_i\in\bigcup_{j=1}^4\Lambda_j$ for $i\in\{1,2\}$. Then* (1.1) *has Hyers–Ulam stability with an HUS constant $B(\lambda_1,h)B(\lambda_2,h)$ on $h\mathbb{Z}$, where $B(\cdot,h)$ is the constant given by* (1.5).

Proof. Assume that a function $\phi:h\mathbb{Z}\to\mathbb{R}$ satisfies

$$\left|\Delta_h^2\phi(t)+\alpha\Delta_h\phi(t)+\beta\phi(t)-f(t)\right|\le\varepsilon$$

for all $t\in h\mathbb{Z}$. Let $\psi(t)=\Delta_h\phi(t)-\lambda_1\phi(t)$ for $t\in h\mathbb{Z}$. From $\lambda_1+\lambda_2=-\alpha$, $\lambda_1\lambda_2=\beta$ and the above assumption, we get the inequality

$$|\Delta_h\psi(t)-\lambda_2\psi(t)-f(t)|=\left|\Delta_h^2\phi(t)+\alpha\Delta_h\phi(t)+\beta\phi(t)-f(t)\right|\le\varepsilon \tag{2.3}$$

for all $t\in h\mathbb{Z}$. Using Theorems A and B, we can find a solution $u:h\mathbb{Z}\to\mathbb{R}$ of

$$\Delta_hu(t)-\lambda_2u(t)=f(t) \tag{2.4}$$

such that $|\psi(t)-u(t)|\le B(\lambda_2,h)\varepsilon$ for all $t\in h\mathbb{Z}$. Namely, we have

$$|\Delta_h\phi(t)-\lambda_1\phi(t)-u(t)|\le B(\lambda_2,h)\varepsilon \tag{2.5}$$

for all $t\in h\mathbb{Z}$. Using Theorems A and B again, there exists a solution $v:h\mathbb{Z}\to\mathbb{R}$ of

$$\Delta_hv(t)-\lambda_1v(t)=u(t) \tag{2.6}$$

such that $|\phi(t) - v(t)| \le B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$. Since $u(t)$ is a solution of (2.4), we have

$$\begin{aligned}\Delta_h^2 v(t) + \alpha\Delta_h v(t) + \beta v(t) &= \Delta_h^2 v(t) - (\lambda_1 + \lambda_2)\Delta_h v(t) + \lambda_1\lambda_2 v(t)\\ &= \Delta_h(\Delta_h v(t) - \lambda_1 v(t)) - \lambda_2(\Delta_h v(t) - \lambda_1 v(t))\\ &= \Delta_h u(t) - \lambda_2 u(t) = f(t)\end{aligned}$$

for all $t \in h\mathbb{Z}$. Therefore we can conclude that $v(t)$ is a solution of (1.1). □

More explicitly, we can obtain the following result.

Theorem 2.2. *Let $\varepsilon > 0$ be a given arbitrary constant, and let $B(\cdot, h)$ be the constant given by (1.5). Define*

$$F(t) = \Delta_h^{-1} f(t)(\lambda_2 h + 1)^{-\frac{t+h}{h}}$$

for $t \in h\mathbb{Z}$. Suppose that (2.1) has real roots λ_1 and λ_2 with $\lambda_i \in \bigcup_{j=1}^{4} \Lambda_j$ for $i \in \{1, 2\}$. If a function $\phi : h\mathbb{Z} \to \mathbb{R}$ satisfies

$$\left|\Delta_h^2\phi(t) + \alpha\Delta_h\phi(t) + \beta\phi(t) - f(t)\right| \le \varepsilon$$

for all $t \in h\mathbb{Z}$, then one of the following holds:

(i) if λ_1, $\lambda_2 \in \Lambda_1 \cup \Lambda_4$, then the limiting values

$$c_1 = \lim_{t\to\infty} \left\{(\Delta_h\phi(t) - \lambda_1\phi(t))(\lambda_2 h + 1)^{-\frac{t}{h}} - F(t)\right\}$$

and

$$d_1 = \lim_{t\to\infty} \left\{\phi(t)(\lambda_1 h + 1)^{-\frac{t}{h}} - \Delta_h^{-1}\left(F(t) + c_1\right)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}}\right\}$$

exist, and there exists a unique solution

$$x(t) = \left\{\Delta_h^{-1}\left(F(t) + c_1\right)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}} + d_1\right\}(\lambda_1 h + 1)^{\frac{t}{h}}$$

of (1.1) such that $|\phi(t) - x(t)| \le B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$;

(ii) if $\lambda_1 \in \Lambda_1 \cup \Lambda_4$ and $\lambda_2 \in \Lambda_2 \cup \Lambda_3$, then the limiting values

$$c_2 = \lim_{t\to-\infty} \left\{(\Delta_h\phi(t) - \lambda_1\phi(t))(\lambda_2 h + 1)^{-\frac{t}{h}} - F(t)\right\}$$

and

$$d_2 = \lim_{t\to\infty} \left\{\phi(t)(\lambda_1 h + 1)^{-\frac{t}{h}} - \Delta_h^{-1}\left(F(t) + c_2\right)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}}\right\}$$

exist, and there exists a unique solution

$$x(t) = \left\{\Delta_h^{-1}\left(F(t) + c_2\right)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}} + d_2\right\}(\lambda_1 h + 1)^{\frac{t}{h}}$$

of (1.1) such that $|\phi(t) - x(t)| \le B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$;

(iii) if λ_1, $\lambda_2 \in \Lambda_2 \cup \Lambda_3$, then the limiting values c_2 and

$$d_3 = \lim_{t\to-\infty} \left\{\phi(t)(\lambda_1 h + 1)^{-\frac{t}{h}} - \Delta_h^{-1}\left(F(t) + c_2\right)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}}\right\}$$

exist, and there exists a unique solution

$$x(t) = \left\{\Delta_h^{-1}\left(F(t) + c_2\right)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}} + d_3\right\}(\lambda_1 h + 1)^{\frac{t}{h}}$$

of (1.1) such that $|\phi(t) - x(t)| \le B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$.

Proof. Assume that a function $\phi : h\mathbb{Z} \to \mathbb{R}$ satisfies

$$\left|\Delta_h^2\phi(t) + \alpha\Delta_h\phi(t) + \beta\phi(t) - f(t)\right| \leq \varepsilon$$

on $h\mathbb{Z}$. Let $\psi(t) = \Delta_h\phi(t) - \lambda_1\phi(t)$ for $t \in h\mathbb{Z}$. Using the above assumption with $\lambda_1 + \lambda_2 = -\alpha$, $\lambda_1\lambda_2 = \beta$, we have (2.3) for $t \in h\mathbb{Z}$.

First we prove case (i). Using $\lambda_2 \in \Lambda_1 \cup \Lambda_4$ and Theorem A, we see that

$$\begin{aligned}&\lim_{t\to\infty}\left\{\psi(t)(\lambda_2 h + 1)^{-\frac{t}{h}} - F(t)\right\}\\ &\qquad = \lim_{t\to\infty}\left\{(\Delta_h\phi(t) - \lambda_1\phi(t))(\lambda_2 h + 1)^{-\frac{t}{h}} - F(t)\right\} = c_1\end{aligned}$$

exists, and there exists a unique solution

$$u(t) = (F(t) + c_1)(\lambda_2 h + 1)^{\frac{t}{h}}$$

of (2.4) such that $|\psi(t) - u(t)| \leq B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$. That is, (2.5) holds on $h\mathbb{Z}$. Using $\lambda_1 \in \Lambda_1 \cup \Lambda_4$ and Theorem A again, we conclude that the limiting value

$$\lim_{t\to\infty}\left\{\phi(t)(\lambda_1 h + 1)^{-\frac{t}{h}} - \Delta_h^{-1}(F(t) + c_1)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}}\right\} = d_1$$

exists, and there exists a unique solution

$$v(t) = \left\{\Delta_h^{-1}(F(t) + c_1)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}} + d_1\right\}(\lambda_1 h + 1)^{\frac{t}{h}}$$

of (2.6) such that $|\phi(t) - v(t)| \leq B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$. Using the same argument as in the proof of Theorem 2.1, we see that $v(t)$ is a solution of (1.1). Noticing that $v(t)$ is a unique solution of (1.1) such that $|\phi(t) - v(t)| \leq B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$.

Next we prove case (ii). Using $\lambda_2 \in \Lambda_2 \cup \Lambda_3$ and Theorem B, we see that the limiting value

$$\begin{aligned}&\lim_{t\to-\infty}\left\{\psi(t)(\lambda_2 h + 1)^{-\frac{t}{h}} - F(t)\right\}\\ &\qquad = \lim_{t\to-\infty}\left\{(\Delta_h\phi(t) - \lambda_1\phi(t))(\lambda_2 h + 1)^{-\frac{t}{h}} - F(t)\right\} = c_2\end{aligned}$$

exists, and there exists a unique solution

$$u(t) = (F(t) + c_2)(\lambda_2 h + 1)^{\frac{t}{h}}$$

of (2.4) such that (2.5) holds for all $t \in h\mathbb{Z}$. Using $\lambda_1 \in \Lambda_1 \cup \Lambda_4$ and Theorem A, we can conclude that the limiting value

$$\lim_{t\to\infty}\left\{\phi(t)(\lambda_1 h + 1)^{-\frac{t}{h}} - \Delta_h^{-1}(F(t) + c_2)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}}\right\} = d_2$$

exists, and there exists a unique solution

$$v(t) = \left\{\Delta_h^{-1}(F(t) + c_2)(\lambda_2 h + 1)^{\frac{t}{h}}(\lambda_1 h + 1)^{-\frac{t+h}{h}} + d_2\right\}(\lambda_1 h + 1)^{\frac{t}{h}}$$

of (2.6) such that $|\phi(t) - v(t)| \leq B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$. Repeating the same argument as in the proof of Theorem 2.1, $v(t)$ is a unique solution of (1.1) such that $|\phi(t) - v(t)| \leq B(\lambda_1, h)B(\lambda_2, h)\varepsilon$ for all $t \in h\mathbb{Z}$.

We prove case (iii). As in the same argument of the preceding paragraph, using $\lambda_2 \in \Lambda_2 \cup \Lambda_3$ and Theorem B, we see that c_2 exists, and there exists a unique solution

$$u(t) = (F(t) + c_2)(\lambda_2 h + 1)^{\frac{t}{h}}$$

of (2.4) such that (2.5) holds on $h\mathbb{Z}$. Using $\lambda_1 \in \Lambda_2 \cup \Lambda_3$ and Theorem B again, we can find the limiting value

$$\lim_{t\to-\infty} \left\{\phi(t)(\lambda_1 h+1)^{-\frac{t}{h}} - \Delta_h^{-1}\left(F(t)+c_2\right)(\lambda_2 h+1)^{\frac{t}{h}}(\lambda_1 h+1)^{-\frac{t+h}{h}}\right\} = d_3$$

and a unique solution

$$v(t) = \left\{\Delta_h^{-1}\left(F(t)+c_2\right)(\lambda_2 h+1)^{\frac{t}{h}}(\lambda_1 h+1)^{-\frac{t+h}{h}} + d_3\right\}(\lambda_1 h+1)^{\frac{t}{h}}$$

of (2.6) such that $|\phi(t)-v(t)| \le B(\lambda_1,h)B(\lambda_2,h)\varepsilon$ for all $t \in h\mathbb{Z}$. By the same argument as in the proof of Theorem 2.1, $v(t)$ is a unique solution of (1.1) such that $|\phi(t)-v(t)| \le B(\lambda_1,h)B(\lambda_2,h)\varepsilon$ for all $t \in h\mathbb{Z}$. □

A natural question now arises. Is $B(\lambda_1,h)B(\lambda_2,h)$ the best HUS constant for (1.1) on $h\mathbb{Z}$? A partial answer to this question is as follows.

Theorem 2.3. *Suppose that* (2.1) *has real roots* λ_1 *and* λ_2 *with* $\lambda_i \in \bigcup_{j=1}^4 \Lambda_j$ *for* $i \in \{1,2\}$. *Then the best HUS constant for* (1.1) *on* $h\mathbb{Z}$ *is greater than or equal to*

$$\max\left\{\frac{1}{|\lambda_1\lambda_2|}, \frac{1}{|(\lambda_1+2/h)(\lambda_2+2/h)|}\right\}.$$

Before to prove this theorem, we will give a lemma.

Lemma 2.1. *Suppose that* (2.1) *has two roots* λ_1 *and* λ_2 *with* $\lambda_i \neq -1/h$ *for* $i \in \{1,2\}$. *Define*

$$F(t) = \Delta_h^{-1} f(t)(\lambda_2 h+1)^{-\frac{t+h}{h}}$$

and

$$Y(t;\lambda_1,\lambda_2) = \left\{\Delta_h^{-1}F(t)(\lambda_2 h+1)^{\frac{t}{h}}(\lambda_1 h+1)^{-\frac{t+h}{h}}\right\}(\lambda_1 h+1)^{\frac{t}{h}} \tag{2.7}$$

for $t \in h\mathbb{Z}$. *Then* $Y(t;\lambda_1,\lambda_2)$ *is a solution of* (1.1).

Proof. Since

$$\begin{aligned}
&F(t)(\lambda_2 h+1)^{\frac{t}{h}}(\lambda_1 h+1)^{-\frac{t+h}{h}}\\
&\quad= \Delta_h Y(t;\lambda_1,\lambda_2)(\lambda_1 h+1)^{-\frac{t}{h}}\\
&\quad= \frac{1}{h}\left\{Y(t+h;\lambda_1,\lambda_2)(\lambda_1 h+1)^{-\frac{t+h}{h}} - Y(t;\lambda_1,\lambda_2)(\lambda_1 h+1)^{-\frac{t}{h}}\right\}\\
&\quad= \frac{1}{h}\{Y(t+h;\lambda_1,\lambda_2) - (\lambda_1 h+1)Y(t;\lambda_1,\lambda_2)\}(\lambda_1 h+1)^{-\frac{t+h}{h}}\\
&\quad= (\Delta_h Y(t;\lambda_1,\lambda_2) - \lambda_1 Y(t;\lambda_1,\lambda_2))(\lambda_1 h+1)^{-\frac{t+h}{h}}
\end{aligned}$$

holds, we have

$$\Delta_h Y(t;\lambda_1,\lambda_2) - \lambda_1 Y(t;\lambda_1,\lambda_2) = F(t)(\lambda_2 h+1)^{\frac{t}{h}}$$

for all $t \in h\mathbb{Z}$. Using this equality, we obtain

$$\begin{aligned}
&\Delta_h^2 Y(t;\lambda_1,\lambda_2) - \lambda_1 \Delta_h Y(t;\lambda_1,\lambda_2)\\
&\quad= \Delta_h F(t)(\lambda_2 h+1)^{\frac{t}{h}}\\
&\quad= \frac{1}{h}\left\{F(t+h)(\lambda_2 h+1)^{\frac{t+h}{h}} - F(t)(\lambda_2 h+1)^{\frac{t}{h}}\right\}\\
&\quad= \frac{1}{h}\left(F(t+h) - \frac{1}{\lambda_2 h+1}F(t)\right)(\lambda_2 h+1)^{\frac{t+h}{h}}\\
&\quad= \left(\Delta_h F(t) + \frac{\lambda_2}{\lambda_2 h+1}F(t)\right)(\lambda_2 h+1)^{\frac{t+h}{h}}\\
&\quad= f(t) + \lambda_2 F(t)(\lambda_2 h+1)^{\frac{t}{h}}\\
&\quad= f(t) + \lambda_2(\Delta_h Y(t;\lambda_1,\lambda_2) - \lambda_1 Y(t;\lambda_1,\lambda_2))
\end{aligned}$$

for all $t \in h\mathbb{Z}$. This means that $Y(t;\lambda_1,\lambda_2)$ is a solution of (1.1). □

Proof of Theorem 2.3. We have only to show that for a given $\varphi(t)$ satisfying

$$\left|\Delta_h^2\varphi(t)+\alpha\Delta_h\varphi(t)+\beta\varphi(t)-f(t)\right|\le\varepsilon$$

on $h\mathbb{Z}$, we find an explicit solution $x(t)$ of (1.1) such that

$$|\varphi(t)-x(t)|=\max\left\{\frac{1}{|\lambda_1\lambda_2|},\frac{1}{|(\lambda_1+2/h)(\lambda_2+2/h)|}\right\}$$

for all $t\in h\mathbb{Z}$.

We now consider the second-order difference equation

$$\Delta_h^2\varphi(t)+\alpha\Delta_h\varphi(t)+\beta\varphi(t)-f(t)=\varepsilon(-1)^{\frac{mt}{h}} \tag{2.8}$$

on $h\mathbb{Z}$, where $\varepsilon>0$ and $m\in\{1,2\}$. Let

$$\varphi_m(t)=\frac{\varepsilon(-1)^{\frac{mt}{h}}}{\left\{\dfrac{(-1)^m-1}{h}-\lambda_1\right\}\left\{\dfrac{(-1)^m-1}{h}-\lambda_2\right\}}$$

and $\varphi(t)=Y(t;\lambda_1,\lambda_2)+\varphi_m(t)$ for all $t\in h\mathbb{Z}$, where $Y(t;\lambda_1,\lambda_2)$ is the function given by (2.7). Note here that $Y(t;\lambda_1,\lambda_2)$ is a solution of (1.1) from Lemma 2.1. Now, we will check that $\varphi(t)$ is a solution of (2.8). From (1.7), we have

$$\Delta_h^2(-1)^{\frac{mt}{h}}=\left\{\frac{(-1)^m-1}{h}\right\}^2(-1)^{\frac{mt}{h}}.$$

Using this, we get

$$\begin{aligned}&\Delta_h^2\varphi_m(t)+\alpha\Delta_h\varphi_m(t)+\beta\varphi_m(t)\\&=\left[\left\{\frac{(-1)^m-1}{h}\right\}^2+\alpha\frac{(-1)^m-1}{h}+\beta\right]\frac{\varepsilon(-1)^{\frac{mt}{h}}}{\left\{\dfrac{(-1)^m-1}{h}-\lambda_1\right\}\left\{\dfrac{(-1)^m-1}{h}-\lambda_2\right\}}\\&=\varepsilon(-1)^{\frac{mt}{h}}\end{aligned}$$

for all $t\in h\mathbb{Z}$. That is, $\varphi_m(t)$ is a solution of (2.8) with $f(t)\equiv 0$. Using the above facts, we obtain

$$\begin{aligned}&\Delta_h^2\varphi(t)+\alpha\Delta_h\varphi(t)+\beta\varphi(t)\\&\quad=\Delta_h^2Y(t;\lambda_1,\lambda_2)+\alpha\Delta_hY(t;\lambda_1,\lambda_2)+\beta Y(t;\lambda_1,\lambda_2)\\&\qquad+\Delta_h^2\varphi_m(t)+\alpha\Delta_h\varphi_m(t)+\beta\varphi_m(t)\\&\quad=f(t)+\varepsilon(-1)^{\frac{mt}{h}}.\end{aligned}$$

This means that $\varphi(t)$ is a solution of (2.8). Therefore,

$$\left|\Delta_h^2\varphi(t)+\alpha\Delta_h\varphi(t)+\beta\varphi(t)-f(t)\right|=\varepsilon$$

holds for all $t\in h\mathbb{Z}$. Let $x_0(t)$ be the general solution of (1.1) with $f(t)\equiv 0$. That is, $x_0(t)$ is written by

$$c_1(\lambda_1h+1)^{\frac{t}{h}}+c_2(\lambda_2h+1)^{\frac{t}{h}}\quad\text{or}\quad c_1(\lambda_1h+1)^{\frac{t}{h}}+c_2t(\lambda_1h+1)^{\frac{t}{h}},$$

where c_1 and c_2 are arbitrary constants. Since $Y(t;\lambda_1,\lambda_2)$ is a solution of (1.1), the general solution of (1.1) is written as

$$x(t)=x_0(t)+Y(t;\lambda_1,\lambda_2)$$

for all $t\in h\mathbb{Z}$. Noticing that $c_1=c_2=0$ holds if and only if $|\varphi(t)-x(t)|$ is bounded on $h\mathbb{Z}$. When $c_1=c_2=0$, we have

$$|\varphi(t)-x(t)|=|\varphi_m(t)|=\frac{\varepsilon}{\left|\lambda_1+\dfrac{1-(-1)^m}{h}\right|\left|\lambda_2+\dfrac{1-(-1)^m}{h}\right|}$$

8

for all $t \in h\mathbb{Z}$ and $m \in \{1, 2\}$. This means that the best HUS constant for (1.1) on $h\mathbb{Z}$ is greater than or equal to

$$\max\left\{\frac{1}{|\lambda_1\lambda_2|}, \frac{1}{|(\lambda_1 + 2/h)(\lambda_2 + 2/h)|}\right\}.$$

□

Theorems 2.1 and 2.3 imply the following result.

Corollary 2.4. *Suppose that* (2.1) *has real roots* λ_1 *and* λ_2. *If* $\lambda_1,\ \lambda_2 \in \Lambda_1 \bigcup \Lambda_2$ *or* $\lambda_1,\ \lambda_2 \in \Lambda_3 \bigcup \Lambda_4$, *then* (1.1) *has Hyers–Ulam stability with the best HUS constant* $B(\lambda_1, h)B(\lambda_2, h)$ *on* $h\mathbb{Z}$, *where* $B(\cdot, h)$ *is the constant given by* (1.5).

Proof. From Theorem 2.1, (1.1) has Hyers–Ulam stability with an HUS constant $B(\lambda_1, h)B(\lambda_2, h)$ on $h\mathbb{Z}$. Since

$$\max\left\{\frac{1}{|\lambda_1\lambda_2|}, \frac{1}{|(\lambda_1 + 2/h)(\lambda_2 + 2/h)|}\right\} = \frac{1}{|\lambda_1\lambda_2|}$$

if $\lambda_1,\ \lambda_2 \in \Lambda_1 \bigcup \Lambda_2$, and

$$\max\left\{\frac{1}{|\lambda_1\lambda_2|}, \frac{1}{|(\lambda_1 + 2/h)(\lambda_2 + 2/h)|}\right\} = \frac{1}{|(\lambda_1 + 2/h)(\lambda_2 + 2/h)|}$$

if $\lambda_1,\ \lambda_2 \in \Lambda_3 \bigcup \Lambda_4$, we conclude that

$$\max\left\{\frac{1}{|\lambda_1\lambda_2|}, \frac{1}{|(\lambda_1 + 2/h)(\lambda_2 + 2/h)|}\right\} = B(\lambda_1, h)B(\lambda_2, h).$$

From Theorem 2.3 it follows that $B(\lambda_1, h)B(\lambda_2, h)$ is the best HUS constant. □

From Corollary 2.4, we obtain the following.

Corollary 2.5. *Suppose that* (2.1) *has exactly one real root* λ *with* $\lambda \in \bigcup_{j=1}^{4} \Lambda_j$. *Then* (1.1) *has Hyers–Ulam stability with the best HUS constant* $B^2(\lambda, h)$ *on* $h\mathbb{Z}$, *where* $B(\cdot, h)$ *is the constant given by* (1.5).

3 Classification of HUS constants by the coefficients

According to Theorem 2.1, we see that the following fact.

Remark 3.1. An HUS constant for (1.1) on $h\mathbb{Z}$ is rewritten as

$$B(\lambda_1, h)B(\lambda_2, h) = \begin{cases} \dfrac{1}{|\lambda_1\lambda_2|} & \text{if } \lambda_1, \lambda_2 \in \Lambda_1 \cup \Lambda_2, \\ \dfrac{1}{|\lambda_1(\lambda_2 + 2/h)|} & \text{if } \lambda_1 \in \Lambda_1 \cup \Lambda_2,\ \lambda_2 \in \Lambda_3 \cup \Lambda_4, \\ \dfrac{1}{|(\lambda_1 + 2/h)(\lambda_2 + 2/h)|} & \text{if } \lambda_1, \lambda_2 \in \Lambda_3 \cup \Lambda_4, \end{cases}$$

where λ_1 and λ_2 are real roots of (2.1) satisfying $\lambda_i \neq 0$, $-1/h$ and $-2/h$ for $i \in \{1, 2\}$.

Unfortunately, HUS constants in the right-hand side of the equation are implicit expressions. In this section, we will decide HUS constants more explicitly. To be specific, we will classify HUS constants for (1.1) on $h\mathbb{Z}$ by coefficients α and β. Let S be the set

$$S = \left\{(\alpha, \beta) \in \mathbb{R}^2 \,\middle|\, \beta \le \frac{\alpha^2}{4},\ \beta \neq \frac{1}{h}\alpha - \frac{1}{h^2},\ \beta \neq \frac{2}{h}\alpha - \frac{4}{h^2},\ \beta \neq 0\right\}.$$

Since $\beta = \alpha/h - 1/h^2$ is the tangent line to the curve $\beta = \alpha^2/4$ at $(2/h, 1/h^2)$, S is divided into three sets as follows (see Figure 1):

$$S_1 = \left\{ (\alpha, \beta) \in \mathbb{R}^2 \middle| \ \frac{1}{h}\alpha - \frac{1}{h^2} < \beta \leq \frac{\alpha^2}{4},\ \alpha < \frac{2}{h},\ \beta \neq 0 \right\},$$

$$S_2 = \left\{ (\alpha, \beta) \in \mathbb{R}^2 \middle| \ \beta < \frac{1}{h}\alpha - \frac{1}{h^2},\ \beta \neq \frac{2}{h}\alpha - \frac{4}{h^2},\ \beta \neq 0 \right\},$$

$$S_3 = \left\{ (\alpha, \beta) \in \mathbb{R}^2 \middle| \ \frac{1}{h}\alpha - \frac{1}{h^2} < \beta \leq \frac{\alpha^2}{4},\ \alpha > \frac{2}{h},\ \beta \neq \frac{2}{h}\alpha - \frac{4}{h^2} \right\}.$$

Note that $\beta = 2\alpha/h - 4/h^2$ is the tangent line to the curve $\beta = \alpha^2/4$ at $(4/h, 4/h^2)$; $S_1 \cap S_2$, $S_2 \cap S_3$ and $S_3 \cap S_1$ are empty sets; $S = S_1 \cup S_2 \cup S_3$ holds. The above-mentioned sets are used without notice in this paper.

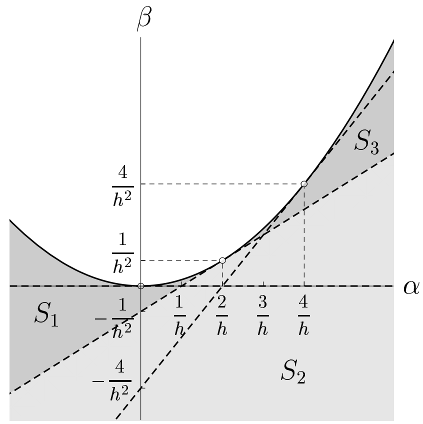


Figure 1: The sets S_1, S_2 and S_3 on the (α, β) plane.

The obtained result is as follows.

Corollary 3.1. *If $(\alpha, \beta) \in S$, then (1.1) has Hyers–Ulam stability with an HUS constant*

$$B\left(\frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2}, h\right) B\left(\frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2}, h\right)$$

on $h\mathbb{Z}$, where $B(\cdot, h)$ is the constant given by (1.5). Furthermore, one of the following holds:

(i) if $(\alpha, \beta) \in S_1$, then the best HUS constant for (1.1) on $h\mathbb{Z}$ is $1/|\beta|$;

(ii) if $(\alpha, \beta) \in S_2$, then an HUS constant for (1.1) on $h\mathbb{Z}$ is

$$\frac{1}{\left|\beta + \left(-\alpha + \sqrt{\alpha^2 - 4\beta}\right)/h\right|};$$

(iii) if $(\alpha, \beta) \in S_3$, then the best HUS constant for (1.1) on $h\mathbb{Z}$ is

$$\frac{1}{|\beta - 2\alpha/h + 4/h^2|}.$$

Proof. Suppose that $(\alpha, \beta) \in S$. Let

$$\mu_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} \quad \text{and} \quad \mu_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2}.$$

Then μ_1 and μ_2 are real roots of (2.1) since $\beta \le \alpha^2/4$ holds. By $\beta \ne \alpha/h - 1/h^2$, (1.2) is satisfied, and therefore, we have $\mu_1 \ne -1/h \ne \mu_2$. From $\beta \ne 2\alpha/h - 4/h^2$ we see that $\mu_1 \ne -2/h \ne \mu_2$. In addition, by $\beta \ne 0$, non of μ_1 and μ_2 are equal to 0. Therefore, μ_1, $\mu_2 \in \bigcup_{j=1}^{4} \Lambda_j$. Using Theorem 2.1, (1.1) has Hyers–Ulam stability with an HUS constant $B(\mu_1, h)\, B(\mu_2, h)$.

Next, we will show that the assertions (i)–(iii). We consider the case $(\alpha, \beta) \in S_1$. From

$$\frac{1}{h}\alpha - \frac{1}{h^2} < \beta \le \frac{\alpha^2}{4}$$

it follows that

$$0 \le \alpha^2 - 4\beta < \alpha^2 - \frac{4}{h}\alpha + \frac{4}{h^2} = \left(\alpha - \frac{2}{h}\right)^2.$$

That is, $0 \le \sqrt{\alpha^2 - 4\beta} < \sqrt{(\alpha - 2/h)^2} = |\alpha - 2/h|$ holds, and therefore, we have

$$\frac{-\alpha - |\alpha - 2/h|}{2} < \mu_2 \le \mu_1 < \frac{-\alpha + |\alpha - 2/h|}{2}. \tag{3.1}$$

By using $\alpha < 2/h$, we obtain $-1/h < \mu_2 \le \mu_1$. This means that μ_1, $\mu_2 \in \Lambda_1 \cup \Lambda_2$. From Corollary 2.4 and Remark 3.1, the best HUS constant for (1.1) on $h\mathbb{Z}$ is

$$\frac{1}{|\mu_1 \mu_2|} = \frac{1}{|\beta|}.$$

Next, we consider the case $(\alpha, \beta) \in S_2$. Since

$$\beta < \frac{1}{h}\alpha - \frac{1}{h^2}$$

holds, we have $\sqrt{\alpha^2 - 4\beta} > \sqrt{(\alpha - 2/h)^2} = |\alpha - 2/h|$. This means that

$$-\sqrt{\alpha^2 - 4\beta} < \alpha - \frac{2}{h} < \sqrt{\alpha^2 - 4\beta}.$$

Using this inequality we obtain

$$\mu_2 < -\frac{1}{h} < \mu_1.$$

That is, $\mu_1 \in \Lambda_1 \cup \Lambda_2$, $\mu_2 \in \Lambda_3 \cup \Lambda_4$. From Remark 3.1, an HUS constant for (1.1) on $h\mathbb{Z}$ is

$$\frac{1}{|\mu_1(\mu_2 + 2/h)|} = \frac{1}{\left|\beta + \left(-\alpha + \sqrt{\alpha^2 - 4\beta}\right)/h\right|}.$$

Finally, we consider the case $(\alpha, \beta) \in S_3$. Using the same argument in the proof of the case $(\alpha, \beta) \in S_1$, we have (3.1). By using $\alpha > 2/h$, we obtain $\mu_2 \le \mu_1 < -1/h$. This and $\beta \ne 2\alpha/h - 4/h^2$ imply that μ_1, $\mu_2 \in \Lambda_3 \cup \Lambda_4$. From Corollary 2.4 and Remark 3.1, the best HUS constant for (1.1) on $h\mathbb{Z}$ is

$$\frac{1}{|(\mu_1 + 2/h)(\mu_2 + 2/h)|} = \frac{1}{|\mu_1\mu_2 + 2(\mu_1 + \mu_2)/h + 4/h^2|} = \frac{1}{|\beta - 2\alpha/h + 4/h^2|}.$$

This completes the proof of Corollary 3.1. □

For illustration of the obtained result, we will present an example.

Example. We consider the second-order linear difference equation

$$\Delta_h^2 x(t) + 3\Delta_h x(t) + x(t) = f(t) \tag{3.2}$$

on $h\mathbb{Z}$, where (1.2) and

$$1 \ne \frac{6}{h} - \frac{4}{h^2}$$

hold. Since $(3,1) \in S$, Corollary 3.1 implies that (3.2) has Hyers–Ulam stability. Moreover, fixing the stepsize gives an HUS constant. For example, if $h = 1/3$ then $(3,1) \in S_1$, and therefore, the best HUS constant for (3.2) is one. If $h = 1$ then $(3,1) \in S_2$. So, we get an HUS constant $1/\left(\sqrt{5}-2\right)$. If $h = 3$ then $(3,1) \in S_3$, and thus, the best HUS constant for (3.2) is 9/5.

Remark 3.2. Under the assumption that (α,β) is included in the first quadrant and S, if the stepsize is sufficiently small, then we can choose a h so that $(\alpha,\beta) \in S_1$. On the other hand, if the stepsize is sufficiently large, then we can choose a h so that $(\alpha,\beta) \in S_3$. From Corollary 3.1 and Example 3, we see that the best HUS constant for (1.1) on $h\mathbb{Z}$ is affected by the stepsize. In other words, it is concluded that the best HUS constant changes by the influence of the stepsize.

Acknowledgment

The author was supported by JSPS KAKENHI Grant Number JP17K14226.

References

[1] R. P. Agarwal, B. Xu and W. Zhang, *Stability of functional equations in single variable*, J. Math. Anal. Appl. **288** (2003), no. 2, 852–869.

[2] C. Alsina and R. Ger, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl. **2** (1998), no. 4, 373–380.

[3] D. R. Anderson and J. Otto, *Hyers–Ulam stability of linear differential equations with vanishing coefficients*, Communications Appl. Anal. **19** (2015), 15–30.

[4] J. Brzdęk, K. Ciepliński and Z. Leśniak, *On Ulam's type stability of the linear equation and related issues*, Discrete Dyn. Nat. Soc. **2014** (2014), Art. ID 536791, 14 pages.

[5] J. Brzdęk, D. Popa and I. Raşa, *Hyers–Ulam stability with respect to gauges*, J. Math. Anal. Appl. **453** (2017), no. 1, 620–628.

[6] J. Brzdęk, D. Popa, I. Raşa and B. Xu, *Ulam stability of operators. Mathematical analysis and its applications*, Academic Press, 2018.

[7] J. Brzdęk, D. Popa and B. Xu, *Remarks on stability of linear recurrence of higher order*, Appl. Math. Lett. **23** (2010), no. 12, 1459–1463.

[8] O. Hatori, K. Kobayashi, T. Miura, H. Takagi and S.-E. Takahasi, *On the best constant of Hyers–Ulam stability*, J. Nonlinear Convex Anal. **5** (2004), no. 3, 387–393.

[9] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A. **27** (1941), 222–224.

[10] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of functional equations in several variables.* (English summary) Progress in Nonlinear Differential Equations and their Applications, 34, Birkhäuser Boston, Inc., Boston, MA, 1998.

[11] S.-M. Jung, *Hyers–Ulam stability of linear differential equations of first order. III*, J. Math. Anal. Appl. **311** (2005), no. 1, 139–146.

[12] S.-M. Jung, *Hyers–Ulam stability of a system of first order linear differential equations with constant coefficients*, J. Math. Anal. Appl. **320** (2006), no. 2, 549–561.

[13] S.-M. Jung, *Hyers–Ulam–Rassias stability of functional equations in nonlinear analysis*, Springer optimization and its applications, 48, Springer, New York, 2011.

[14] S.-M. Jung, *Hyers–Ulam stability of the first-order matrix difference equations*, Adv. Difference Equ. **2015** (2015), Art. ID 2015:170, 13 pages.

[15] S.-M. Jung, Y. W. Nam, *Hyers–Ulam stability of the delayed homogeneous matrix difference equation with constructive method*, J. Comput. Anal. Appl. **22** (2017), no. 5, 941–948.

[16] S.-M. Jung, Y. W. Nam, *Hyers–Ulam stability of the first order inhomogeneous matrix difference equation*, J. Comput. Anal. Appl. **23** (2017), no. 8, 1368–1383.

[17] Y. Li, *Hyers–Ulam stability of linear differential equations $y'' = \lambda^2 y$*, Thai J. Math., **8** (2010), no. 2, 215–219.

[18] Y. Li, Y. Shen, *Hyers–Ulam stability of linear differential equations of second order*, Appl. Math. Lett., **23** (2010), no. 3, 306–309.

[19] T. Li and A. Zada, *Connections between Hyers–Ulam stability and uniform exponential stability of discrete evolution families of bounded linear operators over Banach spaces*, Adv. Difference Equ., **2016** (2016), Art. ID 2016:153, 8 pages.

[20] M. S. Moslehian and D. Popa, *On the stability of the first-order linear recurrence in topological vector spaces*, Nonlinear Anal. **73** (2010), no. 9, 2792–2799.

[21] M. Onitsuka, *Influence of the stepsize on Hyers–Ulam stability of first-order homogeneous linear difference equations*, Int. J. Difference Equ. **12** (2017), no. 2, 281–302.

[22] M. Onitsuka, *Hyers–Ulam stability of first-order nonhomogeneous linear difference equations with a constant stepsize*, Appl. Math. Comput. **330** (2018), 143–151.

[23] M. Onitsuka and T. Shoji, *Hyers–Ulam stability of first-order homogeneous linear differential equations with a real-valued coefficient*, Appl. Math. Lett. **63** (2017), 102–108.

[24] D. Popa, *Hyers–Ulam stability of the linear recurrence with constant coefficients*, Adv. Difference Equ. **2005** (2005), no. 2, 101–107.

[25] D. Popa, *Hyers–Ulam–Rassias stability of a linear recurrence*, J. Math. Anal. Appl. **309** (2005), no. 2, 591–597.

[26] D. Popa and I. Raşa, *On the Hyers–Ulam stability of the linear differential equation*, J. Math. Anal. Appl. **381** (2011), no. 2, 530–537.

[27] D. Popa and I. Raşa, *Hyers–Ulam stability of the linear differential operator with nonconstant coefficients*, Appl. Math. Comput. **219** (2012), no. 4, 1562–1568.

[28] D. Popa and I. Raşa, *On the best constant in Hyers–Ulam stability of some positive linear operators*, J. Math. Anal. Appl. **412** (2014), no. 1, 103–108.

[29] D. Popa and I. Raşa, *Best constant in Hyers–Ulam stability of some functional equations*, Carpathian J. Math. **30** (2014), no. 3, 383–386.

[30] D. Popa and I. Raşa, *Best constant in stability of some positive linear operators*, Aequationes Math. **90** (2016), no. 4, 719–726.

[31] S.-E. Takahasi, H. Takagi, T. Miura and S. Miyajima, *The Hyers–Ulam stability constants of first order linear differential operators*, J. Math. Anal. Appl. **296** (2004), no. 2, 403–409.

[32] S. M. Ulam, *A collection of mathematical problems.* Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience Publishers, New York-London, 1960.

[33] S. M. Ulam, *Problems in modern mathematics.* Science Editions John Wiley & Sons, Inc., New York, 1964.

[34] G. Wang, M. Zhou and L. Sun, *Hyers–Ulam stability of linear differential equations of first order*, Appl. Math. Lett. **21** (2008), no. 10, 1024–1028.

[35] J. Xue, *Hyers–Ulam stability of linear differential equations of second order with constant coefficient*, Ital. J. Pure Appl. Math., **32**, (2014), 419–424.

[36] A. Zada, O. Shah and R. Shah, *Hyers–Ulam stability of non-autonomous systems in terms of boundedness of Cauchy problems*, Appl. Math. Comput., **271**, (2015), 512–518.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Choquet-Iyengar type advanced inequalities

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we extend advanced known Iyengar type inequalities to Choquet integrals setting with respect to distorted Lebesgue measures and for monotone functions.

2010 Mathematics Subject Classification: 26D10, 26D15, 28A25.
Keywords and phrases: Choquet integral, distorted Lebesgue measure, analytic inequality, Iyengar inequality.

1 Background - I

In the year 1938, Iyengar [7] proved the following interesting inequality.

Theorem 1 *Let f be a differentiable function on $[a,b]$ and $|f'(x)| \leq M_1$. Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) \right| \leq \frac{M_1 (b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M_1}. \quad (1)$$

In 2001, X.-L. Cheng [3] proved that

Theorem 2 *Let $f \in C^2([a,b])$ and $|f''(x)| \leq M_2$. Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{8}(b-a)^2 (f'(b)-f'(a)) \right| \leq$$

$$\frac{M_2}{24}(b-a)^3 - \frac{(b-a)}{16M_2}\Delta_1^2, \quad (2)$$

where

$$\Delta_1 = f'(a) - \frac{2(f(b)-f(a))}{(b-a)} + f'(b).$$

In 2006, [6], the authors proved:

1

Theorem 3 *Let $f \in C^2([a,b])$ and $|f''(x)| \leq M$. Set*

$$I = \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{8}(b-a)^2(f'(b)-f'(a)). \quad (3)$$

Then

$$-\frac{M(b-a)^3}{24} + \frac{M}{3}\left(\lambda_a^3 + \lambda_b^3\right) \leq I \leq$$
$$\frac{M(b-a)^3}{24} - \frac{M}{3}\left[\left(\frac{b-a}{2} - \lambda_a\right)^3 + \left(\frac{b-a}{2} - \lambda_b\right)^3\right], \quad (4)$$

where

$$\lambda_a = \frac{1}{2M}\left(f'\left(\frac{a+b}{2}\right) - f'(a)\right) + \frac{b-a}{4}, \quad (5)$$

$$\lambda_b = \frac{1}{2M}\left(f'(b) - f'\left(\frac{a+b}{2}\right)\right) + \frac{b-a}{4}. \quad (6)$$

In 1996, Agarwal and Dragomir [1] obtained a generalization of (1):

Theorem 4 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that for all $x \in [a,b]$ with $M > m$ we have $m \leq f'(x) \leq M$. Then*

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b))\right| \leq$$
$$\frac{(f(b)-f(a)-m(b-a))(M(b-a)-f(b)+f(a))}{2(M-m)}. \quad (7)$$

In [9], Qi proved

Theorem 5 *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function such that for all $x \in [a,b]$ with $M > 0$ we have $|f''(x)| \leq M$. Then*

$$\left|\int_a^b f(x)\,dx - \frac{(f(a)+f(b))}{2}(b-a) + \frac{\left(1+Q^2\right)}{8}(f'(b)-f'(a))(b-a)^2\right| \leq$$
$$\frac{M(b-a)^3}{24}\left(1-3Q^2\right), \quad (8)$$

where

$$Q^2 = \frac{\left(f'(a)+f'(b)-2\left(\frac{f(b)-f(a)}{b-a}\right)\right)^2}{M^2(b-a)^2 - (f'(b)-f'(a))^2}. \quad (9)$$

Finally in 2005, Zheng Liu, [8], proved the following:

Theorem 6 *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is integrable on $[a, b]$ and for all $x \in [a, b]$ with $M > m$ we have*

$$m \leq \frac{f'(x) - f'(a)}{x - a} \leq M \quad and\ m \leq \frac{f'(b) - f'(x)}{b - x} \leq M. \tag{10}$$

Then

$$\left| \int_a^b f(x)\,dx - \frac{(f(a) + f(b))}{2}(b - a) + \left(\frac{1 + P^2}{8}\right)(f'(b) - f'(a))(b - a)^2 \right.$$

$$\left. - \left(\frac{1 + 3P^2}{48}\right)(m + M)(b - a)^3 \right| \leq \frac{(M - m)(b - a)^3}{48}\left(1 - 3P^2\right), \tag{11}$$

where

$$P^2 = \frac{\left(f'(a) + f'(b) - 2\left(\frac{f(b) - f(a)}{b - a}\right)\right)^2}{\left(\frac{M - m}{2}\right)^2 (b - a)^2 - \left(f'(b) - f'(a) - \left(\frac{m + M}{2}\right)(b - a)\right)^2}. \tag{12}$$

In [2] we extended (1) for Choquet integrals. Motivated by these results we extend here Theorems 2-6 to the Choquet integrals setting.

2 Background - II

In the next assume that $(X, \mathcal{F})$ is a measurable space and $(\mathbb{R}^+)$ $\mathbb{R}$ is the set of all (nonnegative) real numbers.

We recall some concepts and some elementary results of capacity and the Choquet integral [4, 5].

Definition 7 *A set function $\mu : \mathcal{F} \to \mathbb{R}^+$ is called a non-additive measure (or capacity) if it satisfies*
(1) $\mu(\varnothing) = 0$;
(2) $\mu(A) \leq \mu(B)$ for any $A \subseteq B$ and $A, B \in \mathcal{F}$.
The non-additive measure μ is called concave if

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B), \tag{13}$$

for all $A, B \in \mathcal{F}$. In the literature the concave non-additive measure is known as submodular or 2-alternating non-additive measure. If the above inequality is reverse, μ is called convex. Similarly, convexity is called supermodularity or 2-monotonicity, too.

First note that the Lebesgue measure λ for an interval $[a, b]$ is defined by $\lambda([a, b]) = b - a$, and that given a distortion function m, which is increasing (or non-decreasing) and such that $m(0) = 0$, the measure $\mu(A) = m(\lambda(A))$ is a distorted Lebesgue measure. We denote a Lebesgue measure with distortion m

by $\mu = \mu_m$. It is known that μ_m is concave (convex) if m is a concave (convex) function.

The family of all the nonnegative, measurable function $f : (X, \mathcal{F}) \to (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ is denoted as L_∞^+, where $\mathcal{B}(\mathbb{R}^+)$ is the Borel σ-field of $\mathbb{R}^+$. The concept of the integral with respect to a non-additive measure was introduced by Choquet [4].

Definition 8 *Let $f \in L_\infty^+$. The Choquet integral of f with respect to non-additive measure μ on $A \in \mathcal{F}$ is defined by*

$$(C)\int_A f d\mu := \int_0^\infty \mu(\{x : f(x) \geq t\} \cap A)\, dt, \tag{14}$$

where the integral on the right-hand side is a Riemann integral.

Instead of $(C)\int_X f d\mu$, we shall write $(C)\int f d\mu$. If $(C)\int f d\mu < \infty$, we say that f is Choquet integrable and we write

$$L_C^1(\mu) = \left\{ f : (C)\int f d\mu < \infty \right\}.$$

The next lemma summarizes the basic properties of Choquet integrals [5].

Lemma 9 *Assume that $f, g \in L_C^1(\mu)$.*

(1) $(C)\int 1_A d\mu = \mu(A)$, $A \in \mathcal{F}$.

(2) (Positive homogeneity) For all $\lambda \in \mathbb{R}^+$, we have $(C)\int \lambda f d\mu = \lambda \cdot (C)\int f d\mu$.

(3) (Translation invariance) For all $c \in \mathbb{R}$, we have $(C)\int_A (f + c)\, d\mu = (C)\int_A f d\mu + c\mu(A)$.

(4) (Monotonicity in the integrand) If $f \leq g$, then we have

$$(C)\int f d\mu \leq (C)\int g d\mu.$$

(Monotonicity in the set function) If $\mu \leq \nu$, then we have $(C)\int f d\mu \leq (C)\int f d\nu$.

(5) (Subadditivity) If μ is concave, then

$$(C)\int (f + g)\, d\mu \leq (C)\int f d\mu + (C)\int g d\mu.$$

(Superadditivity) If μ is convex, then

$$(C)\int (f + g)\, d\mu \geq (C)\int f d\mu + (C)\int g d\mu.$$

(6) (Comonotonic additivity) If f and g are comonotonic, then

$$(C)\int (f + g)\, d\mu = (C)\int f d\mu + (C)\int g d\mu,$$

where we say that f and g are comonotonic, if for any $x, x' \in X$, then

$$(f(x) - f(x'))(g(x) - g(x')) \geq 0.$$

4

We next mention the amazing result from [10], which permits us to compute the Choquet integral when the non-additive measure is a distorted Lebesgue measure.

Theorem 10 *Let f be a nonnegative and measurable function on $\mathbb{R}^+$ and $\mu = \mu_m$ be a distorted Lebesgue measure. Assume that $m(x)$ and $f(x)$ are both continuous and $m(x)$ is differentiable. When f is an increasing (non-decreasing) function on $\mathbb{R}^+$, the Choquet integral of f with respect to μ_m on $[0,t]$ is represented as*

$$(C)\int_{[0,t]} f d\mu_m = \int_0^t m'(t-x) f(x) dx, \tag{15}$$

however, when f is a decreasing (non-increasing) function on $\mathbb{R}^+$, the Choquet integral of f is

$$(C)\int_{[0,t]} f d\mu_m = \int_0^t m'(x) f(x) dx. \tag{16}$$

Remark 11 *We denote by*

$$\gamma(t,x) := \begin{cases} m'(t-x), & \text{when } f \text{ is increasing (non-decreasing)}, \\ m'(x), & \text{when } f \text{ is decreasing (non-increasing)}. \end{cases} \tag{17}$$

So for f continuous and monotone we can combine (15) and (16) into

$$(C)\int_{[0,t]} f d\mu_m = \int_0^t \gamma(t,x) f(x) dx. \tag{18}$$

3 Main Results

We present the following advanced Choquet-Iyengar type inequalities: The next is based on Theorem 2.

Theorem 12 *Here $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone twice continuously differentiable function on $\mathbb{R}^+$, μ_m is a distorted Lebesgue measure, where m is such that $m(0) = 0$, m is increasing and thrice continuously differentiable on $\mathbb{R}^+$, $t \in \mathbb{R}^+$. Then*

i) if f is increasing and $\left|(m'(t-\cdot) f)''(x)\right| \leq M_2$, $\forall\, x \in [0,t]$, $M_2 > 0$, we have that

$$\Bigg|(C)\int_{[0,t]} f(x) d\mu_m(x) - \frac{t}{2}\left[m'(t) f(0) + m'(0) f(t)\right] +$$

$$\frac{t^2}{8}\left[(m'(0) f'(t) - m'(t) f'(0)) + (m''(t) f(0) - m''(0) f(t))\right]\Bigg| \leq$$

$$\frac{M_2}{24} t^3 - \frac{t}{16 M_2} \Delta_1^{*2}, \tag{19}$$

where

$$\Delta_1^* = \left(m'(t) f'(0) + m'(0) f'(t)\right) - \frac{2\left(m'(0) f(t) - m'(t) f(0)\right)}{t} -$$

$$\left(m''(t) f(0) + m''(0) f(t)\right), \tag{20}$$

ii) if f is decreasing and $\left|(m'f)''(x)\right| \leq M_3$, $\forall\ x \in [0,t]$, $M_3 > 0$, we have that

$$\left|(C)\int_{[0,t]} f(x)\, d\mu_m(x) - \frac{t}{2}\left(m'(0) f(0) + m'(t) f(t)\right) + \right.$$

$$\left.\frac{t^2}{8}\left[\left(m''(t) f(t) - m''(0) f(0)\right) + \left(m'(t) f'(t) - m'(0) f'(0)\right)\right]\right| \leq$$

$$\frac{M_3}{24} t^3 - \frac{t}{16 M_3} \Delta_1^{**2}, \tag{21}$$

where

$$\Delta_1^{**} = \left[m''(t) f(t) + m''(0) f(0)\right] - \frac{2\left[m'(t) f(t) - m'(0) f(0)\right]}{t} +$$

$$\left[m'(t) f'(t) + m'(0) f'(0)\right]. \tag{22}$$

Proof. i) If f is increasing and $\left|(m'(t-\cdot) f)''(x)\right| \leq M_2$, $\forall\ x \in [0,t]$, $M_2 > 0$, we have that

$$\left|(C)\int_{[0,t]} f(x)\, d\mu_m(x) - \frac{t}{2}\left(m'(t) f(0) + m'(0) f(t)\right) + \right.$$

$$\left.\frac{t^2}{8}\Big((m'(t-\cdot) f)'(t) - (m'(t-\cdot) f)'(0)\Big)\right| =$$

$$\left|(C)\int_{[0,t]} f(x)\, d\mu_m(x) - \frac{t}{2}\left[m'(t) f(0) + m'(0) f(t)\right] + \right.$$

$$\left.\frac{t^2}{8}\left[\left(m'(0) f'(t) - m'(t) f'(0)\right) + \left(m''(t) f(0) - m''(0) f(t)\right)\right]\right| \overset{(\text{by } (2)\ \&\ (15))}{\leq}$$

$$\frac{M_2}{24} t^3 - \frac{t}{16 M_2} \Delta_1^{*2}, \tag{23}$$

where

$$\Delta_1^* = (m'(t-\cdot) f)'(0) - \frac{2\left(m'(0) f(t) - m'(t) f(0)\right)}{t} + (m'(t-\cdot) f)'(t) =$$

$$\left(m'(t) f'(0) + m'(0) f'(t)\right) - \frac{2\left(m'(0) f(t) - m'(t) f(0)\right)}{t} -$$

$$\left(m''(t) f(0) + m''(0) f(t)\right). \tag{24}$$

6

ii) If f is decreasing and $\left|(m'f)''(x)\right| \leq M_3$, $\forall\ x \in [0,t]$, $M_3 > 0$, we have that

$$\left|(C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}\left(m'(0)f(0) + m'(t)f(t)\right) + \frac{t^2}{8}\left((m'f)'(t) - (m'f)'(0)\right)\right| =$$

$$\left|(C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}\left(m'(0)f(0) + m'(t)f(t)\right) + \frac{t^2}{8}\left[\left(m''(t)f(t) - m''(0)f(0)\right) + \left(m'(t)f'(t) - m'(0)f'(0)\right)\right]\right| \overset{(\text{by } (2)\ \&\ (16))}{\leq}$$

$$\frac{M_3}{24}t^3 - \frac{t}{16M_3}\Delta_1^{**2}, \tag{25}$$

where

$$\Delta_1^{**} = \left[m''(t)f(t) + m''(0)f(0)\right] + \left[m'(t)f'(t) + m'(0)f'(0)\right] - \frac{2\left[(m'f)(t) - (m'f)(0)\right]}{t}. \tag{26}$$

The theorem is proved. ■

The next result is based on Theorem 3.

Theorem 13 *Here $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone twice continuously differentiable function on $\mathbb{R}^+$, μ_m is a distorted Lebesgue measure, where m is such that $m(0) = 0$, m is increasing and thrice continuously differentiable on $\mathbb{R}^+$, $t \in \mathbb{R}^+$. Then*

i) if f is increasing and $\left|(m'(t-\cdot)f)''(x)\right| \leq M_1$, $\forall\ x \in [0,t]$, $M_1 > 0$, we call:

$$I_1 = (C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}\left[m'(t)f(0) + m'(0)f(t)\right] + \frac{t^2}{8}\left[\left(m'(0)f'(t) - m'(t)f'(0)\right) + \left(m''(t)f(0) - m''(0)f(t)\right)\right], \tag{27}$$

and

$$\lambda_0^{(1)} = \frac{1}{2M_1}\left[\left(-m''\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right) + m'\left(\frac{t}{2}\right)f'\left(\frac{t}{2}\right)\right) + \left(m''(t)f(0) - m'(t)f'(0)\right)\right] + \frac{t}{4}, \tag{28}$$

and

$$\lambda_t^{(1)} = \frac{1}{2M_1}\left[\left(-m''(0)f(t) + m'(0)f'(t)\right) + \left(m''\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right) - m'\left(\frac{t}{2}\right)f'\left(\frac{t}{2}\right)\right)\right] + \frac{t}{4}, \tag{29}$$

and we obtain

$$-M_1\frac{t^3}{24}+\frac{M_1}{3}\left(\left(\lambda_0^{(1)}\right)^3+\left(\lambda_t^{(1)}\right)^3\right)\leq I_1\leq$$

$$\frac{M_1t^3}{24}-\frac{M_1}{3}\left[\left(\frac{t}{2}-\lambda_0^{(1)}\right)^3+\left(\frac{t}{2}-\lambda_t^{(1)}\right)^3\right],\tag{30}$$

ii) if f is decreasing and $\left|(m'f)''(x)\right|\leq M_2$, $\forall\ x\in[0,t]$, $M_2>0$, we call:

$$I_2=(C)\int_{[0,t]}f(x)\,d\mu_m(x)-\frac{t}{2}\left(m'(0)f(0)+m'(t)f(t)\right)+$$

$$\frac{t^2}{8}\left[\left(m''(t)f(t)-m''(0)f(0)\right)+\left(m'(t)f'(t)-m'(0)f'(0)\right)\right]\tag{31}$$

and

$$\lambda_0^{(2)}=\frac{1}{2M_2}\left[\left(m''\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)+m'\left(\frac{t}{2}\right)f'\left(\frac{t}{2}\right)\right)\right.$$

$$\left.-\left(m''(0)f(0)+m'(0)f'(0)\right)\right]+\frac{t}{4},\tag{32}$$

and

$$\lambda_t^{(2)}=\frac{1}{2M_2}\left[\left(m''(t)f(t)+m'(t)f'(t)\right)\right.$$

$$\left.-\left(m''\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)+m'\left(\frac{t}{2}\right)f'\left(\frac{t}{2}\right)\right)\right]+\frac{t}{4},\tag{33}$$

and we obtain:

$$-\frac{M_2t^3}{24}+\frac{M_2}{3}\left(\left(\lambda_0^{(2)}\right)^3+\left(\lambda_t^{(2)}\right)^3\right)\leq I_2\leq$$

$$\frac{M_2t^3}{24}-\frac{M_2}{3}\left[\left(\frac{t}{2}-\lambda_0^{(2)}\right)^3+\left(\frac{t}{2}-\lambda_t^{(2)}\right)^3\right].\tag{34}$$

Proof. i) Here f is increasing and $\left|(m'(t-\cdot)f)''(x)\right|\leq M_1$, $\forall\ x\in[0,t]$, $M_1>0$.

We call

$$I_1=(C)\int_{[0,t]}f(x)\,d\mu_m(x)-\frac{t}{2}\left(m'(t)f(0)+m'(0)f(t)\right)+$$

$$\frac{t^2}{8}\left(\left(m'(t-\cdot)f\right)'(t)-\left(m'(t-\cdot)f\right)'(0)\right)=$$

$$(C)\int_{[0,t]}f(x)\,d\mu_m(x)-\frac{t}{2}\left[m'(t)f(0)+m'(0)f(t)\right]+$$

$$\frac{t^2}{8}\left[\left(m'(0)f'(t)-m'(t)f'(0)\right)+\left(m''(t)f(0)-m''(0)f(t)\right)\right].\tag{35}$$

8

We set

$$\lambda_0^{(1)} = \frac{1}{2M_1}\left(\left(m'(t-\cdot)f\right)'\left(\frac{t}{2}\right) - \left(m'(t-\cdot)f\right)'(0)\right) + \frac{t}{4} = \tag{36}$$

$$\frac{1}{2M_1}\left[\left(-m''\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right) + m'\left(\frac{t}{2}\right)f'\left(\frac{t}{2}\right)\right)\right.$$

$$\left. + \left(m''(t)f(0) - m'(t)f'(0)\right)\right] + \frac{t}{4}, \tag{37}$$

and

$$\lambda_t^{(1)} = \frac{1}{2M_1}\left(\left(m'(t-\cdot)f\right)'(t) - \left(m'(t-\cdot)f\right)'\left(\frac{t}{2}\right)\right) + \frac{t}{4} = \tag{38}$$

$$\frac{1}{2M_1}\left[\left(-m''(0)f(t) + m'(0)f'(t)\right)\right.$$

$$\left. + \left(m''\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right) - m'\left(\frac{t}{2}\right)f'\left(\frac{t}{2}\right)\right)\right] + \frac{t}{4}.$$

By Theorem 3 and (15) we get

$$-M_1\frac{t^3}{24} + \frac{M_1}{3}\left(\left(\lambda_0^{(1)}\right)^3 + \left(\lambda_t^{(1)}\right)^3\right) \leq I_1 \leq$$

$$\frac{M_1 t^3}{24} - \frac{M_1}{3}\left[\left(\frac{t}{2} - \lambda_0^{(1)}\right)^3 + \left(\frac{t}{2} - \lambda_t^{(1)}\right)^3\right]. \tag{39}$$

ii) Next f is decreasing and $\left|(m'f)''(x)\right| \leq M_2$, $\forall\, x \in [0,t]$, $M_2 > 0$.
We call

$$I_2 = (C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}\left(m'(0)f(0) + m'(t)f(t)\right) +$$

$$\frac{t^2}{8}\left((m'f)'(t) - (m'f)'(0)\right) =$$

$$(C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}\left(m'(0)f(0) + m'(t)f(t)\right) + \tag{40}$$

$$\frac{t^2}{8}\left[\left(m''(t)f(t) - m''(0)f(0)\right) + \left(m'(t)f'(t) - m'(0)f'(0)\right)\right].$$

We set

$$\lambda_0^{(2)} = \frac{1}{2M_2}\left[\left(m''\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right) + m'\left(\frac{t}{2}\right)f'\left(\frac{t}{2}\right)\right)\right. \tag{41}$$

$$\left. - \left(m''(0)f(0) + m'(0)f'(0)\right)\right] + \frac{t}{4},$$

and

$$\lambda_t^{(2)} = \frac{1}{2M_2}\left[\left(m''(t)f(t) + m'(t)f'(t)\right)\right. \tag{42}$$

9

$$-\left(m''\left(\frac{t}{2}\right)f\left(\frac{t}{2}\right)+m'\left(\frac{t}{2}\right)f'\left(\frac{t}{2}\right)\right)\Bigg]+\frac{t}{4}.$$

By Theorem 3 and (16) we get

$$-\frac{M_2t^3}{24}+\frac{M_2}{3}\left(\left(\lambda_0^{(2)}\right)^3+\left(\lambda_t^{(2)}\right)^3\right)\leq I_2\leq$$

$$\frac{M_2t^3}{24}-\frac{M_2}{3}\left[\left(\frac{t}{2}-\lambda_0^{(2)}\right)^3+\left(\frac{t}{2}-\lambda_t^{(2)}\right)^3\right]. \tag{43}$$

The theorem is proved. ■

The next result is based on Theorem 4.

Theorem 14 *Here $f:\mathbb{R}^+\to\mathbb{R}^+$ is a monotone differentiable function on $\mathbb{R}^+$, μ_m is a distorted Lebesgue measure, where m is such that $m(0)=0$, m is increasing and twice differentiable on $\mathbb{R}^+$, $t\in\mathbb{R}^+$. Then*

i) if f is increasing, and $m_1\leq(m'(t-\cdot)f)'(x)\leq M_1$, $\forall\ x\in[0,t]$, where $M_1>m_1$, we obtain:

$$\left|(C)\int_{[0,t]}f(x)\,d\mu_m(x)-\frac{t}{2}\left(m'(t)f(0)+m'(0)f(t)\right)\right|\leq$$

$$\frac{\left(m'(0)f(t)-m'(t)f(0)-m_1t\right)\left(M_1t-m'(0)f(t)+m'(t)f(0)\right)}{2(M_1-m_1)}. \tag{44}$$

ii) if f is decreasing, and $m_2\leq(m'f)'(x)\leq M_2$, $\forall\ x\in[0,t]$, where $M_2>m_2$, we obtain:

$$\left|(C)\int_{[0,t]}f(x)\,d\mu_m(x)-\frac{t}{2}\left(m'(0)f(0)+m'(t)f(t)\right)\right|\leq$$

$$\frac{\left(m'(t)f(t)-m'(0)f(0)-m_2t\right)\left(M_2t-m'(t)f(t)+m'(0)f(0)\right)}{2(M_2-m_2)}. \tag{45}$$

Proof. i) Here f is increasing and $m_1\leq(m'(t-\cdot)f)'(x)\leq M_1$, $\forall\ x\in[0,t]$, where $M_1>m_1$. We get, by Theorem 4 and (15), that

$$\left|(C)\int_{[0,t]}f(x)\,d\mu_m(x)-\frac{t}{2}\left(m'(t)f(0)+m'(0)f(t)\right)\right|\leq$$

$$\frac{\left(m'(0)f(t)-m'(t)f(0)-m_1t\right)\left(M_1t-m'(0)f(t)+m'(t)f(0)\right)}{2(M_1-m_1)}. \tag{46}$$

ii) Next f is decreasing and $m_2\leq(m'f)'(x)\leq M_2$, $\forall\ x\in[0,t]$, where $M_2>m_2$. We get, by Theorem 4 and (16), that

$$\left|(C)\int_{[0,t]}f(x)\,d\mu_m(x)-\frac{t}{2}\left(m'(0)f(0)+m'(t)f(t)\right)\right|\leq$$

$$\frac{\left(m'\left(t\right)f\left(t\right)-m'\left(0\right)f\left(0\right)-m_2t\right)\left(M_2t-m'\left(t\right)f\left(t\right)+m'\left(0\right)f\left(0\right)\right)}{2\left(M_2-m_2\right)}. \tag{47}$$

The theorem is proved. ■

The next result is based on Theorem 5.

Theorem 15 *Here $f:\mathbb{R}^+\rightarrow\mathbb{R}^+$ is a monotone twice differentiable function on $\mathbb{R}^+$, μ_m is a distorted Lebesgue measure, where m is such that $m(0)=0$, m is increasing and thrice differentiable on $\mathbb{R}^+$, $t\in\mathbb{R}^+$. Then*

i) if f is increasing, and $\left|\left(m'\left(t-\cdot\right)f\right)''\left(x\right)\right|\leq M_1$, $\forall\ x\in[0,t]$, $M_1>0$, we call:

$$Q_1^2=$$

$$\frac{\left[\left(-m''\left(t\right)f\left(0\right)+m'\left(t\right)f'\left(0\right)\right)+\left(-m''\left(0\right)f\left(t\right)+m'\left(0\right)f'\left(t\right)\right)-2\left(\frac{m'(0)f(t)-m'(t)f(0)}{t}\right)\right]^2}{M_1^2t^2-\left(-m''\left(0\right)f\left(t\right)+m'\left(0\right)f'\left(t\right)+m''\left(t\right)f\left(0\right)-m'\left(t\right)f'\left(0\right)\right)^2}$$

and we obtain

$$\left|(C)\int_{[0,t]}f\left(x\right)d\mu_m\left(x\right)-\frac{t}{2}\left[m'\left(t\right)f\left(0\right)+m'\left(0\right)f\left(t\right)\right]+\right.$$

$$\left.\left(\frac{\left(1+Q_1^2\right)t^2}{8}\right)\left(-m''\left(0\right)f\left(t\right)+m'\left(0\right)f'\left(t\right)+m''\left(t\right)f\left(0\right)-m'\left(t\right)f'\left(0\right)\right)\right|\leq$$

$$\frac{M_1t^3}{24}\left(1-3Q_1^2\right), \tag{49}$$

ii) if f is decreasing, and $\left|\left(m'f\right)''\left(x\right)\right|\leq M_2$, $\forall\ x\in[0,t]$, $M_2>0$, we call:

$$Q_2^2=$$

$$\frac{\left[\left(m''\left(0\right)f\left(0\right)+m'\left(0\right)f'\left(0\right)+m''\left(t\right)f\left(t\right)+m'\left(t\right)f'\left(t\right)\right)-2\left(\frac{m'(t)f(t)-m'(0)f(0)}{t}\right)\right]^2}{M_2^2t^2-\left[m''\left(t\right)f\left(t\right)+m'\left(t\right)f'\left(t\right)-m''\left(0\right)f\left(0\right)-m'\left(0\right)f'\left(0\right)\right]^2} \tag{50}$$

and we obtain

$$\left|(C)\int_{[0,t]}f\left(x\right)d\mu_m\left(x\right)-\frac{t}{2}\left[m'\left(0\right)f\left(0\right)+m'\left(t\right)f\left(t\right)\right]+\right.$$

$$\left.\left(\frac{\left(1+Q_2^2\right)t^2}{8}\right)\left(m''\left(t\right)f\left(t\right)+m'\left(t\right)f'\left(t\right)-m''\left(0\right)f\left(0\right)-m'\left(0\right)f'\left(0\right)\right)\right|\leq$$

$$\frac{M_2t^3}{24}\left(1-3Q_2^2\right). \tag{51}$$

11

Proof. i) If f is increasing, and $\left|\left(m'(t-\cdot)f\right)''(x)\right| \leq M_1$, $\forall\ x \in [0,t]$, $M_1 > 0$, we set:

$$Q_1^2 = \frac{\left(\left(m'(t-\cdot)f\right)'(0) + \left(m'(t-\cdot)f\right)'(t) - 2\left(\frac{\left(m'(t-\cdot)f\right)(t) - \left(m'(t-\cdot)f\right)(0)}{t}\right)\right)^2}{M_1^2t^2 - \left(\left(m'(t-\cdot)f\right)'(t) - \left(m'(t-\cdot)f\right)'(0)\right)^2} =$$

$$\frac{\left(\left(-m''(t)f(0) + m'(t)f'(0)\right) + \left(-m''(0)f(t) + m'(0)f'(t)\right) - 2\left(\frac{m'(0)f(t) - m'(t)f(0)}{t}\right)\right)^2}{M_1^2t^2 - \left(-m''(0)f(t) + m'(0)f'(t) + m''(t)f(0) - m'(t)f'(0)\right)^2}. \tag{52}$$

By Theorem 5 and (15) we derive

$$\left|(C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}\left[m'(t)f(0) + m'(0)f(t)\right] + \right.$$

$$\left.\left(\frac{\left(1+Q_1^2\right)t^2}{8}\right)\left(-m''(0)f(t) + m'(0)f'(t) + m''(t)f(0) - m'(t)f'(0)\right)\right| \leq$$

$$\frac{M_1t^3}{24}\left(1 - 3Q_1^2\right). \tag{53}$$

ii) If f is decreasing, and $\left|\left(m'f\right)''(x)\right| \leq M_2$, $\forall\ x \in [0,t]$, $M_2 > 0$, we set:

$$Q_2^2 = \frac{\left(\left(m'f\right)'(0) + \left(m'f\right)'(t) - 2\left(\frac{\left(m'f\right)(t) - \left(m'f\right)(0)}{t}\right)\right)^2}{M_2^2t^2 - \left(\left(m'f\right)'(t) - \left(m'f\right)'(0)\right)^2} = \tag{54}$$

$$\frac{\left[\left(m''(0)f(0) + m'(0)f'(0) + m''(t)f(t) + m'(t)f'(t)\right) - 2\left(\frac{m'(t)f(t) - m'(0)f(0)}{t}\right)\right]^2}{M_2^2t^2 - \left[m''(t)f(t) + m'(t)f'(t) - m''(0)f(0) - m'(0)f'(0)\right]^2}.$$

By Theorem 5 and (16) we derive

$$\left|(C)\int_{[0,t]} f(x)\,d\mu_m(x) - \frac{t}{2}\left[m'(0)f(0) + m'(t)f(t)\right] + \right.$$

$$\left.\left(\frac{\left(1+Q_2^2\right)t^2}{8}\right)\left(m''(t)f(t) + m'(t)f'(t) - m''(0)f(0) - m'(0)f'(0)\right)\right| \leq$$

$$\frac{M_2t^3}{24}\left(1 - 3Q_2^2\right). \tag{55}$$

The theorem is proved. ■

Finally we apply Theorem 6 to obtain:

Theorem 16 *Here $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone and continuously differentiable function on $\mathbb{R}^+$, μ_m is a distorted Lebesgue measure, where m is such that $m(0) = 0$, m is increasing and twice continuously differentiable on $\mathbb{R}^+$, $t \in \mathbb{R}^+$. We have*

i) If f is increasing, and

$$m_1 \leq \frac{(m'(t-\cdot) f)'(x) - (m'(t-\cdot) f)'(0)}{x} \leq M_1, \tag{56}$$

and

$$m_1 \leq \frac{(m'(t-\cdot) f)'(t) - (m'(t-\cdot) f)'(x)}{t-x} \leq M_1, \tag{57}$$

$\forall\ x \in [0,t]$, with $m_1 < M_1$, we set:

$$P_1^2 = \frac{\left((m'(t-\cdot) f)'(0) + (m'(t-\cdot) f)'(t) - 2\left(\frac{(m'(t-\cdot)f)(t) - (m'(t-\cdot)f)(0)}{t}\right)\right)^2}{\left(\frac{M_1-m_1}{2}\right)^2 t^2 - \left((m'(t-\cdot) f)'(t) - (m'(t-\cdot) f)'(0) - \frac{(m_1+M_1)}{2} t\right)^2}. \tag{58}$$

Then

$$\Bigg|(C)\int_{[0,t]} f(x)\, d\mu_m(x) - \frac{((m'(t-\cdot) f)(0) + (m'(t-\cdot) f)(t))}{2} t +$$

$$\left(\frac{1+P_1^2}{8}\right)\Big((m'(t-\cdot) f)'(t) - (m'(t-\cdot) f)'(0)\Big) t^2 - \left(\frac{1+3P_1^2}{48}\right)(m_1+M_1) t^3 \Bigg|$$

$$\leq \frac{(M_1-m_1) t^3}{48}\left(1 - 3P_1^2\right). \tag{59}$$

ii) If f is decreasing, and

$$m_2 \leq \frac{(m' f)'(x) - (m' f)'(0)}{x} \leq M_2, \tag{60}$$

and

$$m_2 \leq \frac{(m' f)'(t) - (m' f)'(x)}{t-x} \leq M_2, \tag{61}$$

$\forall\ x \in [0,t]$, with $m_2 < M_2$, we set:

$$P_2^2 = \frac{\left((m' f)'(0) + (m' f)'(t) - 2\left(\frac{(m'f)(t) - (m'f)(0)}{t}\right)\right)^2}{\left(\frac{M_2-m_2}{2}\right)^2 t^2 - \left((m' f)'(t) - (m' f)'(0) - \frac{(m_2+M_2)}{2} t\right)^2}. \tag{62}$$

Then

$$\Bigg|(C)\int_{[0,t]} f(x)\, d\mu_m(x) - \frac{((m' f)(0) + (m' f)(t))}{2} t +$$

13

$$\left(\frac{1+P_2^2}{8}\right)\left(\left(m'f\right)'(t)-\left(m'f\right)'(0)\right)t^2-\left(\frac{1+3P_2^2}{48}\right)\left(m_2+M_2\right)t^3\Bigg| \leq$$

$$\frac{\left(M_2-m_2\right)t^3}{48}\left(1-3P_2^2\right). \tag{63}$$

Example 17 *A well-known distortion function is* $m(t)=\frac{t}{1+t}$, $t\in\mathbb{R}^+$. *We have* $m(0)=0$, $m(t)\geq 0$, $m'(t)=\frac{1}{(1+t)^2}>0$, *that is* m *is strictly increasing. We have that* $m''(t)=-2(1+t)^{-3}$, $m^{(3)}(t)=6(1+t)^{-4}$, *and in general we get that* $m^{(n)}(t)=(-1)^{n+1}n!(1+t)^{-(n+1)}$, $\forall\ n\in\mathbb{N}$.

References

[1] R.P. Agarwal, S.S. Dragomir, *An application of Hayashi's inequality for differentiable functions*, Computers Math. Applic., 6 (1996), 95-99.

[2] G. Anastassiou, *Choquet integral analytic inequalities*, submitted, 2018.

[3] Xiao-Liang Cheng, *The Iyengar-type inequality*, Applied Math. Letters 14 (2001), 975-978.

[4] G. Choquet, *Theory of capacities*, Ann. Inst. Fourier (Grenoble), 5 (1953), 131-295.

[5] D. Denneberg, *Non-additive Measure and Integral*, Kluwer Academic Publishers, Boston, 1994.

[6] I. Franjic, J. Pecaric, I. Peric, *Note on an Iyengar type inequality*, Applied Math. Letters 19 (2006), 657-660.

[7] K.S.K. Iyengar, *Note on an inequality*, Math. Student 6, (1938), 75-76.

[8] Zheng Liu, *Note on Iyengar's inequality*, Univ. Beograd Publ. Elektrotechn. Fak., Ser. Mat. 16 (2005), 29-35.

[9] F. Qi, *Further generalizations of inequalities for an integral*, Univ. Beograd Publ. Elektrotechn. Fak., Ser. Mat. 8 (1997), 79-83.

[10] M. Sugeno, *A note on derivatives of functions with respect to fuzzy measures*, Fuzzy Sets Syst., 222 (2013), 1-17.

SOME RESULTS ABOUT $\Delta\mathcal{I}-$STATISTICALLY PRE-CAUCHY SEQUENCES WITH AN ORLICZ FUNCTION

HAFİZE GÜMÜŞ, ÖMER KİŞİ, AND EKREM SAVAŞ

Abstract. In this study, we define the concept of $\mathcal{I}-$statistically convergence for difference sequences and we use an Orlicz function to obtain more general results. We also show that an $\Delta\mathcal{I}-$statistically convergent sequence with an Orlicz function is $\Delta\mathcal{I}-$statistically pre-Cauchy .

1. Introduction

In this part, we give a short literature data about $\mathcal{I}-$statistical convergence, statistical pre-Cauchy sequences and difference sequence spaces. As is known, convergence is one of the basic notions of Mathematics and statistical convergence extends the notion. It is easy to see that any convergent sequence is statistically convergent but not conversely. Statistical convergence was given by Zygmund [35] in Warsaw in 1935 and then it was formally introduced by Fast [16] and Steinhaus [33], independently. Later it was reintroduced by Schoenberg [32] . Even now, this concept has very much applications in different areas such as number theory by Erdös and Tenenbaum [10] , measure theory by Miller [26] and summability theory by Freedman and Sember [17] . Statistical convergence is also applied to approximation theory by Gadjiev and Orhan [18], Anastassiou and Duman [1] and Sakaoğlu and Ünver [19]. If we want to briefly remember this concept by using the characteristic function, we should give the following definitions:

Definition 1.1. *Let E be a subset of $\mathbb{N}$, the set of all natural numbers. The natural density of E is defined by*

$$d(E) := \lim_n \frac{1}{n} \sum_{j=1}^{n} \chi_E(j)$$

whenever the limit exists where $\chi(E)$ is characteristic function of E.

Definition 1.2. ([16]) *A number sequence (x_n) is statistically convergent to x provided that for every $\varepsilon > 0$,*

$$d\{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} = \lim_n \frac{1}{n} |\{k \leq n : |x_k - x| \geq \varepsilon\}| = 0$$

2000 *Mathematics Subject Classification.* 40G15, 40A35.

Key words and phrases. $\mathcal{I}-$convergence, difference sequences, statistical pre-cauchy sequences, Orlicz function.

A small summary of this article was presented at the International Conference on Recent Advances in Pure and Applied Mathematics held in Kuşadası in 2017 and the first author supported by Necmettin Erbakan University Scientific Research Commission under research project number 172518001-186.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

or equivalently there exists a subset $K \subseteq \mathbb{N}$ with $d(E) = 1$ and $n_0(\varepsilon)$ such that $n > n_0(\varepsilon)$ and $n \in K$ imply that $|x_n - x| < \varepsilon$. In this case we write $st - \lim x_n = x$. Statistical convergent sequences are generally denoted by S.

$\mathcal{I}-$convergence has emerged as a kind of generalization form of many types of convergence. This means that, if we choose different ideals we will have different convergences such as usual convergence and statistical convergence as we will see from the examples below. In 2000, Koystro et. al. [24] introduced this concept in a metric space and then many concepts studied for statistical convergence have moved to ideal convergence. Before defining $\mathcal{I}-$convergence, the definitions of ideal and filter will be needed.

Definition 1.3. *A non-empty family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if i) $\emptyset \in \mathcal{I}$, ii) for each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$ and iii) for each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.*

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Definition 1.4. *A non-empty family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if i) $\emptyset \notin \mathcal{F}$, ii) for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$ and iii) for each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.*

If $\mathcal{I}$ is a non-trivial ideal in $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets

$$F(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$$

is a filter in $\mathbb{N}$.

Remark 1.1. *Generally we will use ideals in our proofs but if the notion is more familiar for filters, we will use the notion of filter.*

Definition 1.5. ([24]) *Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be a proper ideal on $\mathbb{N}$. The real sequence $x = (x_n)$ is said to be $\mathcal{I}-$convergent to $x \in \mathbb{R}$ provided that for each $\varepsilon > 0$,*

$$A(\varepsilon) = \{k \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in \mathcal{I}.$$

The set of all $\mathcal{I}-$convergent sequences usually denoted by $c_{\mathcal{I}}$.

More investigations in this direction and more applications can be found in Kostyrko, Salát and Wilezyński's paper. We just want to give some well known examples which we mentioned before.

Example 1.1. *If $\mathcal{I} = \mathcal{I}_f$ $=\{A \subseteq \mathbb{N} : A$ is finite$\}$ then we have the usual convergence.*

Example 1.2. *If $\mathcal{I} = \mathcal{I}_d$ $=\{A \subseteq \mathbb{N} : d(A\ ) = 0\}$ then we have the statistical convergence where d is the asymptotic density of A.*

Following the statistical convergence and $\mathcal{I}-$convergence located an important role in this area, Das, Savaş and Ghosal [6] have introduced the concept of $\mathcal{I}-$statistical convergence as follows and they extend the important summability methods statistical convergence and $\mathcal{I}-$convergence using ideals.

Definition 1.6. ([6])*A sequence $x = (x_n)$ is said to be $\mathcal{I}-$statistically convergent to L for each $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

We will denote the set of all $\mathcal{I}-$statistically convergent sequences by $S_{\mathcal{I}}$.

Before giving information about the definitions and works of pre-Cauchy sequences, lets remember the definition of an Orlicz function. Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. An Orlicz function M satisfies the Δ_2-condition if there exits a constant $K > 0$ such that $M(2u) \leq KM(u)$ for all $u \geq 0$. We want to give a little note here that if convexity of Orlicz function M is replaced by $M(x + y) = M(x) + M(y)$ then we get the modulus function which is familiar to us.

Lindendstrauss and Tzafriri [25] used the idea of Orlicz function to define the following sequence space.

$$l_M := \left\{ x \in w : \sum_{n=1}^{\infty} M\left(\frac{|x_n|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}$$

which called an Orlicz sequence space. l_M is a Banach space with the norm

$$\|x\| := \inf \left\{ \rho > 0 : \sum_{n=1}^{\infty} M\left(\frac{|x_n|}{\rho}\right) \leq 1 \right\}.$$

The notion of statistically pre-Cauchy for real sequences was introduced by Connor, Fridy and Kline [4] in 1994. They proved that statistically convergent sequences are statistically pre-Cauchy and any bounded statistical pre-Cauchy sequence with nowhere dense set of limit points is statistically convergent. Khan and Lohani [20] handled this concept in a different way with the Orlicz function. More works on statistically pre-Cauchy sequences are found in Dutta, Eşi and Tripathy [8] , Dutta and Tripathy [9] and Khan and Tabassum [21] .

As an expected result, in 2012, Khan, Ebedullah and Ahmad [22] defined pre-Cauchy sequences for $\mathcal{I}-$convergence and they introduced the concept of $\mathcal{I}-$pre-Cauchy sequence. They established the criterion for arbitrary sequence to be $\mathcal{I}-$pre-Cauchy and they also gave another criterion for $\mathcal{I}-$convergence.

Definition 1.7. ([22]) *Let $x = (x_n)$ be a sequence and let M be an Orlicz function then x is $\mathcal{I}-$pre-Cauchy if and only if*

$$\mathcal{I} - \lim_n \frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|x_k - x_j|}{\rho}\right) = 0$$

for some $\rho > 0$.

Yamancı and Gürdal [34] , Ojha and Srivastava [27] and Saha et. al. [28] have some studies about this new definiton.

Definition 1.8. ([7]) *A sequence $x = (x_n)$ is said to be $\mathcal{I}-$statistically pre-Cauchy if for any $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n^2} \left| \{(j,k) : |x_k - x_j| \geq \varepsilon\} \, , \ j, k \leq n \right| \geq \delta \right\} \in \mathcal{I}.$$

In another direction, in 1981, $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ difference sequence spaces defined by Kızmaz [23] where l_∞, c and c_0 are bounded, convergent and null sequence spaces, respectively. In this study the sequence $\Delta x = (\Delta x_n)$ defined by $(\Delta x_n) = (x_n - x_{n+1})$ for all $n \in \mathbb{N}$ and some relations between these spaces for example $c_0(\Delta) \subseteq c(\Delta) \subseteq l_\infty(\Delta)$ were obtained. In Et and Çolak's paper [11] Kızmaz's

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

results generalized for Δ^m sequences such that,

$$\begin{aligned} c_0(\Delta^m) &= \{x = (x_n) : \Delta^m x \in c_0\} \\ c(\Delta^m) &= \{x = (x_n) : \Delta^m x \in c\} \\ l_\infty(\Delta^m) &= \{x = (x_n) : \Delta^m x \in l_\infty\} \end{aligned}$$

where $m \in \mathbb{N}$ and $\Delta^m x = (\Delta^m x_n) = (\Delta^{m-1} x_n - \Delta^{m-1} x_{n+1})$ i.e.
$\Delta^m x_n = \sum\limits_{v=0}^{m} (-1)^v \binom{m}{v} x_{n+v}$. They proved that these spaces are Banach spaces with the norm

$$\|.\|_\Delta = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty .$$

Following these definitions, Et [12], Et and Çolak [11], Et and Başarır [13], Aydın and Başar [2], Bektaş et. al. [3], Et and Eşi [14], Savaş [31] and many others searched various properties of this concept. Et and Nuray [15] have introduced the Δ^m-statistical convergence and the set of all Δ^m-statistical convergent sequences was denoted by $S(\Delta^m)$. Following this study, Gümüş and Nuray [19] have extended Δ^m-statistical convergence to Δ^m-ideal convergence.

Definition 1.9. ([19]) *Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be a non-trivial ideal in $\mathbb{N}$. The sequence $x = (x_n)$ of real numbers is said to be $\Delta\mathcal{I}-$convergent to $x \in \mathbb{R}$ if for each $\varepsilon > 0$ the set*

$$\{n \in \mathbb{N} : |\Delta x_n - x| \geq \varepsilon\} \in \mathcal{I}.$$

The space of all $\Delta\mathcal{I}-$convergent sequences is denoted by $c_{\mathcal{I}}(\Delta)$.

Before we get to the part where our main results are, we would like to give some expressions that have already been proved before about $\mathcal{I}-$convergence and $\Delta\mathcal{I}-$convergence, without moving away from our aim. At the same time it will be interesting to move these expressions to $\mathcal{I}-$statistical convergence.

Proposition 1.1. *Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an ideal in $\mathbb{N}$ and (Δx_n) be a real sequence. Then*

$$c(\Delta) \subseteq c_{\mathcal{I}}(\Delta).$$

Note that the inverse of this proposition is not generally true as can be seen from the following example.

Example 1.3. *For the difference sequence $\Delta x = (\Delta x_n) = \begin{cases} 1, & n \text{ is square} \\ 0, & n \text{ is not square} \end{cases}$, $x \in c_{\mathcal{I}_d}(\Delta)$ but $x \notin c(\Delta)$.*

Definition 1.10. *Let $\mathcal{I}$ be an ideal in $\mathbb{N}$. If $\{n+1 : n \in \mathbb{N}\} \in \mathcal{I}$ for any $A \in \mathcal{I}$, then $\mathcal{I}$ is said to be a translation invariant ideal.*

Corollary 1.1. *If $\mathcal{I}$ is translation invariant and $(x_n) \in c_{\mathcal{I}}$ then $(x_{n+1}) \in c_{\mathcal{I}}$.*

Example 1.4. *$\mathcal{I}_d$ is a translation invariant ideal.*

Proposition 1.2. *If $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is an admissible translation invariant ideal then $c_{\mathcal{I}} \subseteq c_{\mathcal{I}}(\Delta)$.*

2. Main Results

In this section, we define $\Delta\mathcal{I}-$statistical convergent and $\Delta\mathcal{I}-$statistically pre-Cauchy sequences and we give some inclusion theorems.

Definition 2.1. *A sequence $x = (x_n)$ is said to be $\Delta\mathcal{I}-$statistically convergent to L provided that*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : |\Delta x_k - L| \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In our paper, the set of all $\Delta\mathcal{I}-$statistically convergent sequences will be denoted by $S_{\mathcal{I}}(\Delta)$.

Now, lets evaluate this new definition for the $\mathcal{I}_f$ ideal in the example mentioned above.

Example 2.1. *For the ideal $\mathcal{I} = \mathcal{I}_f$, $S_{\mathcal{I}_f}(\Delta) = S(\Delta)$.*

Definition 2.2. *A sequence $x = (x_n)$ is said to be $\Delta\mathcal{I}-$statistically pre-Cauchy if, for any $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n^2} \left|\{(j,k) : |\Delta x_k - \Delta x_j| \geq \varepsilon, \quad j,k \leq n\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Theorem 2.1. *An $\Delta\mathcal{I}-$statistically convergent sequence is $\Delta\mathcal{I}-$statistically pre-Cauchy.*

Proof. Let $x = (x_n)$ be $\Delta\mathcal{I}-$statistically convergent to L. Let $\varepsilon > 0$ and $\delta > 0$ be given. We know that

$$A = \left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{k \leq n : |\Delta x_k - L| \geq \frac{\varepsilon}{2}\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Then for all $n \in A^c$ where c stands for the complement,

$$\frac{1}{n} \left|\left\{k \leq n : |\Delta x_k - L| \geq \frac{\varepsilon}{2}\right\}\right| < \delta \text{ i.e. } \frac{1}{n} \left|\left\{k \leq n : |\Delta x_k - L| < \frac{\varepsilon}{2}\right\}\right| > 1 - \delta.$$

Writing $B_n = \left\{k \leq n : |\Delta x_k - L| < \frac{\varepsilon}{2}\right\}$ we observe that for $j, k \in B_n$,

$$|\Delta x_k - \Delta x_j| \leq |\Delta x_k - L| + |\Delta x_j - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $B_n \times B_n \subset \{(j,k) : |\Delta x_k - \Delta x_j| < \varepsilon, \quad j,k \leq n\}$ which implies

$$\left[\frac{|B_n|}{n}\right]^2 \leq \frac{1}{n^2} \left|\{(j,k) : |\Delta x_k - \Delta x_j| < \varepsilon, \quad j,k \leq n\}\right|.$$

Thus for all $n \in A^c$,

$$\frac{1}{n^2} \left|\{(j,k) : |\Delta x_k - \Delta x_j| < \varepsilon, \quad j,k \leq n\}\right| \geq \left[\frac{|B_n|}{n}\right]^2 > (1-\delta)^2$$

i.e.

$$\frac{1}{n^2} \left|\{(j,k) : |\Delta x_k - \Delta x_j| \geq \varepsilon, \quad j,k \leq n\}\right| < 1 - (1-\delta)^2.$$

Let $\delta_1 > 0$ be given. Choosing $\delta > 0$ so that $1 - (1-\delta)^2 < \delta_1$ we see that $\forall n \in A^c$,

$$\frac{1}{n^2} \left|\{(j,k) : |\Delta x_k - \Delta x_j| \geq \varepsilon, \quad j,k \leq n\}\right| < \delta_1$$

and so

$$\left\{n \in \mathbb{N} : \frac{1}{n^2} \left|\{(j,k) : |\Delta x_k - \Delta x_j| \geq \varepsilon, \quad j,k \leq n\}\right| \geq \delta_1\right\} \subset A.$$

Since $A \in \mathcal{I}$, we have the proof. $\square$

Theorem 2.2. *Let $x = (x_n)$ be a sequence and M be Orlicz function. Then x is $\Delta\mathcal{I}-$statistically pre-Cauchy if and only if*

$$\mathcal{I} - \lim_n \frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) = 0 \quad \text{for some } \rho > 0.$$

Proof. First suppose that $\mathcal{I} - \lim_n \frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) = 0$ for some $\rho > 0$. For each $\varepsilon > 0$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}
\frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) &= \frac{1}{n^2} \sum_{\substack{k,j \leq n \\ |\Delta x_k - \Delta x_j| < \varepsilon}} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) \\
&+ \frac{1}{n^2} \sum_{\substack{k,j \leq n \\ |\Delta x_k - \Delta x_j| \geq \varepsilon}} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) \\
&\geq \frac{1}{n^2} \sum_{\substack{k,j \leq n \\ |\Delta x_k - \Delta x_j| \geq \varepsilon}} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) \\
&\geq M(\varepsilon)\left(\frac{1}{n^2} \left|\{(j,k) : |\Delta x_k - \Delta x_j| \geq \varepsilon, \quad j,k \leq n\}\right|\right)
\end{aligned}$$

Then for any $\delta > 0$,

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \tfrac{1}{n^2} \left|\{(j,k) : |\Delta x_k - \Delta x_j| \geq \varepsilon, \quad j,k \leq n\}\right| \geq \delta\right\} \\
&\subset n \in \mathbb{N} : \frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) \geq \delta M(\varepsilon)
\end{aligned}$$

Thus x is $\Delta\mathcal{I}-$statistically pre-Cauchy.

Now conversely assume that x is $\Delta\mathcal{I}-$statistically pre-Cauchy and $\varepsilon > 0$ be given. Let $\eta > 0$ be such that $M(\eta) < \frac{\varepsilon}{2}$. Since Orlicz function is bounded, there exists an integer B such that $M(x) < \frac{B}{2}$ for all $x \geq 0$. Then for each $n \in \mathbb{N}$,

$$\begin{aligned}
\frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) &= \frac{1}{n^2} \sum_{\substack{k,j \leq n \\ |\Delta x_k - \Delta x_j| < \eta}} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) \\
&+ \frac{1}{n^2} \sum_{\substack{k,j \leq n \\ |\Delta x_k - \Delta x_j| \geq \eta}} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) \\
&\leq M(\eta) + \frac{1}{n^2} \sum_{\substack{k,j \leq n \\ |\Delta x_k - \Delta x_j| \geq \eta}} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho}\right) \\
&\leq \frac{\varepsilon}{2} + \frac{B}{2}\left(\frac{1}{n^2} \left|\{(j,k) : |\Delta x_k - \Delta x_j| \geq \eta, \quad j,k \leq n\}\right|\right)
\end{aligned}$$

Since x is $\Delta\mathcal{I}-$statistically pre-Cauchy, for $\delta > 0$,

$$A = \left\{ n \in \mathbb{N} : \frac{1}{n^2} \left| \{ (j,k) : |\Delta x_k - \Delta x_j| \geq \eta, \quad j,k \leq n \} \right| \geq \delta \right\} \in \mathcal{I}.$$

Then for $n \in A^c$,

$$\frac{1}{n^2} \left| \{ (j,k) : |\Delta x_k - \Delta x_j| \geq \eta, \quad j,k \leq n \} \right| < \delta$$

and so

$$\frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho} \right) \leq \frac{\varepsilon}{2} + \frac{B}{2}\delta.$$

Let $\delta_1 > 0$ be given. Then choosing $\varepsilon, \delta > 0$ such that $\frac{\varepsilon}{2} + \frac{B}{2}\delta < \delta_1$ we see that for each $n \in A^c$,

$$\frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho} \right) < \delta_1$$

i.e.

$$\left\{ n \in \mathbb{N} : \frac{1}{n^2} \sum_{k,j \leq n} M\left(\frac{|\Delta x_k - \Delta x_j|}{\rho} \right) \geq \delta_1 \right\} \subset A \in \mathcal{I}.$$

□

Theorem 2.3. *Let $x = (x_n)$ be a sequence and M be Orlicz function. Then x is $\Delta\mathcal{I}-$statistically convergent to L if and only if*

$$\mathcal{I} - \lim_n \frac{1}{n} \sum_{k=1}^{n} M\left(\frac{|\Delta x_k - L|}{\rho} \right) = 0 \quad \textit{for some } \rho > 0.$$

Proof. Suppose that $\mathcal{I} - \lim_n \frac{1}{n} \sum\limits_{k=1}^{n} M\left(\frac{|\Delta x_k - L|}{\rho} \right) = 0$ for some $\rho > 0$. We have,

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} M\left(\frac{|\Delta x_k - L|}{\rho} \right) &= \frac{1}{n} \sum_{\substack{k=1 \\ |\Delta x_k - L| < \varepsilon}}^{n} M\left(\frac{|\Delta x_k - L|}{\rho} \right) + \frac{1}{n} \sum_{\substack{k=1 \\ |\Delta x_k - L| \geq \varepsilon}}^{n} M\left(\frac{|\Delta x_k - L|}{\rho} \right) \\ &\geq M(\varepsilon) \frac{1}{n} \left| \{ k \leq n : |\Delta x_k - L| \geq \varepsilon \} \right|. \end{aligned}$$

Then for any $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \{ k \leq n : |\Delta x_k - L| \geq \varepsilon \} \right| \geq \delta \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} M\left(\frac{|\Delta x_k - L|}{\rho} \right) \geq M(\varepsilon).\delta \right\}$$

Due to the statement we accepted at the beginning of the theorem, right hand side belongs to the ideal. As we know from the second expression of ideal, left hand side is also in ideal and this proves the theorem.

Since the second part of the theory is very similar to the second part of the previous theorem, we can easily prove. □

References

[1] G. A. Anastassiou and O. Duman, Statistical Korovkin theory for multivariate stochastic processes, *Stoch. Anal. Appl. 28* (2010), no. 4, 648-661.

[2] C. Aydın and F. Başar, Some new difference sequence spaces, *Appl. Math.Comput.*, 157(3) (2004), 677-693.

[3] Ç.A. Bektaş, M. Et and R. Çolak, Generalized difference sequence spaces and their dual spaces, *J.Math.Anal.Appl.* 292 (2004), 423-432.

[4] J. Connor, J. A. Fridy and J. Kline, Statistically pre-Cauchy sequences, Analysis, 14 (1994), 311-317.

[5] J. Connor, R-type summability methods, Cauchy criteria, P-sets and statistical convergence, Proc. Amer. Math. Soc. 115(2), (1992), 319-327.

[6] P. Das, E. Savaş and S. K. Ghosal, On generalizations of certain summability methods using ideals, Appl. Math. Letters 24 (2011), 1509-1514.

[7] P. Das and E. Savaş, On $\mathcal{I}-$statistically pre-Cauchy sequences, Taiwanese Jour. of Math., 18 (1), (2014), 115-126.

[8] A. J. Dutta, a. Eşi and B. C. Tripathy, Statistically convergence triple sequence spaces defined by Orlicz function, Jour. of Math. Anal. 4(2) (2013), 16-22.

[9] A. J. Dutta and B. C. Tripathy, Statistically pre-Cauchy Fuzzy real-valued sequences defined by Orlicz function, Proyecciones Jour. of Math, 33(3) (2014), 235-243.

[10] P.Erdös and G. Tenenbaum, Sur les densities de certaines suites d'entiers, *Proc. London. Math. Soc.3(59)*, (1989), 417-438.

[11] M. Et and R. Çolak, On some generalized difference sequence spaces, *Soochow Journal Of Mathematics*, 21(4) (1995), 377-386

[12] M. Et, On some difference sequence spaces, *Doğa-Tr. J.of Mathematics* 17 (1993), 18-24.

[13] M. Et and M. Başarır, On some new generalized difference sequence spaces, *Periodica Math. Hungarica* 35 (3) (1997), 169-175.

[14] M. Et. and A. Eşi, On Köthe- Toeplitz duals of generalized difference sequence spaces, *Malaysian Math. Sci. Soc.* 23 (2000), 25-32.

[15] M. Et and F. Nuray, Δ^m-Statistical convergence, *Indian J.Pure Appl. Math.* 32(6) (2001), 961-969.

[16] H. Fast, Sur la Convergence Statistique, *Coll. Math.* 2 (1951), 241-244.

[17] A.R. Freedman, J. Sember and M. Raphael, Some Cesàro-type summability spaces, *Proc. London Math. Soc. (3) 37 no. 3* (1978), 508–520.

[18] A. D. Gadjiev and C. Orhan, Some approximation theorems via statistical convergence*, Rocky Mountain J. Math. 32* (2002), no. 1, 129-138.

[19] H. Gümüş and F. Nuray, Δ^m-Ideal Convergence, *Selçuk J. Appl. Math.*12(2) (2011), 101-110.

[20] V. A. Khan and Q. M. D. Lohani, Statistically pre-Cauchy sequences and Orlicz functions, Southeast Asian Bulletin of Math., 31 (2007), 1107-1112.

[21] V. A. Khan and S. Tabassum, Statistically pre-Cauchy double sequences, Southeast Asian Bulletin of Mathematics 36: (2012), 249–254.

[22] V. A. Khan, K. Ebadullah and A. Ahmad, $\mathcal{I}-$pre-Cauchy sequences and Orlicz functions, Jour. of Mat. Anal. 3 (1) (2012), 21-26.

[23] H. Kızmaz, On certain sequence spaces, *Canad. Math. Bull.* 24(2) (1981), 169-176.

[24] P. Kostyrko, T.Šalát and W. Wilezyński, $\mathcal{I}-$convergence, *Real Anal. Exchange,* 26, 2 (2000), 669-686.

[25] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, Israel J. Math., 101 (1971), 379-390.

[26] H. I. Miller, A measure theoretical subsequence characterization of statistical convergence, *Trans. of the Amer. Math. Soc.Vol. 347, No.5*, (1995), 1811-1819.

[27] S. Ojha and P. D. Srivastava, $\mathcal{I}-$statistically pre-Cauchy sequences of fuzzy numbers, Annals of Fuzzy Math. and Informatics

[28] S. Saha, B. Nath and S. Roy, $\mathcal{I}-$statistically pre-Cauchy triple sequences of fuzzy real numbers, Inter. Jour. of Management and Fuzzy Systems, 2 (2) (2016), 15-21.

[29] İ. Sakaoğlu and M. Ünver, Statistical approximation for multivariable integrable functions*, Miskolc Math. Notes 13* (2012), no.2, 485-491.

[30] E. Savaş, On some new sequence spaces in 2-normed spaces using ideal convergence and Orlicz function, Jour. of Inequal. and Appl. (2010), 8 pages.

[31] E. Savaş and P. Das, A generalized statistical convergence via ideals, *Appl. Math. Lett.* 24 (2011), 826-830.

[32] I. J. Schoenberg, The integrability of certain functions and related summability methods, *The American Mathematical Monthly 66* (1959), 361-375.

[33] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique*, Colloquium Athematicum 2* (1951), 73-74.

[34] U. Yamancı and M. Gürdal, $\mathcal{I}-$statistically pre-Cauchy double sequences, Global Journal of Mathematical Analysis, 2 (4) (2014) 297-303.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.1, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

[35] Zygmund, A., Trigonometric Series, Cam. Uni. Press, Cambridge, UK., (1979).

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO. 1, 2020

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO. 1, 2020

(continued)

Volume 28, Number 2 March 2020
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(six times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, St.Martin Univ.,Olympia,WA,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Jong Kyu Kim
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis, Variational Inequalities, Nonlinear Ergodic Theory, ODE, PDE, Functional Equations.

Robert Kozma
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel Hilbert Spaces,
Neural Percolation Theory

Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference Equations

Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, lasiecka@memphis.edu

Burkhard Lenze
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis, Approximation Theory

Hrushikesh N. Mhaskar
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials, Approximation Theory, Splines, Wavelets, Neural Networks

Ram N. Mohapatra
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis, Approximation Th., Fourier Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations, Abstract Harmonic Analysis, Fractional Differential Equations, Almost Periodicity & Almost Automorphy

M.Zuhair Nashed
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems, Numerical Functional Analysis, Integral Equations, Optimization, Signal Analysis

Mubenga N. Nkashama
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations, Partial Differential Equations

Vassilis Papanicolaou
Department of Mathematics
National Technical University of Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations, Probability

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

Journal Article

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus homepage.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

On invariance and solutions of some fifth-order rational recursive sequences

M. Folly-Gbetoula * and D. Nyirenda †

Abstract

We study the fifth-order difference equations of the form

$$x_{n+1} = \frac{x_{n-4}x_{n-2}}{x_{n-1}(a_n + b_n x_{n-4}x_{n-2})}, n = 0, 1, \ldots,$$

where a_n and b_n are real sequences, using the method of Lie group analysis. In particular, nontrivial vector fields associated with the group of point transformations are derived and exact solutions obtained. Closed form formulas for the solutions to the recursive sequences are given explicitly. This work is a generalization of a result by Elsayed [E. M. Elsayed, Behavior and expression of the solutions of some rational difference equations, J. Computational Analysis and Applications, 15(1) (2013), 73–81].

Keywords: Difference equation; Symmetry; Reduction; Group invariant; Periodicity
Mathematics Subjet Classification: 39A10; 39A13; 39A90

1 Introduction

Over a century ago, Sophus Lie [7] developed an algorithm based on the invariance of the ordinary differential equations under their symmetry group. Maeda [8, 9] observed that the Lie Symmetry approach can be applied to ordinary difference equations. Recently, Hydon [3] utilized a similar method to come up with some interest-provoking results. It is now a foregone conclusion that Lie's method can be used to find symmetries and conservation laws of recursive sequences, even in the context of variational equations.
In this paper, we obtain the vector fields of

$$x_{n+1} = \frac{x_{n-4}x_{n-2}}{x_{n-1}(a_n + b_n x_{n-4}x_{n-2})}, \tag{1}$$

where a_n and b_n are random real sequences, and then proceed to find the solutions in closed form. Our work extends the work by Elsayed [1], where the formulas of the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-4}x_{n-2}}{x_{n-1}(\pm \pm x_{n-4}x_{n-2})} \quad n = 0, 1, \ldots, \tag{2}$$

in which the initial conditions $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary non-zero real numbers, were obtained.
For related work, see [2, 4, 10].

*School of Mathematics, University of the Witwatersrand, Johannesburg, X3, Wits 2050, South Africa Email: Mensah.Folly-Gbetoula@wits.ac.za
†School of Mathematics, University of the Witwatersrand, Johannesburg, X3, Wits 2050, South Africa Email: Darlison.Nyirenda@wits.ac.za

1.1 Background on Lie analysis

In this section, we briefly discuss some key ideas on Lie group analysis of difference equations. For a broader comprehension of the concepts, refer to [3, 11]. The definitions and notation are taken from the same source [3, 11].
Let

$$x^{\star} = X(x; \varepsilon) \tag{3}$$

be a one parameter Lie group of transformations.

Definition 1.1 *An infinitely differentiable function F is an invariant function of the Lie group of point transformation* (3) *if and only if, for any group transformations,*

$$F(x) = F(x^{\star}). \tag{4}$$

Definition 1.2 *The infinitesimal generator of the one-parameter Lie group of point transformation* (3) *is the operator*

$$X = X(x) = \xi(x) \times \Delta = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i}, \tag{5}$$

where Δ is the gradient operator.

Theorem 1.1 *$F(x)$ is invariant under the Lie group of transformations* (3) *if and only if*

$$XF(x) = 0. \tag{6}$$

Consider the forward fifth-order recursive sequence

$$u_{n+5} = \Phi(n, u_n, \ldots, u_{n+4}) \tag{7}$$

for some smooth function Φ. Suppose the one-parameter Lie group of point transformations is of the form

$$n^* = n, \quad u^*_{n+k} = u_{n+k} + \varepsilon S^k \xi(n, u_n) +)(\varepsilon^2), \quad k = 0, \ldots, 5, \tag{8}$$

where ξ denotes the characteristic, ε (ε is small enough) is the group parameter and $S : n \mapsto n+1$ is the shift forward operator. The symmetry condition is given by

$$u^*_{n+5} = \Phi(n, u^*_n, \ldots, u^*_{n+4}), \tag{9}$$

whenever (7) is true. The substitution of (8) in (9) yields the linearized symmetry condition:

$$S^5 \xi(n, u_n) - X\Phi = 0 \tag{10}$$

where X, the vector fields of (7), is given by

$$X = \xi(n, u_n) \frac{\partial}{\partial u_n} + \xi(n+1, u_{n+1}) \frac{\partial}{\partial u_{n+1}} + \cdots + \xi(n+4, u_{n+4}) \frac{\partial}{\partial u_{n+4}}. \tag{11}$$

Despite the fact that (10) looks simple, its solution finding process is highly involving.
In our work, we will use the canonical coordinate [5]

$$S_n = \int \frac{du_n}{\xi(n, u_n)} \tag{12}$$

to lower the order of the difference equation under investigation.

2 Main results

Let

$$u_{n+5} = \Phi = \frac{u_n u_{n+2}}{u_{n+3}(A_n + B_n u_n u_{n+2})}, \tag{13}$$

where A_n and B_n are random real sequences, be the forward recursive equation equivalent to (2).
Substituting (13) in (10), we have that

$$S^5\xi + \frac{u_n u_{n+2}(S^3\xi)}{u_{n+3}^2(A_n + B_n u_n u_{n+2})} - \frac{A_n u_n(S\xi)}{u_{n+3}(A_n + B_n u_n u_{n+2})^2} - \frac{A_n u_{n+2}\xi}{u_{n+3}(A_n + B_n u_n u_{n+2})^2} = 0. \tag{14}$$

We act the differential operator

$$L = \frac{\partial}{\partial u_n} - \frac{\Phi_{u_n}}{\Phi_{u_{n+3}}}\frac{\partial}{\partial u_{n+3}}$$

to eliminate the first term in (14). This leads to

$$(A_n + B_n u_n u_{n+2})\left[(S^3\xi)' - (S^3\xi)\right] + B_n u_n(S\xi) - (A_n + B_n u_n u_{n+2})\xi' + \frac{A_n}{u_n}\xi = 0 \tag{15}$$

after simplification. The differentiation of (15) with respect to u_n twice, keeping u_{n+3} fixed, yields

$$-(A_n + B_n u_n u_{n+2})\xi^{(3)} + \frac{A_n}{u_n}\xi^{(2)} - \frac{2A_n}{{u_n}^2}\xi' + \frac{2A_n}{{u_n}^3}\xi = 0. \tag{16}$$

Split (16) by comparing powers of u_{n+2}; we have

$$\begin{cases} u_{n+2} \quad \text{term}: & u_n^3\xi^{(3)} - {u_n}^2\xi^{(2)} + 2u_n\xi' - 2\xi = 0, \\ \text{other terms}: & \xi^{(3)} = 0. \end{cases} \tag{17}$$

Equations in (17) further simplify to

$${u_n}^2\xi^{(2)} - 2u_n\xi' + 2\xi = 0. \tag{18}$$

It is clear that the solution of (16) is

$$\xi\,(n, u_n) = f_n u_n + g_n {u_n}^2 \tag{19}$$

for some arbitrary functions f_n and g_n of n. Using characteristic's expression as given in (19), we reduce equation (14) to the following difference equation

$$\begin{aligned} &B_n g_{n+3} u_n u_{n+2} {u_{n+3}}^2 + B_n(f_{n+3} + f_{n+5}) u_n u_{n+2} u_{n+3} - A_n g_n u_n u_{n+3} + g_{n+5} u_n u_{n+2} \\ &- A_n(f_n + f_{n+2} + f_{n+3} + f_{n+5}) u_{n+3} - A_n g_{n+1} u_{n+2} u_{n+3} + A_n g_{n+3} {u_{n+3}}^2 = 0. \end{aligned} \tag{20}$$

which then splits into a system (by comparing products of powers of shifts of u_n) as follows:

$$u_{n+3} \text{ terms} : f_n + f_{n+2} + f_{n+3} + f_{n+5} = 0 \tag{21a}$$
$$u_n u_{n+2} \text{ terms} : g_{n+5} = 0 \tag{21b}$$
$$u_n u_{n+3} \text{ terms} : g_n = 0 \tag{21c}$$
$$u_n u_{n+2} u_{n+3} \text{ terms} : f_{n+3} + f_{n+5} = 0 \tag{21d}$$
$$u_n u_{n+2} {u_{n+3}}^2 \text{ terms} : g_{n+3} = 0 \tag{21e}$$
$$u_{n+2} u_{n+3} \text{ terms} : g_{n+1} = 0 \tag{21f}$$
$${u_{n+3}}^2 \text{ terms} : g_{n+3} = 0. \tag{21g}$$

Thus, the 'final constraint' is given by:

$$f_n + f_{n+2} = 0, \tag{22a}$$
$$g_n = 0. \tag{22b}$$

Solving (22) for f, we obtain two independent solutions given by $\exp(\pm n\pi/2)$. Therefore, the characteristics are

$$\xi_1 = \alpha^n u_n, \ \xi_2 = \bar{\alpha}^n u_n, \tag{23}$$

and so the prolonged infinitesimal generators admitted by (13) are

$$X_1 = \alpha^n u_n \partial_{u_n} + \alpha^{n+1} u_{n+1} \partial_{u_{n+1}} + \alpha^{n+2} u_{n+2} \partial_{u_{n+2}} + \alpha^{n+3} u_{n+3} \partial_{u_{n+3}} + \alpha^{n+4} u_{n+4} \partial_{u_{n+4}}, \tag{24a}$$

$$X_2 = \bar{\alpha}^n u_n \partial_{u_n} + \bar{\alpha}^{n+1} u_{n+1} \partial_{u_{n+1}} + \bar{\alpha}^{n+2} u_{n+2} \partial_{u_{n+2}} + \bar{\alpha}^{n+3} u_{n+3} \partial_{u_{n+3}} + \bar{\alpha}^{n+4} u_{n+4} \partial_{u_{n+4}}. \tag{24b}$$

Observe that $\alpha = \exp(i\pi/2)$ and $\bar{\alpha}$ is its complex conjugate. Using the generator X_1, we have the canonical coordinate

$$S_n = \int \frac{du_n}{\alpha^n u_n} = \frac{1}{\alpha^n} \ln |u_n|. \tag{25}$$

Thanks to the form of (22), the invariant function $\tilde{V}_n$ is constructed as follows

$$\tilde{V}_n = S_n \alpha^n + S_{n+2} \alpha^{n+2} \tag{26}$$

since $X_1 \tilde{V}_n = \alpha^n + \alpha^{n+2} = 0$ and $X_2 \tilde{V}_n = \bar{\alpha}^n + \bar{\alpha}^{n+2} = 0$. For rational difference equations, it is convenience to use

$$|V_n| = \exp\{-\tilde{V}_n\}, \tag{27}$$

i.e., $V_n = \pm 1/(u_n u_{n+2})$ but we will be using the one with plus sign: $V_n = 1/u_n u_{n+2}$. We then substitute (27) into equation (13) to get the third-order linear difference equation

$$V_{n+3} = A_n V_n + B_n. \tag{28}$$

The iteration of equation (28) leads to

$$V_{3n+j} =V_j \left(\prod_{k_1=0}^{n-1} A_{3k_1+j}\right) + \sum_{l=0}^{n-1}\left(B_{3l+j}\prod_{k_2=l+1}^{n-1} A_{3k_2+j}\right), \quad j=0,1,2. \tag{29}$$

Invoking (25), (26) and (27), we have that

$$\begin{aligned} |u_n| &= \exp\left(\alpha_n S_n\right) \\ &= \exp\left(\alpha^n c_1 + \bar{\alpha}^n c_2 - \frac{1}{2}\sum_{k_1=0}^{n-1} \alpha^n \bar{\alpha}^{k_1} \tilde{V}_{k_1} - \frac{1}{2}\sum_{k_2=0}^{n-1} \bar{\alpha}^n \alpha^{k_2} \tilde{V}_{k_2}\right) \\ &= \exp\left(\alpha^n c_1 + \bar{\alpha}^n c_2 + \frac{1}{2}\sum_{k_1=0}^{n-1} \alpha^n \bar{\alpha}^{k_1} \ln|V_{k_1}| + \frac{1}{2}\sum_{k_2=0}^{n-1} \bar{\alpha}^n \alpha^{k_2} \ln|V_{k_2}|\right) \\ &= \exp\left(H_n + \sum_{k_1=0}^{n-1} \mathrm{Re}(\gamma(n,k_1)) \ln|V_{k_1}|\right), \end{aligned} \tag{30}$$

in which $H_n = \alpha^n c_1 + \bar{\alpha}^n c_2$ and $\gamma(n,k) = \alpha^n \bar{\alpha}^k$.
It is worthwhile to mention that the function γ satisfies the following:

$$\begin{aligned} &\gamma(0,1) = \bar{\alpha}, \gamma(1,0) = \alpha, \gamma(n,n) = 1, \gamma(n+2,k) = -\gamma(n,k), \\ &\gamma(n,k+2) = -\gamma(n,k), \gamma(4n,k) = \gamma(0,k), \gamma(n,4k) = \gamma(n,0). \end{aligned} \tag{31}$$

From the expression of u_n given in (30) and from the above properties (31), note that

$$|u_{4n+j}| = \exp\left(H_j + \sum_{k_1=0}^{4n+j-1} \mathrm{Re}(\gamma(j,k_1)) \ln|V_{k_1}|\right). \tag{32}$$

For $j=0$, we have

$$\begin{aligned} |u_{4n}| &= \exp(H_0 + \ln|V_0| - \ln|V_2| + \ldots + \ln|V_{4n-4}| - \ln|V_{4n-2}|) \\ &= \exp(H_0) \prod_{s=0}^{n-1} \left|\frac{V_{4s}}{V_{4s+2}}\right|. \end{aligned} \tag{33}$$

By setting $n=0$ in (30), we get $\exp(H_0) = u_0$ and so

$$u_{4n} = u_0 \prod_{s=0}^{n-1} \frac{V_{4s}}{V_{4s+2}}. \tag{34}$$

We have omitted the absolute function because it can be shown, using (27), that there is no need for it. In a similar way, we have that

$$u_{4n+j} =u_j \prod_{s=0}^{n-1} \frac{V_{4s+j}}{V_{4s+j+2}}, \quad \text{for any } j=0,1,2,3. \tag{35}$$

This equation implies that

$$\begin{aligned} u_{12n+j} &= u_j \prod_{s=0}^{3n-1} \frac{V_{4s+j}}{V_{4s+j+2}} \\ &= u_j \prod_{s=0}^{n-1} \frac{V_{12s+j}}{V_{12s+j+2}} \frac{V_{12s+4+j}}{V_{12s+j+6}} \frac{V_{12s+j+8}}{V_{12s+j+10}} \end{aligned}$$

which now holds for $j = 0, 1, 2, \ldots, 11$.
For $j = 0$, we have

$$u_{12n} = u_0 \prod_{s=0}^{n-1} \frac{V_{12s}}{V_{12s+2}} \frac{V_{12s+4}}{V_{12s+6}} \frac{V_{12s+8}}{V_{12s+10}}. \tag{36}$$

Using (29) in (36), we have that

$$\begin{aligned} u_{12n} = u_0 \prod_{s=0}^{n-1} & \frac{V_0 \prod_{k_1=0}^{4s-1} A_{3k_1} + \sum_{l=0}^{4s-1} B_{3l} \prod_{k_2=l+1}^{4s-1} A_{3k_2}}{V_2 \prod_{k_1=0}^{4s-1} A_{3k_1+2} + \sum_{l=0}^{4s-1} B_{3l+2} \prod_{k_2=l+1}^{4s-1} A_{3k_2+2}} \frac{V_1 \prod_{k_1=0}^{4s} A_{3k_1+1} + \sum_{l=0}^{4s} B_{3l+1} \prod_{k_2=l+1}^{4s} A_{3k_2+1}}{V_0 \prod_{k_1=0}^{4s+1} A_{3k_1} + \sum_{l=0}^{4s+1} B_{3l} \prod_{k_2=l+1}^{4s+1} A_{3k_2}} \\ & \times \frac{V_2 \prod_{k_1=0}^{4s+1} A_{3k_1+2} + \sum_{l=0}^{4s+1} \left(B_{3l+2} \prod_{k_2=l+1}^{4s+1} A_{3k_2+2} \right)}{V_1 \prod_{k_1=0}^{4s+2} A_{3k_1+1} + \sum_{l=0}^{4s+2} \left(B_{3l+1} \prod_{k_2=l+1}^{4s+2} A_{3k_2+1} \right)} \end{aligned}$$

$$\begin{aligned} = u_0 \prod_{s=0}^{n-1} & \frac{\prod_{k_1=0}^{4s-1} A_{3k_1} + u_0 u_2 \sum_{l=0}^{4s-1} B_{3l} \prod_{k_2=l+1}^{4s-1} A_{3k_2}}{\prod_{k_1=0}^{4s-1} A_{3k_1+2} + u_2 u_4 \sum_{l=0}^{4s-1} B_{3l+2} \prod_{k_2=l+1}^{4s-1} A_{3k_2+2}} \frac{\prod_{k_1=0}^{4s} A_{3k_1+1} + u_1 u_3 \sum_{l=0}^{4s} B_{3l+1} \prod_{k_2=l+1}^{4s} A_{3k_2+1}}{\prod_{k_1=0}^{4s+1} A_{3k_1} + u_0 u_2 \sum_{l=0}^{4s+1} B_{3l} \prod_{k_2=l+1}^{4s+1} A_{3k_2}} \\ & \times \frac{\prod_{k_1=0}^{4s+1} A_{3k_1+2} + u_2 u_4 \sum_{l=0}^{4s+1} \left(B_{3l+2} \prod_{k_2=l+1}^{4s+1} A_{3k_2+2} \right)}{\prod_{k_1=0}^{4s+2} A_{3k_1+1} + u_1 u_3 \sum_{l=0}^{4s+2} \left(B_{3l+1} \prod_{k_2=l+1}^{4s+2} A_{3k_2+1} \right)}. \end{aligned}$$

Hence x_{12n-4} is equal to

$$x_{-4}\prod_{s=0}^{n-1}\frac{\prod_{k_1=0}^{4s-1}a_{3k_1}+x_{-4}x_{-2}\sum_{l=0}^{4s-1}b_{3l}\prod_{k_2=l+1}^{4s-1}a_{3k_2}}{\prod_{k_1=0}^{4s-1}a_{3k_1+2}+x_{-2}x_0\sum_{l=0}^{4s-1}b_{3l+2}\prod_{k_2=l+1}^{4s-1}a_{3k_2+2}}\frac{\prod_{k_1=0}^{4s}a_{3k_1+1}+x_{-3}x_{-1}\sum_{l=0}^{4s}b_{3l+1}\prod_{k_2=l+1}^{4s}a_{3k_2+1}}{\prod_{k_1=0}^{4s+1}a_{3k_1}+x_{-4}x_{-2}\sum_{l=0}^{4s+1}b_{3l}\prod_{k_2=l+1}^{4s+1}a_{3k_2}}$$
$$\times\frac{\prod_{k_1=0}^{4s+1}a_{3k_1+2}+x_{-2}x_0\sum_{l=0}^{4s+1}\left(b_{3l+2}\prod_{k_2=l+1}^{4s+1}a_{3k_2+2}\right)}{\prod_{k_1=0}^{4s+2}a_{3k_1+1}+x_{-3}x_{-1}\sum_{l=0}^{4s+2}\left(b_{3l+1}\prod_{k_2=l+1}^{4s+2}a_{3k_2+1}\right)}.$$

For $j=1$, we have

$$u_{12n+1}=u_1\prod_{s=0}^{n-1}\frac{V_1\prod_{k_1=0}^{4s-1}A_{3k_1+1}+\sum_{l=0}^{4s-1}B_{3l+1}\prod_{k_2=l+1}^{4s-1}A_{3k_2+1}}{V_0\prod_{k_1=0}^{4s}A_{3k_1}+\sum_{l=0}^{4s}B_{3l}\prod_{k_2=l+1}^{4s}A_{3k_2}}\frac{V_2\prod_{k_1=0}^{4s}A_{3k_1+2}+\sum_{l=0}^{4s}B_{3l+2}\prod_{k_2=l+1}^{4s}A_{3k_2+2}}{V_1\prod_{k_1=0}^{4s+1}A_{3k_1+1}+\sum_{l=0}^{4s+1}B_{3l+1}\prod_{k_2=l+1}^{4s+1}A_{3k_2+1}}$$
$$\times\frac{V_0\prod_{k_1=0}^{4s+2}A_{3k_1}+\sum_{l=0}^{4s+2}B_{3l}\prod_{k_2=l+1}^{4s+2}A_{3k_2}}{V_2\prod_{k_1=0}^{4s+2}A_{3k_1+2}+\sum_{l=0}^{4s+2}B_{3l+2}\prod_{k_2=l+1}^{4s+2}A_{3k_2+2}}$$

so that x_{12n-3} is equal to

$$x_{-3}\prod_{s=0}^{n-1}\frac{\prod_{k_1=0}^{4s-1}a_{3k_1+1}+x_{-3}x_{-1}\sum_{l=0}^{4s-1}b_{3l+1}\prod_{k_2=l+1}^{4s-1}a_{3k_2+1}}{\prod_{k_1=0}^{4s}a_{3k_1}+x_{-4}x_{-2}\sum_{l=0}^{4s}b_{3l}\prod_{k_2=l+1}^{4s}a_{3k_2}}\frac{\prod_{k_1=0}^{4s}a_{3k_1+2}+x_{-2}x_0\sum_{l=0}^{4s}b_{3l+2}\prod_{k_2=l+1}^{4s}A_{3k_2+2}}{\prod_{k_1=0}^{4s+1}A_{3k_1+1}+x_{-3}x_{-1}\sum_{l=0}^{4s+1}b_{3l+1}\prod_{k_2=l+1}^{4s+1}a_{3k_2+1}}$$
$$\times\frac{\prod_{k_1=0}^{4s+2}a_{3k_1}+x_{-4}x_{-2}\sum_{l=0}^{4s+2}b_{3l}\prod_{k_2=l+1}^{4s+2}a_{3k_2}}{\prod_{k_1=0}^{4s+2}a_{3k_1+2}+x_{-2}x_0\sum_{l=0}^{4s+2}b_{3l+2}\prod_{k_2=l+1}^{4s+2}a_{3k_2+2}}.$$

For $j=2$, we have

$$u_{12n+2}=u_2\prod_{s=0}^{n-1}\frac{V_2\prod_{k_1=0}^{4s-1}A_{3k_1+2}+\sum_{l=0}^{4s-1}B_{3l+2}\prod_{k_2=l+1}^{4s-1}A_{3k_2+2}}{V_1\prod_{k_1=0}^{4s}A_{3k_1+1}+\sum_{l=0}^{4s}B_{3l+1}\prod_{k_2=l+1}^{4s}A_{3k_2+1}}\frac{V_0\prod_{k_1=0}^{4s+1}A_{3k_1}+\sum_{l=0}^{4s+1}B_{3l}\prod_{k_2=l+1}^{4s+1}A_{3k_2}}{V_2\prod_{k_1=0}^{4s+1}A_{3k_1+2}+\sum_{l=0}^{4s+1}B_{3l+2}\prod_{k_2=l+1}^{4s+1}A_{3k_2+2}}$$
$$\times\frac{V_1\prod_{k_1=0}^{4s+2}A_{3k_1+1}+\sum_{l=0}^{4s+2}B_{3l+2}\prod_{k_2=l+1}^{4s+2}A_{3k_2+1}}{V_0\prod_{k_1=0}^{4s+3}A_{3k_1}+\sum_{l=0}^{4s+3}B_{3l}\prod_{k_2=l+1}^{4s+3}A_{3k_2}}$$

7

so that

$$x_{12n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{4s-1}a_{3k_1+2}+x_{-2}x_0\sum\limits_{l=0}^{4s-1}b_{3l+2}\prod\limits_{k_2=l+1}^{4s-1}a_{3k_2+2}}{\prod\limits_{k_1=0}^{4s}a_{3k_1+1}+x_{-3}x_{-1}\sum\limits_{l=0}^{4s}b_{3l+1}\prod\limits_{k_2=l+1}^{4s}a_{3k_2+1}}\frac{\prod\limits_{k_1=0}^{4s+1}a_{3k_1}+x_{-4}x_{-2}\sum\limits_{l=0}^{4s+1}b_{3l}\prod\limits_{k_2=l+1}^{4s+1}a_{3k_2}}{\prod\limits_{k_1=0}^{4s+1}a_{3k_1+2}+x_{-2}x_0\sum\limits_{l=0}^{4s+1}b_{3l+2}\prod\limits_{k_2=l+1}^{4s+1}a_{3k_2+2}}$$
$$\times\frac{\prod\limits_{k_1=0}^{4s+2}a_{3k_1+1}+x_{-3}x_{-1}\sum\limits_{l=0}^{4s+2}b_{3l+2}\prod\limits_{k_2=l+1}^{4s+2}a_{3k_2+1}}{\prod\limits_{k_1=0}^{4s+3}a_{3k_1}+x_{-4}x_{-2}\sum\limits_{l=0}^{4s+3}b_{3l}\prod\limits_{k_2=l+1}^{4s+3}a_{3k_2}}.$$

Following similar substitutions as above where $u_i = x_{i-4}$ and $V_i = \frac{1}{x_{i-4}x_{i-2}}$, we deduce that for $x_{12n+j-4}$ with $j = 3, 4, 5, \ldots, 11$;

$$x_{12n-1} =$$
$$x_{-1}\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{4s}a_{3k_1}+x_{-4}x_{-2}\sum\limits_{l=0}^{4s}b_{3l}\prod\limits_{k_2=l+1}^{4s}a_{3k_2}}{\prod\limits_{k_1=0}^{4s}a_{3k_1+2}+x_{-2}x_0\sum\limits_{l=0}^{4s}b_{3l+2}\prod\limits_{k_2=l+1}^{4s}a_{3k_2+2}}\frac{\prod\limits_{k_1=0}^{4s+1}a_{3k_1+1}+x_{-3}x_{-1}\sum\limits_{l=0}^{4s+1}b_{3l+1}\prod\limits_{k_2=l+1}^{4s+1}a_{3k_2+1}}{\prod\limits_{k_1=0}^{4s+2}a_{3k_1}+x_{-4}x_{-2}\sum\limits_{l=0}^{4s+2}b_{3l}\prod\limits_{k_2=l+1}^{4s+2}a_{3k_2}}$$
$$\times\frac{\prod\limits_{k_1=0}^{4s+2}a_{3k_1+2}+x_{-2}x_0\sum\limits_{l=0}^{4s+2}b_{3l+2}\prod\limits_{k_2=l+1}^{4s+2}a_{3k_2+2}}{\prod\limits_{k_1=0}^{4s+3}a_{3k_1+1}+x_{-3}x_{-1}\sum\limits_{l=0}^{4s+3}b_{3l+1}\prod\limits_{k_2=l+1}^{4s+3}a_{3k_2+1}},$$

$$x_{12n} =$$
$$x_0\prod_{s=0}^{n-1}\frac{\prod\limits_{k_1=0}^{4s}a_{3k_1+1}+x_{-3}x_{-1}\sum\limits_{l=0}^{4s}b_{3l+1}\prod\limits_{k_2=l+1}^{4s}a_{3k_2+1}}{\prod\limits_{k_1=0}^{4s+1}a_{3k_1}+x_{-4}x_{-2}\sum\limits_{l=0}^{4s+1}b_{3l}\prod\limits_{k_2=l+1}^{4s+1}a_{3k_2}}\frac{\prod\limits_{k_1=0}^{4s+1}a_{3k_1+2}+x_{-2}x_0\sum\limits_{l=0}^{4s+1}b_{3l+2}\prod\limits_{k_2=l+1}^{4s+1}a_{3k_2+2}}{\prod\limits_{k_1=0}^{4s+2}a_{3k_1+1}+x_{-3}x_{-1}\sum\limits_{l=0}^{4s+2}b_{3l+1}\prod\limits_{k_2=l+1}^{4s+2}a_{3k_2+1}}$$
$$\times\frac{\prod\limits_{k_1=0}^{4s+3}a_{3k_1}+x_{-4}x_{-2}\sum\limits_{l=0}^{4s+3}b_{3l}\prod\limits_{k_2=l+1}^{4s+3}a_{3k_2}}{\prod\limits_{k_1=0}^{4s+3}a_{3k_1+2}+x_{-2}x_0\sum\limits_{l=0}^{4s+3}b_{3l+2}\prod\limits_{k_2=l+1}^{4s+3}a_{3k_2+2}},$$

$$x_{12n+1} =$$

$$x_1 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s} b_{3l+2} \prod_{k_2=l+1}^{4s} a_{3k_2+2}}{\prod_{k_1=0}^{4s+1} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+1} b_{3l+1} \prod_{k_2=l+1}^{4s+1} a_{3k_2+1}} \frac{\prod_{k_1=0}^{4s+2} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+2} b_{3l} \prod_{k_2=l+1}^{4s+2} a_{3k_2}}{\prod_{k_1=0}^{4s+2} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+2} b_{3l+2} \prod_{k_2=l+1}^{4s+2} a_{3k_2+2}}$$
$$\times \frac{\prod_{k_1=0}^{4s+3} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+3} b_{3l+1} \prod_{k_2=l+1}^{4s+3} a_{3k_2+1}}{\prod_{k_1=0}^{4s+4} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+4} b_{3l} \prod_{k_2=l+1}^{4s+4} a_{3k_2}},$$

$$x_{12n+2} =$$

$$x_2 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+1} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+1} b_{3l} \prod_{k_2=l+1}^{4s+1} a_{3k_2}}{\prod_{k_1=0}^{4s+1} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+1} b_{3l+2} \prod_{k_2=l+1}^{4s+1} a_{3k_2+2}} \frac{\prod_{k_1=0}^{4s+2} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+2} b_{3l+1} \prod_{k_2=l+1}^{4s+2} a_{3k_2+1}}{\prod_{k_1=0}^{4s+3} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+3} b_{3l} \prod_{k_2=l+1}^{4s+3} a_{3k_2}}$$
$$\times \frac{\prod_{k_1=0}^{4s+3} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+3} b_{3l+2} \prod_{k_2=l+1}^{4s+3} a_{3k_2+2}}{\prod_{k_1=0}^{4s+4} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+4} b_{3l+1} \prod_{k_2=l+1}^{4s+4} a_{3k_2+1}},$$

$$x_{12n+3} =$$

$$x_3 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+1} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+1} b_{3l+1} \prod_{k_2=l+1}^{4s+1} a_{3k_2+1}}{\prod_{k_1=0}^{4s+2} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+2} b_{3l} \prod_{k_2=l+1}^{4s+2} a_{3k_2}} \frac{\prod_{k_1=0}^{4s+2} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+2} b_{3l+2} \prod_{k_2=l+1}^{4s+2} a_{3k_2+2}}{\prod_{k_1=0}^{4s+3} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+3} b_{3l+1} \prod_{k_2=l+1}^{4s+3} a_{3k_2+1}}$$
$$\times \frac{\prod_{k_1=0}^{4s+4} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+4} b_{3l} \prod_{k_2=l+1}^{4s+4} a_{3k_2}}{\prod_{k_1=0}^{4s+4} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+4} b_{3l+2} \prod_{k_2=l+1}^{4s+4} a_{3k_2+2}},$$

$$x_{12n+4} = x_4 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+1} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+1} b_{3l+2} \prod_{k_2=l+1}^{4s+1} a_{3k_2+2}}{\prod_{k_1=0}^{4s+2} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+2} b_{3l+1} \prod_{k_2=l+1}^{4s+2} a_{3k_2+1}} \frac{\prod_{k_1=0}^{4s+3} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+3} b_{3l} \prod_{k_2=l+1}^{4s+3} a_{3k_2}}{\prod_{k_1=0}^{4s+3} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+3} b_{3l+2} \prod_{k_2=l+1}^{4s+3} a_{3k_2+2}} \times \frac{\prod_{k_1=0}^{4s+4} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+4} b_{3l+1} \prod_{k_2=l+1}^{4s+4} a_{3k_2+1}}{\prod_{k_1=0}^{4s+5} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+5} b_{3l} \prod_{k_2=l+1}^{4s+5} a_{3k_2}},$$

$$x_{12n+5} = x_5 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+2} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+2} b_{3l} \prod_{k_2=l+1}^{4s+2} a_{3k_2}}{\prod_{k_1=0}^{4s+2} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+2} b_{3l+2} \prod_{k_2=l+1}^{4s+2} a_{3k_2+2}} \frac{\prod_{k_1=0}^{4s+3} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+3} b_{3l+1} \prod_{k_2=l+1}^{4s+3} a_{3k_2+1}}{\prod_{k_1=0}^{4s+4} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+4} b_{3l} \prod_{k_2=l+1}^{4s+4} a_{3k_2}} \times \frac{\prod_{k_1=0}^{4s+4} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+4} b_{3l+2} \prod_{k_2=l+1}^{4s+4} a_{3k_2+2}}{\prod_{k_1=0}^{4s+5} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+5} b_{3l+1} \prod_{k_2=l+1}^{4s+5} a_{3k_2+1}},$$

$$x_{12n+6} = x_6 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+2} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+2} b_{3l+1} \prod_{k_2=l+1}^{4s+2} a_{3k_2+1}}{\prod_{k_1=0}^{4s+3} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+3} b_{3l} \prod_{k_2=l+1}^{4s+3} a_{3k_2}} \frac{\prod_{k_1=0}^{4s+3} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+3} b_{3l+2} \prod_{k_2=l+1}^{4s+3} a_{3k_2+2}}{\prod_{k_1=0}^{4s+4} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+4} b_{3l+1} \prod_{k_2=l+1}^{4s+4} a_{3k_2+1}} \times \frac{\prod_{k_1=0}^{4s+5} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+5} b_{3l} \prod_{k_2=l+1}^{4s+5} a_{3k_2}}{\prod_{k_1=0}^{4s+5} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+5} b_{3l+2} \prod_{k_2=l+1}^{4s+5} a_{3k_2+2}},$$

$$x_{12n+7} = $$
$$x_7 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{4s+2} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+2} b_{3l+2} \prod_{k_2=l+1}^{4s+2} a_{3k_2+2}}{\prod_{k_1=0}^{4s+3} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+3} b_{3l+1} \prod_{k_2=l+1}^{4s+3} a_{3k_2+1}} \frac{\prod_{k_1=0}^{4s+4} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+4} b_{3l} \prod_{k_2=l+1}^{4s+4} a_{3k_2}}{\prod_{k_1=0}^{4s+4} a_{3k_1+2} + x_{-2}x_0 \sum_{l=0}^{4s+4} b_{3l+2} \prod_{k_2=l+1}^{4s+4} a_{3k_2+2}}$$
$$\times \frac{\prod_{k_1=0}^{4s+5} a_{3k_1+1} + x_{-3}x_{-1} \sum_{l=0}^{4s+5} b_{3l+1} \prod_{k_2=l+1}^{4s+5} a_{3k_2+1}}{\prod_{k_1=0}^{4s+6} a_{3k_1} + x_{-4}x_{-2} \sum_{l=0}^{4s+6} b_{3l} \prod_{k_2=l+1}^{4s+6} a_{3k_2}},$$

where $x_1, x_2, x_3, x_4, x_5, x_6$ and x_7 are given as follows:

$$x_1 = \frac{x_{-4}x_{-2}}{x_{-1}(a_0 + b_0x_{-4}x_{-2})},\ x_2 = \frac{x_{-3}x_{-1}}{x_0(a_1 + b_1x_{-3}x_{-1})},\ x_3 = \frac{x_{-1}x_0(a_0 + b_0x_{-4}x_{-2})}{x_{-4}(a_2 + b_2x_{-2}x_0)},$$

$$x_4 = \frac{x_{-4}x_{-2}x_0(a_1 + b_1x_{-3}x_{-1})}{x_{-3}x_{-1}(a_0a_3 + (b_0a_3 + b_3)x_{-4}x_{-2})},\ x_5 = \frac{x_{-3}x_{-4}(a_2 + b_2x_{-2}x_0)}{x_0(a_0 + b_0x_{-4}x_{-2})(a_1a_4 + (b_1a_4 + b_4)x_{-3}x_{-1})},$$
$$x_6 = \frac{x_{-3}x_{-1}(a_0a_3 + (b_0a_3 + b_3)x_{-4}x_{-2})}{x_{-4}(a_1 + b_1x_{-3}x_{-1})(a_5a_2 + (b_2a_5 + b_5)x_{-2}x_0)},$$

and

$$x_7 = \frac{x_{-2}x_0(a_0 + b_0x_{-4}x_{-2})(a_1a_4 + (b_1a_4 + b_4)x_{-3}x_{-1})}{x_{-3}(a_2 + b_2x_{-2}x_0)(a_6a_3a_0 + (a_6a_3b_0 + a_6b_3 + b_6)x_{-4}x_{-2})}.$$

We now turn our attention to special cases in the subsequent sections.

3 The case a_n and b_n are 1-periodic

Let $a_n = a$ and $b_n = b$, where $a, b \in \mathbb{R}$. We simply carry out a substitution and find the following solution:

$$x_{12n-4} = x_{-4} \prod_{s=0}^{n-1} \frac{a^{4s} + bx_{-4}x_{-2} \sum_{l=0}^{4s-1} a^l}{a^{4s} + bx_{-2}x_0 \sum_{l=0}^{4s-1} a^l} \frac{a^{4s+1} + bx_{-3}x_{-1} \sum_{l=0}^{4s} a^l}{a^{4s+2} + bx_{-4}x_{-2} \sum_{l=0}^{4s+1} a^l} \frac{a^{4s+2} + bx_{-2}x_0 \sum_{l=0}^{4s+1} a^l}{a^{4s+3} + bx_{-3}x_{-1} \sum_{l=0}^{4s+2} a^l},$$

$$x_{12n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{a^{4s} + bx_{-3}x_{-1} \sum_{l=0}^{4s-1} a^l}{a^{4s+1} + bx_{-4}x_{-2} \sum_{l=0}^{4s} a^l} \frac{a^{4s+1} + bx_{-2}x_0 \sum_{l=0}^{4s} a^l}{a^{4s+2} + bx_{-3}x_{-1} \sum_{l=0}^{4s+1} a^l} \frac{a^{4s+3} + bx_{-4}x_{-2} \sum_{l=0}^{4s+2} a^l}{a^{4s+3} + bx_{-2}x_0 \sum_{l=0}^{4s+2} a^l},$$

11

$$x_{12n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{a^{4s}+bx_{-2}x_0\sum\limits_{l=0}^{4s-1}a^l}{a^{4s+1}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s}a^l}\,\frac{a^{4s+2}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+1}a^l}{a^{4s+2}+bx_{-2}x_0\sum\limits_{l=0}^{4s+1}a^l}\,\frac{a^{4s+3}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+2}a^l}{a^{4s+4}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+3}a^l},$$

$$x_{12n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{a^{4s+1}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s}a^l}{a^{4s+1}+bx_{-2}x_0\sum\limits_{l=0}^{4s}a^l}\,\frac{a^{4s+2}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+1}a^l}{a^{4s+3}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+2}a^l}\,\frac{a^{4s+3}+bx_{-2}x_0\sum\limits_{l=0}^{4s+2}a^l}{a^{4s+4}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+3}a^l},$$

$$x_{12n} = x_0\prod_{s=0}^{n-1}\frac{a^{4s+1}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s}a^l}{a^{4s+2}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+1}a^l}\,\frac{a^{4s+2}+bx_{-2}x_0\sum\limits_{l=0}^{4s+1}a^l}{a^{4s+3}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+2}a^l}\,\frac{a^{4s+4}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+3}a^l}{a^{4s+4}+bx_{-2}x_0\sum\limits_{l=0}^{4s+3}a^l},$$

$$x_{12n+1} = x_1\prod_{s=0}^{n-1}\frac{a^{4s+1}+bx_{-2}x_0\sum\limits_{l=0}^{4s}a^l}{a^{4s+2}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+1}a^l}\,\frac{a^{4s+3}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+2}a^l}{a^{4s+3}+bx_{-2}x_0\sum\limits_{l=0}^{4s+2}a^l}\,\frac{a^{4s+4}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+3}a^l}{a^{4s+5}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+4}a^l},$$

$$x_{12n+2} = x_2\prod_{s=0}^{n-1}\frac{a^{4s+2}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+1}a^l}{a^{4s+2}+bx_{-2}x_0\sum\limits_{l=0}^{4s+1}a^l}\,\frac{a^{4s+3}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+2}a^l}{a^{4s+4}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+3}a^l}\,\frac{a^{4s+4}+bx_{-2}x_0\sum\limits_{l=0}^{4s+3}a^l}{a^{4s+5}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+4}a^l},$$

$$x_{12n+3} = x_3\prod_{s=0}^{n-1}\frac{a^{4s+2}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+1}a^l}{a^{4s+3}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+2}a^l}\,\frac{a^{4s+3}+bx_{-2}x_0\sum\limits_{l=0}^{4s+2}a^l}{a^{4s+4}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+3}a^l}\,\frac{a^{4s+5}+x_{-4}x_{-2}\sum\limits_{l=0}^{4s+4}a^l}{a^{4s+5}+bx_{-2}x_0\sum\limits_{l=0}^{4s+4}a^l},$$

$$x_{12n+4} = x_4\prod_{s=0}^{n-1}\frac{a^{4s+2}+bx_{-2}x_0\sum\limits_{l=0}^{4s+1}a^l}{a^{4s+3}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+2}a^l}\,\frac{a^{4s+4}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+3}a^l}{a^{4s+4}+bx_{-2}x_0\sum\limits_{l=0}^{4s+3}a^l}\,\frac{a^{4s+5}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+4}a^l}{a^{4s+6}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+5}a^l},$$

$$x_{12n+5} = x_5\prod_{s=0}^{n-1}\frac{a^{4s+3}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+2}a^l}{a^{4s+3}+bx_{-2}x_0\sum\limits_{l=0}^{4s+2}a^l}\,\frac{a^{4s+4}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+3}a^l}{a^{4s+5}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+4}a^l}\,\frac{a^{4s+5}+bx_{-2}x_0\sum\limits_{l=0}^{4s+4}a^l}{a^{4s+6}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+5}a^l},$$

12

$$x_{12n+6}=x_6\prod_{s=0}^{n-1}\frac{a^{4s+3}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+2}a^l}{a^{4s+4}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+3}a^l}\,\frac{a^{4s+4}+bx_{-2}x_0\sum\limits_{l=0}^{4s+3}a^l}{a^{4s+5}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+4}a^l}\,\frac{a^{4s+6}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+5}a^l}{a^{4s+6}+bx_{-2}x_0\sum\limits_{l=0}^{4s+5}a^l},$$

$$x_{12n+7}=x_7\prod_{s=0}^{n-1}\frac{a^{4s+3}+bx_{-2}x_0\sum\limits_{l=0}^{4s+2}a^l}{a^{4s+4}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+3}a^l}\,\frac{a^{4s+5}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+4}a^l}{a^{4s+5}+bx_{-2}x_0\sum\limits_{l=0}^{4s+4}a^l}\,\frac{a^{4s+6}+bx_{-3}x_{-1}\sum\limits_{l=0}^{4s+5}a^l}{a^{4s+7}+bx_{-4}x_{-2}\sum\limits_{l=0}^{4s+6}a^l},$$

where $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ are given by

$$x_1=\frac{x_{-4}x_{-2}}{x_{-1}(a+bx_{-4}x_{-2})},\quad x_2=\frac{x_{-3}x_{-1}}{x_0(a+bx_{-3}x_{-1})},\quad x_3=\frac{x_{-1}x_0(a+bx_{-4}x_{-2})}{x_{-4}(a+bx_{-2}x_0)},$$

$$x_4=\frac{x_{-4}x_{-2}x_0(a+bx_{-3}x_{-1})}{x_{-3}x_{-1}(a^2+(ab+b)x_{-4}x_{-2})},\quad x_5=\frac{x_{-4}x_{-3}(a+bx_{-2}x_0)}{x_0(a+bx_{-4}x_{-2})(a^2+(ab+b)x_{-3}x_{-1})},$$

$$x_6=\frac{x_{-3}x_{-1}(a^2+(ab+b)x_{-4}x_{-2})}{x_{-4}(a+bx_{-3}x_{-1})(a^2+(ab+b)x_{-2}x_0)}$$

and

$$x_7=\frac{x_{-2}x_0(a+bx_{-4}x_{-2})(a^2+(ab+b)x_{-3}x_{-1})}{x_{-3}(a+bx_{-2}x_0)(a^3+(a^2b+ab+b)x_{-4}x_{-2})}.$$

3.1 The case $a = 1$

The solution, which appears for $b = \pm 1$ in Theorems 1 and 6 of [1], is given by

$$x_{12n-4}=x_{-4}\prod_{s=0}^{n-1}\frac{1+4sbx_{-4}x_{-2}}{1+4sbx_{-2}x_0}\,\frac{1+(4s+1)bx_{-3}x_{-1}}{1+(4s+2)bx_{-4}x_{-2}}\,\frac{1+(4s+2)bx_{-2}x_0}{1+(4s+3)bx_{-3}x_{-1}},$$

$$x_{12n-3}=x_{-3}\prod_{s=0}^{n-1}\frac{1+4sbx_{-3}x_{-1}}{1+(4s+1)bx_{-4}x_{-2}}\,\frac{1+(4s+1)bx_{-2}x_0}{1+(4s+2)bx_{-3}x_{-1}}\,\frac{1+(4s+3)bx_{-4}x_{-2}}{1+(4s+3)bx_{-2}x_0},$$

$$x_{12n-2}=x_{-2}\prod_{s=0}^{n-1}\frac{1+4sbx_{-2}x_0}{1+(4s+1)bx_{-3}x_{-1}}\,\frac{1+(4s+2)bx_{-4}x_{-2}}{1+(4s+2)bx_{-2}x_0}\,\frac{1+(4s+3)bx_{-3}x_{-1}}{1+(4s+4)bx_{-4}x_{-2}},$$

$$x_{12n-1}=x_{-1}\prod_{s=0}^{n-1}\frac{1+(4s+1)bx_{-4}x_{-2}}{1+(4s+1)bx_{-2}x_0}\,\frac{1+(4s+2)bx_{-3}x_{-1}}{1+(4s+3)bx_{-4}x_{-2}}\,\frac{1+(4s+3)bx_{-2}x_0}{1+(4s+4)bx_{-3}x_{-1}},$$

$$x_{12n} = x_0 \prod_{s=0}^{n-1} \frac{1+(4s+1)bx_{-3}x_{-1}}{1+(4s+2)bx_{-4}x_{-2}} \frac{1+(4s+2)bx_{-2}x_0}{1+(4s+3)bx_{-3}x_{-1}} \frac{1+(4s+4)bx_{-4}x_{-2}}{1+(4s+4)bx_{-2}x_0},$$

$$x_{12n+1} = x_1 \prod_{s=0}^{n-1} \frac{1+(4s+1)bx_{-2}x_0}{1+(4s+2)bx_{-3}x_{-1}} \frac{1+(4s+3)bx_{-4}x_{-2}}{1+(4s+3)bx_{-2}x_0} \frac{1+(4s+4)bx_{-3}x_{-1}}{1+(4s+5)bx_{-4}x_{-2}},$$

$$x_{12n+2} = x_2 \prod_{s=0}^{n-1} \frac{1+(4s+2)bx_{-4}x_{-2}}{1+(4s+2)bx_{-2}x_0} \frac{1+(4s+3)bx_{-3}x_{-1}}{1+(4s+4)bx_{-4}x_{-2}} \frac{1+(4s+4)bx_{-2}x_0}{1+(4s+5)bx_{-3}x_{-1}},$$

$$x_{12n+3} = x_3 \prod_{s=0}^{n-1} \frac{1+(4s+2)bx_{-3}x_{-1}}{1+(4s+3)bx_{-4}x_{-2}} \frac{1+(4s+3)bx_{-2}x_0}{1+(4s+4)bx_{-3}x_{-1}} \frac{1+(4s+5)bx_{-4}x_{-2}}{1+(4s+5)bx_{-2}x_0},$$

$$x_{12n+4} = x_4 \prod_{s=0}^{n-1} \frac{1+(4s+2)bx_{-2}x_0}{1+(4s+3)bx_{-3}x_{-1}} \frac{1+(4s+4)bx_{-4}x_{-2}}{1+(4s+4)bx_{-2}x_0} \frac{1+(4s+5)bx_{-3}x_{-1}}{1+(4s+6)bx_{-4}x_{-2}},$$

$$x_{12n+5} = x_5 \prod_{s=0}^{n-1} \frac{1+(4s+3)bx_{-4}x_{-2}}{1+(4s+3)bx_{-2}x_0} \frac{1+(4s+4)bx_{-3}x_{-1}}{1+(4s+5)bx_{-4}x_{-2}} \frac{1+(4s+5)bx_{-2}x_0}{1+(4s+6)bx_{-3}x_{-1}},$$

$$x_{12n+6} = x_6 \prod_{s=0}^{n-1} \frac{1+(4s+3)bx_{-3}x_{-1}}{1+(4s+4)bx_{-4}x_{-2}} \frac{1+(4s+4)bx_{-2}x_0}{1+(4s+5)bx_{-3}x_{-1}} \frac{1+(4s+6)bx_{-4}x_{-2}}{1+(4s+6)bx_{-2}x_0},$$

$$x_{12n+7} = x_7 \prod_{s=0}^{n-1} \frac{1+(4s+3)bx_{-2}x_0}{1+(4s+4)bx_{-3}x_{-1}} \frac{1+(4s+5)bx_{-4}x_{-2}}{1+(4s+5)bx_{-2}x_0} \frac{1+(4s+6)bx_{-3}x_{-1}}{1+(4s+7)bx_{-4}x_{-2}},$$

where $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ are given by

$$x_1 = \frac{x_{-4}x_{-2}}{x_{-1}(1+bx_{-4}x_{-2})}, \quad x_2 = \frac{x_{-3}x_{-1}}{x_0(1+bx_{-3}x_{-1})}, \quad x_3 = \frac{x_{-1}x_0(1+bx_{-4}x_{-2})}{x_{-4}(1+bx_{-2}x_0)},$$

$$x_4 = \frac{x_{-4}x_{-2}x_0(1+bx_{-3}x_{-1})}{x_{-3}x_{-1}(1+2bx_{-4}x_{-2})}, \quad x_5 = \frac{x_{-4}x_{-3}(1+bx_{-2}x_0)}{x_0(1+bx_{-4}x_{-2})(1+2bx_{-3}x_{-1})},$$

$$x_6 = \frac{x_{-3}x_{-1}(1+2bx_{-4}x_{-2})}{x_{-4}(1+bx_{-3}x_{-1})(1+2bx_{-2}x_0)} \quad \text{and} \quad x_7 = \frac{x_{-2}x_0(1+bx_{-4}x_{-2})(1+2bx_{-3}x_{-1})}{x_{-3}(1+bx_{-2}x_0)(1+3bx_{-4}x_{-2})}.$$

3.2 The case $a = -1$

The solution, which appears for $b = \pm$ in Theorems 3 and 8 of [1], is given by

$$x_{12n-4} = x_{-4},\ x_{12n-3} = x_{-3},\ x_{12n-2} = x_{-2},\ x_{12n-1} = x_{-1},\ x_{12n} = x_0,$$

$$x_{12n+1} = \frac{x_{-4}x_{-2}}{x_{-1}(-1+bx_{-4}x_{-2})},\ x_{12n+2} = \frac{x_{-3}x_{-1}}{x_0(-1+bx_{-3}x_{-1})},\ x_{12n+3} = \frac{x_{-1}x_0(-1+bx_{-4}x_{-2})}{x_{-4}(-1+bx_{-2}x_0)},$$

$$x_{12n+4} = \frac{x_{-4}x_{-2}x_0(-1+bx_{-3}x_{-1})}{x_{-3}x_{-1}},\ x_{12n+5} = \frac{x_{-4}x_{-3}(-1+bx_{-2}x_0)}{x_0(-1+bx_{-4}x_{-2})},$$

$$x_{12n+6} = \frac{x_{-3}x_{-1}}{x_{-4}(-1+bx_{-3}x_{-1})},\ x_{12n+7} = \frac{x_{-2}x_0}{x_{-3}(-1+bx_{-2}x_0)}.$$

4 The case a_n and b_n are 3-periodic

The 3-periodicity of the sequences yields the following solution:

$$x_{12n-4} = x_{-4}\prod_{s=0}^{n-1}\frac{a_0^{4s}+b_0x_{-4}x_{-2}\sum_{l=0}^{4s-1}a_0^l}{a_2^{4s}+b_2x_{-2}x_0\sum_{l=0}^{4s-1}a_2^l}\,\frac{a_1^{4s+1}+b_1x_{-3}x_{-1}\sum_{l=0}^{4s}a_1^l}{a_0^{4s+2}+b_0x_{-4}x_{-2}\sum_{l=0}^{4s+1}a_0^l}\,\frac{a_2^{4s+2}+b_2x_{-2}x_0\sum_{l=0}^{4s+1}a_2^l}{a_1^{4s+3}+b_1x_{-3}x_{-1}\sum_{l=0}^{4s+2}a_1^l},$$

$$x_{12n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{a_1^{4s}+b_1x_{-3}x_{-1}\sum_{l=0}^{4s-1}a_1^l}{a_0^{4s+1}+b_0x_{-4}x_{-2}\sum_{l=0}^{4s}a_0^l}\,\frac{a_2^{4s+1}+b_2x_{-2}x_0\sum_{l=0}^{4s}a_2^l}{a_1^{4s+2}+b_1x_{-3}x_{-1}\sum_{l=0}^{4s+1}a_1^l}\,\frac{a_0^{4s+3}+b_0x_{-4}x_{-2}\sum_{l=0}^{4s+2}a_0^l}{a_2^{4s+3}+b_2x_{-2}x_0\sum_{l=0}^{4s+2}a_2^l},$$

$$x_{12n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{a_2^{4s}+b_2x_{-2}x_0\sum_{l=0}^{4s-1}a_2^l}{a_1^{4s+1}+b_1x_{-3}x_{-1}\sum_{l=0}^{4s}a_1^l}\,\frac{a_0^{4s+2}+b_0x_{-4}x_{-2}\sum_{l=0}^{4s+1}a_0^l}{a_2^{4s+2}+b_2x_{-2}x_0\sum_{l=0}^{4s+1}a_2^l}\,\frac{a_1^{4s+3}+b_1x_{-3}x_{-1}\sum_{l=0}^{4s+2}a_1^l}{a_0^{4s+4}+b_0x_{-4}x_{-2}\sum_{l=0}^{4s+3}a_0^l},$$

$$x_{12n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{a_0^{4s+1}+b_0x_{-4}x_{-2}\sum_{l=0}^{4s}a_0^l}{a_2^{4s+1}+b_2x_{-2}x_0\sum_{l=0}^{4s}a_2^l}\,\frac{a_1^{4s+2}+b_1x_{-3}x_{-1}\sum_{l=0}^{4s+1}a_1^l}{a_0^{4s+3}+b_0x_{-4}x_{-2}\sum_{l=0}^{4s+2}a_0^l}\,\frac{a_2^{4s+3}+b_2x_{-2}x_0\sum_{l=0}^{4s+2}a_2^l}{a_1^{4s+4}+b_1x_{-3}x_{-1}\sum_{l=0}^{4s+3}a_1^l},$$

$$x_{12n} = x_0\prod_{s=0}^{n-1}\frac{a_1^{4s+1}+b_1x_{-3}x_{-1}\sum_{l=0}^{4s}a_1^l}{a_0^{4s+2}+b_0x_{-4}x_{-2}\sum_{l=0}^{4s+1}a_0^l}\,\frac{a_2^{4s+2}+b_2x_{-2}x_0\sum_{l=0}^{4s+1}a_2^l}{a_1^{4s+3}+b_1x_{-3}x_{-1}\sum_{l=0}^{4s+2}a_1^l}\,\frac{a_0^{4s+4}+b_0x_{-4}x_{-2}\sum_{l=0}^{4s+3}a_0^l}{a_2^{4s+4}+b_2x_{-2}x_0\sum_{l=0}^{4s+3}a_2^l},$$

$$x_{12n+1} = x_1 \prod_{s=0}^{n-1} \frac{a_2^{4s+1} + b_2 x_{-2} x_0 \sum_{l=0}^{4s} a_2^l}{a_1^{4s+2} + b_1 x_{-3} x_{-1} \sum_{l=0}^{4s+1} a_1^l} \frac{a_0^{4s+3} + b_0 x_{-4} x_{-2} \sum_{l=0}^{4s+2} a_0^l}{a_2^{4s+3} + b_2 x_{-2} x_0 \sum_{l=0}^{4s+2} a_2^l} \frac{a_1^{4s+4} + b_1 x_{-3} x_{-1} \sum_{l=0}^{4s+3} a_1^l}{a_0^{4s+5} + b_0 x_{-4} x_{-2} \sum_{l=0}^{4s+4} a_0^l},$$

$$x_{12n+2} = x_2 \prod_{s=0}^{n-1} \frac{a_0^{4s+2} + b_0 x_{-4} x_{-2} \sum_{l=0}^{4s+1} a_0^l}{a_2^{4s+2} + b_2 x_{-2} x_0 \sum_{l=0}^{4s+1} a_2^l} \frac{a_1^{4s+3} + b_1 x_{-3} x_{-1} \sum_{l=0}^{4s+2} a_1^l}{a_0^{4s+4} + b_0 x_{-4} x_{-2} \sum_{l=0}^{4s+3} a_0^l} \frac{a_2^{4s+4} + b_2 x_{-2} x_0 \sum_{l=0}^{4s+3} a_2^l}{a_1^{4s+5} + b_1 x_{-3} x_{-1} \sum_{l=0}^{4s+4} a_1^l},$$

$$x_{12n+3} = x_3 \prod_{s=0}^{n-1} \frac{a_1^{4s+2} + b_1 x_{-3} x_{-1} \sum_{l=0}^{4s+1} a_1^l}{a_0^{4s+3} + b_0 x_{-4} x_{-2} \sum_{l=0}^{4s+2} a_0^l} \frac{a_2^{4s+3} + b_2 x_{-2} x_0 \sum_{l=0}^{4s+2} a_2^l}{a_1^{4s+4} + b_1 x_{-3} x_{-1} \sum_{l=0}^{4s+3} a_1^l} \frac{a_0^{4s+5} + b_0 x_{-4} x_{-2} \sum_{l=0}^{4s+4} a_0^l}{a_2^{4s+5} + b_2 x_{-2} x_0 \sum_{l=0}^{4s+4} a_2^l},$$

$$x_{12n+4} = x_4 \prod_{s=0}^{n-1} \frac{a_2^{4s+2} + b_2 x_{-2} x_0 \sum_{l=0}^{4s+1} a_2^l}{a_1^{4s+3} + b_1 x_{-3} x_{-1} \sum_{l=0}^{4s+2} a_1^l} \frac{a_0^{4s+4} + b_0 x_{-4} x_{-2} \sum_{l=0}^{4s+3} a_0^l}{a_2^{4s+4} + b_2 x_{-2} x_0 \sum_{l=0}^{4s+3} a_2^l} \frac{a_1^{4s+5} + b_1 x_{-3} x_{-1} \sum_{l=0}^{4s+4} a_1^l}{a_0^{4s+6} + b_0 x_{-4} x_{-2} \sum_{l=0}^{4s+5} a_0^l},$$

$$x_{12n+5} = x_5 \prod_{s=0}^{n-1} \frac{a_0^{4s+3} + b_0 x_{-4} x_{-2} \sum_{l=0}^{4s+2} a_0^l}{a_2^{4s+3} + b_2 x_{-2} x_0 \sum_{l=0}^{4s+2} a_2^l} \frac{a_1^{4s+4} + b_1 x_{-3} x_{-1} \sum_{l=0}^{4s+3} a_1^l}{a_0^{4s+5} + b_0 x_{-4} x_{-2} \sum_{l=0}^{4s+4} a_0^l} \frac{a_2^{4s+5} + b_2 x_{-2} x_0 \sum_{l=0}^{4s+4} a_2^l}{a_1^{4s+6} + b_1 x_{-3} x_{-1} \sum_{l=0}^{4s+5} a_1^l},$$

$$x_{12n+6} = x_6 \prod_{s=0}^{n-1} \frac{a_1^{4s+3} + b_1 x_{-3} x_{-1} \sum_{l=0}^{4s+2} a_1^l}{a_0^{4s+4} + b_0 x_{-4} x_{-2} \sum_{l=0}^{4s+3} a_0^l} \frac{a_2^{4s+4} + b_2 x_{-2} x_0 \sum_{l=0}^{4s+3} a_2^l}{a_1^{4s+5} + b_1 x_{-3} x_{-1} \sum_{l=0}^{4s+4} a_1^l} \frac{a_0^{4s=6} + b_0 x_{-4} x_{-2} \sum_{l=0}^{4s+5} a_0^l}{a_2^{4s+6} + b_0 x_{-2} x_0 \sum_{l=0}^{4s+5} a_0^l},$$

$$x_{12n+7} = x_7 \prod_{s=0}^{n-1} \frac{a_2^{4s+3} + b_2 x_{-2} x_0 \sum_{l=0}^{4s+2} a_2^l}{a_1^{4s+4} + b_1 x_{-3} x_{-1} \sum_{l=0}^{4s+3} a_1^l} \frac{a_0^{4s+5} + b_0 x_{-4} x_{-2} \sum_{l=0}^{4s+4} a_0^l}{a_2^{4s+5} + b_2 x_{-2} x_0 \sum_{l=0}^{4s+4} a_2^l} \frac{a_1^{4s+6} + b_1 x_{-3} x_{-1} \sum_{l=0}^{4s+5} a_1^l}{a_0^{4s+7} + b_0 x_{-4} x_{-2} \sum_{l=0}^{4s+6} a_0^l}$$

where $x_1, x_2, x_3, x_4, x_5, x_6$ and x_7 are given as follows:

16

$$x_1 = \frac{x_{-4}x_{-2}}{x_{-1}(a_0 + b_0 x_{-4}x_{-2})}, \quad x_2 = \frac{x_{-3}x_{-1}}{x_0(a_1 + b_1 x_{-3}x_{-1})}, \quad x_3 = \frac{x_{-1}x_0(a_0 + b_0 x_{-4}x_{-2})}{x_{-4}(a_2 + b_2 x_{-2}x_0)},$$

$$x_4 = \frac{x_{-4}x_{-2}x_0(a_1 + b_1 x_{-3}x_{-1})}{x_{-3}x_{-1}(a_0^2 + (b_0 a_0 + b_0)x_{-4}x_{-2})}, \quad x_5 = \frac{x_{-3}x_{-4}(a_2 + b_2 x_{-2}x_0)}{x_0(a_0 + b_0 x_{-4}x_{-2})(a_1^2 + (b_1 a_1 + b_1)x_{-3}x_{-1})},$$

$$x_6 = \frac{x_{-3}x_{-1}(a_0^2 + (b_0 a_0 + b_0)x_{-4}x_{-2})}{x_{-4}(a_1 + b_1 x_{-3}x_{-1})(a_2^2 + (b_2 a_2 + b_2)x_{-2}x_0)},$$

and

$$x_7 = \frac{x_{-2}x_0(a_0 + b_0 x_{-4}x_{-2})(a_1^2 + (b_1 a_1 + b_1)x_{-3}x_{-1})}{x_{-3}(a_2 + b_2 x_{-2}x_0)(a_0^3 + (a_0^2 b_0 + a_0 b_0 + b_0)x_{-4}x_{-2})}.$$

5 Conclusion

In this paper, we derived symmetry generators for the difference equations (2) and explicit formulas for the solutions of the equations were also obtained. Our solution generalised Theorems 1, 3, 6 and 8 of Elsayed [1].

References

[1] E. M. Elsayed, Behavior and expression of the solutions of some rational difference equations, J. Computational Analysis and Applications, 15(1) (2013), 73–81.

[2] M. Folly-Gbetoula, Symmetry, reductions and exact solutions of the difference equation $u_{n+2} = (au_n)/(1 + bu_n u_{n+1})$, J. of Diff. Equations Appl., 23(6) (2017).

[3] P. E. Hydon, Difference Equations by Differential Equation Methods, Cambridge University Press (2014).

[4] T. F. Ibrahim and M. A. El-Moneam, Global stability of a higher-order difference equation, Iran J. Sci. Technol. Trans. Sci., 41(1) (2017), 51–58.

[5] N. Joshi and P. Vassiliou, The existence of Lie Symmetries for First-Order Analytic Discrete Dynamical Systems, J. of Math. Anal. Appl., 195 (1995), 872-887 (1995).

[6] D. Levi, L. Vinet and P. Winternitz, Lie group formalism for difference equations, J. Phys. A: Math. Gen. , 30 (1997), 633-649.

[7] S. Lie, Classification und Integration von gewohnlichen Differentialgleichungen zwischen xy, die eine Gruppe von Transformationen gestatten I , Math. Ann., 22 (1888), 213–253.

[8] S. Maeda, Canonical structure and symmetries for discrete systems, Math. Japonica, 25 (1980) 405–420.

[9] S. Maeda, The similarity method for difference equations, IMA J. Appl. Math., 38 (1987), 129–134.

[10] D. Nyirenda and M. Folly-Gbetoula, Invariance analysis and exact solutions of some sixth-order difference equations, J. Nonlinear Sci. Appl., **10** (2017), 6262-6273.

[11] P. J. Olver, Applications of Lie Groups to Differential Equations, Second Edition, Springer, New York (1993).

[12] G. R. W. Quispel and R. Sahadevan, Lie symmetries and the integration of difference equations, Physics Letters A, 184 (1993), 64-70.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

On some conditions for p-valency

Mamoru Nunokawa[1], **Janusz Sokół**[2]

[1] University of Gunma, Hoshikuki-cho 798-8, Chuou-Ward, Chiba, 260-0808, Japan
E-Mail:mamoru_nuno@doctor.nifty.jp
[2] University of Rzeszów, Faculty of Mathematics and Natural Sciences, ul. Prof. Pigonia 1, 35-310 Rzeszów, Poland **E-Mail:jsokol@ur.edu.pl**

Abstract

In this paper we consider analytic functions in the unit disc $\mathbb{D}$ such that $|f^{(p)}(z)|$ is bounded in $\mathbb{D}$. We present several sufficient conditions for function to be p-valent starlike, convex or strongly starlike of a certain order.

Key Words and Phrases. univalent functions; starlike; convex; close-to-convex
2010 Mathematics Subject Classification. Primary 30C45, Secondary 30C80

1. Introduction

A function f analytic in a domain $D \in \mathbb{C}$ is called p-valent in D, if for every complex number w, the equation $f(z) = w$ has at most p roots in D, so that there exists a complex number w_0 such that the equation $f(z) = w_0$ has exactly p roots in D. We denote by $\mathcal{H}$ the class of functions $f(z)$ which are holomorphic in the open unit unit $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{A}(p)$, $p \in \mathbb{N} = \{1, 2, \ldots\}$, the class of functions $f(z) \in \mathcal{H}$ given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (z \in \mathbb{D}).$$

Let $\mathcal{A} = \mathcal{A}(1)$. Let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent. Also let $\mathcal{S}_p^*(\alpha)$ and $\mathcal{C}_p(\alpha)$ be the subclasses of $\mathcal{A}(p)$ consisting of all p-valent functions which are starlike and convex of order α, $0 \le \alpha < 1$, defined as

$$\begin{aligned} \mathcal{S}_p^*(\alpha) &= \left\{ f(z) \in \mathcal{A}(p) : \mathfrak{Re}\left\{ \frac{zf'(z)}{pf(z)} \right\} > \alpha,\ z \in \mathbb{D} \right\}, \\ \mathcal{C}_p(\alpha) &= \left\{ f(z) \in \mathcal{A}(p) : zf'(z)/p \in \mathcal{S}_p^*(\alpha) \right\}. \end{aligned}$$

Note that $\mathcal{S}_1^*(0) = \mathcal{S}^*$ and $\mathcal{C}_1(0) = \mathcal{C}$, where $\mathcal{S}^*$ and $\mathcal{C}$ are usual classes of starlike and convex functions respectively.

The well-known Noshiro-Warschawski theorem [1, 10], says that if $f \in \mathcal{H}$ satisfies

$$\mathfrak{Re}\left\{e^{i\alpha} f'(z)\right\} > 0, \quad (z \in \mathbb{D}) \tag{1.1}$$

for some real α, then $f(z)$ is univalent in $\mathbb{D}$. Ozaki [5], generalized the above theorem for $f \in \mathcal{A}(p)$: if

$$\mathfrak{Re}\left\{e^{i\alpha} f^{(p)}(z)\right\} > 0, \quad (z \in \mathbb{D}) \tag{1.2}$$

for some real α, then $f(z)$ is at most p-valent in $\mathbb{D}$. Also in [3, 454] it was shown that if $f \in \mathcal{A}(p)$, $p \ge 2$, and

$$|\arg\{f^{(p)}(z)\}| < \frac{3\pi}{4} \quad (z \in \mathbb{D}), \tag{1.3}$$

then f is at most p-valent in $\mathbb{D}$.

The above results (1.1), (1.2) and (1.3) describe some consequences of a certain conditions on $\mathfrak{Re}\{f^{(p)}(z)\}$, or $|\arg\{f^{(p)}(z)\}|$. It is the purpose of this paper is to consider analytic functions with bounded modulus of a certain order of derivative, like $|f''(z)|$, and to present some implications of this hypothesis.

2. Main results

A function $f(z) \in \mathcal{H}$ is said to subordinate a function $g \in \mathcal{H}$ in the unit disc E, written $f \prec g$ if and only if there exits an analytic function $w \in \mathcal{H}$ such that $w(0) = 0, |w(z)| < 1$ and $f(z) = g[w(z)]$ for $z \in E$. Therefore $f \prec g$ in E implies $f(E) \subset g(E)$. In particular if g is univalent in E then $f \prec g$ if and only if $f(0) = g(0)$ and $f(E) \subset g(E)$. The idea of subordination was used for defining many of the classes of functions studied in geometric function theory. In [9] Tuneski proved the following theorem.

Theorem 2.1. *If $f(z) \in \mathcal{A}$, $0 < k \leq 1$*

$$|f''(z)| \leq k, \quad (z \in \mathbb{D}),$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{kz}{2-k}, \quad (z \in \mathbb{D}).$$

In [6] it was proved a weaker result

$$|f''(z)| \leq 1, \quad (z \in \mathbb{D})$$

implies that $f(z)$ is univalent in $\mathbb{D}$. Applying Theorem 2.1, Tuneski in [9] obtained the following corollaries.

Corollary 2.2. *If $f(z) \in \mathcal{A}$, $0 \leq \alpha < 1$ and*

$$|f''(z)| \leq \frac{2(1-\alpha)}{2-\alpha}, \quad (z \in \mathbb{D}),$$

then

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad (z \in \mathbb{D}).$$

The result is sharp.

Corollary 2.3. *If $f(z) \in \mathcal{A}$, $0 < \alpha \leq 1$ and*

$$|f''(z)| \leq \frac{2\sin(\alpha\pi/2)}{1+\sin(\alpha\pi/2)}, \quad (z \in \mathbb{D}),$$

then

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}).$$

The result is sharp.

In [9] Tuneski proved also the following result.

Theorem 2.4. *If $f(z) \in \mathcal{A}$, $0 < k \leq 1$*

$$|f''(z)| \leq k, \quad (z \in \mathbb{D}),$$

then

$$f'(z) \prec 1 + kz, \quad (z \in \mathbb{D}).$$

Theorem 2.4 implies the following corollary.

Corollary 2.5. *If $f(z) \in \mathcal{A}$, $0 \leq \alpha < 1$ and*

$$|f''(z)| \leq \frac{1-\alpha}{2-\alpha}, \quad (z \in \mathbb{D}),$$

then

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (z \in \mathbb{D}).$$

The result is sharp.

We also need the following result.

Theorem 2.6. [9] *If $f(z) \in \mathcal{A}$, $0 < \lambda \le 1$*

$$|f'(z) - 1| \le \lambda, \quad (z \in \mathbb{D}),$$

then

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}),$$

where

$$\alpha = \frac{2}{\pi}\sin^{-1}\left(\lambda\sqrt{1-(\lambda^2/4)} + \frac{\lambda}{2}\sqrt{1-\lambda^2}\right).$$

In [2] it was proved the following result.

Theorem 2.7. [2] *Let $f(z) \in \mathcal{A}(p)$. Suppose that there exists a positive integer j, $1 \le j \le p$, such that*

$$j + \mathfrak{Re}\left\{\frac{zf^{(j+1)}(z)}{f^{(j)}(z)}\right\} > 0, \quad (z \in \mathbb{D}).$$

Then we have

$$j - 1 + \mathfrak{Re}\left\{\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right\} > 0, \quad (z \in \mathbb{D}).$$

3. Main results

Now we are going to make use of Theorem 2.1, Corollary 2.2 and of Theorem 2.7 to obtain the following theorem.

Theorem 3.1. [2] *Let $f(z) \in \mathcal{A}(p)$. Suppose that*

$$|f^{(p+1)}(z)| < p!, \quad (z \in \mathbb{D}).$$

Then $f(z)$ is p-valently convex and p-valently starlike in $\mathbb{D}$.

Proof. If we put

$$g(z) = \frac{1}{p!}f^{(p-1)}(z), \quad g(0) = g'(0) - 1 = 0, \quad (z \in \mathbb{D}),$$

then it follows that

$$|g''(z)| < \frac{|f^{(p+1)}(z)|}{p!} < 1, \quad (z \in \mathbb{D}).$$

From Theorem 2.1 and Corollary 2.2, we have

$$\mathfrak{Re}\left\{\frac{zg'(z)}{g(z)}\right\} = \mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0, \quad (z \in \mathbb{D})$$

and so, we have

$$p - 1 + \mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > p - 1 \ge 0, \quad (z \in \mathbb{D}).$$

From Theorem 2.7, it follows that

$$1 + \mathfrak{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} > 0 \quad \text{and} \quad \mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in \mathbb{D}).$$

This shows that $f(z)$ is p-valently convex and p-valently starlike in $\mathbb{D}$. □

For real α, $0 \le \alpha < 1$, if $f(z) \in \mathcal{A}(p)$ satisfies

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}),$$

then $f(z)$ is called a strongly starlike function of order α. Applying Corollary 2.3 and the method of proving from [2, Th.5] give us the following theorems.

Theorem 3.2. *If $f(z) \in \mathcal{A}(p)$ and if there exists a α, $0 < \alpha \leq 1$, such that*

$$|f^{(p+1)}(z)| \leq \frac{2\sin(\alpha\pi/2)}{1+\sin(\alpha\pi/2)}, \quad (z \in \mathbb{D}), \tag{3.1}$$

then

$$\left|\arg\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}),$$

or $f^{(p-1)}(z)/p!$ is strongly starlike of order α in $\mathbb{D}$.

Proof. For the case $p = 1$ Theorem 3.2 becomes Tuneski's result 2.3. Suppose that $p \geq 2$.

If we put

$$g(z) = \frac{1}{p!}f^{(p-1)}(z), \quad g(0) = g'(0) - 1 = 0, \quad (z \in \mathbb{D}),$$

then it follows that

$$\frac{zg'(z)}{g(z)} = \frac{zf^{(p)}(z)}{f^{(p-1)}(z)}, \quad (z \in \mathbb{D}).$$

From Corollary 2.3, we have

$$\left|\arg\left\{\frac{zg'(z)}{g(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}),$$

and so, we have

$$\left|\arg\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}),$$

This shows that $f^{(p-1)}(z)/p!$ is strongly starlike of order α in $\mathbb{D}$. $\square$

Again, applying [2, Th.5] yields us that if $f(z) \in \mathcal{A}(p)$, then for all $z \in \mathbb{D}$, we have

$$\mathfrak{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0 \quad \Rightarrow \quad \forall k \in \{1, \ldots, p\}: \quad \mathfrak{Re}\left\{\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}\right\} > 0. \tag{3.2}$$

Therefore, if we put

$$\frac{2f^{(p-2)}(z)}{p!} := G(z) = z^2 + \cdots \in \mathcal{A}(2),$$

then

$$\frac{zG'(z)}{G(z)} = \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}, \quad (z \in \mathbb{D})$$

and so (3.1) also implies that

$$\mathfrak{Re}\left\{\frac{zG'(z)}{G(z)}\right\} = \mathfrak{Re}\left\{\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}\right\} > 0, \quad (z \in \mathbb{D}).$$

This shows that $G(z)$ or $2f^{(p-2)}(z)/p!$ is 2-valently starlike in $\mathbb{D}$.

Theorem 3.3. *If $f(z) \in \mathcal{A}(p)$, $0 < \alpha \leq 1$, $1 \leq p$ and*

$$|f^{(p+1)}(z)| \leq \frac{1}{2}, \quad (z \in \mathbb{D}),$$

then

$$k + \mathfrak{Re}\left\{\frac{zf^{(k+1)}(z)}{f^{(k)}(z)}\right\} > 0, \quad (z \in \mathbb{D})$$

for all k, $k \in \{1, 2, \ldots, p-1\}$.

Proof. If we put

$$g(z) = \frac{1}{p!} f^{(p-1)}(z), \quad g(0) = g'(0) - 1 = 0, \quad (z \in \mathbb{D}),$$

then it follows that

$$\frac{zg''(z)}{g'(z)} = \frac{zf^{(p+1)}(z)}{f^{(p)}(z)}, \quad (z \in \mathbb{D}).$$

From Corollary 2.5, we have

$$1 + \mathfrak{Re}\left\{\frac{zg''(z)}{g'(z)}\right\} = 1 + \mathfrak{Re}\left\{\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\} > 0, \quad (z \in \mathbb{D})$$

and so, we have

$$p + \mathfrak{Re}\left\{\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right\} > 0, \quad (z \in \mathbb{D}).$$

Applying Theorem 2.7 gives finally

$$k + \mathfrak{Re}\left\{\frac{zf^{(k+1)}(z)}{f^{(k)}(z)}\right\} > 0, \quad (z \in \mathbb{D})$$

for all k, $k \in \{1, 2, \ldots, p-1\}$. It completes the proof. □

From Theorem 3.3, we have

$$|f^{(p+1)}(z)| \leq \frac{1}{2}, \quad (z \in \mathbb{D}) \quad \Rightarrow \quad \mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad (z \in \mathbb{D}),$$

this suggests the following question.

Open problem. What is the best value of $\alpha(p)$ such that

$$|f^{(p+1)}(z)| \leq \frac{1}{2}, \quad (z \in \mathbb{D}) \quad \Rightarrow \quad \mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha(p), \quad (z \in \mathbb{D}).$$

If $p = 1$, then the function $f(z) = z + z^2/4$ shows that the best value of $\alpha(p)$ is 0.

Theorem 3.4. *If $f(z) \in \mathcal{A}(p)$, $0 < \lambda \leq 1$ and if*

$$|f^{(p)}(z) - p!| < p!\lambda, \quad (z \in \mathbb{D}), \tag{3.3}$$

then

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}), \tag{3.4}$$

where

$$\alpha = \frac{2}{\pi}\sin^{-1}\left(\lambda\sqrt{1-(\lambda^2/4)} + \frac{\lambda}{2}\sqrt{1-\lambda^2}\right). \tag{3.5}$$

This means that $f(z)$ is strongly starlike of order α in $\mathbb{D}$.

Proof. If we put

$$g(z) = \frac{1}{p!} f^{(p-1)}(z), \quad g(0) = g'(0) - 1 = 0, \quad (z \in \mathbb{D}),$$

then from (3.3), we have

$$|g'(z) - 1| = \left|\frac{f^{(p)}(z)}{p!} - 1\right| < \lambda, \quad (z \in \mathbb{D}).$$

From Theorem 2.6, we have

$$\left|\arg\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}),$$

where α has the form (3.5). Let us put

$$p(z)=\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)},\quad (z\in\mathbb{D}).$$

Then it follows that

$$\frac{zp'(z)}{p(z)}=1+\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}-\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}$$

or

$$1+\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}=p(z)+\frac{zp'(z)}{p(z)}.$$

From Theorem 2.6, we have

$$\left|\arg\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}\right|<\frac{\alpha\pi}{2},\quad (z\in\mathbb{D}),$$

this gives

(3.6) $$\left|\arg\left\{1+\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}\right|<\left|\arg\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\}\right|<\frac{\alpha\pi}{2},\quad (z\in\mathbb{D}).$$

If there exists a point $z_0\in\mathbb{D}$, such that

$$|\arg\{p(z)\}|<\frac{\alpha\pi}{2},\quad (|z|<|z_0|)$$

and

$$|\arg\{p(z_0)\}|=\frac{\alpha\pi}{2},$$

then from [4], we have

$$\frac{z_0p'(z_0)}{p(z_0)}=i\alpha k,$$

where k is a real number such that

$$k\geq\frac{1}{2}\left(a+\frac{1}{a}\right)$$

when $p(z_0)=ia$, while for $p(z_0)=-ia$, such that

$$k\leq-\frac{1}{2}\left(a+\frac{1}{a}\right),$$

where $p^{1/\alpha}(z_0)=\pm ia$, $a>0$. For the case $p^{1/\alpha}(z_0)=ia$, we have

$$\begin{aligned}1+\frac{z_0f''(z_0)}{f'(z_0)} &= p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\\ &= p(z_0)\left\{1+\frac{z_0p'(z_0)}{p^2(z_0)}\right\}\\ &= (ia)^{\alpha}\left\{1+i\alpha k\frac{1}{(ia)^{\alpha}}\right\}\\ &= a^{\alpha}e^{i\alpha\pi/2}\left\{1+e^{i\pi(1-\alpha)/2}\alpha k\frac{1}{a^{\alpha}}\right\}.\end{aligned}$$

Thus, it is trivial that

$$\arg\left\{1+\frac{z_0f^{(p)}(z_0)}{f^{(p-1)}(z_0)}\right\}\geq\frac{\alpha\pi}{2}$$

since we have

$$\arg\left\{1+e^{i\pi(1-\alpha)/2}\alpha k\frac{1}{a^{\alpha}}\right\}>0,$$

where
$$k \geq \frac{1}{2}\left(a+\frac{1}{a}\right).$$
This contradicts (3.6) and for the case $p^{1/\alpha}(z_0) = -ia$, applying the same method as the above, we would have
$$\arg\left\{1+\frac{z_0 f^{(p)}(z_0)}{f^{(p-1)}(z_0)}\right\} \leq -\frac{\alpha\pi}{2}$$
which also contradicts (3.6). Applying the same method repeatedly once again, we can complete the proof of Theorem 3.4. □

We now note that Pommerenke [7] and Sakaguchi [8] showed the following.

Lemma 3.5. [7] *If f and h are analytic in $\mathbb{D}$, and h is convex and univalent in $\mathbb{D}$, with*
$$\left|\arg\left\{\frac{f'(z)}{h'(z)}\right\}\right| \leq \frac{\alpha\pi}{2}, \quad (z\in\mathbb{D}),$$
for some real α, $0\leq\alpha\leq 1$, then
$$\left|\arg\left\{\frac{f(z_2)-f(z_1)}{h(z_2)-h(z_1)}\right\}\right| \leq \frac{\alpha\pi}{2}, \quad (z\in\mathbb{D}),$$
for all $z_1, z_2\in\mathbb{D}$.

Putting $z_1 = 0$, $z_2 = z$ in Lemma 3.5 gives

(3.7)
$$\left|\arg\left\{\frac{f'(z)}{h'(z)}\right\}\right| \leq \frac{\alpha\pi}{2}, \quad z\in\mathbb{D} \quad \Rightarrow \quad \left|\arg\left\{\frac{f(z)}{h(z)}\right\}\right| \leq \frac{\alpha\pi}{2}, \quad (z\in\mathbb{D}).$$

Therefore, applying Theorem 3.4 and (3.7) we can deduce the following corollary.

Corollary 3.6. *If $f(z)\in\mathcal{A}(p)$ is such that*
$$\int_0^z \frac{f(t)}{t}\mathrm{d}t$$
is a convex function, and if

(3.8)
$$|f^{(p)}(z)-p!| < p!\lambda, \quad (z\in\mathbb{D}),$$
for some λ, $0<\lambda\leq 1$, then

(3.9)
$$\left|\arg\left\{\frac{f(z)}{\int_0^z \frac{f(t)}{t}\mathrm{d}t}\right\}\right| < \frac{\alpha\pi}{2}, \quad (z\in\mathbb{D}),$$
where α is given in (3.5).

References

[1] K. Noshiro, On the theory of schlicht functions, J. Fac. Sci. Hokkaido Univ. Jap., 2(1)(1934-35) 129–135.
[2] M. Nunokawa, On the theory of multivalent functions, Tsukuba J. Math. 11(2)(1987) 273–286. 35–43.
[3] M. Nunokawa, A note on multivalent functions, Tsukuba J. Math. 13(2)(1989) 453–455. 35–43.
[4] M. Nunokawa, On the order of strongly starlikeness of strongly convex functions, Proc. Japan Acad. Ser. A 69(7)(1993) 234–237.
[5] S. Ozaki, On the theory of multivalent functions , Sci. Rep. Tokyo Bunrika Daigaku Sect. A 2(1935) 167–188.
[6] S. Ozaki, I. Ono, T. Umezawa, On a General Second Order Derivative, Science Reports of the Tokyo Kyoiku Daigaku, 5(124)(1956) 111–114.
[7] Ch. Pommerenke, On close to-convex functions, Trans. Amer. Math. Soc. 114(1)(1965) 176–186.
[8] K. Sakaguchi, On certain univalent mapping, J. Math. Soc. Japan, 11(1959) 72-75.
[9] N. Tuneski, On some simple sufficient conditions for univalence. Math. Bohemica, 126(1)(2001) 229–236.
[10] S. Warschawski, On the higher derivatives at the boundary in conformal mapping, Trans. Amer. Math. Soc. 38(1935) 310–340.

Fractional Cauchy Euler Differential Equation

By

Al-Horani, M. Khalil, R. and Aldarawi, I.

Department of Mathematics, The University of Jordan, Amman, Jordan

e-mail: horani@ju.edu.jo, roshdi@ju.edu.jo, i.aldarawi@ju.edu.jo

Abstract

In this paper we give general solution of fractional linear differential equations and fractional Cauchy Euler equation. Since there are many definitions for fractional derivatives, we use the conformable derivative to get exact solutions. Factorizing polynomials of the fractional differential operators is the key method to get such solutions. Some specific examples on both types of equations are presented.

Key Words and Phrases:Conformable, Cauchy Euler, Conformable Linear Differential equations, Conformable Cauchy Euler Equation.

AMS Classification Number : 26A33

1. Introduction

Many authors have solved many well known differential equations like the Conformable Fractional Heat equation, Bessel equation, Legendre equation and many more. [1], [4], [5], [6], [7], [9] and [10]. The Cauchy Euler equation is a well known important type of ordinary differential equation. In This paper we give the procedure and justification of how to handle the Cauchy Euler equation, but the fractional one.

However, there are many definitions available in the literature for fractional derivatives. The main ones are the Riemann Liouville definition and the Caputo definition, see [8] .

(i) Riemann - Liouville Definition. For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

(ii) Caputo Definition. For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

Such definitions have many setbacks such as

(i) The Riemann-Liouville derivative does not satisfy $D_a^{\alpha}(1) = 0$ ($D_a^{\alpha}(1) = 0$ for the Caputo derivative), if α is not a natural number.

(ii) All fractional derivatives do not satisfy the known formula of the derivative of the product of two functions:

$$D_a^{\alpha}(fg) = fD_a^{\alpha}(g) + gD_a^{\alpha}(f).$$

(iii) All fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions:

$$D_a^{\alpha}(f/g) = \frac{gD_a^{\alpha}(f) - fD_a^{\alpha}(g)}{g^2}.$$

(iv) All fractional derivatives do not satisfy the chain rule:

$$D_a^{\alpha}(f \circ g)(t) = f^{(\alpha)}(g(t))\, g^{(\alpha)}(t).$$

(v) All fractional derivatives do not satisfy: $D^{\alpha}D^{\beta}f = D^{\alpha+\beta}f$, in general.

(vi) All fractional derivatives, specially Caputo definition, assumes that the function f is differentiable.

We refer the reader to [3] for more results on Caputo and Riemann - Liouville Definitions.

Recently, the authors in [2], gave a new definition of fractional derivative which is a natural extension to the usual first derivative. So many papers since then were written, and many equations were solved using such definition. The definition goes as follows:

Given a function $f : [0, \infty) \longrightarrow \mathbb{R}$. Then for all $t > 0$, $\alpha \in (0, 1)$, let

$$T_{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

T_{α} is called the **conformable fractional derivative of f of order α.**
Let $f^{(\alpha)}(t)$ stands for $T_{\alpha}(f)(t)$.

If f is $\alpha-$differentiable in some $(0, b)$, $b > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, **then define**

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

According to this definition, we have the following properties, [2],

1. $T_\alpha(1) = 0$,
2. $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$,
3. $T_\alpha(\sin at) = at^{1-\alpha} \cos at, \quad a \in \mathbb{R}$,
4. $T_\alpha(\cos at) = -at^{1-\alpha} \sin at, \quad a \in \mathbb{R}$
5. $T_\alpha(e^{at}) = at^{1-\alpha} e^{at}, \quad a \in \mathbb{R}$.

Further, many functions behave as in the usual derivative. Here are some formulas

$$
\begin{aligned}
&T_\alpha(\frac{1}{\alpha}t^\alpha) = 1 \\
&T_\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}, \\
&T_\alpha(\sin \frac{1}{\alpha}t^\alpha) = \cos(\frac{1}{\alpha}t^\alpha), \\
&T_\alpha(\cos \frac{1}{\alpha}t^\alpha) = -\sin(\frac{1}{\alpha}t^\alpha).
\end{aligned}
$$

We will use the conformable fractional derivative for the Cauchy Euler equation. But first, we present the linear fractional case with constant coefficients.

2. Conformable Linear Differential equations

Let us write $y^{(n\alpha)}$ to denote the $\alpha-$derivative of y, $n-$times. That is $y^{(n\alpha)} = T_\alpha T_\alpha ... T_\alpha(y)$, n-times.

Theorem 1. *Let*

$$y^{(n\alpha)} + a_{n-1}y^{(n-1)\alpha} + ... + a_1 y^\alpha + a_0 y = 0 \tag{1}$$

Consider the equation

$$r^{(n\alpha)} + a_{n-1}r^{(n-1)\alpha} + ... + a_1 r^\alpha + a_0 = 0 \tag{$*$}$$

If $r_1^\alpha = \lambda_1, ..., r_n^\alpha = \lambda_n$ *are the real roots of* $(*)$ *then* $y_h = c_1 y_1 + ... + c_n y_n$ *where* $y_k = e^{r_k^\alpha e^{\frac{t^\alpha}{\alpha}}}$.

Proof. Let $T^{n\alpha} = T^\alpha T^\alpha ... T^\alpha$ n-times. Then equation (1) can be written in the form

$$(T^{(n\alpha)} + a_{n-1}T^{(n-1)\alpha} + ... + a_1 r^\alpha + a_0 I)y = 0 \tag{2}$$

(where $T^\alpha = \frac{d^\alpha}{dx^\alpha}$).

Now, if we let $D = T^\alpha$ then (2) becomes

$$(D^n + a_{n-1}D^{n-1} + ... + a_0 I)y = 0$$

The polynomial $(D^n + a_{n-1}D^{n-1} + ... + a_0 I)y = 0$, factorizes to

$$(D - \lambda_1)(D - \lambda_2)...(D - \lambda_n)y = 0 \tag{3}$$

Now, y will be a solution to (3) if $y \in \ker(D-\lambda_k) \forall 1 \le k \le n$, noting that $(D-\lambda_i)$ commutes with $(D-\lambda_j)$ for all i and j . Thus y is the solution for (3) if

$$(D-\lambda_1)y = 0 \text{ or } (D-\lambda_2)y = 0 \text{ or ...or } (D-\lambda_n)y = 0$$

However $(D-\lambda_k)y = 0$ implies

$Dy - \lambda_k y = 0$

So $y^\alpha - \lambda_k y = 0$

Hence $y_k = e^{\lambda_k e^{\frac{t^\alpha}{\alpha}}}$ if λ_k is real.

Consequently, $y_h = c_1 e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}} + ... + c_n e^{\lambda_n e^{\frac{t^\alpha}{\alpha}}}$, if all the roots are real and distinct.

Now, replacing T^α by r^α we get

$$(r^\alpha - \lambda_1)y = 0 \text{ or } (r^\alpha - \lambda_2)y = 0 \text{ or ...or } (r^\alpha - \lambda_n)y = 0$$

Thus the roots are

$$r_1^\alpha = \lambda_1 \ , \ r_2^\alpha = \lambda_2 \ , ..., r_n^\alpha = \lambda_n$$

and the general solution is

$$y_h = c_1 e^{r_1^\alpha e^{\frac{t^\alpha}{\alpha}}} + ... + c_n e^{r_n^\alpha e^{\frac{t^\alpha}{\alpha}}}$$

There are two other cases for the roots to be considered:

(1) (i)If one of the root is repeated, say λ_1, 2-times. That is $(T^\alpha - \lambda_1)^2$ is a factor of (3). Then $y_1 = e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}}, y_2 = \frac{t^\alpha}{\alpha} e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}}$, are two independent solutions for the differential equation (3).

Proof. We have to show

$$(T^\alpha - \lambda_1)^2 \ \frac{t^\alpha}{\alpha} e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}} = 0$$

Indeed:

$$(T^\alpha - \lambda_1) \ (T^\alpha - \lambda_1) \frac{t^\alpha}{\alpha} e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}} = 0$$

$$= (T^\alpha - \lambda_1) \ \left[T^\alpha (\frac{t^\alpha}{\alpha} e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}}) - \lambda_1 (\frac{t^\alpha}{\alpha} e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}}) = 0 \right]$$

$$= (T^\alpha - \lambda_1) \ \left[e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}} + \lambda_1 (\frac{t^\alpha}{\alpha} e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}}) - \lambda_1 (\frac{t^\alpha}{\alpha} e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}}) = 0 \right]$$

$$= (T^\alpha - \lambda_1) \ e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}} = 0$$

Similarly one can show that if λ_1 is repeated k-times then

$$y_1 = e^{\lambda_1 e^{\frac{t^\alpha}{\alpha}}}, \frac{t^\alpha}{\alpha} y_1, ..., (\frac{t^\alpha}{\alpha})^{k-1} y_1$$

are independent solutions.

(ii)There is a root, say $\lambda_1 = a + ib$, $a, b \in \mathbb{R}$. Then

$$y_1 = e^{a\frac{t^\alpha}{\alpha}} \cos b \frac{t^\alpha}{\alpha} \text{ and } \quad y_2 = e^{a\frac{t^\alpha}{\alpha}} \sin b \frac{t^\alpha}{\alpha}$$

are two solutions of (3) associated with λ_1.

Indeed:

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Since $\lambda_1 = a + ib$ is a root, then $\overline{\lambda_1} = a - ib$ is a root.
Then

$y_1 = e^{(a+ib)\frac{t^\alpha}{\alpha}}$ and $y_2 = e^{(a-ib)\frac{t^\alpha}{\alpha}}$ are solutions of (3)

But

$$y_1 = e^{a\frac{t^\alpha}{\alpha}}(\cos b\frac{t^\alpha}{\alpha} + i\sin b\frac{t^\alpha}{\alpha})$$

$$y_2 = e^{a\frac{t^\alpha}{\alpha}}(\cos b\frac{t^\alpha}{\alpha} - i\sin b\frac{t^\alpha}{\alpha})$$

Place $y_1 + y_2$ is a solution (the equation being homogenous) and $y_1 - y_2$ is a solution too. So

$$\widetilde{y_1} = y_1 + y_2 = 2e^{a\frac{t^\alpha}{\alpha}}\cos b\frac{t^\alpha}{\alpha} \text{ and } \widetilde{y_2} = y_1 - y_2 = 2ie^{a\frac{t^\alpha}{\alpha}}\sin b\frac{t^\alpha}{\alpha}$$

are solutions of the homogenous equation (3).

Consequentely

$$\widetilde{\widetilde{y_1}} = \frac{1}{2}\widetilde{y_1} = e^{a\frac{t^\alpha}{\alpha}}\cos b\frac{t^\alpha}{\alpha} \text{ and } \widetilde{\widetilde{y_2}} = \frac{1}{2i}\widetilde{y_2} = e^{a\frac{t^\alpha}{\alpha}}\sin b\frac{t^\alpha}{\alpha}$$

are two independent solutions for the equation.

Example 1

$$T^{2\alpha}y + T^\alpha y - 2y = 0 \qquad (i)$$

Solution. Consider the associated equation

$$r^{2\alpha} + r^\alpha - 2 = 0$$

$$(r^\alpha - 2)(r^\alpha + 1) = 0$$

Hence $\lambda_1 = 2, \lambda_2 = -1$

Thus $y_1 = c_1 e^{2\frac{t^\alpha}{\alpha}}$ *and* $y_2 = c_2 e^{-\frac{t^\alpha}{\alpha}}$

One can easily check that these are solutions of (i).See figure (1)

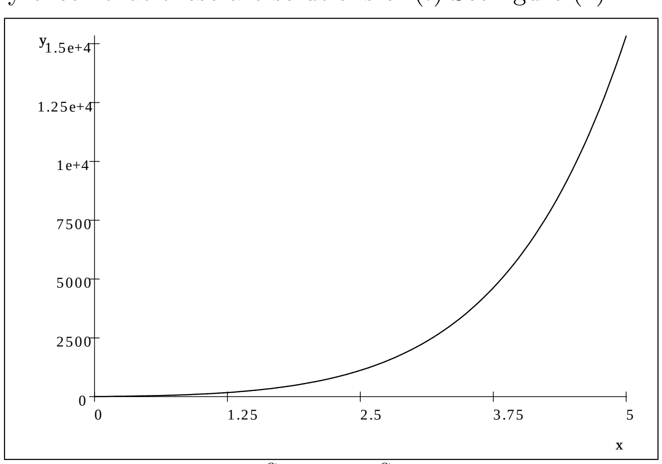


Fig.1 $y = c_1 e^{2\frac{t^\alpha}{\alpha}} + c_2 e^{-\frac{t^\alpha}{\alpha}}, \alpha = 0.5, c_1 > 0$

3. Conformable Cauchy Euler Equation

The standard form of the classical homogenous Cauchy Euler equation of order 2 is:

$$x^2 y'' + a_1 x y' + a_0 y = 0$$

Now the conformable Cauchy Euler equation of order 2 can be written as

$$x^{2\alpha} T^{2\alpha} y + a_1 x^{\alpha} T^{\alpha} y + a_0 y = 0 \tag{1}$$

Now we will give the procedure how to solve (1).

Procedure

Put $y = x^{\alpha r}$

Then

$$\begin{aligned} T^{2\alpha} y &= T^{\alpha}(T^{\alpha} y) \\ &= T^{\alpha}(T^{\alpha} x^{\alpha r}) \\ &= T^{\alpha}(\alpha r x^{\alpha r - \alpha}) \\ &= \alpha r(\alpha r - \alpha) x^{\alpha r - 2\alpha} \\ a_1 T^{\alpha} y &= a_1 (T^{\alpha} x^{\alpha r}) \\ &= a_1 \alpha r x^{\alpha r - \alpha} \end{aligned}$$

Thus

$$x^{2\alpha} T^{2\alpha} y = x^{2\alpha}(\alpha^2 r(r-1)) x^{\alpha r} x^{-2\alpha}$$

$$a_1 x^{\alpha} T^{\alpha} y = a_1 \alpha r x^{\alpha(r-1)} x^{\alpha r}$$

$$a_0 y = a_0 x^{\alpha r}$$

Hence

$$x^{2\alpha}(\alpha^2 r(r-1)) x^{-2\alpha}.x^{\alpha r} + a_1 \alpha r x^{\alpha r} x^{\alpha r} x^{-\alpha} x^{\alpha} + a_0 x^{\alpha r} = 0$$

So

$$x^{\alpha r}\left[\alpha^2 r(r-1) + a_1 \alpha r + a_0\right] = 0$$

Solve

$$\alpha^2 r(r-1) + a_1 \alpha r + a_0 = 0$$

to get $r = r_1, r = r_2$. Assume r_1, r_2 are reals . Then

$y_1 = x^{\alpha r_1},\ y_2 = x^{\alpha r_2}$ are two independent solutions of (1) and

$$y_h = c_1 x^{\alpha r_1} +\ c_2 x^{\alpha r_2}$$

Remark. The case of conformable Cauchy Euler Equation of any order can be handled in the same way as the case of order 2.

Example 2. Solve

$$x^{2\alpha} y^{(2\alpha)} + x^{2\alpha} y^{(\alpha)} - \frac{y}{2} = 0, y(1) = 1,\ y^{(\alpha)}(1) = 1$$

Solution. Put $y = x^{\alpha r}$ and substitute in the equation to get

$\alpha^2 r(r-1) + \alpha r - \frac{1}{2} = 0$

Take $\alpha = \frac{1}{2}$ we get

$\frac{1}{4} r(r-1) + \frac{1}{2} r - \frac{1}{2} = 0$

$r(r-1) + 2r - 2 = 0$

$r^2 + r - 2 = 0$

$(r+2)(r-1) = 0$

$r_1 = 2, r_2 = 1$

$y_1 = x^{-\frac{2}{2}},\ y_2 = x^{\frac{1}{2}}$ $\qquad (\alpha = \frac{1}{2})$

$y_h = c_1 \frac{1}{x} + c_2 \sqrt[2]{x}$

$y(1) = c_1 + c_2$

$y^{\frac{1}{2}}(x) = c_1(-1)x^{-1-\frac{1}{2}} + c_2 \frac{1}{2}$

So $y^{\frac{1}{2}}(1) = -c_1 + \frac{c_2}{2} = 1$. Hence $\frac{3}{2} c_2 = 2 \Longrightarrow c\ c_1 = -\frac{1}{3}$

$y_h = -\frac{1}{3x} + \frac{4}{3} \sqrt[2]{x}$. See figure (2).

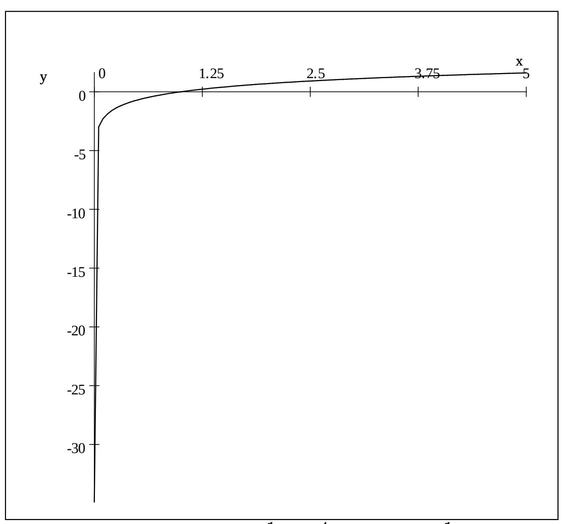


Fig.2 $y_h = -\frac{1}{3x} + \frac{4}{3} \sqrt[2]{x}$, $\alpha = \frac{1}{2}$

4. Conclusion

Conformable fractional derivative can be applied to solve linear differential equation with variable coefficients as an example Cauchy Euler equation.

References

[1] Anderson, Dr, Positive Green's Functions for Boundary Value Problems with Conformable Derivatives.Mathematical Analysis, Approximation Theory and Their Applications 1(2016)63-74

[2] Hosseini, K., Bekir, A., Kaplan, M. and Güner,O. On a new technique for solving the nonlinear conformable time-fractional differential equations.Optical and Quantum Electronics, 49(2017)

[3] Kaplan, M. Applications of two reliable methods for solving a nonlinear conformable time-fractional equation.Optical and Quantum Electronics, 49(2017)312-

[4] Khalil, R. and Abu Hammad, M.Abel. S Formula And Wronskian For Conformable Fractional Differential Equations. I.J. Differential Equations and Applications, 13 (2014) 177-183.

[5] Khalil,R. M. Al horani, M. Yousef, A. Sababheh,M. Anew definition of fractional derivative, Journal of Computational Apllied Mathematics, 264 (2014), 65-70.

[6]Abu Hammad, M. and Khalil, R. Legendre fractional differential equation and Legender fractional polynomials I.J of Applied Mathematical Research, 3 (3) (2014) 214-219.

[7]Abu Hammad, M. and Khalil, R..Conformable fractional Heat differential equation M Abu Hammad, R Khalil - Int. J. Pure Appl. Math, Volume 94 (2014,) 215-221

[8] Kilbas, A. Srivastava, H. and Trujillo,J. Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[9] Kurt, A. Cenesiz, Y. and Taşbozan, O. Exact Solution for the Conformable Burgers' Equation by the Hopf-Cole Transform,Çankaya University Journal of Science and Engineering ,V.13, (2016)

018–023

[10] Hosseini, K., Bekir, A., Kaplan, M. and Güner,O. On a new technique for solving the nonlinear conformable time-fractional differential equations.Optical and Quantum Electronics, 49(2017)

Applications of neutrosophic sets in B-algebras

Sun Shin Ahn

Department of Mathematics Education, Dongguk University, Seoul 04620, Korea

Abstract. The notions of a neutrosophic subalgebra and a neutrosophic normal subalgebra of a B-algebra are introduced and characterizations of them are discussed. We show that the homomorphic preimage of a neutrosophic subalgebra of a B-algebra is a neutrosophic subalgebra, and the onto homomorphic image of a neutrosophic subalgebra of a B-algebra is a neutrosophic subalgebra.

1. Introduction

Zadeh [12] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [2] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality (i) as independent component in 1995 (published in 1998) and defined the neutrosophic set on three components (t, i, f) = (truth, indeterminacy, falsehood). Y. B. Jun, E. H. Roh and H. S. Kim [4] introduced a new notion, called a BH-algebra. J. Neggers and H. S. Kim [9] introduced a new notion, called a B-algebra. C. B. Kim and H. S. Kim [7] introduced the notion of a BG-algebra which is a generalization of B-algebras. S. S. Ahn and H. D. Lee [1] classified the subalgebras by their family of level subalgebras in BG-algebras.

In this paper, we introduce the notions of a neutrosophic subalgebra and a neutrosophic normal subalgebra of a B-algebra and discuss characterizations of them. We show that the homomorphic preimage of a neutrosophic subalgebra of a B-algebra is a neutrosophic subalgebra, and the onto homomorphic image of neutrosophic image of a neutrosophic subalgebra of a B-algebra is a neutrosophic subalgebra.

2. Preliminaries

A *B-algebra* ([9]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying axioms:

(B1) $x * x = 0$,

(B2) $x * 0 = x$,

(B) $(x * y) * z = x * (z * (0 * y))$

for any x, y, z in X. For brevity we call X a *B-algebra.* In X we can define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BH-algebra* if it satisfies (B1), (B2) and

(BH) $x * y = y * x = 0$ imply $x = y$ for any $x, y \in X$.

---

[0] **2010 Mathematics Subject Classification**: 06F35; 03G25; 03B52.

[0] **Keywords**: B-algebra; (normal) subalgebra; neutrosophic subalgebra; neutrosophic normal subalgebra.

[0]**E-mail:** `sunshine@dongguk.edu`

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BG-algebra* if it satisfies (B1), (B2) and

(BG) $(x*y)*(0*y)=x$ for any $x,y\in X$.

Proposition 2.1.([3, 9]) *Let $(X;*,0)$ be a B-algebra. Then*

(i) *the left cancellation law holds in X, i.e., $x*y=x*z$ implies $y=z$,*
(ii) *if $x*y=0$, then $x=y$ for any $x,y\in X$,*
(iii) *if $0*x=0*y$, then $x=y$ for any $x,y\in X$,*
(iv) *$0*(0*x)=x$, for all $x\in X$,*
(v) *$x*(y*z)=(x*(0*z))*y$ for all $x,y,z\in X$.*

Theorem 2.2.([7]) *If $(X;*,0)$ is a B-algebra, then it is a BG-algebra.*

Proposition 2.3.([7]) *Every BG-algebra is a BH-algebra.*

Let $(X;*_X,0_X)$ and $(Y;*_Y,0_Y)$ be B-algebras. A mapping $\varphi:X\to Y$ is called a *homomorphism* if $\varphi(x*_X y)=\varphi(x)*_Y\varphi(y)$ for any $x,y\in X$. A non-empty subset S of X is called a *subalgebra* of X if $x*y\in S$ for any $x,y\in X$. A non-empty subset N of X is said to be *normal* if $(x*a)*(y*b)\in N$ for any $x*y,a*b\in N$. Then any normal subset N of a B-algebra X is a subalgebra of X, but the converse need not be true ([10]). A non-empty subset X of a B-algebra X is a called a *normal subalgebra* of X if it is both a subalgebra and a normal set.

Definition 2.4. Let X be a space of points (objects) with generic elements in X denoted by x. A simple valued neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. Then a simple valued neutrosophic set A can be denoted by

$$A:=\{\langle x,T_A(x),I_A(x),F_A(x)\rangle|x\in X\},$$

where $T_A(x),I_A(x),F_A(x)\in[0,1]$ for each point x in X. Therefore the sum of $T_A(x),I_A(x)$, and $F_A(x)$ satisfies the condition $0\le T_A(x)+I_A(x)+F_A(x)\le 3$.

For convenience, "simple valued neutrosophic set" is abbreviated to "neutrosophic set" later.

Definition 2.5. Let A be a neutrosophic set in a B-algebra X and $\alpha,\beta,\gamma\in[0,1]$ with $0\le\alpha+\beta+\gamma\le 3$ and an (α,β,γ)-level set of X denoted by $A^{(\alpha,\beta,\gamma)}$ is defined as

$$A^{(\alpha,\beta,\gamma)}=\{x\in X|T_A(x)\le\alpha,I_A(x)\ge\beta,F_A(x)\le\gamma\}.$$

For any family $\{a_i|i\in\Lambda\}$, we define

$$\bigvee\{a_i|i\in\Lambda\}:=\begin{cases}\max\{a_i|i\in\Lambda\} & \text{if }\Lambda\text{ is finite,}\\ \sup\{a_i|i\in\Lambda\} & \text{otherwise}\end{cases}$$

and

$$\bigwedge\{a_i|i\in\Lambda\}:=\begin{cases}\min\{a_i|i\in\Lambda\} & \text{if }\Lambda\text{ is finite,}\\ \inf\{a_i|i\in\Lambda\} & \text{otherwise.}\end{cases}$$

3. Neutrosophic subalgebras in B-algebras

Definition 3.1. A neutrosophic set A in a B-algebra X is called a *neutrosophic subalgebra* of X if it satisfies:

(NSS) $T_A(x*y) \leq \max\{T_A(x), T_A(y)\}, I_A(x*y) \geq \min\{I_A(x), I_A(y)\},$ and $F_A(x*y) \leq \max\{F_A(x), F_A(y)\}$, for any $x, y \in X$.

Proposition 3.2. *Every neutrosophic subalgebra of a B-algebra X satisfies the following conditions:*

(3.1) $T_A(0) \leq T_A(x), I_A(0) \geq I_A(x)$, *and* $F_A(0) \leq F_A(x)$ *for any* $x \in X$.

Proof. Straightforward. □

Example 3.3. Let $X := \{0, 1, 2, 3\}$ be a B-algebra with the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Define a neutrosophic set A in X as follows:

$$T_A(x) = \begin{cases} 0.13, & \text{if } x \in \{0, 2\} \\ 0.84, & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} 0.82, & \text{if } x \in \{0, 2\} \\ 0.15, & \text{otherwise,} \end{cases}$$

$$F_A(x) = \begin{cases} 0.13, & \text{if } x \in \{0, 2\} \\ 0.84, & \text{otherwise.} \end{cases}$$

It is easy to check that A is a neutrosophic subalgebra of X.

Theorem 3.4. *Let A be a neutrosophic set in a B-algebra X and let $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$. Then A is a neutrosophic subalgebra of X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.*

Proof. Assume that A is a neutrosophic subalgebra of X. Let $\alpha, \beta, \gamma \in [0, 1]$ be such that $0 \leq \alpha + \beta + \gamma \leq 3$ and $A^{(\alpha,\beta,\gamma)} \neq \emptyset$. Let $x, y \in A^{(\alpha,\beta,\gamma)}$. Then $T_A(x) \leq \alpha, T_A(y) \leq \alpha, I_A(x) \geq \beta, I_A(y) \geq \beta$ and $F_A(x) \leq \gamma, F_A(y) \leq \gamma$. Using (NSS), we have $T_A(x*y) \leq \max\{T_A(x), T_A(y)\} \leq \alpha$, $I_A(x*y) \geq \min\{I_A(x), I_A(y)\} \geq \beta$, and $F_A(x*y) \leq \max\{F_A(x), F_A(y)\} \leq \gamma$. Hence $x*y \in A^{(\alpha,\beta,\gamma)}$. Therefore $A^{(\alpha,\beta,\gamma)}$ is a subalgebra of X.

Conversely, all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$. Assume that there exist $a_t, b_t, a_i, b_i \in X$ and $a_f, b_f \in X$ such that $T_A(a_t*b_t) > \max\{T_A(a_t), T_A(b_t)\}, I_A(a_i*b_i) < \min\{I_A(a_i), I_A(b_i)\}$ and $F_A(a_f*b_f) > \max\{F_A(a_f), F_A(b_f)\}$. Then $T_A(a_t*b_t) > \alpha_1 \geq \max\{T_A(a_t), T_A(b_t)\}, I_A(a_i*b_i) < \beta_1 \leq \min\{I_A(a_i), I_A(b_i)\}$ and $F_A(a_f*b_f) > \gamma_1 \geq \max\{F_A(a_f), F_A(b_f)\}$ for some $\alpha_1, \gamma_1 \in [0, 1)$ and $\beta_1 \in (0, 1]$. Hence $a_t, b_t, a_i, b_i \in A^{(\alpha_1,\beta_1,\gamma_1)}$, and $a_f, b_f \in A^{(\alpha_1,\beta_1,\gamma_1)}$. But $a_t*b_t, a_i*b_i \notin A^{(\alpha_1,\beta_1,\gamma_1)}$, and $a_f*b_f \notin A^{(\alpha_1,\beta_1,\gamma_1)}$, which is a contradiction. Hence $T_A(x*y) \leq \max\{T_A(x), T_A(y)\}, I_A(x*y) \geq \min\{I_A(x), I_A(y)\}$, and $F_A(x*y) \leq \max\{T_A(x), T_A(y)\}$, for any $x, y \in X$. Therefore A is a neutrosophic subalgebra of X. □

Since $[0, 1]$ is a completely distributive lattice with respect to the usual ordering, we have the following theorem.

Theorem 3.5. *If $\{A_i | i \in \mathbb{N}\}$ is a family of neutrosopic subalgebras of a B-algebra X, then $(\{A_i | i \in \mathbb{N}\}, \subseteq)$ forms a complete distributive lattice.*

Theorem 3.6. *Let A be a neutrosophic subalgebra of a B-algebra X. If there exists a sequence $\{a_n\}$ in X such that $\lim_{n\to\infty} T_A(a_n) = 0, \lim_{n\to\infty} I_A(a_n) = 1$, and $\lim_{n\to\infty} F_A(a_n) = 0$, then $T_A(0) = 0, I_A(0) = 1$, and $F_A(0) = 0$.*

Proof. By Proposition 3.2, we have $T_A(0) \le T_A(x), I_A(0) \ge I_A(x)$, and $F_A(0) \le F_A(x)$ for all $x \in X$. Hence we have $T_A(0) \le T_A(a_n), I_A(0) \ge I_A(a_n)$, and $F_A(0) \le F_A(a_n)$ for every positive integer n. Therefore $0 \le T_A(0) \le \lim_{n\to\infty} T_A(a_n) = 0, 1 = \lim_{n\to\infty} I_A(a_n) \le I_A(0) \le 1$, and $0 \le F_A(0) \le \lim_{n\to\infty} F_A(a_n) = 0$. Thus we have $T_A(0) = 0, I_A(0) = 1$, and $F_A(0) = 0$. □

Proposition 3.7. *If every neutrosophic subalgebra A of a B-algebra X satisfies the condition*

(3.2) $T_A(x * y) \le T_A(y), I_A(x * y) \ge I_A(y), F_A(x * y) \le F_A(y)$, *for any $x, y \in X$,*

then T_A, I_A, and F_A are constant functions.

Proof. It follows from (3.2) that $T_A(x) = T_A(x * 0) \le T_A(0), I_A(x) = I_A(x * 0) \ge I_A(0)$, and $F_A(x) = F_A(x * 0) \le F_A(0)$ for any $x \in X$. By Proposition 3.2, we have $T_A(x) = T_A(0), I_A(x) = I_A(0)$, and $F_A(x) = F_A(0)$ for any $x \in X$. Hence T_A, I_A, and F_A are constant functions. □

Definition 3.8. A neutrosophic set A in a B-algebra X is said to be *neutrosophic normal* of X if it satisfies:

(NSN) $T_A((x*a)*(y*b)) \le \max\{T_A(x*y), T_A(a*b)\}, I_A((x*a)*(y*b)) \ge \min\{I_A(x*y), I_A(a*b)\}$, and $F_A((x*a)*(y*b)) \le \max\{F_A(x*y), F_A(a*b)\}$, for any $x, y, a, b \in X$.

A neutrosophic set A in a B-algebra X is called a *neutrosophic normal subalgebra* of X if it satisfies (NSS) and (NSN).

Example 3.9. Let $X := \{0, 1, 2, 3\}$ be a B-algebra ([8]) with the following table:

$*$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Define a neutrosophic set A in X as follows:

$$T_A(x) = \begin{cases} 0.12, & \text{if } x \in \{0, 3\} \\ 0.76, & \text{otherwise}, \end{cases}$$

$$I_A(x) = \begin{cases} 0.73, & \text{if } x \in \{0, 3\} \\ 0.14, & \text{otherwise}, \end{cases}$$

$$F_A(x) = \begin{cases} 0.12, & \text{if } x \in \{0, 3\} \\ 0.76, & \text{otherwise}. \end{cases}$$

It is easy to check that A is a neutrosophic normal subalgebra of X.

Proposition 3.10. *Every neutrosophic normal of a B-algebra X is a neutrosophic subalgebra of X.*

Proof. Let A be neutrosophic normal of X. Put $y := 0, b := 0$ and $a := y$ in (NSN). Then $T_A((x*y)*(0*0)) \leq \max\{T_A(x*0), T_A(y*0)\}, I_A((x*y)*(0*0)) \geq \min\{I_A(x*0), I_A(y*0)\}$, and $F_A((x*y)*(0*0)) \leq \max\{F_A(x*0), F_A(y*0)\}$. Using (B2) and (B1), we have $T_A(x*y) \leq \max\{T_A(x), T_A(y)\}, I_A(x*y) \geq \min\{I_A(x), I_A(y)\}$, and $F_A(x*y) \leq \max\{F_A(x), F_A(y)\}$, for any $x, y \in X$. Hence A is a neutrosophic subalgebra of X. $\square$

The converse of Proposition 3.10 may not be true in general (see Example 3.11).

Example 3.11. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a B-algebra ([10]) with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Define a neutrosophic set A in X as follows:

$$T_A(x) = \begin{cases} 0.12, & \text{if } x = 0 \\ 0.23, & \text{if } x = 5 \\ 0.52 & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} 0.58, & \text{if } x = 0 \\ 0.13, & \text{if } x = 5 \\ 0.11, & \text{otherwise,} \end{cases}$$

$$F_A(x) = \begin{cases} 0.12, & \text{if } x = 0 \\ 0.23, & \text{if } x = 5 \\ 0.52 & \text{otherwise.} \end{cases}$$

It is easy to check that A is a neutrosophic subalgebra of X. But it is not neutrosophic normal of X, since $T_A(1) = T_A((1*3)*(4*2)) = 0.52 \nleq \max\{T_A(1*4), T_A(3*2)\} = \max\{T_A(5), T_A(5)\} = 0.23$, and/or $I_A(1) = I_A((1*3)*(4*2)) = 0.11 \ngeq \min\{I_A(1*4), I_A(3*2)\} = \min\{I_A(5), I_A(5)\} = 0.13$, and/or $F_A(1) = F_A((1*3)*(4*2)) = 0.52 \nleq \max\{F_A(1*4), F_A(3*2)\} = \max\{F_A(5), F_A(5)\} = 0.23$.

Theorem 3.12. *Let A be a neutrosophic set in a B-algebra X and let $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha+\beta+\gamma \leq 3$. Then A is a neutrosophic normal subalgebra of X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are normal subalgebras of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.*

Proof. Similar to Theorem 3.4. $\square$

Proposition 3.13. *Let A be a neutrosophic normal subalgebra of a B-algebra X. Denote that $X_T := \{x \in X | T_A(x) = T_A(0)\}, X_I := \{x \in X | I_A(x) = I_A(0)\}$, and $X_F := \{x \in X | F_A(x) = F_A(0)\}$. Then X_T, X_I, and X_F are normal subalgebras of X.*

Proof. It is sufficient to show that X_T, X_I, and X_F are normal. Let $a, b, x, y \in X$ be such that $x * y, a * b \in X_T$. Then $T_A(x*y) = T_A(0) = T_A(a*b)$. Since A is a neutrosophic normal subalgebra of X, we have $T_A((x*a)*(y*b)) \leq \max\{T_A(x*y), T_A(a*b)\} = T_A(0)$. By Proposition 3.2, we get $T_A((x*a)*(y*b)) = T_A(0)$. Hence $(x*a)*(y*b) \in X_T$. Therefore X_T is a normal subalgebra of X. Similarly, X_I, X_F are normal subalgebras of X. This completes the proof. □

Definition 3.14. Let A and B be neutrosophic sets of a set X. The *union* of A and B is defined to be a neutrosophic set

$$A\tilde{\cup}B := \{\langle x, T_{A\cup B}(x), I_{A\cup B}(x), F_{A\cup B}(x)\rangle | x \in X\},$$

where $T_{A\cup B}(x) = \min\{T_A(x), T_B(x)\}, I_{A\cup B}(x) = \max\{I_A(x), I_B(x)\}, F_{A\cup B}(x) = \min\{F_A(x), F_B(x)\}$, for all $x \in X$. The *intersection* of A and B is defined to be a neutrosophic set

$$A\tilde{\cap}B := \{\langle x, T_{A\cap B}(x), I_{A\cap B}(x), F_{A\cap B}(x)\rangle | x \in X\},$$

where $T_{A\cap B}(x) = \max\{T_A(x), T_B(x)\}, I_{A\cap B}(x) = \min\{I_A(x), I_B(x)\}, F_{A\cap B}(x) = \max\{F_A(x), F_B(x)\}$, for all $x \in X$.

Theorem 3.15. *The intersection of two neutrosophic subalgebras of a B-algebra X is a also a neutrosophic subalgebra of X.*

Proof. Let A and B be neutrosophic subalgebras of X. For any $x, y \in X$, we have

$$\begin{aligned}
T_{A\cap B}(x*y) &= \max\{T_A(x*y), T_B(x*y)\}\\
&\leq \max\{\max\{T_A(x), T_A(y)\}, \max\{T_B(x), T_B(y)\}\}\\
&= \max\{\max\{T_A(x), T_B(x)\}, \max\{T_A(y), T_B(y)\}\}\\
&= \max\{T_{A\cap B}(x), T_{A\cap B}(y)\},\\
I_{A\cap B}(x*y) &= \min\{I_A(x*y), I_B(x*y)\}\\
&\geq \min\{\min\{I_A(x), I_A(y)\}, \min\{I_B(x), I_B(y)\}\}\\
&= \min\{\min\{I_A(x), I_B(x)\}, \min\{I_A(y), I_B(y)\}\}\\
&= \min\{I_{A\cap B}(x), I_{A\cap B}(y)\},
\end{aligned}$$

and

$$\begin{aligned}
F_{A\cap B}(x*y) &= \max\{F_A(x*y), F_B(x*y)\}\\
&\leq \max\{\max\{F_A(x), F_A(y)\}, \max\{F_B(x), F_B(y)\}\}\\
&= \max\{\max\{F_A(x), F_B(x)\}, \max\{F_A(y), F_B(y)\}\}\\
&= \max\{F_{A\cap B}(x), F_{A\cap B}(y)\}.
\end{aligned}$$

Hence $A\tilde{\cap}B$ is a neutrosophic subalgebra of X. □

Corollary 3.16. *If $\{A_i | i \in \mathbb{N}\}$ is a family of neutrosophic subalgebras of a B-algebra X, then so is $\tilde{\cap}_{i\in\mathbb{N}} A_i$.*

The union of any set of neutrosophic subalgebras of a B-algebra X need not be a neutrosophic subalgebra of X.

Example 3.17. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a B-algebra as in Example 3.11. Define neutrosophic sets A and B of X as follows:

$$T_A(x) = \begin{cases} 0.11, & \text{if } x \in \{0, 4\} \\ 0.73 & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} 0.82, & \text{if } x \in \{0, 4\} \\ 0.12, & \text{otherwise,} \end{cases}$$

$$F_A(x) = \begin{cases} 0.11, & \text{if } x \in \{0, 4\} \\ 0.73 & \text{otherwise,} \end{cases}$$

$$T_B(x) = \begin{cases} 0.13, & \text{if } x \in \{0, 5\} \\ 0.74 & \text{otherwise,} \end{cases}$$

$$I_B(x) = \begin{cases} 0.83, & \text{if } x \in \{0, 5\} \\ 0.13, & \text{otherwise,} \end{cases}$$

and

$$F_B(x) = \begin{cases} 0.13, & \text{if } x \in \{0, 5\} \\ 0.74 & \text{otherwise.} \end{cases}$$

It is easy to check that A and B are neutrosophic subalgebras of X. But $A\tilde{\cup}B$ is not a neutrosophic subalgebra of X, since

$$\begin{aligned} T_{A\cup B}(4 * 5) =& T_{A\cup B}(2) = \min\{T_A(2), T_B(2)\} = 0.73 \\ &\nleq \max\{T_{A\cup B}(4), T_{A\cup B}(5)\} \\ &= \max\{\min\{T_A(4), T_B(4)\}, \min\{T_A(5), T_B(5)\}\} = 0.13, \end{aligned}$$

and/or

$$\begin{aligned} I_{A\cup B}(4 * 5) =& I_{A\cup B}(2) = \max\{I_A(2), I_B(2)\} = 0.13 \\ &\ngeq \min\{I_{A\cup B}(4), I_{A\cup B}(5)\} \\ &= \min\{\max\{I_A(4), I_B(4)\}, \max\{I_A(5), I_B(5)\}\} = 0.82, \end{aligned}$$

and/or

$$\begin{aligned} F_{A\cup B}(4 * 5) =& F_{A\cup B}(2) = \min\{F_A(2), F_B(2)\} = 0.73 \\ &\nleq \max\{F_{A\cup B}(4), F_{A\cup B}(5)\} \\ &= \max\{\min\{F_A(4), F_B(4)\}, \min\{F_A(5), F_B(5)\}\} = 0.13. \end{aligned}$$

Let $f : X \to Y$ be a function of sets. If $M = \{\langle y, T_M(y), I_M(y), F_M(y)\rangle | y \in Y\}$ is a neutrosophic set of a set Y, then the preimage of M under f is defined to be a neutrosophic set

$$f^{-1}(M) := \{\langle x, f^{-1}(T_M)(x), f^{-1}(I_M)(x), f^{-1}(F_M)(x)\rangle | x \in X\}$$

of X, where $f^{-1}(T_M)(x) = T_M(f(x)), f^{-1}(I_M)(x) = I_M(f(x))$ and $f^{-1}(F_M)(x) = F_M(f(x))$ for all $x \in X$.

Theorem 3.18. *Let $f : X \to Y$ be a homomorphism of B-algebras. If $M = \{\langle y, T_M(y), I_M(y), F_M(y)\rangle | y \in Y\}$ is a neutrosophic subalgebra of Y, then the preimage of M under f is a neutrosophic subalgebra of X.*

Proof. Let $f^{-1}(M)$ be the preimage of M under f. For any $x, y \in X$, we have

$$\begin{aligned} f^{-1}(T_M(x*y)) =& T_M(f(x*y)) = T_M(f(x)*f(y)) \\ & \leq \max\{T_M(f(x)), T_M(f(y))\} = \max\{f^{-1}(T_M)(x), f^{-1}(T_M)(y)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(I_M(x*y)) =& I_M(f(x*y)) = I_M(f(x)*f(y)) \\ & \geq \min\{I_M(f(x)), I_M(f(y))\} = \min\{f^{-1}(I_M)(x), f^{-1}(I_M)(y)\}, \end{aligned}$$

and

$$\begin{aligned} f^{-1}(F_M(x*y)) =& F_M(f(x*y)) = F_M(f(x)*f(y)) \\ & \leq \max\{F_M(f(x)), F_M(f(y))\} = \max\{f^{-1}(F_M)(x), f^{-1}(F_M)(y)\}. \end{aligned}$$

Hence $f^{-1}(M)$ is a neutrosophic subalgebra of X. □

Let $f : X \to Y$ be an onto function of sets. If A is a neutrosophic set of X, then the image of A under f is defined to be a neutrosophic set

$$f(A) := \{\langle y, f(T_A)(y), f(I_A)(y), f(F_A)(y)\rangle | y \in Y\}$$

of Y, where $f(T_A)(y) = \bigwedge_{x\in f^{-1}(y)} T_A(x), f(I_A)(y) = \bigvee_{x\in f^{-1}(y)} I_A(x)$, and $f(F_A)(y) = \bigwedge_{x\in f^{-1}(y)} F_A(x)$.

Theorem 3.19. *For an onto homomorphism $f : X \to Y$ of B-algebras, let A be a neutrosophic set of X such that*

(3.3) $(\forall C \subseteq X)(\exists x_0 \in C)(T_A(x_0) = \bigwedge_{z\in C} T_A(z), I_A(x_0) = \bigvee_{z\in C} I_A(z), F_A(x_0) = \bigwedge_{z\in C} F_A(z))$.

If A is a neutrosophic subalgebra of a B-algebra X, then the image of A under f is a neutrosophic subalgebra of Y.

Proof. Let $f(A)$ be the image of A under f. Let $a, b \in Y$. Then $f^{-1}(a) \neq \emptyset$ and $f^{-1}(b) \neq \emptyset$ in X. By (3.3), there exist $x_a \in f^{-1}(a)$ and $x_b \in f^{-1}(b)$ such that

$$T_A(x_a) = \bigwedge_{z\in f^{-1}(a)} T_A(z), I_A(x_a) = \bigvee_{z\in f^{-1}(a)} I_A(z), F_A(x_a) = \bigwedge_{z\in f^{-1}(a)} F_A(z),$$

$$T_A(x_b) = \bigwedge_{w\in f^{-1}(b)} T_A(w), I_A(x_b) = \bigvee_{w\in f^{-1}(b)} I_A(w), F_A(x_b) = \bigwedge_{w\in f^{-1}(b)} F_A(w).$$

Thus

$$\begin{aligned} f(T_A)(a*b) =& \bigwedge_{x\in f^{-1}(a*b)} T_A(x) \leq T_A(x_a*x_b) \leq \max\{T_A(x_a), T_A(x_b)\} \\ & = \max\{\bigwedge_{z\in f^{-1}(a)} T_A(z), \bigwedge_{w\in f^{-1}(b)} T_A(w)\} = \max\{f(T_A)(a), f(T_A)(b)\}, \end{aligned}$$

$$\begin{aligned} f(I_A)(a*b) =& \bigvee_{x\in f^{-1}(a*b)} I_A(x) \geq I_A(x_a*x_b) \geq \min\{I_A(x_a), I_A(x_b)\} \\ & = \min\{\bigvee_{z\in f^{-1}(a)} I_A(z), \bigvee_{w\in f^{-1}(b)} I_A(w)\} = \min\{f(I_A)(a), f(I_A)(b)\}, \end{aligned}$$

and

$$\begin{aligned} f(F_A)(a*b) &= \bigwedge_{x\in f^{-1}(a*b)} F_A(x) \le F_A(x_a * x_b) \le \max\{F_A(x_a), F_A(x_b)\} \\ &= \max\{\bigwedge_{z\in f^{-1}(a)} F_A(z), \bigwedge_{w\in f^{-1}(b)} F_A(w)\} = \max\{f(F_A)(a), f(F_A)(b)\}. \end{aligned}$$

Hence $f(A)$ is a neutrosophic subalgebra of Y. $\square$

References

[1] S. S. Ahn and H. D. Lee, *Fuzzy subalgebras of BG-algebras*, Comm. Kore. Math. Soc. **19** (2004), 243-251.

[2] K. Atanassov, *Intuitionistic fuzzy sets,* Fuzzy sets and Systems 20 (1986), 87–96.

[3] J. R. Cho and H. S. Kim, *On B-algebras and Related Systems,* **8**(2001), 1-6.

[4] Y. B. Jun, E. H. Roh and H. S. Kim, *On BH-algebras*, Sci. Mathematica **1** (1998), 347-354.

[5] Y. B. Jun, E. H. Roh and H. S. Kim, *On fuzzy B-algebras*, Czech. Math. J. **52** (2002), 375-384.

[6] M. Khan, S. Anis, F. Smarandache and Y. B. Jun, *Neutrosophic N-structures and their applications in semigroups*, Ann. Fuzzy Math. Inform., (to appear).

[7] C. B. Kim and H. S. Kim, *On BG-algebras*, Demon. Math. **41** (2008), 497-505.

[8] Y. H. Kim and S. J. Yeom, *Qutient B-algebras via fuzzy normal B-algebras*, Honam Math. J. **30** (2008), 21-32.

[9] J. Neggers and H. S. Kim, *On B-algebras*, Mate. Vesnik **54**(2002), 21-29.

[10] J. Neggers and H. S. Kim, *A fundamental theorem of B-homomorphism for B-algebras*, Intern. Math. J. **2**(2002), 207-214.

[11] F. Smarandache, Neutrosophy, Neutrosophic Probablity, Sets, and Logic, Amer. Res. Press, Rehoboth, USA, 1998.

[12] L. A. Zadeh, *Fuzzy sets*, Information and Control 8 (1965), 338-353.

.

Investigating Some Properties of a Fourth Order Difference Equation

M. B. Almatrafi[1], E. M. Elsayed[1,2] and Faris Alzahrani.[1]
[1]Mathematics Department, Faculty of Science,
King Abdulaziz University,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.
[2]Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt.
E-mails: mmutrafi@taibahu.edu.sa, emmelsayed@yahoo.com,
faris.kau@hotmail.com.

ABSTRACT

The principal purpose of this paper is to present some qualitative behavior of the following fourth order difference equation:

$$x_{n+1} = ax_{n-1} - \frac{bx_{n-1}}{cx_{n-1} - dx_{n-3}}, \quad n = 0, 1, ...,$$

where the parameters $a,\ b,\ c$ and d are positive real numbers and the initial conditions $x_{-3},\ x_{-2},\ x_{-1}$ and x_0 are arbitrary non zero real numbers.

Keywords: stability, periodicity, global attractor, difference equations.

Mathematics Subject Classification: 39A10.

1. INTRODUCTION

This paper will provide a detailed study in terms of the local, global stability and obtain the form of the solutions of the following difference equation

$$x_{n+1} = ax_{n-1} - \frac{bx_{n-1}}{cx_{n-1} - dx_{n-3}}, \quad n = 0, 1, ..., \tag{1}$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1}$ and x_0 are arbitrary non zero real numbers and a, b, c, d are positive constants..

A huge number of researchers has concentrated on studying and investigating nonlinear difference equations in recent years. In particular, they have highlighted the boundedness, the global attractivity and the periodic behaviour of some certain types of difference equations. For instance: Elsayed et al.[19] studied the global attractor, local stability, periodic solutions and boundedness of the following recursive equation:

$$x_{n+1} = \frac{ax_n x_{n-2}}{bx_n + cx_{n-3}}.$$

Cinar[5] investigated the solution of the difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}.$$

Ibrahim[24] presented some relevant results of the difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + bx_n x_{n-2})}.$$

Elsayed[16] analyzed the global stability and examined the periodic solution of the following difference equation:

$$x_{n+1} = ax_{n-l} + \frac{bx_{n-l}}{cx_{n-l} - dx_{n-k}}.$$

Elabbasy et al.[8] investigated the global stability, periodicity character and gave the solution of special case of the difference equation

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}.$$

Yang et al.[36] examined the global and local stability of the equilibrium points of the following recursive equation:

$$x_{n+1} = \frac{ax_{n-1} + bx_{n-2}}{c + dx_{n-1}x_{n-2}}.$$

Simsek et al.[33] obtained the solution of the following difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

Abo-Zeid et al.[1] gave a detailed study about the convergence and the periodicity of the solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{(B - Cx_n x_{n-2})}.$$

Tolly et al.[35] illustrated some properties of the solution of the following recursive equation:

$$y_{n+1} = \frac{ay_{n-1}}{by_n y_{n-1} + cy_{n-1}y_{n-2} + d}.$$

Other relevant consequences of rational difference equations can be obtained in refs.[9]-.[12]

Now, some relevant results and definitions will be introduced here to be used in our discussion.

Let I be some interval of real numbers and the function f has continuous partial derivatives on I^{k+1} where $I^{k+1} = I \times I \times \cdots \times I$ $(k+1-$ times). Then, for initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, it is easy to see that the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 1.1. *(Stability)*

(i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1},..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).

(v) The equilibrium point $\overline{x}$ of Eq.(2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Now assume that the characteristic equation associated with Eq.(3) is

$$p(\lambda) = p_0\lambda^k + p_1\lambda^{k-1} + ... + p_{k-1}\lambda + p_k = 0, \tag{4}$$

where $p_i = \dfrac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}}$.

Theorem A [12]: *Assume that* $p_i \in R$, $i = 1, 2, ...$ *and* $k \in \{0, 1, 2, ...\}$. *Then*

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+k} + p_1 y_{n+k-1} + ... + p_k y_n = 0, \ \ n = 0, 1, ...$$

Next, we introduce a fundamental theorem to prove the global attractor of the fixed points.

Theorem B [26]: Let $g : [a, b]^{k+1} \to [a, b]$, be a continuous function, where k is a positive integer, and where $[a, b]$ is an interval of real numbers. Consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}, ..., x_{n-k}), \qquad n = 0, 1, ... \ . \tag{5}$$

Suppose that g satisfies the following conditions.

(1) For each integer i with $1 \leq i \leq k+1$; the function $g(z_1, z_2, ..., z_{k+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.

(2) If m, M is a solution of the system

$$m = g(m_1, m_2, ..., m_{k+1}), \qquad M = g(M_1, M_2, ..., M_{k+1}),$$

then $m = M$, where for each $i = 1, 2, ..., k+1$, we set

$$m_i = \begin{Bmatrix} m, & \text{if } g \text{ is non-decreasing in } z_i, \\ M, & \text{if } g \text{ is non-increasing in } z_i, \end{Bmatrix}, \qquad Mi = \begin{Bmatrix} M, & \text{if } g \text{ is non-decreasing in } z_i, \\ m, & \text{if } g \text{ is non-increasing in } z_i. \end{Bmatrix}.$$

Then there exists exactly one equilibrium point $\bar{x}$ of Equation (5), and every solution of Equation (5) converges to $\bar{x}$.

2. LOCAL STABILITY OF THE EQUILIBRIUM POINT

This section is devoted to give a detailed description about the local stability of the fixed point.

The equilibrium point of Eq.(1) is given by the following equation:

$$\overline{x} = a\overline{x} - \frac{b\overline{x}}{c\overline{x} - d\overline{x}},$$

from which we have

$$\overline{x} = \frac{b}{(a-1)(c-d)},$$

where $a \neq 1$ and $c \neq d$. Suppose that $f : (0,\infty)^2 \longrightarrow (0,\infty)$ defined as following:

$$f(u,v) = au - \frac{bu}{cu - dv}. \tag{6}$$

Then,

$$\frac{\partial f(u,v)}{\partial u} = a - \frac{b(cu - dv) - bcu}{(cu - dv)^2} = a + \frac{bdv}{(cu - dv)^2}, \tag{7}$$

$$\frac{\partial f(u,v)}{\partial v} = -\frac{-bu(-d)}{(cu - dv)^2} = -\frac{bdu}{(cu - dv)^2}. \tag{8}$$

Next, we calculate equations (7) and (8) at the equilibrium point as follows:

$$\frac{\partial f(\overline{x},\overline{x})}{\partial u} = a + \frac{bd\overline{x}}{(c\overline{x} - d\overline{x})^2} = a + \frac{bd}{(c-d)^2\overline{x}} = a + \frac{d(a-1)}{(c-d)} = -p_0,$$

$$\frac{\partial f(\overline{x},\overline{x})}{\partial v} = -\frac{bd\overline{x}}{(c\overline{x} - d\overline{x})^2} = -\frac{bd}{(c-d)^2\overline{x}} = -\frac{d(a-1)}{(c-d)} = -p_1.$$

Now, the linearized difference equation of Eq.(1) about the fixed point is given by

$$y_{n+1} + p_0 y_{n-1} + p_1 y_{n-3} = 0.$$

Theorem 1. Assume that

$$|ac - d| + d\,|a - 1| < |c - d|\,.$$

Then the fixed point of Eq.(1) is locally asymptotically stable.

Proof. By using Theorem A we notice that Eq.(1) is asymptotically stable if

$$|p_0| + |p_1| < 1.$$

Hence, we have

$$\left|a + \frac{d(a-1)}{(c-d)}\right| + \left|-\frac{d(a-1)}{(c-d)}\right| < 1,$$

which can be rearranged as follows:

$$|a(c-d)+d(a-1)|+|-d(a-1)|<|(c-d)|.$$

Therefore,

$$|ac-d|+d\,|a-1|<|c-d|.$$

This completes the proof.

3. GLOBAL STABILITY OF THE EQUILIBRIUM POINT

The global attractivity character of the considered equation will be presented in this section.

Theorem 2. The equilibrium point of Eq.(1) is a global attractor if $a<1$.

Proof. Suppose that p and q are two real numbers and let $f:[p,q]^2\longrightarrow[p,q]$ be a function defined by Eq.(6). Then, equations (7) and (8) tell us that $f(u,v)$ is increasing in u and decreasing in v. Now, we assume that (m,M) is a solution of the following system:

$$m=f(m,M),\quad\text{and}\quad M=f(M,m).$$

Substituting this into Eq.(6) gives

$$\begin{aligned} m &= am-\frac{bm}{cm-dM}, \\ M &= aM-\frac{bM}{cM-dm}. \end{aligned}$$

Then,

$$cm^2-dmM=acm^2-admM-bm, \tag{9}$$

$$cM^2-dmM=acM^2-admM-bM. \tag{10}$$

Subtracting Eq.(9) from Eq.(10) yields

$$c(m^2-M^2)=ac(m^2-M^2)+b\,(M-m)\,.$$

Hence, we obtain

$$(m-M)\,[c(1-a)(m+M)+b]=0.$$

Thus, when $a<1$, then we have

$$m=M.$$

We conclude from Theorem B that the equilibrium point is a global attractor of Eq.(1).

4. PERIODICITY OF THE SOLUTION

This section will present a theorem which shows that Eq.(1) has no periodic solution.

Theorem 3. Eq.(1) has no prime period two solutions.

Proof. We will use contradiction to prove this theorem. Assume that Eq.(1) has a positive prime period two solutions given as follows:

$$..., p, q, p, q, ...$$

Then,

$$p = ap - \frac{bp}{cp - dp}. \tag{11}$$

$$q = aq - \frac{bq}{cq - dq}. \tag{12}$$

Equations (11) and (12) can be written as follows:

$$p(a-1) = \frac{b}{c-d},$$

$$q(a-1) = \frac{b}{c-d},$$

which implies that $p = q$ and this contradicts the fact that $p \neq q$.

5. SPECIAL CASE OF EQ.(1)

In this section we will study the solution of the following special case:

$$x_{n+1} = x_{n-1} - \frac{x_{n-1}}{x_{n-1} - x_{n-3}}, \qquad n = 0, 1, 2, ..., \tag{13}$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1}$ and x_0 are nonzero real numbers with $x_{-3} \neq x_{-1}$ and $x_{-2} \neq x_0$.

Theorem 4. Let $\{x_n\}_{n=-3}^{\infty}$ be the solution of Eq. (13) satisfying $x_{-3} = r,\ x_{-2} = l,\ x_{-1} = k$ and $x_0 = h$. Then for $n = 0, 1, ...$

$$\begin{aligned}
x_{4n-3} &= nk - (n-1)r - n(n-1) - \frac{nk}{k-r}, \\
x_{4n-2} &= nh - (n-1)l - n(n-1) - \frac{nh}{h-l}, \\
x_{4n-1} &= (n+1)k - nr - n^2 - \frac{nk}{k-r}, \\
x_{4n} &= (n+1)h - nl - n^2 - \frac{nh}{h-l}.
\end{aligned}$$

Proof. For $n = 0$ the result holds. Now, we assume that $n > 0$ and our assumption satisfies for $n-1$. That is

$$x_{4n-7} = (n-1)k - (n-2)r - (n-1)(n-2) - \frac{(n-1)k}{k-r},$$

$$x_{4n-6} = (n-1)h - (n-2)l - (n-1)(n-2) - \frac{(n-1)h}{h-l},$$

$$x_{4n-5} = nk - (n-1)r - (n-1)^2 - \frac{(n-1)k}{k-r},$$

$$x_{4n-4} = nh - (n-1)l - (n-1)^2 - \frac{(n-1)h}{h-l}.$$

Next, it follows form Eq. (13) that

$$\begin{aligned}
x_{4n-3} &= x_{4n-5} - \frac{x_{4n-5}}{x_{4n-5} - x_{4n-7}}, \\
&= nk - (n-1)r - (n-1)^2 - \frac{(n-1)k}{k-r} \\
&\quad - \frac{nk - (n-1)r - (n-1)^2 - \frac{(n-1)k}{k-r}}{nk - (n-1)r - (n-1)^2 - \frac{(n-1)k}{k-r} - ((n-1)k - (n-2)r - (n-1)(n-2) - \frac{(n-1)k}{k-r})}, \\
&= nk - (n-1)r - (n-1)^2 - \frac{(n-1)k}{k-r} - \frac{(nk - nr + r)(k - n - r + 1)}{(k-r)(k-n-r+1)}, \\
&= nk - (n-1)r - (n-1)^2 - \frac{2nk - k - nr + r}{k-r}, \\
&= nk - (n-1)r - \frac{n(nk - rn + r)}{k-r}, \\
&= nk - (n-1)r - n(n-1) - \frac{nk}{k-r}.
\end{aligned}$$

Also, we obtain from Eq. (13)

$$\begin{aligned}
x_{4n-2} &= x_{4n-4} - \frac{x_{4n-4}}{x_{4n-4} - x_{4n-6}}, \\
&= nh - (n-1)l - (n-1)^2 - \frac{(n-1)h}{h-l} \\
&\quad - \frac{nh - (n-1)l - (n-1)^2 - \frac{(n-1)h}{h-l}}{nh - (n-1)l - (n-1)^2 - \frac{(n-1)h}{h-l} - (n-1)h + (n-2)l + (n-1)(n-2) + \frac{(n-1)h}{h-l}}, \\
&= nh - (n-1)l - (n-1)^2 - \frac{(n-1)h}{h-l} - \frac{(nh - nl + l)(h - l - n + 1)}{(h-l)(h-l-n+1)}, \\
&= nh - (n-1)l - (n-1)^2 - \frac{2nh - h - nl + l}{h-l}, \\
&= nh - (n-1)l - \frac{n(nh - nl + l)}{h-l}, \\
&= nh - (n-1)l - n(n-1) - \frac{nh}{h-l}.
\end{aligned}$$

Next, we will prove the third part of the theorem. Eq.(13) gives

$$\begin{aligned}
x_{4n-1} &= x_{4n-3} - \frac{x_{4n-3}}{x_{4n-3} - x_{4n-5}}, \\
&= nk - (n-1)r - n(n-1) - \frac{nk}{k-r} \\
&\quad - \frac{nk - (n-1)r - n(n-1) - \frac{nk}{k-r}}{[nk - (n-1)r - n(n-1) - \frac{nk}{k-r}] - [nk - (n-1)r - (n-1)^2 - \frac{(n-1)k}{k-r}]}, \\
&= nk - (n-1)r - n(n-1) - \frac{nk}{k-r} + \frac{(k-n-r)(nk-nr+r)(k-r)}{(k-r)(nk-nr+r)}, \\
&= nk - (n-1)r - n(n-1) - \frac{nk}{k-r} + k - n - r = (n+1)k - nr - n^2 - \frac{nk}{k-r}.
\end{aligned}$$

Finally, we prove the last part of the theorem. Eq.(13) leads to

$$\begin{aligned}
x_{4n} &= x_{4n-2} - \frac{x_{4n-2}}{x_{4n-2} - x_{4n-4}}, \\
&= nh - (n-1)l - n(n-1) - \frac{nh}{h-l} \\
&\quad - \frac{nh - (n-1)l - n(n-1) - \frac{nh}{h-l}}{[nh - (n-1)l - n(n-1) - \frac{nh}{h-l}] - [nh - (n-1)l - (n-1)^2 - \frac{(n-1)h}{h-l}]}, \\
&= nh - (n-1)l - n(n-1) - \frac{nh}{h-l} + \frac{(nh-nl+l)(h-l-n)(h-l)}{(h-l)(nh-nl+1)}, \\
&= nh - (n-1)l - n(n-1) - \frac{nh}{h-l} + h - l - n = (n+1)h - nl - n^2 - \frac{nh}{h-l}.
\end{aligned}$$

Hence, the proof has done.

6. NUMERICAL SOLUTIONS

This section shows some numerical examples that confirm the results we obtained in this paper.

Example 1. Let $x_{-3} = 0.2,\ x_{-2} = 5,\ x_{-1} = 1,\ x_0 = 2,\ a = 0.5,\ b = 1,\ c = 6$ and $d = 1$. Then, the local stability is shown as follows:

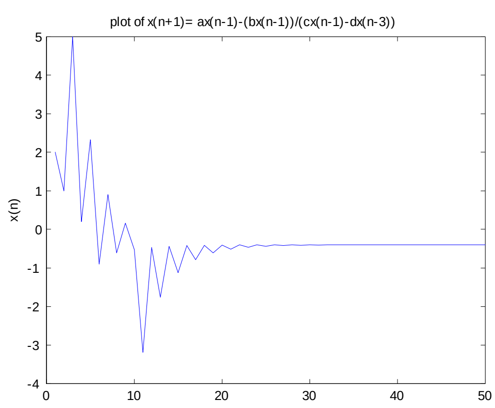


Figure 1. This figure shows the local stability of Eq.(1).

Example 2. Assume that $x_{-3} = 0.2,\ x_{-2} = 3,\ x_{-1} = 0.1,\ x_0 = 2,\ a = 0.1,\ b = 1,\ c = 2$ and $d = 9$. Then, the global stability is illustrated as follows:

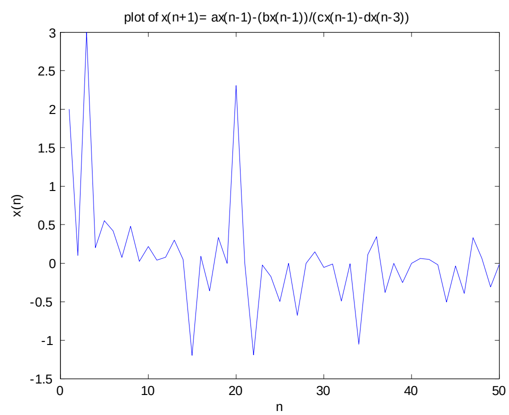


Figure 2. This figure presents a global stability of Eq.(1).

Example 3. This example presents the solution of Eq.(1) when we suppose that $x_{-3} = 0.2,\ x_{-2} = 3,\ x_{-1} = 1,\ x_0 = 0.5,\ a = b = 1,\ c = 0.5$ and $d = 9$. See Figure 3.

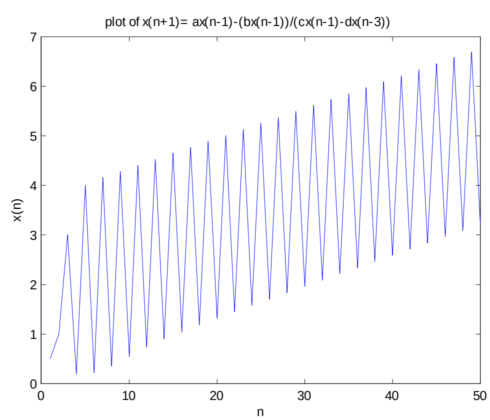


Figure 3. This figure shows the solutions of Eq.(1) when $x_{-3} = 0.2,\ x_{-2} = 3,\ x_{-1} = 1,\ x_0 = 0.5,\ a = b = 1,\ c = 0.5$ and $d = 9$.

Example 4. This example illustrates the solution of Eq.(13) when we assume that $x_{-3} = -7,\ x_{-2} =$

5, $x_{-1} = 0.5,\ x_0 = 8$. See Figure 4.

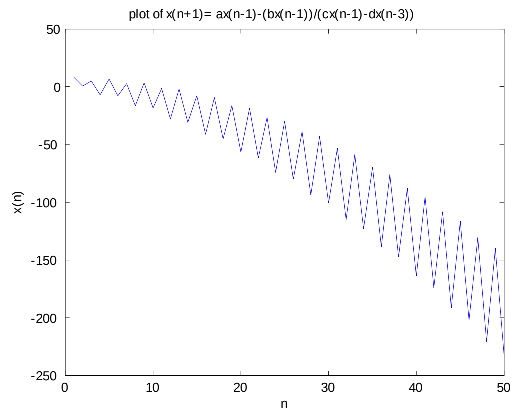


Figure 4.

REFERENCES

1. R. Abo-Zeid and C. Cinar, Global Behavior of The Difference Equation $x_{n+1} = Ax_{n-1}/(B - Cx_nx_{n-2})$, Bol. Soc. Paran. Mat.,31 (1) (2013), 43-49.
2. M. Aloqeili, Dynamics of a Rational Difference Equation, Appl. Math. Comp., 176 (2) (2006), 768-774.
3. A. Asiri, M. M. El-Dessoky and E. M. Elsayed, Solution of a third order fractional system of difference equations, Journal of Computational Analysis and Applications, 24 (3) (2018), 444-453.
4. F. Belhannache, N. Touafek and R. Abo-zeid, On a Higher-Order Rational Difference Equation, J. Appl. Math. & Informatics, 34 (5-6) (2016),369-382.
5. C. Cinar, On The Positive Solutions of The Difference Equation $x_{n+1} = ax_{n-1}/(1 + bx_nx_{n-1})$, Applied Mathematics and Computation, 156 (2004) 587-590.
6. C. Cinar, On The Positive Solutions of The Difference Equation $x_{n+1} = x_{n-1}/(1 + x_nx_{n-1})$, Applied Mathematics and Computation, 150 (2004) 21-24.
7. C. Cinar, On the difference equation $x_{n+1} = x_{n-1}/(-1 + x_nx_{n-1})$, Applied Mathematics and Computation, 158 (2004) 813-816.
8. E. M. Elabbasy, H. El-Metawally and E. M. Elsayed, On The Difference Equation $x_{n+1} = ax_n - bx_n/(cx_n - dx_{n-1})$, Advances in Difference Equations, 2006 (2006), 1-10.
9. E. M. Elabbasy, H. El-Metawally and E. M. Elsayed, On The Difference Equation $x_{n+1} = (ax_n^2 + bx_{n-1}x_{n-k})/(cx_n^2 + dx_{n-1}x_{n-k})$, Sarajevo Journal of Mathematics, 4 (17) (2008), 1-10.
10. E. M. Elabbasy, H. El-Metawally and E. M. Elsayed, On The Difference Equation $x_{n+1} = (\alpha x_{n-l} + \beta x_{n-k})/(Ax_{n-l} + Bx_{n-k})$, Acta Mathematica Vietnamica, 33 (1) (2008), 85-94.
11. E. M. Elabbasy, H. El-Metawally and E. M. Elsayed, Qualitative Behavior of Higher Order Difference Equation, Soochow Journal of Mathematics, 33 (4) (2007), 861-873.
12. E. M. Elabbasy and E. M. Elsayed, Global Attractivity and Periodic Nature of a Difference Equation, World Applied Sciences Journal, 12 (1), (2011), 39-47.
13. M. M. El-Dessoky and M. El-Moneam, On The Higher Order Difference Equation $x_{n+1} = Ax_n + Bx_{n-l} + Cx_{n-k} + (\gamma x_{n-k})/(Dx_{n-s} + Ex_{n-t})$, J. Computational Analysis and Applications, 25 (2) (2018), 342-354.
14. H. El-Metwally and E. M. Elsayed, Solution and Behavior of a Third Rational Difference Equation, Utilitas Mathematica, 88 (2012), 27-42.
15. E. M. Elsayed, Behavior and Expression of The Solutions of Some Rational Difference Equations, Journal of Computational Analysis and Applications, 15 (1) (2013), 73-81.

16. E. M. Elsayed, Dynamics of a Recursive Sequence of Higher Order, Communications on Applied Nonlinear Analysis, 16 (2) (2009), 37-50.
17. E. M. Elsayed, On The Global Attractivity and The Periodic Character of a Recursive Sequence, Opuscula Mathematica, 30 (4) (2010), 431-446.
18. E. M. Elsayed, A. Alghamdi, Dynamics and Global Stability of Higher Order Nonlinear Difference Equation, Journal of Computational Analysis and Applications, 21 (3) (2016),493-503.
19. E. M. Elsayed, A. Alotaibi and H. A. Almaylabi, The Behavior and Closed Form of The Solutions of Some Difference Equations, Journal of Computational and Theoretical Nanoscience, 13 (1-10) (2016).
20. E. M. Elsayed, F. Alzahrani, and H. S. Alayachi, Formulas and properties of some class of nonlinear difference equations, Journal of Computational Analysis and Applications, 24 (8) (2018), 1517-1531.
21. E. M. Elsayed and M. M. El-Dessoky, Dynamics and Behavior of a Higher Order Rational Recursive Sequence, Advances in Difference Equations, 2012 (69) (2012), 1-16.
22. E. M. Elsayed, M. Ghazel, and A. E. Matouk, Dynamical Analysis Of The Rational Difference Equation $x_{n+1} = Cx_{n-3}/(A + Bx_{n-1}x_{n-3})$, Journal of Computational Analysis and Applications, 23 (3) (2017), 496-507.
23. E. M. Elsayed, S. R. Mahmoud and A. T. Ali, Expression and Dynamics of The Solutions of Some Rational Recursive Sequences, Iranian Journal of Science & Technology, 38A3 (2014), 295-303.
24. T. Ibrahim, On The Third Order Rational Difference Equation $x_{n+1} = (x_nx_{n-2})/(x_{n-1}(a+bx_nx_{n-2}))$, Int. J. Contemp. Math. Sciences, 4 (27) (2009), 1321-1334.
25. R. Karatas, Global Behavior of a Higher Order Difference Equation, International Journal of Contemporary Mathematical Sciences, 12 (3) (2017),133-138.
26. A. Khaliq, F. Alzahrani and E. M. Elsayed, Global Attractivity of a Rational Difference Equation of Order Ten, Journal of Nonlinear Science and Applications, 9 (2016), 4465-4477.
27. A. Khaliq, and E. Elsayed, The Dynamics and Solution of Some Difference Equations, Journal of Nonlinear Sciences and Applications, 9 (3) (2016), 1052-1063.
28. Y. Kostrov, On a Second-Order Rational Difference Equation with a Quadratic Term, International Journal of Difference Equations, 11 (2) (2016),179-202.
29. M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman & Hall / CRC Press, 2001.
30. K. Liu, P. Li, F. Han, and W. Zhong, Global Dynamics of Nonlinear Difference Equation $x_{n+1} = A + x_n/x_{n-1}x_{n-2}$, Journal of Computational Analysis and Applications, 24 (6) (2018), 1125-1132.
31. M. A. Obaid, E. M. Elsayed and M. M. El-Dessoky, Global Attractivity and Periodic Character of Difference Equation of Order Four, Disc. Dyn. Nat. Soc., Volume 2012 (2012), Article ID 746738, 20 pages.
32. M. Saleh and M. Aloqeili, On The Difference Equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$, Appl. Math. Comput., 176 (1), (2006), 359 363.
33. D. Simsek, C. Cinar and I. Yalcinkaya, On The Recursive Sequence $x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}}$, Int. J. Contemp. Math. Sci., 1 (10) (2006), 475-480.
34. Y. Su and W. Li, Global Asymptotic Stability of a Second-Order Nonlinear Difference Equation, Applied Mathematics and Computation, 168 (2005), 981-989.
35. D. Tolly, Y. Yazlik and N. Taskara, Behavior of Positive Solutions of a Difference Equation, 35 (3-4) (2017), 217-230.
36. X. Yang, W. Su, B. Chen, G. M. Megson and D. J. Evans, On The Recursive Sequence $x_{n+1} = (ax_{n-1} + bx_{n-2})/(c + dx_{n-1}x_{n-2})$, Appl. Math. Comp., 162 (2005), 1485-1497.
37. X. Yan and W. Li , Global Attractivity In The Recursive Sequence $x_{n+1} = \alpha - \beta x_n/(\gamma - x_{n-1})$, Appl. Math. Comp., 138 (2-3) (2003), 415-423.
38. E. Zayed and M. El-Moneam, On The Rational Recursive Sequence $x_{n+1} = Ax_n + Bx_{n-k} + (\beta x_n + \gamma x_{n-k})/(Cx_n + Dx_{n-k})$, Acta Applicandae Mathematicae, 111 (3) (2010), 287–301.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

A necessary condition for eventually equilibrium or periodic to a system of difference equations

Wirot Tikjha[a,b,⋆] and Kunrasatree Piasu [a]
[a] Faculty of Science and Technology,Pibulsongkram Rajabhat University, Phitsanulok,Thailand 65000.
[b]Centre of Excellence in Mathematics, PERDO, CHE, Thailand.
⋆Corresponding author. E-mail: wirottik@psru.ac.th

March 1, 2018

Abstract

In this paper we consider the behavior of a special case of piecewise linear systems of difference equations with initial condition in first quadrant. We found a necessary condition that the solutions become equilibrium point or periodic with prime period 4 without using stability theorems. We constructed inductive statement to represent the behavior of the system and we apply useful lemmas in the proof of main theorem.

Key words: Difference equation, Periodic solution, Stability, Equilibrium point, Piecewise linear system of difference equation.
2010 Mathematics Subject Classification: 39A10 and 65Q10.

1 Introduction

To investigate stability of system of difference equations requires theorems that involve Jacobian matrix. So the functions of the system must are differentiable. Unfortunately, piecewise linear systems of difference equations are the system with absolute value. So we can not apply the stability theorem to the piecewise linear systems. In 1978 Lozi [1] hypothesized a simplified version of Hénon's transformation by using system of difference equation with absolute value and Lozi's Piecewise Linear Model admits a strange attractor with a specific parameter and initial condition. Then, Devaney [2, 3] investigated Gingerbreadman map and he was shown Gingerbreadman map, a map with absolute value, being chaotic in certain regions. Moreover, Ladas's open problem was mentioned an in article [4] as the system of difference equations:

$$x_{n+1} = |x_n| + ay_n + b, y_{n+1} = x_n + c|y_n| + d, n = 0, 1, \ldots$$

where the initial condition $(x_0, y_0) \in \mathbf{R}^2$ and the parameters $a, b, c,$ and $d \in \{-1, 0, 1\}$. He suggests to investigate boundedness character of solutions, the

global stability, and periodic nature of the solutions.There are several authors studied this open problem such as Grove et. al [4] found that every solution of a specific system is eventually periodic with period 3, Tikjha et. al [5, 6] found that the character of system is eventually periodic with some period and equilibrium point respectively. As mentioned above, we can not apply the stability theorems to this open problem. The common idea of proofs of the above systems of piecewise linear articles is to separate initial condition into few regions and find some characters of solution to the system of each region and then establishing lemmas and finally summarizing the behaviors of each system to be a theorem. Our ultimate goals is to know complete global character of system:

$$x_{n+1} = |x_n| - y_n - b, y_{n+1} = x_n - |y_n| + 1, n = 0, 1, \ldots \tag{1}$$

where the initial condition $(x_0, y_0) \in \mathbf{R}^2$ and the parameters b is any positive number. In this article we will focus to a special case of System(1) when $b = 3$ with initial condition are some points in the first quadrant.

2 Preliminaries

The following definitions [7] are used in this article. A *system of difference equations of the first order* is a system of the form

$$x_{n+1} = f(x_n, y_n), y_{n+1} = g(x_n, y_n), n = 0, 1, ... \tag{2}$$

where f and g are continuous functions which map $\mathbf{R}^2$ into $\mathbf{R}$.

A *solution* of the System(2) is a sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ which satisfies the system for all $n \geq 0$. If we prescribe an *initial condition* $(x_0, y_0) \in \mathbf{R}^2$ then

$$x_1 = f(x_0, y_0), y_1 = g(x_0, y_0)$$
$$x_2 = f(x_1, y_1), y_2 = g(x_1, y_1)$$
$$\vdots$$

and so the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of the System(2) exists for all $n \geq 0$ and is uniquely determined by the initial condition (x_0, y_0).

A solution of the System(2) which is constant for all $n \geq 0$ is called an *equilibrium solution.* If

$$(x_n, y_n) = (\bar{x}, \bar{y}) \text{ for all } n \geq 0$$

is an equilibrium solution of the System(2), then $(\bar{x}, \bar{y})$ is called an *equilibrium point*, or simply an *equilibrium* of the System(2).

A solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of a system of difference equations is called *eventually periodic with prime period p* or *eventually prime period p solution* if

there exists an integer $N > 0$ and p is the smallest positive integer such that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is periodic with period p; that is,

$$(x_{n+p}, y_{n+p}) = (x_n, y_n) \text{ for all } n \geq N. \tag{3}$$

The p consecutive point of the solution is called a *p-cycle* of System(2). We denote

$$\begin{pmatrix} x_0, y_0 \\ x_1, y_1 \\ x_2, y_2 \\ x_3, y_3 \end{pmatrix}$$

as 4-cycle which consists of $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ and (x_3, y_3) in xy plain.

3 Main Results

In this section we will investigate behaviors of the following system:

$$x_{n+1} = |x_n| - y_n - 3, y_{n+1} = x_n - |y_n| + 1, n = 0, 1, \ldots . \tag{4}$$

From System(4) and by simple calculations,

$$P_{4.1} = \begin{pmatrix} -5, & -1 \\ 3, & -5 \\ 5, & -1 \\ 3, & 5 \end{pmatrix} \quad \text{and} \quad P_{4.2} = \begin{pmatrix} 1, & -3 \\ 1, & -1 \\ -1, & 1 \\ -3, & -1 \end{pmatrix}$$

are two 4-cycles of System(4) and equilibrium point is $(-1, -1)$. For convenience in the later part of the proof, we let $S := \{(x, y)|x + \frac{1}{2} \leq y \leq x + 1\}, a_n := \frac{2^{2n+3}-1}{2^{2n+3}}, u_n := \frac{2^{2n+2}+1}{2^{2n+2}}, l_n := \frac{2^{2n+2}-1}{2^{2n+2}}, \delta_n = 2^{2n+4} - 1, B_{n+2} := \{(x, y)|x + \frac{2^{2n+3}-1}{2^{2n+3}} \leq y < x + \frac{2^{2n+4}-1}{2^{2n+4}}\}$. The proof of main theorem requires the following two lemmas.

Lemma 1. *Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be a solution of System(4) If there is positive integer N such that $x_N = -y_N - 2 < 0$ and $y_N < 0$ then (x_{N+1}, y_{N+1}) is equilibrium point $(-1, -1)$.*

Proof. The proof is obvious. □

Lemma 2. *Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be a solution of System(4) If there is positive integer N such that $x_N = y_N - 2 \geq 0$ then $\{(x_n, y_n)\}_{n=N+6}^{\infty}$ are in $P_{4.1}$.*

Proof. With condition $x_N = y_N - 2 \geq 0$ by simple calculation, we have $(x_{N+1}, y_{N+1}) = (-5, -1) \in P_{4.1}$. □

The following theorem provides a necessary condition of equilibrium point or prime period 4 to System(4) with initial condition in first quadrant.

Theorem 1. *Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System(4) and $x_0, y_0 \geq 0$. If*

$$(x_0, y_0) \in S - B_{n+2} \tag{5}$$

for all integer $n \geq -1$, then $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually equilibrium point or the prime period 4 solution($P_{4.1}$ or $P_{4.2}$).

Proof. Let $(x_0, y_0) \in S - B_{n+2}$ for all integer $n \geq -1$. Then $x_0 + \frac{1}{2} \leq y_0$ and $y_0 \leq x_0 + 1$, so we have $x_1 = x_0 - y_0 - 3 < 0$ and $y_1 = x_0 - y_0 + 1 \geq 0$, $x_2 = -2x_0 + 2y_0 - 1 \geq 0$ and $y_2 = -3$, $x_3 = -2x_0 + 2y_0 - 1 \geq 0$ and $y_3 = -2x_0 + 2y_0 - 3$.

If $y_0 \geq x_0 + \frac{3}{2}$ then $y_3 \geq 0$ and so $(x_4, y_4) = (-1, 3)$ and $(x_6, y_6) = (-5, -1) \in P_{4.1}$. Suppose that $y_0 < x_0 + \frac{3}{2}$ then $y_3 < 0$ and so $(x_4, y_4) = (-1, -4x_0 + 4y_0 - 3)$

If $y_4 = -4x_0 + 4y_0 - 3 < 0$ then we have $(x_5, y_5) \in B_1$. This contradicts Condition(5). Suppose that $y_4 \geq 0$, so $x_5 = 4x_0 - 4y_0 + 1 < 0$ and $y_5 = 4x_0 - 4y_0 + 3 \leq 0$, $x_6 = -8x_0 + 8y_0 - 7$ and $y_6 = 8x_0 - 8y_0 + 5 < 0$.

If $x_6 < 0$, that is $x_6 = -y_6 - 2 < 0$, then applying Lemma(1), we have $(x_7, y_7) = (-1, -1)$. Suppose that $x_6 \geq 0$, that is $x_0 + \frac{7}{8} \leq y_0 < x_0 + \frac{3}{2}$, then $(x_7, y_7) = (-16x_0 + 16y_0 - 15, -1)$.

If $x_7 < 0$, then $x_0 + \frac{7}{8} \leq y_0 < x_0 + \frac{15}{16}$, and so $(x_8, y_8) \in B_2$. This contradicts Condition(5). Suppose that $x_7 \geq 0$. That is $x_0 + \frac{15}{16} \leq y_0 < x_0 + \frac{3}{2}$, so $x_8 = -16x_0 + 16y_0 - 17$ and $y_8 = -16x_0 + 16y_0 - 15 \geq 0$.

If $x_8 \geq 0$ that is $x_0 + \frac{17}{16} \leq y_0 < x_0 + \frac{3}{2}$. Applying Lemma(2), $(x_9, y_9) \in P_{4.1}$. Suppose that $x_8 < 0$ that is $x_0 + \frac{15}{16} \leq y_0 < x_0 + \frac{17}{16}$. We have $x_9 = 32x_0 - 32y_0 + 29 < 0$ and $y_9 = -1$, $x_{10} = -32x_0 + 32y_0 - 31$ and $y_{10} = 32x_0 - 32y_0 + 29 < 0$.

If $x_{10} < 0$ that is $x_0 + \frac{15}{16} \leq y_0 < x_0 + \frac{31}{32}$. We have $(x_{11}, y_{11}) = (-1, -1)$. Suppose that $x_{10} \geq 0$ that is $x_0 + \frac{31}{32} \leq y_0 < x_0 + \frac{17}{16}$. We have a closed form of inductive statement on $n \geq 1$ and let $P(n)$ be the following statement:
For $(x_0, y_0) \in R_n = \{(x, y) | x + a_n \leq y < x + u_n\}$, then $x_{4n+6} \geq 0$ and so

$x_{4n+7} = -2^{2n+4}x_0 + 2^{2n+4}y_0 - \delta_n$

$y_{4n+7} = -1$.

If $(x_0, y_0) \in B_{n+2} = \{(x, y) | x + a_n \leq y < x + l_{n+1}\}$, then $x_{4n+7} < 0$.

If $(x_0, y_0) \in R_n - B_{n+2} = \{(x, y) | x + l_{n+1} \leq y < x + u_n\}$, then $x_{4n+7} \geq 0$ and so

$x_{4n+8} = -2^{2n+4}x_0 + 2^{2n+4}y_0 - \delta_n - 2$

$y_{4n+8} = -2^{2n+4}x_0 + 2^{2n+4}y_0 - \delta_n \geq 0$.

If $(x_0, y_0) \in R_n^* = \{(x, y) | x + u_{n+1} \leq y < x + u_n\}$, then $x_{4n+8} \geq 0$ and so

$x_{4n+9} = -5$

$y_{4n+9} = -1$.

If $(x_0, y_0) \in (R_n - B_{n+2}) - R_n^* = \{(x, y) | x + l_{n+1} \leq y < x + u_{n+1}\}$, then $x_{4n+8} < 0$ and so

$x_{4n+9} = 2^{2n+5}x_0 - 2^{2n+5}y_0 + 2\delta_n - 1 < 0$

$y_{4n+9} = -1$

$x_{4n+10} = -2^{2n+5}x_0 + 2^{2n+5}y_0 - 2\delta_n - 1$

$y_{4n+10} = 2^{2n+5}x_0 - 2^{2n+5}y_0 + 2\delta_n - 1 < 0$.

4

If $(x_0, y_0) \in \tilde{R}_n = \{(x,y)|x + l_{n+1} \le y < x + a_{n+1}\}$, then $x_{4n+10} < 0$ and so
$x_{4n+11} = -1$
$y_{4n+11} = -1$.
If $(x_0, y_0) \in R_{n+1} = \{(x,y)|x + a_{n+1} \le y < x + u_{n+1}\}$, then $x_{4n+10} \ge 0$.
We shall first show that $P(1)$ is true. For $(x_0, y_0) \in R_1 = \{(x,y)|x + \frac{31}{32} \le y < x + \frac{17}{16}\}$ and $\delta_1 = 63$, we have $x_{10} = -32x_0 + 32y_0 - 31 \ge 0$ and so
$x_{4(1)+7} = x_{11} = -2^{2(1)+4}x_0 + 2^{2(1)+4}y_0 - \delta_1$
$y_{4(1)+7} = y_{11} = -1$.
If $(x_0, y_0) \in B_3 = \{(x,y)|x + \frac{31}{32} \le y < x + \frac{63}{64}\}$, then $x_{11} = -64x_0 + 64y_0 - 63 < 0$.
If $(x_0, y_0) \in R_1 - B_3 = \{(x,y)|x + \frac{63}{64} \le y < x + \frac{17}{16}\}$, then $x_{11} = -64x_0 + 64y_0 - 63 \ge 0$ and so
$x_{4(1)+8} = x_{12} = -64x_0 + 64y_0 - 65 = -2^{2(1)+4}x_0 + 2^{2(1)+4}y_0 - \delta_1 - 2$
$y_{4(1)+8} = y_{12} = -64x_0 + 64y_0 - 63 = -2^{2(1)+4}x_0 + 2^{2(1)+4}y_0 - \delta_1 \ge 0$.
If $(x_0, y_0) \in R_1^* = \{(x,y)|x + \frac{65}{64} \le y < x + \frac{17}{16}\}$, then $x_{12} = -64x_0 + 64y_0 - 65 \ge 0$ and so
$x_{4(1)+9} = x_{13} = -5$
$y_{4(1)+9} = y_{13} = -1$.
If $(x_0, y_0) \in (R_1 - B_3) - R_1^* = \{(x,y)|x + \frac{63}{64} \le y < x + \frac{65}{64}\}$, then $x_{12} = -64x_0 + 64y_0 - 65 < 0$ and so
$x_{4(1)+9} = x_{13} = 128x_0 - 128y_0 + 125 = 2^{2(1)+5}x_0 - 2^{2(1)+5}y_0 + 2\delta_1 - 1 < 0$
$y_{4(1)+9} = y_{13} = -1$
$x_{4(1)+10} = x_{14} = -128x_0 + 128y_0 - 127 = -2^{2(1)+5}x_0 + 2^{2(1)+5}y_0 - 2\delta_1 - 1$
$y_{4(1)+10} = y_{14} = 128x_0 - 128y_0 + 125 = 2^{2(1)+5}x_0 - 2^{2(1)+5}y_0 + 2\delta_1 - 1 < 0$.
If $(x_0, y_0) \in \tilde{R}_1 = \{(x,y)|x + \frac{63}{64} \le y < x + \frac{127}{128}\}$, then $x_{14} = -128x_0 + 128y_0 - 127 < 0$ and so
$x_{4(1)+11} = x_{15} = -1$
$y_{4(1)+11} = y_{15} = -1$.
If $(x_0, y_0) \in R_2 = \{(x,y)|x + \frac{127}{128} \le y < x + \frac{65}{64}\}$, then $x_{4(1)+10} = -128x_0 + 128y_0 - 127 \ge 0$. Therefore $P(1)$ is true, as required.
Suppose $P(k)$ is true for a positive integer k. If $(x_0, y_0) \in R_{k+1} = \{(x,y)|x + \frac{2^{2k+5}-1}{2^{2k+5}} \le y < x + \frac{2^{2k+4}+1}{2^{2k+4}}\}$, then
$x_{4k+10} = -2^{2k+5}x_0 + 2^{2k+5}y_0 - 2\delta_k - 1 \ge 0$ and $y_{4k+10} = 2^{2k+5}x_0 - 2^{2k+5}y_0 + 2\delta_k - 1 < 0$ and so

$$\begin{aligned} x_{4(k+1)+7} = x_{4k+11} &= -2^{2k+6}x_0 + 2^{2k+6}y_0 - (4\delta_k + 3) \\ &= -2^{2(k+1)+4}x_0 + 2^{2(k+1)+4}y_0 - \delta_{k+1} \end{aligned}$$

$y_{4(k+1)+7} = y_{4k+11} = -1$.
If $(x_0, y_0) \in B_{k+3} = \left\{(x,y)|x + \frac{2^{2k+5}-1}{2^{2k+5}} \le y < x + \frac{2^{2k+6}-1}{2^{2k+6}}\right\}$, then
$x_{4k+11} = -2^{2k+6}x_0 + 2^{2k+6}y_0 - \delta_{k+1} < 0$.
If $(x_0, y_0) \in (R_{k+1} - B_{k+3}) = \left\{(x,y)|x + \frac{2^{2k+6}-1}{2^{2k+6}} \le y < x + \frac{2^{2k+4}+1}{2^{2k+4}}\right\}$, then
$x_{4k+11} = -2^{2k+6}x_0 + 2^{2k+6}y_0 - \delta_{k+1} \ge 0$, and so

$$\begin{aligned} x_{4(k+1)+8} = x_{4k+12} &= -2^{2k+6}x_0 + 2^{2k+6}y_0 - \delta_{k+1} - 2 \\ &= -2^{2k+6}x_0 + 2^{2k+6}y_0 - 2^{2k+6} - 1 \end{aligned}$$

5

$$\begin{aligned}y_{4(k+1)+8} = y_{4k+12} &= -2^{2k+6}x_0 + 2^{2k+6}y_0 - \delta_{k+1}\\ &= -2^{2(k+1)+4}x_0 + 2^{2(k+1)+4}y_0 - 2^{2(k+1)+4} + 1 \geq 0.\end{aligned}$$

If $(x_0, y_0) \in R^*_{k+1} = \{(x,y)|x + u_{k+2} \leq y < x + u_{k+1}\}$
$= \left\{(x,y)|x + \frac{2^{2k+6}+1}{2^{2k+6}} \leq y < x + \frac{2^{2k+4}+1}{2^{2k+4}}\right\}$, then

$$\begin{aligned}x_{4k+12} &= -2^{2k+6}x_0 + 2^{2k+6}y_0 - \delta_{k+1} - 2 \geq 0, \text{ and so}\\ x_{4(k+1)+9} &= x_{4k+13} = -5\\ y_{4(k+1)+9} &= x_{4k+13} = -1.\end{aligned}$$

If $(x_0, y_0) \in (R_{k+1} - B_{k+3}) - R^*_{k+1} = \left\{(x,y)|x + \frac{2^{2k+6}-1}{2^{2k+6}} \leq y < x + \frac{2^{2k+6}+1}{2^{2k+6}}\right\}$, then

$$\begin{aligned}x_{4k+12} &= -2^{2k+6}x_0 + 2^{2k+6}y_0 - \delta_{k+1} - 2 < 0, \text{ and so}\\ x_{4(k+1)+9} &= 2\left(2^{2k+6}x_0\right) - 2\left(2^{2k+6}y_0\right) + 2\delta_{k+1} - 1\\ &= 2^{2k+7}x_0 - 2^{2k+7}y_0 + 2\delta_{k+1} - 1\\ &= 2^{2(k+1)+5}x_0 - 2^{2(k+1)+5}y_0 + 2^{2(k+1)+5} - 3 < 0\\ y_{4(k+1)+9} &= x_{4k+13} = -1\\ x_{4(k+1)+10} &= -2^{2k+7}x_0 + 2^{2k+7}y_0 - 2\delta_{k+1} - 1\\ &= -2^{2(k+1)+5}x_0 + 2^{2(k+1)+5}y_0 - 2^{2(k+1)+5} + 1\\ y_{4(k+1)+10} &= x_{4k+14} = 2^{2k+7}x_0 - 2^{2k+7}y_0 + 2\delta_{k+1} - 1\\ &= 2^{2(k+1)+5}x_0 - 2^{2(k+1)+5}y_0 + 2^{2(k+1)+5} - 3 < 0.\end{aligned}$$

If $(x_0, y_0) \in \tilde{R}_{k+1} = \left\{(x,y)|x + \frac{2^{2k+6}-1}{2^{2k+6}} \leq y < x + \frac{2^{2k+7}-1}{2^{2k+7}}\right\}$, then

$$x_{4k+14} = -2^{2k+7}x_0 + 2^{2k+7}y_0 - 2\delta_{k+1} - 1 < 0,$$

and so

$$\begin{aligned}x_{4(k+1)+11} &= x_{4k+15} = -1\\ y_{4(k+1)+11} &= x_{4k+15} = -1.\end{aligned}$$

If $(x_0, y_0) \in R_{k+2} = \{(x,y)|x + a_{k+2} \leq y < x + u_{k+2}\}$
$= \left\{(x,y)|x + \frac{2^{2k+7}-1}{2^{2k+7}} \leq y < x + \frac{2^{2k+6}+1}{2^{2k+6}}\right\}$, then

$$x_{4k+14} = -2^{2k+7}x_0 + 2^{2k+7}y_0 - 2\delta_{k+1} - 1 \geq 0.$$

Hence $P(k+1)$ is also true. By mathematical induction $P(n)$ is true for any positive integer n. Note that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} l_n = \lim_{n\to\infty} u_n = 1.$$

If $y_0 = x_0 + 1$, then $(x_1, y_1) = (-4, 0)$ and so $(x_2, y_2) = (1, -3) \in P_{4.2}$ and the proof is complete. □

4 Conclusion

In this paper we showed that solution of System(4) with initial condition being a specific region in first quadrant is eventually equilibrium point or prime period 4. We described the behavior of solution to the system by using inductive statement. If initial conditions are in R^*_n then the solution is eventually prime period 4 ($P_{4.1}$). If initial conditions are in $\tilde{R}_1$ then the solution is eventually equilibrium point. The limit of R_n tend to a line $y = x + 1$ and if we choose

6

initial condition in the line $y = x + 1$, then solution is eventually prime period 4 ($P_{4.2}$).

Acknowledgement

This work was supported by National Research Council of Thailand and Pibulsongkram Rajabhat University. The first author is supported by the Centre of Excellence in Mathematics, CHE, Thailand.

References

[1] R. Lozi, Un attracteur etrange du type attracteur de Henon., J. Phys., **39**, 9-10, 1978.

[2] M.F. Barnsley, R.L. Devaney, B.B. Mandelbrot, H.O. Peitgen, D. Saupe and R.F.Voss, The Science of Fractal Images, Springer-Verlag, New York, 1991.

[3] R.L. Devaney, A piecewise linear model of the the zones of instability of an area-preserving map, Phys. D., **10D**, 387-393, 1984.

[4] E.A. Grove, E. Lapierre and W. Tikjha, On the global behavior of $x_{n+1} = |x_n| - y_n - 1$ and $y_{n+1} = x_n + |y_n|$. Cubo., **14**, 125 - 166, 2012.

[5] W. Tikjha, E.G. Lapierre and Y. Lenbury, On the global character of the system of piecewise linear difference equations $x_{n+1} = |x_n| - y_n - 1$ and $y_{n+1} = x_n - |y_n|$, Adv. Difference Equ., **2010**, (2010). doi: 10.1155/2010/573281

[6] W. Tikjha, E. Lapierre and T. Sitthiwirattham, The stable equilibrium of a system of piecewise linear difference equations, Adv. Difference Equ., **2017**, (2017). doi: 10.1186/s13662-017-1117-2

[7] E.A.Grove, G. Ladas, Periodicities in Nonlinear Difference Equations, Chapman Hall/CRC Press, New York, 2005.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

BI-UNIVALENT FUNCTIONS ASSOCIATED WITH WRIGHT HYPERGEOMETRIC FUNCTIONS

E. ANALOUEI ADEGANI[1], N. E. CHO[2,*], A. MOTAMEDNEZHAD[1] AND M. JAFARI[3]

Abstract. In this work, using of the Faber polynomial expansions we find upper bounds for $|a_n|$ $(n \geq 3)$ coefficients of functions in subclasses $\mathcal{G}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ and $\mathcal{B}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$, which were defined with Wright hypergeometric functions and quasi-subordinate conditions in the open unit disk. Our results generalize and improve some of the previously known results.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Many derivative and integral operators can be written in terms of convolution of certain analytic functions. This formalism facilitates further mathematical explorations and helps deep understanding of the geometric properties of such operators. For functions $f, h \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $h(z) = z + \sum\limits_{n=2}^{\infty} c_n z^n$, Hadamard product (or convolution) of $f(z)$ and $h(z)$ is denoted by $f * h$ and is defined by

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n c_n z^n = (h * f)(z).$$

Now, we recall and state some concepts of the special functions and operators as follows:

For complex parameters $\alpha_1, \ldots, \alpha_\ell$ $(\frac{\alpha_j}{A_j} \neq 0, -1, \ldots; j = 1, 2, \ldots, \ell)$ and $\beta_1, \ldots, \beta_m$ $(\frac{\beta_j}{B_j} \neq 0, -1, \ldots; j = 1, 2, \ldots, m)$, Fox's H-functions (for details, see [19]) which mean the Wright's generalized hypergeometric functions ${}_\ell\Psi_m$ with $A_j, B_j > 0$, give (rather general and typical examples of H-functions, not reducible to G-functions):

$$\begin{aligned} &{}_\ell\Psi_m\left(\begin{array}{c} (\alpha_1, A_1), \ldots, (\alpha_\ell, A_\ell) \\ (\beta_1, B_1), \ldots, (\beta_m, B_m) \end{array} : z \right) = {}_\ell \Psi_m\left[(\alpha_j, A_j)_{1,\ell}; (\beta_j, B_j)_{1,m}; z\right] \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + nA_1) \ldots \Gamma(\alpha_\ell + nA_\ell)}{\Gamma(\beta_1 + nB_1) \ldots \Gamma(\beta_m + nB_m)} \frac{z^n}{n!}, \end{aligned}$$

2010 *Mathematics Subject Classification*: Primary 30C45, 30C50; Secondary 30C80
Key words and phrases: Coefficient estimates, Faber polynomial expansion, Wright hypergeometric functions, Subordinate
* Corresponding author

where $1+\sum\limits_{n=1}^{m} B_n - \sum\limits_{n=1}^{\ell} A_n \geq 0\,(\ell, m \in \mathbb{N} = \{1,2,\ldots\})$ and for suitably bounded values of $|z|$.

Now the linear operator is introduced comprising of the generalized hypergeometric function from Srivastava [19] (see [7]) and Wright [24]. Let $\ell, m \in \mathbb{N}$ and suppose that the parameters $\alpha_1, A_1, \ldots, \alpha_\ell, A_\ell$ and $\beta_1, B_1, \ldots, \beta_m, B_m$ are also positive real numbers. Then, corresponding to a function

$$ {}_\ell\Phi_m\big[(\alpha_j, A_j)_{1,\ell}; (\beta_j, B_j)_{1,m}; z\big] $$

defined by

$$ {}_\ell\Phi_m\big[(\alpha_j, A_j)_{1,\ell}; (\beta_j, B_j)_{1,m}; z\big] = \Omega z\, {}_\ell\Psi_m\big[(\alpha_j, A_j)_{1,\ell}; (\beta_j, B_j)_{1,m}; z\big], $$

where $\Omega = \left(\prod\limits_{j=1}^{\ell}\Gamma(\alpha_j)\right)^{-1}\left(\prod\limits_{j=1}^{m}\Gamma(\beta_j)\right)$, we consider a linear operator

$$ \mathcal{W}\big[(\alpha_j, A_j)_{1,\ell}; (\beta_j, B_j)_{1,m}\big] : \mathcal{A} \longrightarrow \mathcal{A} $$

defined by the following Hadamard product

$$ \mathcal{W}\big[(\alpha_j, A_j)_{1,\ell}; (\beta_j, B_j)_{1,m}\big] f(z) := z\, {}_\ell\Phi_m\big[(\alpha_j, A_j)_{1,\ell}; (\beta_j, B_j)_{1,m}; z\big] * f(z). $$

We observe that, for $f(z)$ of the form (1.1), we have

$$ \mathcal{W}\big[(\alpha_j, A_j)_{1,\ell}; (\beta_j, B_j)_{1,m}\big] f(z) := z + \sum_{n=2}^{\infty} \varphi_n a_n z^n, $$

where

$$ \varphi_n = \frac{\Omega\,\Gamma(\alpha_1 + A_1(n-1))\ldots\Gamma(\alpha_\ell + A_\ell(n-1))}{(n-1)!\,\Gamma(\beta_1 + B_1(n-1))\ldots\Gamma(\beta_m + B_m(n-1))}. $$

If, for convenience, we write

$$ \mathcal{W}_m^\ell f(z) = \mathcal{W}\big[(\alpha_1, A_1), \ldots, (\alpha_\ell, A_\ell); (\beta_1, B_1), \ldots, (\beta_m, B_m)\big] f(z). $$

The Koebe one-quarter theorem [6] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Therefore, every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$ f^{-1}\big(f(z)\big) = z \;\; (z \in \mathbb{U}) \quad \text{and} \quad f\big(f^{-1}(w)\big) = w \;\; \left(|w| < r_0(f);\; r_0(f) \geqq \frac{1}{4}\right), $$

where

(1.2)

$$ \begin{aligned} g(w) = f^{-1}(w) &= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \\ &=: w + \sum_{n=2}^{\infty} b_n w^n. \end{aligned} $$

A function $f \in \mathcal{S}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1).

Determination of the bounds for the coefficients a_n is an important problem in geometric function theory as they give information about the geometric properties of these functions. For a brief history and interesting examples in the class Σ, see [13]. Recently, many researchers introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients, see,

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

for example, [4, 15, 20–22, 25, 27]. But the coefficient problem for each of the Taylor-Maclaurin coefficients $|a_n|$ $(n \in \mathbb{N}\backslash\{1,2,3\}$, is still an open problem.

A function $f(z)$ is said to be quasi-subordinate to $\phi(z)$ in the open unit disk $\mathbb{U}$ if there exist analytic functions $\psi(z)$ and $w(z)$, with $w(0) = 0$ such that $|\psi(z)| \leq 1$, $|w(z)| < 1$ and $f(z) = \psi(z)\phi(w(z))$. Denote this quasi-subordination by $f(z) \prec_{\tilde{q}} \phi(z)$. For $\psi(z) = 1$, the quasi-subordination reduces to the subordination (see [17, 18]).

Throughout this paper, we let $\phi(z)$ is analytic function in the unit disk $\mathbb{U}$ with $\phi(0) = 1$ such that

$$\phi(z) = 1 + C_1 z + C_2 z^2 + C_3 z^3 + \cdots \qquad (C_1 > 0)$$

and assume that the function $\psi(z)$ is analytic in the unit disk $\mathbb{U}$ and $|\psi(z)| \leq 1$ such that

$$\psi(z) = D_0 + D_1 z + D_2 z^2 + D_3 z^3 + \cdots .$$

Recently, Cho et al., [5] introduced subclasses $\mathcal{G}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ and $\mathcal{B}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ of Σ and only obtained estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses.

Definition 1.1. [5] A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{G}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z(\mathcal{W}_m^l f(z))'}{(1-\lambda)\mathcal{W}_m^l f(z) + \lambda z(\mathcal{W}_m^l f(z))'} - 1\right) \prec_{\tilde{q}} (\phi(z) - 1)$$

and

$$\frac{1}{\gamma}\left(\frac{w(\mathcal{W}_m^l g(w))'}{(1-\lambda)\mathcal{W}_m^l g(w) + \lambda w(\mathcal{W}_m^l g(w))'} - 1\right) \prec_{\tilde{q}} (\phi(w) - 1),$$

where $\gamma \in \mathbb{C}\backslash\{0\}$, $0 \leq \lambda < 1$, $z, w \in \mathbb{U}$ and the function g is given by (2.1).

Definition 1.2. [5] A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{B}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z^{1-\lambda}(\mathcal{W}_m^l f(z))'}{[\mathcal{W}_m^l f(z)]^{1-\lambda}} - 1\right) \prec_{\tilde{q}} (\phi(z) - 1)$$

and

$$\frac{1}{\gamma}\left(\frac{w^{1-\lambda}(\mathcal{W}_m^l g(w))'}{[\mathcal{W}_m^l g(w)]^{1-\lambda}} - 1\right) \prec_{\tilde{q}} (\phi(w) - 1),$$

where $\gamma \in \mathbb{C}\backslash\{0\}$, $\lambda \geq 0$, $z, w \in \mathbb{U}$ and the function g is given by (2.1).

Lemma 1.3. [6] *Let $u(z)$ be analytic in the unit disk $\mathbb{U}$ with $u(0) = 0$ and $|u(z)| < 1$ and suppose that $u(z) = \sum_{n=1}^{\infty} p_n z^n$. Then $|p_n| \leqq 1$ $(n \in \mathbb{N})$.*

Lemma 1.4. [9] *Let the function w in the Schwarz function is given by $w(z) = \sum\limits_{n=1}^{\infty} w_n z^n$, where $z \in \mathbb{U}$. Then for every complex number s,*

$$|w_2 + s w_1^2| \leq 1 + (|s| - 1)|w_1^2|.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Faber [8] introduced the Faber polynomials, which play an important role in various areas of mathematical sciences, especially in geometric function theory. By using the Faber polynomial expansion of functions $f \in \mathcal{S}$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed, (see for details [1] and [2]),

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n) w^n, \tag{1.3}$$

where $b_n = \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots, a_n)$, and

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 + \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} \\ &\times [a_5 + (-n+2) a_3^2] + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5) a_3 a_4] \\ &+ \sum_{j \geq 7} a_2^{n-j} V_j \end{aligned}$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $a_2, a_3, \cdots, a_n$, (see for details [2]). In particular, the first three terms of K_{n-1}^{-n} are

$$\frac{1}{2} K_1^{-2} = -a_2, \quad \frac{1}{3} K_2^{-3} = 2a_2^2 - a_3, \quad \frac{1}{4} K_3^{-4} = -(5a_2^3 - 5a_2 a_3 + a_4).$$

In general, for any $p \in \mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$, an expansion of K_n^p is (see for details [1, 23] or [2, page 349])

$$K_n^p = p a_{n+1} + \frac{p(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!3!} D_n^3 + \cdots + \frac{p!}{(p-n)!n!} D_n^n,$$

where

$$D_n^m(a_2, a_3, \cdots, a_n) = \sum_{n=1}^{\infty} \frac{m!(a_2)^{\mu_1} \cdots (a_n)^{\mu_n}}{\mu_1! \cdots \mu_n!} \tag{1.4}$$

and the sum is taken over all nonnegative integers $\mu_1, ..., \mu_n$ satisfying

$$\begin{cases} \mu_1 + \mu_2 + \cdots + \mu_n = m, \\ \mu_1 + 2\mu_2 + \cdots + n\mu_n = n. \end{cases}$$

We noth that it is clear that $D_n^n(a_2, a_3, \cdots, a_n) = a_2^n$.

Lemma 1.5. [2, Equation (1.6) and (1.7)] *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{S}$, and $k \in \mathbb{Z}$ then we have the following expansion*

$$\begin{aligned} \frac{z f'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^k &= 1 - \sum_{n=1}^{\infty} F_{n-1}^{n+k-1}(a_2, a_3, \ldots, a_n) z^{n-1} \\ &= 1 + \sum_{n=2}^{\infty} \left(1 + \frac{n-1}{k} \right) K_{n-1}^k (a_2, a_3, \ldots, a_n) z^{n-1}, \end{aligned}$$

where the first Faber polynomials $F_{n-1}^{n+k-1}(a_2, a_3, \ldots, a_n)$ are given by

$$F_1^{k+1}(a_2) = (1+\lambda) a_2, \quad F_2^{k+2}(a_2, a_3) = \frac{(\lambda-1)(\lambda+2)}{2} a_2^2 + (\lambda+2) a_3, \ldots.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Several researchers have solved coefficient estimates problem for various subclasses of bi-univalent functions by using Faber polynomial expansions, see for example [10, 11, 20, 26]. In the present paper, by using the Faber polynomial expansions we obtain estimates of coefficients $|a_n|$ where $n \geq 3$, of functions in the subclasses $\mathcal{G}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ and $\mathcal{B}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ of Σ with various special cases.

2. Main results

First, we can write that the Faber polynomial expansion for $f \in \mathcal{G}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ given by (1.1) is in the form of

(2.1)
$$\frac{1}{\gamma}\left(\frac{z(\mathcal{W}_m^l f(z))'}{(1-\lambda)\mathcal{W}_m^l f(z) + \lambda z(\mathcal{W}_m^l f(z))'} - 1\right) = \frac{1}{\gamma}\sum_{n=2}^{\infty} F_{n-1}(\varphi_2 a_2, \varphi_3 a_3, \cdots, \varphi_n a_n) z^{n-1},$$

where

$$F_1(\varphi_2 a_2) = (1-\lambda)\varphi_2 a_2, \quad F_2(\varphi_2 a_2, \varphi_3 a_3) = (\lambda^2 - 1)(\varphi_2 a_2)^2 + 2(1-\lambda)\varphi_3 a_3.$$

In general,

$$F_{n-1}(\varphi_2 a_2, \varphi_3 a_3, \cdots, \varphi_n a_n) = (1-\lambda)(n-1)\varphi_n a_n$$
$$+ \sum_{l=1}^{n-2} K_l^{-1}\Big((1+\lambda)\varphi_2 a_2, (1+2\lambda)\varphi_3 a_3, \cdots, (1+l\lambda)\varphi_{l+1} a_{l+1}\Big)(1-\lambda)(n-l-1)\varphi_{n-l} a_{n-l}.$$

Then to simplify, we define:

$$F(z) \prec_{\tilde{q}} (\phi(z) - 1) \quad \text{and} \quad G(w) \prec_{\tilde{q}} (\phi(w) - 1), \tag{2.2}$$

where

$$F(z) = \frac{1}{\gamma}\left(\frac{z(\mathcal{W}_m^l f(z))'}{(1-\lambda)\mathcal{W}_m^l f(z) + \lambda z(\mathcal{W}_m^l f(z))'} - 1\right) \text{ and } G(w) = \frac{1}{\gamma}\left(\frac{w(\mathcal{W}_m^l g(w))'}{(1-\lambda)\mathcal{W}_m^l g(w) + \lambda w(\mathcal{W}_m^l g(w))'} - 1\right),$$

$$F(z) = \frac{1}{\gamma}\left(\frac{z^{1-\lambda}(\mathcal{W}_m^l f(z))'}{[\mathcal{W}_m^l f(z)]^{1-\lambda}} - 1\right) \text{ and } G(w) = \frac{1}{\gamma}\left(\frac{w^{1-\lambda}(\mathcal{W}_m^l g(w))'}{[\mathcal{W}_m^l g(w)]^{1-\lambda}} - 1\right).$$

In addition, by definition of quasi-subordinate there exist analytic functions ψ and $u, v : \mathbb{U} \to \mathbb{U}$, where $u(z) = \sum_{n=1}^{\infty} p_n z^n$ and $v(z) = \sum_{n=1}^{\infty} q_n z^n$, so that

$$F(z) = \psi(z)[\phi(u(z)) - 1] \quad \text{and} \quad G(w) = \psi(w)[\phi(v(w)) - 1], \tag{2.3}$$

where by equation (1.4) we have

(2.4)
$$\psi(z)[\phi(u(z)) - 1] = [C_1 p_1 z + (C_1 p_2 + C_2 p_1^2) z^2 + \cdots][D_0 + D_1 z + D_2 z^2 + \cdots]$$
$$= \left(\sum_{n=1}^{\infty}\sum_{k=1}^{n} C_k D_n^k(p_1, p_2, \cdots, p_n) z^n\right)\sum_{n=0}^{\infty} D_n z^n$$

and

$$\varphi(w) h(v(w)) = \left(\sum_{n=1}^{\infty}\sum_{k=1}^{n} C_k D_n^k(q_1, q_2, \cdots, q_n) w^n\right)\sum_{n=0}^{\infty} D_n w^n. \tag{2.5}$$

Now, we obtain the following coefficient estimates for these subclasses.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Theorem 2.1. *Let the function $f \in \mathcal{G}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ be given by (1.1) and $D_0 \neq 0$. If $a_k = 0$ for $2 \leq k \leq n-1$, then*

$$|a_n| \leq \frac{|\gamma|\big(C_1 + |D_{n-1}|\big)}{(1-\lambda)(n-1)\varphi_n}, \qquad n \geq 3.$$

Theorem 2.2. *Let the function $f \in \mathcal{B}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ be given by (1.1) and $D_0 \neq 0$. If $a_k = 0$ for $2 \leq k \leq n-1$, then*

$$|a_n| \leq \frac{|\gamma|\big(C_1 + |D_{n-1}|\big)}{\big(\lambda + (n-1)\big)\varphi_n}, \qquad n \geq 3.$$

Theorem 2.3. *Let the function $f \in \mathcal{G}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ be given by (1.1) and $C_1 \geq |C_2|$. Then*

$$|a_2| \leq \frac{|\gamma D_0|\ C_1\sqrt{C_1}}{\sqrt{|\gamma D_0| C_1^2 \big|(\lambda^2-1)\varphi_2^2 + 2(1-\lambda)\varphi_3\big| + (C_1 - |C_2|)(1-\lambda)^2\varphi_2^2}}.$$

Theorem 2.4. *Let the function $f \in \mathcal{B}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ be given by (1.1) and $C_1 \geq |C_2|$. Then*

$$|a_2| \leq \frac{|\gamma D_0| C_1\sqrt{2C_1}}{\sqrt{|\gamma D_0| C_1^2 \big|(\lambda-1)(\lambda+2)\varphi_2^2 + 2(\lambda+2)\varphi_3\big| + 2(C_1 - |C_2|)(1+\lambda)^2\varphi_2^2}}.$$

Remark 2.5.
(1) If we take $\psi(z) = 1$ in Theorem 2.1, then we obtain estimates of coefficients $|a_n|$ $(n \geq 3)$ for subclass defined by Murugusundaramoorthy in [14, Theorem 2.2].
(2) If we take $\psi(z) = 1$ in Theorem 2.3, then we obtain an improvement of the estimates obtained for $|a_2|$ by Murugusundaramoorthy in [14, Theorem 2.2].
(3) By setting $\lambda = 0, \gamma = 1$ and $\ell = 2,\ m = 1$ with $A_1 = A_2 = B_1 = \alpha_1 = \alpha_2 = \beta_1 = 1$, in Theorem 2.3, we get $\varphi_n = 1$ and then we obtain an improvement of the estimates obtained for $|a_2|$ by Algahtani in [3, Theorem 2.5].
(4) By setting $\lambda = \gamma = 1$ and $\ell = 2,\ m = 1$ with $A_1 = A_2 = B_1 = \alpha_1 = \alpha_2 = \beta_1 = 1$, in Theorem 2.4, we get $\varphi_n = 1$ and then we obtain an improvement of the estimates obtained for $|a_2|$ by Algahtani in [3, Theorem 2.2].
(5) By setting $\lambda = 0, \gamma = 1$ and $\ell = 2,\ m = 1$ with $\alpha_1 = 2$ and $A_1 = A_2 = B_1 = \alpha_2 = \beta_1 = 1$, $(\mathcal{W}_1^2 f(z) = zf'(z))$ in Theorem 2.3, then we obtain an improvement of the estimates obtained for $|a_2|$ by Algahtani in [3, Theorem 2.8].
(6) By setting $\psi(z) = 1, \lambda = 0, \gamma = 1$ and $\ell = 2,\ m = 1$ with $\alpha_1 = 2$ and $A_1 = A_2 = B_1 = \alpha_2 = \beta_1 = 1$, in Theorem 2.3, then we obtain an improvement of the estimates obtained for $|a_2|$ by Algahtani in [3, Theorem 2.9].
(7) By setting $\psi(z) = 1, \lambda = 1, \gamma = 1$ and $\ell = 2,\ m = 1$ with $\alpha_1 = a, \alpha_2 = b, \beta_1 = c$, in Theorem 2.4, then we obtain an improvement of the estimates obtained for $|a_2|$ by Omar et al., in [16, Theorem 1].
(8) By setting $\psi(z) = 1, \lambda = 0, \gamma = 1$ and $\ell = 2,\ m = 1$ with $A_1 = A_2 = B_1 = \alpha_1 = \alpha_2 = \beta_1 = 1$, in Theorem 2.3, we get $\varphi_n = 1$ and then we

obtain an improvement of the estimates obtained for $|a_2|$ by Ali et al., in [4, Corollary 2.1].

(9) By setting $\psi(z) = 1$, $\lambda = \gamma = 1$ and $\ell = 2$, $m = 1$ with $A_1 = A_2 = B_1 = \alpha_1 = \alpha_2 = \beta_1 = 1$, in Theorem 2.4, we get $\varphi_n = 1$ and then we obtain an improvement of the estimates obtained for $|a_2|$ by Ali et al., in [4, Theorem 2.1].

(10) Theorem 2.3 and Theorem 2.4 are improvements of the results obtained by Cho et al. [5], respectively.

3. Proof of Theorems

Proof of Theorem 2.1. For this work, let

$$F(z) = \frac{1}{\gamma}\Big(\frac{z(\mathcal{W}_m^l f(z))'}{(1-\lambda)\mathcal{W}_m^l f(z) + \lambda z(\mathcal{W}_m^l f(z))'} - 1\Big)$$

and

$$G(w) = \frac{1}{\gamma}\Big(\frac{w(\mathcal{W}_m^l g(w))'}{(1-\lambda)\mathcal{W}_m^l g(w) + \lambda w(\mathcal{W}_m^l g(w))'} - 1\Big).$$

For the function $f \in \mathcal{G}_\Sigma^{l,m}(\gamma, \lambda, \phi)$, we have the expansion (2.1) and for the inverse map $g = f^{-1}$, considering (1.2), we get that

$$G(w) = \frac{1}{\gamma}\sum_{n=2}^{\infty} F_{n-1}(\varphi_2 b_2, \varphi_3 b_3, \cdots, \varphi_n b_n) w^{n-1}. \tag{3.1}$$

Comparing the coefficients of (2.1) and (2.4), we conclude

$$\frac{1}{\gamma}\Bigg[(1-\lambda)(n-1)\varphi_n a_n + \sum_{l=1}^{n-2} K_l^{-1}\Big((1+\lambda)\varphi_2 a_2, (1+2\lambda)\varphi_3 a_3, \cdots, (1+l\lambda)\varphi_{l+1} a_{l+1}\Big) \times (1-\lambda)(n-l-1)\varphi_{n-l} a_{n-l}\Bigg] = D_{n-1} + \sum_{t=1}^{n-1}\sum_{k=1}^{t} C_k D_t^k(p_1, p_2, \cdots, p_t) D_{n-(t+1)}. \tag{3.2}$$

Similarly, from (3.1) and (2.5), we have

$$\frac{1}{\gamma}\Bigg[(1-\lambda)(n-1)\varphi_n b_n + \sum_{l=1}^{n-2} K_l^{-1}\Big((1+\lambda)\varphi_2 b_2, (1+2\lambda)\varphi_3 b_3, \cdots, (1+l\lambda)\varphi_{l+1} b_{l+1}\Big) \times (1-\lambda)(n-l-1)\varphi_{n-l} b_{n-l}\Bigg] = D_{n-1} + \sum_{t=1}^{n-1}\sum_{k=1}^{t} C_k D_t^k(q_1, q_2, \cdots, q_t) D_{n-(t+1)}. \tag{3.3}$$

For $a_k = 0$ where $2 \le k \le n-1$ and $D_0 \neq 0$, we have $p_2 = p_3 = \cdots = p_{n-2} = 0$ and $q_2 = q_3 = \cdots = q_{n-2} = 0$. So from (3.2) and also from equation (1.3) and (3.3) we get, respectively,

$$\frac{1}{\gamma}(1-\lambda)(n-1)\varphi_n a_n = C_1 p_{n-1} + D_{n-1} \tag{3.4}$$

and

$$\frac{1}{\gamma}(1-\lambda)(n-1)\varphi_n b_n = -\frac{1}{\gamma}(1+\lambda)(n-1)\varphi_n a_n = C_1 q_{n-1} + D_{n-1}. \tag{3.5}$$

By solving either of the equations (3.4) and (3.5) for a_n and using Lemma 1.3, we obtain

$$|a_n| = \frac{|\gamma||C_1 p_{n-1} + D_{n-1}|}{(1-\lambda)(n-1)\varphi_n} \leq \frac{|\gamma|\big(C_1 + |D_{n-1}|\big)}{(1-\lambda)(n-1)\varphi_n}$$

and this completes the proof. □

Proof of Theorem 2.2. Let

$$F(z) = \frac{1}{\gamma}\Big(\frac{z^{1-\lambda}(\mathcal{W}_m^l f(z))'}{[\mathcal{W}_m^l f(z)]^{1-\lambda}} - 1\Big)$$

and

$$G(w) = \frac{1}{\gamma}\Big(\frac{w^{1-\lambda}(\mathcal{W}_m^l g(w))'}{[\mathcal{W}_m^l g(w)]^{1-\lambda}} - 1\Big).$$

For the function $f \in \mathcal{B}_\Sigma^{l,m}(\gamma, \lambda, \phi)$, by Lemma1.5 we have

$$F(z) = \frac{1}{\gamma}\sum_{n=2}^{\infty}\left(1 + \frac{n-1}{\lambda}\right) K_{n-1}^{\lambda}(\varphi_2 a_2, \varphi_3 a_3, \ldots, \varphi_n a_n) z^{n-1}. \tag{3.6}$$

For its inverse map $g = f^{-1}$, regarding the equality (1.2) we have

$$G(w) = \frac{1}{\gamma}\sum_{n=2}^{\infty}\left(1 + \frac{n-1}{\lambda}\right) K_{n-1}^{\lambda}(\varphi_2 b_2, \varphi_3 b_3, \ldots, \varphi_n b_n) w^{n-1}. \tag{3.7}$$

Comparing the coefficients of (3.6), and (2.4), we conclude that

$$\frac{1}{\gamma}\left(1 + \frac{n-1}{\lambda}\right) K_{n-1}^{\lambda}(\varphi_2 a_2, \varphi_3 a_3, \ldots, \varphi_n a_n) = D_{n-1} + \sum_{t=1}^{n-1}\sum_{k=1}^{t} C_k D_t^k(p_1, p_2, \cdots, p_t) D_{n-(t+1)}. \tag{3.8}$$

Similarly, from (3.7) and (2.5), we have

$$\frac{1}{\gamma}\left(1 + \frac{n-1}{\lambda}\right) K_{n-1}^{\lambda}(\varphi_2 b_2, \varphi_3 b_3, \ldots, \varphi_n b_n) = D_{n-1} + \sum_{t=1}^{n-1}\sum_{k=1}^{t} C_k D_t^k(q_1, q_2, \cdots, q_t) D_{n-(t+1)}. \tag{3.9}$$

Since $a_k = 0$ where $2 \leq k \leq n-1$, and $D_0 \neq 0$ from (3.8) and (3.9) we get, respectively,

$$\frac{1}{\gamma}\big(\lambda + (n-1)\big)\varphi_n a_n = C_1 p_{n-1} + D_{n-1}$$

and

$$-\quad \frac{1}{\gamma}\big(\lambda + (n-1)\big)\varphi_n a_n = C_1 q_{n-1} + D_{n-1}.$$

By solving either of the above equations for a_n and using Lemma 1.3, we conclude the desired results and this completes the proof. □

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Proof of Theorem 2.3. For $n = 2$ and $n = 3$ in (3.2) and (3.3), respectively, we obtain

$$(1-\lambda)\varphi_2 a_2 = \gamma D_0 C_1 p_1, \tag{3.10}$$

$$(\lambda^2-1)\varphi_2^2 a_2^2 + 2(1-\lambda)\varphi_3 a_3 = \gamma D_0[C_1 p_2 + C_2 p_1^2] + \gamma D_1 C_1 p_1, \tag{3.11}$$

$$-(1-\lambda)\varphi_2 a_2 = \gamma D_0 C_1 q_1, \tag{3.12}$$

$$(\lambda^2-1)\varphi_2^2 a_2^2 + 2(1-\lambda)\varphi_3(2a_2^2 - a_3) = \gamma D_0[C_1 q_2 + C_2 q_1^2] + \gamma D_1 C_1 q_1. \tag{3.13}$$

From (3.10) and (3.12), we get

$$p_1 = -q_1. \tag{3.14}$$

Adding (3.11) and (3.13) and using (3.14) we obtain

$$\big[2(\lambda^2-1)\varphi_2^2 + 4(1-\lambda)\varphi_3\big]a_2^2 = \gamma D_0 C_1\big[p_2 + \frac{C_2}{C_1}p_1^2 + q_2 + \frac{C_2}{C_1}q_1^2\big].$$

By using Lemma 1.4 we have

$$\begin{aligned}\big|2(\lambda^2-1)\varphi_2^2 + 4(1-\lambda)\varphi_3\big||a_2|^2 &\le |\gamma D_0|C_1\big[|p_2 + \frac{|C_2|}{C_1}p_1^2| + |q_2 + \frac{C_2}{C_1}q_1^2|\big]\\ &\le |\gamma D_0|C_1\big[2 + 2(\frac{|C_2|-C_1}{C_1})|p_1^2|\big]\\ &= |\gamma D_0|C_1\left[2 + 2\left(\frac{(|C_2|-C_1)(1-\lambda)^2\varphi_2^2|a_2^2|}{|\gamma D_0|^2 C_1^3}\right)\right].\end{aligned}$$

After simplification we have

$$\left(|\gamma D_0|C_1^2\left|2(\lambda^2-1)\varphi_2^2 + 4(1-\lambda)\varphi_3\right| + 2(C_1 - |C_2|)(1-\lambda)^2\varphi_2^2\right)|a_2|^2 \le 2|\gamma D_0|^2 C_1^3,$$

which implies that

$$|a_2|^2 \le \frac{|\gamma D_0|^2 C_1^3}{|\gamma D_0|C_1^2\big|(\lambda^2-1)\varphi_2^2 + 2(1-\lambda)\varphi_3\big| + (C_1 - |C_2|)(1-\lambda)^2\varphi_2^2}$$

and this completes the proof. □

Proof of Theorem 2.4. For $n = 2$ and $n = 3$ in (3.8) and (3.9), respectively, we obtain

$$\begin{aligned}(1+\lambda)\varphi_2 a_2 &= \gamma D_0 C_1 p_1,\\ \frac{(\lambda-1)(\lambda+2)}{2}\varphi_2^2 a_2^2 + (\lambda+2)\varphi_3 a_3 &= \gamma D_0[C_1 p_2 + C_2 p_1^2] + \gamma D_1 C_1 p_1,\\ -(1+\lambda)\varphi_2 a_2 &= \gamma D_0 C_1 q_1,\\ \left(\frac{(\lambda-1)(\lambda+2)}{2}\varphi_2^2 + 2(\lambda+2)\varphi_3\right)a_2^2 - (\lambda+2)\varphi_3 a_3 &= \gamma D_0[C_1 q_2 + C_2 q_1^2] + \gamma D_1 C_1 q_1.\end{aligned}$$

With similar method to Theorem 2.3 we get the desired results and this completes the proof. □

Acknowledgements

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2016R1D1A1A09916450).

References

[1] H. Airault and A. Bouali, Differential calculus on the Faber polynomials, Bull. Sci. Math. **130** (2006), 179-222.

[2] H. Airault and J. Ren, An algebra of differential operators and generating functions on the set of univalent functions, Bull. Sci. Math. **126** (2002), 343-367.

[3] O. Algahtani, Estimates of initial coefficients for certain subclasses of bi-univalent functions involving quasi-subordination, J. Nonlinear Sci. Appl. **10** (2017), 1004-1011.

[4] R. M. Ali, S. K. Lee, V. Ravichandran and S. Subramaniam, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, Appl. Math. Lett. **25** (2012), 344-351.

[5] N. E. Cho, G. Murugusundaramoorthy and K. Vijaya, Bi-univalent functions of complex order based on quasi-subordinate conditions involving Wright hypergeometric functions, J. Comput. Anal. Appl. **24** (2018), 58-70.

[6] P. L. Duren, *Univalent functions*, Grundlehren der Mathematischen Wissenschaften, 259, Springer, New York, 1983.

[7] J. Dziok and R. K. Raina, Families of analytic functions associated with the Wright generalized hypergeometric function, Demonstratio Math. **37** (2004), 533-542.

[8] G. Faber, Über polynomische Entwickelungen, Math. Ann. **57** (1903) 389-408.

[9] F. R. Keogh and E. P. Merkes, A coeffcient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc. 20 (1969), 8-12.

[10] S. G. Hamidi and J. M. Jahangiri, Faber polynomial coefficients of bi-subordinate functions, C. R. Math. Acad. Sci. Paris. **354** (2016), 365-370.

[11] J. M. Jahangiri, S. G. Hamidi and S. A. Halim, Coefficients of bi-univalent functions with positive real part derivatives, Bull. Malays. Math. Sci. Soc. **37** (2014), 633-640.

[12] S. Kanas, Seong-A. Kim and S. Sivasubramanian, Verification of Brannan and Clunie's conjecture for certain subclasses of bi-univalent function, Ann. Polonici Mathematici. **113** (3) (2015), 295-304.

[13] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc. **18** (1967), 63-68.

[14] G. Murugusundaramoorthy, Subclasses of bi-univalent functions of complex order based on subordination conditions involving Wright hypergeometric, J. Math. Fund. Sci. **47** (2015), 60-75.

[15] G. Murugusundaramoorthy, T. Janani and N. E. Cho, Bi-univalent functions of complex order based on subordinate conditions involving Hurwitz-Lerch zeta function, East Asian Math. J. **32** (2016), 47-59.

[16] R. Omar, S. A. Halim and A. Janteng, Subclasses of bi-univalent functions associated with Hohlov operator, WASET Int. J. Math. Comput. Sci. **11** (2017), 428-431.

[17] M. S. Robertson, Quasi-subordinate functions, In: Mathematical Essays Dedicated to A. J. MacIntyre, Ohio University Press, Athens, OH (1970), 311-330.

[18] M. S. Robertson, Quasi-subordination and coefficient conjecture, Bull. Amer. Math. Soc. **76** (1970), 1-9.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

[19] H. M. Srivastava, Some Fox's-Wright generalized hypergeometric functions and associated families of convolution operators, Appl. Anal. Discrete Math. **1** (2007), 56-71.

[20] H. M. Srivastava, S. S. Eker and R. M. Ali, Coeffcient bounds for a certain class of analytic and bi-univalent functions, Filomat. **29** (2015), 1839-1845.

[21] H. M. Srivastava, S. Gaboury and F. Ghanim, Coefficient estimates for a general subclass of analytic and bi-univalent functions of the Ma-Minda type, RACSAM. (2017). https://doi.org/10.1007/s13398-017-0416-5

[22] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and biunivalent functions, Appl. Math. Lett. **23** (2010), 1188-1192.

[23] P. G. Todorov, On the Faber polynomials of the univalent functions of class Σ, J. Math. Anal. Appl. **162** (1991), 268-276.

[24] E. M. Wright, The asymptotic expansion of the generalized hypergeometric function, Proc. London. Math. Soc. **46** (1946), 389-408.

[25] Q-H. Xu, H-G. Xiao and H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, Appl. Math. Comput. **218** (2012), 11461-11465.

[26] A. Zireh, E. Analouei Adegani and S. Bulut, Faber polynomial coefficient estimates for a comprehensive subclass of analytic bi-univalent functions defined by subordination, Bull. Belg. Math. Soc. Simon Stevin. **23** (2016), 487-504.

[27] A. Zireh and E. Analouei Adegani, Coefficient estimates for a subclass of analytic and bi-univalent functions, Bull. Iranian Math. Soc. **42** (2016), 881-889.

[1] Faculty of Mathematical Sciences, University of Shahrood, P.O.Box 316-36155, Shahrood, Iran

E-mail address: analoey.ebrahim@gmail.com, a.motamedne@gmail.com

[2] Department of Applied Mathematics Pukyong National University Busan 608-737, Korea

E-mail address: necho@pknu.ac.kr

[3] Department of Mathematics, Najafabad Branch, Islamic Azad University, Najafabad, Iran

E-mail address: mostafajafari83@gmail.com

Conformable Fractional Approximation by Choquet integrals

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we present the conformable fractional quantitative approximation of positive sublinear operators to the unit operator. These are given a precise Choquet integral interpretation. Initially we start with the study of the conformable fractional rate of the convergence of the well-known Bernstein-Kantorovich-Choquet and Bernstein-Durrweyer-Choquet polynomial Choquet-integral operators. Then we study in the fractional sense the very general comonotonic positive sublinear operators based on the representation theorem of Schmeidler (1986) [11]. We continue with the conformable fractional approximation by the very general direct Choquet-integral form positive sublinear operators. The case of convexity is also studied throughly and the estimates become much simpler. All approximations are given via inequalities involving the modulus of continuity of the approximated function's higher order conformable fractional derivative.

2010 AMS Mathematics Subject Classification: 26A33, 41A17, 41A25, 41A35, 41A36.

Keywords and Phrases: Jackson type inequality, Choquet integral, Conformable Fractional derivative, comonotonicity of functions and operators, Bernstein-Kantorovich-Choquet and Bernstein-Durrmeyer-Choquet operators, convexity.

1 Introduction

G. Choquet (1953) ([4]), introduced the capacities and his integral. Initially these were applied to statistical mechanics and potential theory, and they gave rise to the study of non-additive measure theory. Slowly but steady these ideas of Choquet started to attract economists especially after the very important

work of Shapley (1953) ([13]) in the study of cooperative games. Capacities and Choquet integrals became main stream to Decision theorists since 1989 when D. Schmeidler ([12]) was the first to use them in an axiomatic model of choice with non-additive beliefs. The expected utility results are strengthned by the use of Choquet capacities instead of probability measures.

In now days Choquet integral has wide applications, among others, to decision making under risk and uncertainty, in finance, in economics, in portofolio problems and in insurance.

Our motivation also comes from the foundations of Bayesian decision theory and subjective probability.

Because of the paramount importance of Choquet integral, we decided to research the related positive sublinear operators approximation, part of it is exhibited in this work in the conformable fractional sense.

2 Background - I

Next we present briefly about the Choquet integral, see also [8].

We make

Definition 1 *Consider $\Omega \neq \varnothing$ and let $\mathcal{C}$ be a σ-algebra of subsets in Ω.*

(i) (see, e.g., [14], p. 63) The set function $\mu : \mathcal{C} \to [0, +\infty]$ is called a monotone set function (or capacity) if $\mu(\varnothing) = 0$ and $\mu(A) \leq \mu(B)$ for all $A, B \in \mathcal{C}$, with $A \subset B$. Also, μ is called submodular if

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B), \text{ for all } A, B \in \mathcal{C}.$$

μ is called bounded if $\mu(\Omega) < +\infty$ and normalized if $\mu(\Omega) = 1$.

(ii) (see, e.g., [14], p. 233, or [4]) If μ is a monotone set function on $\mathcal{C}$ and if $f : \Omega \to \mathbb{R}$ is $\mathcal{C}$-measurable (that is, for any Borel subset $B \subset \mathbb{R}$ it follows $f^{-1}(B) \in \mathcal{C}$), then for any $A \in \mathcal{C}$, the Choquet integral is defined by

$$(C)\int_A f d\mu = \int_0^{+\infty} \mu(F_\beta(f) \cap A)\, d\beta + \int_{-\infty}^0 [\mu(F_\beta(f) \cap A) - \mu(A)]\, d\beta,$$

where we used the notation $F_\beta(f) = \{\omega \in \Omega : f(\omega) \geq \beta\}$. Notice that if $f \geq 0$ on A, then in the above formula we get $\int_{-\infty}^0 = 0$.

The integrals on the right-hand side are the usual Riemann integral.

The function f will be called Choquet integrable on A if $(C)\int_A f d\mu \in \mathbb{R}$.

Next we list some well known properties of the Choquet integral.

Remark 2 *If $\mu : \mathcal{C} \to [0, +\infty]$ is a monotone set function, then the following properties hold:*

2

(i) For all $a \geq 0$ we have $(C)\int_A af d\mu = a \cdot (C)\int_A f d\mu$ (if $f \geq 0$ then see, e.g., [14], Theorem 11.2, (5), p. 228 and if f is arbitrary sign, then see, e.g., [5], p. 64, Proposition 5.1, (ii)).

(ii) For all $c \in \mathbb{R}$ and f of arbitrary sign, we have (see, e.g., [14], pp. 232-233, or [5], p. 65) $(C)\int_A (f+c)\,d\mu = (C)\int_A f d\mu + c \cdot \mu(A)$.

If μ is submodular too, then for all f, g of arbitrary sign and lower bounded, we have (see, e.g., [5], p. 75, Theorem 6.3)

$$(C)\int_A (f+g)\,d\mu \leq (C)\int_A f d\mu + (C)\int_A g d\mu.$$

(iii) If $f \leq g$ on A then $(C)\int_A f d\mu \leq (C)\int_A g d\mu$ (see, e.g., [14], p. 228, Theorem 11.2, (3) if $f, g \geq 0$ and p. 232 if f, g are of arbitrary sign).

(iv) Let $f \geq 0$. If $A \subset B$ then $(C)\int_A f d\mu \leq (C)\int_B f d\mu$. In addition, if μ is finitely subadditive, then

$$(C)\int_{A\cup B} f d\mu \leq (C)\int_A f d\mu + (C)\int_B f d\mu.$$

(v) It is immediate that $(C)\int_A 1 \cdot d\mu(t) = \mu(A)$.

(vi) The formula $\mu(A) = \gamma(M(A))$, where $\gamma : [0,1] \to [0,1]$ is an increasing and concave function, with $\gamma(0) = 0$, $\gamma(1) = 1$ and M is a probability measure (or only finitely additive) on a σ-algebra on Ω (that is, $M(\varnothing) = 0$, $M(\Omega) = 1$ and M is countably additive), gives simple examples of normalized, monotone and submodular set functions (see, e.g., [5], pp. 16-17, Example 2.1). Such of set functions μ are also called distorsions of countably normalized, additive measures (or distorted measures). For a simple example, we can take $\gamma(t) = \frac{2t}{1+t}$, $\gamma(t) = \sqrt{t}$.

If the above γ function is increasing, concave and satisfies only $\gamma(0) = 0$, then for any bounded Borel measure m, $\mu(A) = \gamma(m(A))$ gives a simple example of bounded, monotone and submodular set function.

(vii) If μ is a countably additive bounded measure, then the Choquet integral $(C)\int_A f d\mu$ reduces to the usual Lebesgue type integral (see, e.g., [5], p. 62, or [14], p. 226).

(viii) If $f \geq 0$, then $(C)\int_A f d\mu \geq 0$.

(ix) Let $\mu = \sqrt{M}$, where M is the Lebesgue measure on $[0, +\infty)$, then μ is a monotone and submodular set function, furthermore μ is strictly positive, see [7].

(x) If $\Omega = \mathbb{R}^N$, $N \in \mathbb{N}$, we call μ strictly positive if $\mu(A) > 0$, for any open subset $A \subseteq \mathbb{R}^N$.

We need some possibility theory:

Definition 3 *([6]) For the $\Omega \neq \varnothing$, the power set $\mathcal{P}(\Omega)$ denotes the family of all subsets of Ω.*

3

(i) A function $\lambda : \Omega \rightarrow [0,1]$ with the property $\sup\{\lambda(s) : s \in \Omega\} = 1$, is called possibility distribution on Ω.

(ii) $P : \mathcal{P}(\Omega) \rightarrow [0,1]$ is called possibility measure, if it satisfies $P(\varnothing) = 0$, $P(\Omega) = 1$, and $P(\cup_{i \in I} A_i) = \sup\{P(A_i) : i \in I\}$ for all $A_i \subset \Omega$, and any I, an at most countable family of indices. Note that if $A, B \subset \Omega$, $A \subset B$, then the last property implies $P(A) \leq P(B)$ and that $P(A \cup B) \leq P(A) + P(B)$.

Any possibility distribution λ on Ω, induces the possibility measure $P_\lambda : \mathcal{P}(\Omega) \rightarrow [0,1]$, $P_\lambda(A) = \sup\{\lambda(s) : s \in A\}$, $A \subset \Omega$. Also, if $f : \Omega \rightarrow \mathbb{R}_+$, then the possibilistic integral of f on $A \subset \Omega$ with respect to P_λ is defined by $(Pos) \int_A f dP_\lambda = \sup\{f(t)\lambda(t) : t \in A\}$ (see [6], chapter 1).

Note that any possiblity measure μ is normalized, monotone and submodular. From $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ we get monotonicity, and from $\mu(A \cap B) \leq \min\{\mu(A), \mu(B)\}$ we derive the submodularity.

3 Background - II

We make

Definition 4 *([2]) Let $f : [a,b] \subseteq [0,\infty) \rightarrow \mathbb{R}$ and $\alpha \in (0,1]$. We say that f is an α-fractional continuous function, iff $\forall\ \varepsilon > 0\ \exists\ \delta > 0$: for any $x, y \in [a,b]$ such that $|x^\alpha - y^\alpha| \leq \delta$ we get that $|f(x) - f(y)| \leq \varepsilon$.*

We mention

Theorem 5 *([2]) Over $[a,b] \subseteq [0,\infty)$, $\alpha \in [0,1]$, an α-fractional continuous function is a uniformly continuous function and vice versa, a uniformly continuous function is an α-fractional continuous function.*

We need

Definition 6 *([2]) Let $[a,b] \subseteq [0,\infty)$, $\alpha \in [0,1]$. We define the α-fractional modulus of continuity:*

$$\omega_1^\alpha(f,\delta) := \sup_{\substack{x,y \in [a,b]: \\ |x^\alpha - y^\alpha| \leq \delta}} |f(x) - f(y)|, \quad \delta > 0. \tag{1}$$

Properties ([2]):

1) $\omega_1^\alpha(f,0) = 0$.

2) $\omega_1^\alpha(f,\delta) \rightarrow 0$ as $\delta \downarrow 0$, iff f is in the set of all α-fractional continuous functions, denoted as $f \in C_\alpha([a,b],\mathbb{R})$ $(= C([a,b],\mathbb{R}))$.

3) ω_1^α is ≥ 0 and non-decreasing on $\mathbb{R}_+$.

4) ω_1^α is subadditive:

$$\omega_1^\alpha (f, t_1 + t_2) \leq \omega_1^\alpha (f, t_1) + \omega_1^\alpha (f, t_2) . \tag{2}$$

5) ω_1^α is continuous on $\mathbb{R}_+$.

6) Clearly it holds

$$\omega_1^\alpha (f, t_1 + ... + t_n) \leq \omega_1^\alpha (f, t_1) + ... + \omega_1^\alpha (f, t_n) , \tag{3}$$

for $t = t_1 = ... = t_n$, we obtain

$$\omega_1^\alpha (f, nt) = n\omega_1^\alpha (f, t) . \tag{4}$$

7) Let $\lambda \geq 0$, $\lambda \notin \mathbb{N}$, we get

$$\omega_1^\alpha (f, \lambda t) \leq (\lambda + 1) \omega_1^\alpha (f, t) . \tag{5}$$

We notice that $\omega_1^\alpha (f, \delta)$ is finite when f is uniformly continuous on $[a, b] \subseteq [0, \infty)$.

We need

Definition 7 *([9], [10]) Let $f : [0, \infty) \to \mathbb{R}$. The conformable α-fractional derivative for $\alpha \in (0, 1]$ is given by*

$$D_\alpha f (t) := \lim_{\varepsilon \to 0} \frac{f \left(t + \varepsilon t^{1-\alpha} \right) - f (t)}{\varepsilon}, \tag{6}$$

$$D_\alpha f (0) = \lim_{t \to 0+} D_\alpha f (t) .$$

If f is differentiable, then

$$D_\alpha f (t) = t^{1-\alpha} f' (t) , \tag{7}$$

where f' is the usual derivative.

We define $D_\alpha^n f = D_\alpha^{n-1} (D_\alpha f)$, $D_\alpha^0 f = f$.

If $f : [0, \infty) \to \mathbb{R}$ is α-differentiable at $t_0 > 0$, $\alpha \in (0, 1]$, then f is continuous at t_0, see [10].

We need

Definition 8 *([2]) Here $C_+ ([a, b]) := \{f : [a, b] \subseteq [0, \infty) \to \mathbb{R}_+$, continuous functions$\}$. Let $L_N : C_+ ([a, b]) \to C_+ ([a, b])$, operators, $\forall\ N \in \mathbb{N}$, such that*

(i)

$$L_N (\alpha f) = \alpha L_N (f) , \ \forall \alpha \geq 0, \forall f \in C_+ ([a, b]) , \tag{8}$$

(ii) if $f, g \in C_+ ([a, b]) : f \leq g$, then

$$L_N (f) \leq L_N (g) , \ \ \forall N \in \mathbb{N}, \tag{9}$$

(iii)

$$L_N (f + g) \leq L_N (f) + L_N (g) , \ \ \forall\ f, g \in C_+ ([a, b]) . \tag{10}$$

We call $\{L_N\}_{N \in \mathbb{N}}$ positive sublinear operators.

We need

Theorem 9 *([2]) Let $\alpha \in (0,1]$, $[a,b] \subseteq [0,\infty)$. Suppose f is α-conformable fractional differentiable on $[a,b]$. $D_\alpha f$ is continuous on $[a,b]$. Let an $x \in [a,b]$ such that $D_\alpha f(x) = 0$, and L_N from $C_+([a,b])$ into itself, positive sublinear operators. Assume that $L_N(1) = 1$ and $L_N\left(|\cdot - x|^{\alpha+1}\right)(x)$, $L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x) > 0$, $\forall\ N \in \mathbb{N}$.*

Then

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1^\alpha\left(D_\alpha f, \left(L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{2(\alpha+1)}}\right)}{\alpha} \cdot$$

$$\left[\left(L_N\left(|\cdot - x|^{\alpha+1}\right)(x)\right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{2}\left(L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{2(\alpha+1)}}\right], \quad \forall\ N \in \mathbb{N}. \tag{11}$$

We make

Remark 10 *([2]) By [2], we get that*

$$L_N\left(|\cdot - x|^{\alpha+1}\right)(x) \leq \left(L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x)\right)^{\frac{1}{2}}. \tag{12}$$

As $N \to +\infty$, by (11) and (12), and $L_N\left((\cdot - x)^{2(\alpha+1)}\right)(x) \to 0$, we obtain that $L_N(f)(x) \to f(x)$.

We need

Theorem 11 *([2]) Let $\alpha \in (0,1]$, $n \in \mathbb{N}$. Suppose f is n times conformable α-fractional differentiable on $[a,b] \subseteq [0,\infty)$, and $D_\alpha^n f$ is continuous on $[a,b]$. For a fixed $x \in [a,b]$ we have $D_\alpha^k f(x) = 0$, $k = 1, ..., n$. Let positive sublinear operators $\{L_N\}_{N \in \mathbb{N}}$ from $C_+([a,b])$ into itself, such that $L_N(1) = 1$, and $L_N\left(|\cdot - x|^{n(\alpha+1)}\right)(x)$, $L_N\left(|\cdot - x|^{(n+1)(\alpha+1)}\right)(x) > 0$, $\forall\ N \in \mathbb{N}$. Then*

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1^\alpha\left(D_\alpha^n f, \left(L_N\left(|\cdot - x|^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{(n+1)(\alpha+1)}}\right)}{\alpha^n n!} \cdot \tag{13}$$

$$\left[\left(L_N\left(|\cdot - x|^{n(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{(n+1)}\left(L_N\left(|\cdot - x|^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{n\alpha}{(n+1)(\alpha+1)}}\right],$$

$\forall\ N \in \mathbb{N}$.

We make

Remark 12 *([2]) By [2], we get that*

$$L_N\left(|\cdot - x|^{n(\alpha+1)}\right)(x) \leq \left(L_N\left(|\cdot - x|^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{n}{n+1}}. \tag{14}$$

As $N \to +\infty$, *by (13), (14), and* $L_N\left(|\cdot - x|^{(n+1)(\alpha+1)}\right)(x) \to 0$, *we derive that* $L_N(f)(x) \to f(x)$.

We also need

Definition 13 *Let* $f \in C([a,b])$. *We define the usual first modulus of continuity of* f *as:*

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in[a,b]: \\ |x-y|\leq\delta}} |f(x) - f(y)|, \quad \delta > 0. \tag{15}$$

We need

Theorem 14 *([3]) Let* $\alpha \in (0,1]$ *and* $n \in \mathbb{N}$. *Suppose* $f \in C_+([a,b])$ *is* n *times conformable* α*-fractional differentiable on* $[a,b] \subseteq [0,\infty)$, *and* $x \in (a,b)$, *and* $D_\alpha^n f$ *is continuous on* $[a,b]$. *Let* $0 < h \leq \min(x-a, b-x)$ *and assume* $|D_\alpha^n f|$ *is convex over* $[a,b]$. *Furthermore assume that* $D_\alpha^k f(x) = 0$, $k = 1,...,n$. *Let* $\{L_N\}_{N\in\mathbb{N}}$ *from* $C_+([a,b])$ *into itself, positive sublinear operators such that:* $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. *Then*

$$|L_N(f)(x) - f(x)| \leq \left(\frac{\omega_1(D_\alpha^n f, h)\, b^{1-\alpha}}{(n+1)!\alpha^{n+1} h}\right) L_N\left(|\cdot - x|^{(n+1)\alpha}\right)(x), \quad \forall\ N \in \mathbb{N}. \tag{16}$$

We have

Theorem 15 *([3]) All as in Theorem 14. Additionally assume that* $L_N\left(|\cdot - x|^{(n+1)(\alpha+1)}\right)(x) > 0, \ \forall\ N \in \mathbb{N}$. *Then*

$$|L_N(f)(x) - f(x)| \leq \left(\frac{\omega_1(D_\alpha^n f, h)\, b^{1-\alpha}}{(n+1)!\alpha^{n+1} h}\right) \left(L_N\left(|\cdot - x|^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{\alpha+1}}, \tag{17}$$

$\forall\ N \in \mathbb{N}$.

An application of Theorem 15 follows:

Theorem 16 *([3])Let* $\{L_N\}_{N\in\mathbb{N}}$ *from* $C_+([a,b])$ *into itself, positive sublinear operators:* $L_N(1) = 1, \forall\ N \in \mathbb{N}$. *Also* $x \in (a,b)$ *and* $L_N\left(|\cdot - x|^{(n+1)(\alpha+1)}\right)(x) > 0, \ \forall\ N \in \mathbb{N}$. *Here* $\alpha \in (0,1]$ *and* $n \in \mathbb{N}$. *Suppose* $f \in C_+([a,b])$ *is* n *times conformable* α*-fractional differentiable on* $[a,b] \subseteq [0,\infty)$, *and* $D_\alpha^n f$ *is*

continuous on $[a,b]$. *Assume here that* $0 < \left(L_N\left(|\cdot - x|^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{\alpha+1}} \leq \min(x-a, b-x)$, $\forall\ N \in \mathbb{N} : N \geq N^* \in \mathbb{N}$, *and assume* $|D_\alpha^n f|$ *is convex over* $[a,b]$. *Furthermore assume that* $D_\alpha^k f(x) = 0$, $k = 1, ..., n$. *Then*

$$|L_N(f)(x) - f(x)| \leq \frac{b^{1-\alpha}\omega_1\left(D_\alpha^n f, \left(L_N\left(|\cdot - x|^{(n+1)(\alpha+1)}\right)(x)\right)^{\frac{\alpha}{\alpha+1}}\right)}{(n+1)!\alpha^{n+1}}, \quad (18)$$

$\forall\ N \in \mathbb{N} : N \geq N^* \in \mathbb{N}$.

If $L_N\left(|\cdot - x|^{(n+1)(\alpha+1)}\right)(x) \to 0$, *then* $L_N(f)(x) \to f(x)$, *as* $N \to +\infty$.

An application of Theorem 14 follows:

Theorem 17 *([3]) Let* $\{L_N\}_{N\in\mathbb{N}}$ *from* $C_+([a,b])$ *into itself, positive sublinear operators:* $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. *Also* $L_N\left(|\cdot - x|^{(n+1)\alpha}\right)(x) > 0$, $\forall\ N \in \mathbb{N}$. *Here* $\alpha \in (0,1]$, $n \in \mathbb{N}$ *and* $x \in (a,b)$; $[a,b] \subseteq [0,\infty)$. *Suppose* $f \in C_+([a,b])$ *is* n *times conformable* α*-fractional differentiable on* $[a,b]$, *and* $D_\alpha^n f$ *is continuous on* $[a,b]$. *Let* $0 < L_N\left(|\cdot - x|^{(n+1)\alpha}\right)(x) \leq \min(x-a, b-x)$, $\forall\ N \geq N^*$; N, $N^* \in \mathbb{N}$, *and assume* $|D_\alpha^n f|$ *is convex over* $[a,b]$. *Furthermore assume that* $D_\alpha^k f(x) = 0$, $k = 1, ..., n$. *Then*

$$|L_N(f)(x) - f(x)| \leq \frac{b^{1-\alpha}\omega_1\left(D_\alpha^n f, L_N\left(|\cdot - x|^{(n+1)\alpha}\right)(x)\right)}{(n+1)!\alpha^{n+1}}, \quad (19)$$

$\forall\ N \geq N^*$, *where* $N, N^* \in \mathbb{N}$.

If $L_N\left(|\cdot - x|^{(n+1)\alpha}\right)(x) \to 0$, *then* $L_N(f)(x) \to f(x)$, *as* $N \to +\infty$.

4 Background - III

We mention

Definition 18 *([7]) Let* $I = [0,1]$, $\mathcal{B}_I$ *the* σ*-algebra of all Borel measurable subsets of* I, $(\Gamma_{N,x})_{N\in\mathbb{N},\ x\in I}$ *will be the collection of the family* $\Gamma_{N,x} = \{\mu_{N,k,x}\}_{k=0}^N$, *of monotone, submodular and strictly positive set functions* $\mu_{N,k,x}$ *on* $\mathcal{B}_I$.

Let $f : [0,1] \to \mathbb{R}_+$ *be a* B_I*-measurable function which is bounded, and call* $p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}$, *for any* $x \in [0,1]$.

The Bernstein-Kantorovich-Choquet operators are defined by the formula

$$K_{N,\Gamma_{N,x}}(f)(x) = \sum_{k=0}^{N} p_{N,k}(x) \frac{(C)\int_{\frac{k}{(N+1)}}^{\frac{(k+1)}{(N+1)}} f(t)\, d\mu_{N,k,x}(t)}{\mu_{N,k,x}\left(\left[\frac{k}{(N+1)}, \frac{(k+1)}{(N+1)}\right]\right)}, \quad \forall\ x \in [0,1]. \quad (20)$$

If $\mu_{N,k,x} = \mu$, *for all* N, x, k, *we will denote* $K_{N,\Gamma_{N,x}}(f) := K_{N,\mu}(f)$.

Theorem 19 *([7]) Suppose that $\mu_{N,k,x} = \mu := \sqrt{M}$, for all N, k and x, where M is the Lebesgue measure on $[0,1]$. Then*

$$\left| K_{N,\mu}(f)(x) - f(x) \right| \leq 2\omega_1 \left(f, \frac{\sqrt{x(1-x)}}{\sqrt{N}} + \frac{1}{N} \right), \tag{21}$$

$\forall\ N \in \mathbb{N}$, $x \in [0,1]$, $f \in C_+([0,1])$, above ω_1 is over $[0,1]$.

Remark 20 *By [7] we have that*

$$K_{N,\mu}(|\cdot - x|)(x) \leq \frac{\sqrt{x(1-x)}}{\sqrt{N}} + \frac{1}{N}, \quad \forall\ N \in \mathbb{N}. \tag{22}$$

Let $m > 1$, notice that $|\cdot - x|^{m-1} \leq 1$, therefore

$$|\cdot - x|^m = |\cdot - x|\,|\cdot - x|^{m-1} \leq |\cdot - x|,$$

hence

$$K_{N,\mu}(|\cdot - x|^m)(x) \leq K_{N,\mu}(|\cdot - x|)(x),$$

that is

$$K_{N,\mu}(|\cdot - x|^m)(x) \leq \frac{\sqrt{x(1-x)}}{\sqrt{N}} + \frac{1}{N}, \quad \forall\ x \in [0,1],\ N \in \mathbb{N},\ m \geq 1. \tag{23}$$

Notice that $K_{N,\mu}(1) = 1$, $\forall\ N \in \mathbb{N}$.

Clearly $K_{N,\mu}$ operators are positive sublinear operators from $C_+([0,1])$ into itself.

We mention

Definition 21 *([8]) Here we consider measures of possibility. Denoting $p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}$, let us defined*

$$\lambda_{N,k}(t) := \frac{p_{N,k}(t)}{k^k N^{-N} (N-k)^{N-k} \binom{N}{k}} = \frac{t^k (1-t)^{N-k}}{k^k N^{-N} (N-k)^{N-k}}, \quad k = 0, ..., N. \tag{24}$$

By convention we assume that $0^0 = 1$, so that the cases $k = 0$, and $k = N$ make sense. By considering the root $\frac{k}{N}$ of $p'_{N,k}(x)$, it is clear that

$$\max\{p_{N,k}(t) : t \in [0,1]\} = k^k N^{-N} (N-k)^{N-k} \binom{N}{k},$$

which implies that each $\lambda_{N,k}$ is a possibility distribution on $[0,1]$.

Denoting by $P_{\lambda_{N,k}}$ the possibility measure induced by $\lambda_{N,k}$ and $\Gamma_{n,x} := \Gamma_N := \{P_{\lambda_{N,k}}\}_{k=0}^N$ (that is Γ_N is independent of x), we define the nonlinear Bernstein-Durrmeyer-Choquet polynomial operators with respect to the set functions in Γ_N given by the formula

$$D_{N,\Gamma_N}(f)(x) := \sum_{k=0}^{N} p_{N,k}(x) \frac{(C)\int_0^1 f(t)\, t^k (1-t)^{N-k} dP_{\lambda_{N,k}}(t)}{(C)\int_0^1 t^k (1-t)^{N-k} dP_{\lambda_{N,k}}(t)}, \quad (25)$$

$\forall\ x \in [0,1]$, $N \in \mathbb{N}$, $f \in C_+([0,1])$.

Remark 22 *Above $P_{\lambda_{N,k}}$ is bounded, monotone, submodular and strictly positive, $N \in \mathbb{N}$, $k = 0, 1, ..., N$. Notice that $D_{N,\Gamma_N}(1) = 1$, $\forall\ N \in \mathbb{N}$.*

Clearly D_{N,Γ_N} operators are positive sublinear operators mapping $C_+([0,1])$ into itself.

We mention

Theorem 23 *([8]) For every $f \in C_+([0,1])$, $x \in [0,1]$ and $N \in \mathbb{N} - \{1\}$, we have*

$$|D_{N,\Gamma_N}(f)(x) - f(x)| \le 2\omega_1\left(f, \frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)} + \sqrt{2}\sqrt{x}}{\sqrt{N}} + \frac{1}{N}\right), \quad (26)$$

where ω_1 is on $[0,1]$.

Remark 24 *By [8] we have that*

$$D_{N,\Gamma_N}(|\cdot - x|)(x) \le \frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)} + \sqrt{2}\sqrt{x}}{\sqrt{N}} + \frac{1}{N}, \quad \forall\ N \in \mathbb{N} - \{1\}. \quad (27)$$

Let $m > 1$, notice that $|\cdot - x|^{m-1} \le 1$, therefore

$$|\cdot - x|^m = |\cdot - x|\,|\cdot - x|^{m-1} \le |\cdot - x|,$$

hence

$$D_{N,\Gamma_N}(|\cdot - x|^m)(x) \le D_{N,\Gamma_N}(|\cdot - x|)(x),$$

that is

$$D_{N,\Gamma_N}(|\cdot - x|^m)(x) \le \frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)} + \sqrt{2}\sqrt{x}}{\sqrt{N}} + \frac{1}{N}, \quad (28)$$

$\forall\ N \in \mathbb{N} - \{1\}$, $\forall\ x \in [0,1]$, $m \ge 1$.

We make

Remark 25 *When $x \in [0,1]$, then the* $\max(x(1-x)) = \frac{1}{4}$, *at $x = \frac{1}{2}$. Therefore it holds*

$$\frac{\sqrt{x(1-x)}}{\sqrt{N}} + \frac{1}{N} \leq \frac{1}{2\sqrt{N}} + \frac{1}{N}, \tag{29}$$

$\forall\, x \in [0,1]$, $\forall\, N \in \mathbb{N}$.

Similarly, it holds

$$\frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)} + \sqrt{2}\sqrt{x}}{\sqrt{N}} + \frac{1}{N} \leq \frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}, \tag{30}$$

$\forall\, x \in [0,1]$, $\forall\, N \in \mathbb{N} - \{1\}$.

Corollary 26 *(to Theorem 19) It holds*

$$\|K_{N,\mu}(f) - f\|_\infty \leq 2\omega_1\left(f, \frac{1}{2\sqrt{N}} + \frac{1}{N}\right), \tag{31}$$

$\forall\, N \in \mathbb{N}$, $f \in C_+([0,1])$.

Corollary 27 *(to Theorem 23) It holds*

$$\|D_{N,\Gamma_N}(f) - f\|_\infty \leq 2\omega_1\left(f, \frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N}\right), \tag{32}$$

$\forall\, N \in \mathbb{N} - \{1\}$, $f \in C_+([0,1])$.

5 Main Results

Here first we apply some of the main theorems mentioned in section 3 to the Bernstein-Kantorovich-Choquet operators $K_{N,\mu}$, where $\mu := \sqrt{M}$, with M the Lebesgue measure on $[0,1]$. More precisely here it is

$$K_{N,\mu}(f)(x) = \sum_{k=0}^{N} p_{N,k}(x) \frac{(C)\int_{\frac{k}{(N+1)}}^{\frac{(k+1)}{(N+1)}} f(t)\, d\mu(t)}{\mu\left(\left[\frac{k}{(N+1)}, \frac{(k+1)}{(N+1)}\right]\right)}, \tag{33}$$

$\forall\, x \in [0,1]$, $\forall\, N \in \mathbb{N}$, $f \in C_+([0,1])$.

It follows applications to Bernstein-Durremeyer-Choquet operators D_{N,Γ_N}, see (25).

In particular we need (a variation of Theorem 11):

Theorem 28 *([2]) Let $\alpha \in (0,1]$ and $n \in \mathbb{N} : n\alpha \geq 1$. That is $\frac{1}{n} \leq \alpha \leq 1$. Suppose f is n times conformable α-fractional differentiable on $[a,b] \subseteq [0,\infty)$, and $D_\alpha^n f$ is continuous on $[a,b]$. For a fixed $x \in [a,b]$ we have $D_\alpha^k f(x) = 0$,*

11

$k = 1, ..., n$. *Let positive sublinear operators* $\{L_N\}_{N \in \mathbb{N}}$ *from* $C_+([a,b])$ *into itself, such that* $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$, *and* $\delta > 0$. *Then*

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1^\alpha(D_\alpha^n f, \delta)}{\alpha^n n!} \cdot$$

$$\left[L_N\left(|\cdot - x|^{n\alpha}\right)(x) + \frac{1}{(n+1)\delta} L_N\left(|\cdot - x|^{(n+1)\alpha}\right)(x) \right], \tag{34}$$

$\forall\ N \in \mathbb{N}$.

We present

Theorem 29 *Let* $\alpha \in (0,1]$ *and* $n \in \mathbb{N} : n\alpha \geq 1$. *Suppose* f *is* n *times conformable* α*-fractional differentiable on* $[0,1]$, *and* $D_\alpha^n f$ *is continuous on* $[0,1]$. *For a fixed* $x \in [0,1]$ *we have* $D_\alpha^k f(x) = 0$, $k = 1, ..., n$. *Then*

$$|K_{N,\mu}(f)(x) - f(x)| \leq \frac{\omega_1^\alpha\left(D_\alpha^n f, \left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}\right)^{\frac{1}{n+1}}\right)}{\alpha^n n!} \cdot$$

$$\left[\left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}\right) + \frac{1}{(n+1)} \left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}\right)^{\frac{n}{n+1}} \right] \leq$$

$$\frac{\omega_1^\alpha\left(D_\alpha^n f, \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{1}{n+1}}\right)}{\alpha^n n!} \left[\left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right) + \frac{1}{(n+1)} \left(\frac{1}{2\sqrt{N}} + \frac{1}{N}\right)^{\frac{n}{n+1}} \right], \tag{35}$$

$\forall\ N \in \mathbb{N}$.

Notice that $\lim_{N \to \infty} K_{N,\mu}(f)(x) = f(x)$.

Proof. By (34) we have

$$|K_{N,\mu}(f)(x) - f(x)| \leq \frac{\omega_1^\alpha(D_\alpha^n f, \delta)}{\alpha^n n!} \cdot$$

$$\left[K_{N,\mu}\left(|\cdot - x|^{n\alpha}\right)(x) + \frac{1}{(n+1)\delta} K_{N,\mu}\left(|\cdot - x|^{(n+1)\alpha}\right)(x) \right] \overset{(23)}{\leq}$$

$$\frac{\omega_1^\alpha(D_\alpha^n f, \delta)}{\alpha^n n!} \left[\left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}\right) + \frac{1}{(n+1)\delta} \left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}\right) \right] \tag{36}$$

(choose $\delta := \left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}\right)^{\frac{1}{n+1}} > 0$, then $\delta^{n+1} = \sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}$, and $\delta^n = \left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}\right)^{\frac{n}{n+1}}$)

$$= \frac{\omega_1^\alpha\left(D_\alpha^n f, \left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N}\right)^{\frac{1}{n+1}}\right)}{\alpha^n n!} \cdot$$

12

$$\left[\left(\sqrt{\frac{x(1-x)}{N}}+\frac{1}{N}\right)+\frac{1}{(n+1)}\left(\sqrt{\frac{x(1-x)}{N}}+\frac{1}{N}\right)^{\frac{n}{n+1}}\right] \overset{(29)}{\leq}$$

$$\frac{\omega_1^{\alpha}\left(D_{\alpha}^n f, \left(\frac{1}{2\sqrt{N}}+\frac{1}{N}\right)^{\frac{1}{n+1}}\right)}{\alpha^n n!}\left[\left(\frac{1}{2\sqrt{N}}+\frac{1}{N}\right)+\frac{1}{(n+1)}\left(\frac{1}{2\sqrt{N}}+\frac{1}{N}\right)^{\frac{n}{n+1}}\right], \tag{37}$$

proving the claim. ■

We continue with

Theorem 30 *All as in Theorem 29. Then*

$$|(D_{N,\Gamma_N}(f))(x)-f(x)| \leq \frac{\omega_1^{\alpha}\left(D_{\alpha}^n f, \left(\frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}}+\frac{1}{N}\right)^{\frac{1}{n+1}}\right)}{\alpha^n n!}$$

$$\left[\left(\frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}}+\frac{1}{N}\right)+\right.$$

$$\left.\frac{1}{(n+1)}\left[\frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}}+\frac{1}{N}\right]^{\frac{n}{n+1}}\right] \leq$$

$$\frac{\omega_1^{\alpha}\left(D_{\alpha}^n f, \left(\frac{1+3\sqrt{2}}{2\sqrt{N}}+\frac{1}{N}\right)^{\frac{1}{n+1}}\right)}{\alpha^n n!}\cdot \tag{38}$$

$$\left[\left(\frac{1+3\sqrt{2}}{2\sqrt{N}}+\frac{1}{N}\right)+\frac{1}{(n+1)}\left[\frac{1+3\sqrt{2}}{2\sqrt{N}}+\frac{1}{N}\right]^{\frac{n}{n+1}}\right],$$

$\forall\ N \in \mathbb{N}-\{1\}$.

Notice that $\lim_{N\to+\infty} D_{N,\Gamma_N}(f)(x) = f(x)$.

Proof. By (34) we have

$$|D_{N,\Gamma_N}(f)(x)-f(x)| \leq \frac{\omega_1^{\alpha}(D_{\alpha}^n f, \delta)}{\alpha^n n!}\cdot$$

$$\left[D_{N,\Gamma_N}\left(|\cdot-x|^{n\alpha}\right)(x)+\frac{1}{(n+1)\delta}D_{N,\Gamma_N}\left(|\cdot-x|^{(n+1)\alpha}\right)(x)\right] \overset{(28)}{\leq}$$

$$\frac{\omega_1^{\alpha}(D_{\alpha}^n f, \delta)}{\alpha^n n!}\left[\left(\frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}}+\frac{1}{N}\right)+\right.$$

$$\left.\frac{1}{(n+1)\delta}\left(\frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}}+\frac{1}{N}\right)\right] \tag{39}$$

13

(choose $\delta := \left(\frac{(1+\sqrt{2})\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}} + \frac{1}{N} \right)^{\frac{1}{n+1}} > 0$, then

$\delta^{n+1} = \frac{(1+\sqrt{2})\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}} + \frac{1}{N}$, and $\delta^{n} = \left(\frac{(1+\sqrt{2})\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}} + \frac{1}{N} \right)^{\frac{n}{n+1}}$)

$$= \frac{\omega_1^{\alpha}\left(D_{\alpha}^{n} f, \left(\frac{(1+\sqrt{2})\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}} + \frac{1}{N} \right)^{\frac{1}{n+1}} \right)}{\alpha^n n!} \cdot$$

$$\left[\left(\frac{\left(1+\sqrt{2}\right)\sqrt{x\left(1-x\right)}+\sqrt{2x}}{\sqrt{N}} + \frac{1}{N} \right) + \right.$$

$$\left. \frac{1}{(n+1)} \left[\frac{\left(1+\sqrt{2}\right)\sqrt{x\left(1-x\right)}+\sqrt{2x}}{\sqrt{N}} + \frac{1}{N} \right]^{\frac{n}{n+1}} \right] \overset{(30)}{\leq} \tag{40}$$

$$\frac{\omega_1^{\alpha}\left(D_{\alpha}^{n} f, \left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N} \right)^{\frac{1}{n+1}} \right)}{\alpha^n n!} \cdot$$

$$\left[\left(\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N} \right) + \frac{1}{(n+1)} \left[\frac{1+3\sqrt{2}}{2\sqrt{N}} + \frac{1}{N} \right]^{\frac{n}{n+1}} \right],$$

$\forall\ N \in \mathbb{N} - \{1\}$, proving the claim. ■

Next we apply Theorem 14.

We give

Theorem 31 *Let $\alpha \in (0,1]$ and $n \in \mathbb{N}$ such that $(n+1)\alpha \geq 1$, that is $\frac{1}{n+1} \leq \alpha \leq 1$. Suppose $f \in C_+([0,1])$ is n times conformable α-fractional differentiable on $[0,1]$, and $x \in (0,1)$, and $D_{\alpha}^{n} f$ is continuous on $[0,1]$. Let $N^* \in \mathbb{N}$ such that $\frac{1}{2\sqrt{N^*}} + \frac{1}{N^*} \leq \min(x, 1-x)$ and assume $|D_{\alpha}^{n} f|$ is convex over $[0,1]$. Furthermore assume that $D_{\alpha}^{k} f(x) = 0$, $k = 1, ..., n$. Then*

$$\left| \left(K_{N,\mu}(f) \right)(x) - f(x) \right| \leq \frac{\omega_1 \left(D_{\alpha}^{n} f, \frac{1}{2\sqrt{N}} + \frac{1}{N} \right)}{(n+1)!\alpha^{n+1}}, \tag{41}$$

$\forall\ N \geq N^*$, $N \in \mathbb{N}$.

It holds $\lim_{N \to +\infty} K_{N,\mu}(f)(x) = f(x)$.

Proof. By (16) we get

$$\left| K_{N,\mu}(f)(x) - f(x) \right| \leq \frac{\omega_1\left(D_{\alpha}^{n} f, h \right)}{(n+1)!\alpha^{n+1} h} K_{N,\mu}\left(|\cdot - x|^{(n+1)\alpha} \right)(x) \overset{(23)}{\leq}$$

$$\frac{\omega_1\left(D_{\alpha}^{n} f, h \right)}{(n+1)!\alpha^{n+1} h} \left(\sqrt{\frac{x(1-x)}{N}} + \frac{1}{N} \right) \overset{(29)}{\leq}$$

14

$$\frac{\omega_1\left(D_\alpha^n f, h\right)}{(n+1)!\alpha^{n+1} h}\left(\frac{1}{2\sqrt{N}}+\frac{1}{N}\right)= \tag{42}$$

(setting $h:=\frac{1}{2\sqrt{N}}+\frac{1}{N}>0$)

$$\frac{\omega_1\left(D_\alpha^n f, \frac{1}{2\sqrt{N}}+\frac{1}{N}\right)}{(n+1)!\alpha^{n+1}},$$

proving the claim. ∎

We continue with

Theorem 32 *Let $x\in(0,1)$ and $N^*\in\mathbb{N}-\{1\}:\frac{(1+3\sqrt{2})}{2\sqrt{N^*}}+\frac{1}{N^*}\leq\min(x,1-x)$. The rest as in Theorem 31. Then*

$$\left|\left(D_{N,\Gamma_N}(f)\right)(x)-f(x)\right|\leq\frac{\omega_1\left(D_\alpha^n f, \frac{(1+3\sqrt{2})}{2\sqrt{N}}+\frac{1}{N}\right)}{(n+1)!\alpha^{n+1}}, \tag{43}$$

$\forall\ N\geq N^*,\ N\in\mathbb{N}-\{1\}.$

It holds $\lim_{N\to+\infty} D_{N,\Gamma_N}(f)(x)=f(x)$.

Proof. We use Theorem 14:

By (16) we get

$$\left|D_{N,\Gamma_N}(f)(x)-f(x)\right|\leq\frac{\omega_1\left(D_\alpha^n f, h\right)}{(n+1)!\alpha^{n+1} h}D_{N,\Gamma_N}\left(|\cdot-x|^{(n+1)\alpha}\right)(x)\overset{(28)}{\leq}$$

$$\frac{\omega_1\left(D_\alpha^n f, h\right)}{(n+1)!\alpha^{n+1} h}\left(\frac{\left(1+\sqrt{2}\right)\sqrt{x(1-x)}+\sqrt{2x}}{\sqrt{N}}+\frac{1}{N}\right)\overset{(30)}{\leq}$$

$$\frac{\omega_1\left(D_\alpha^n f, h\right)}{(n+1)!\alpha^{n+1} h}\left(\frac{\left(1+3\sqrt{2}\right)}{2\sqrt{N}}+\frac{1}{N}\right) \tag{44}$$

(setting $h:=\frac{(1+3\sqrt{2})}{2\sqrt{N}}+\frac{1}{N}>0$)

$$=\frac{\omega_1\left(D_\alpha^n f, \frac{(1+3\sqrt{2})}{2\sqrt{N}}+\frac{1}{N}\right)}{(n+1)!\alpha^{n+1}},$$

proving the claim. ∎

We need

Definition 33 *Let Ω be a set, and let $f,g:\Omega\to\mathbb{R}$ be bounded functions. We say that f and g are comonotonic, if for every $\omega,\omega'\in\Omega$,*

$$\left(f(\omega)-f(\omega')\right)\left(g(\omega)-g(\omega')\right)\geq 0. \tag{45}$$

15

We also need the famous Schmeidler's Representation Theorem (Schmeidler 1986)

Theorem 34 *([11]) Denote with $\mathcal{L}_{\infty}(\mathcal{A})$ the vector space of $\mathcal{A}$-measurable bounded real valued functions on Ω, where $\mathcal{A} \subset 2^{\Omega}$ is a σ-algebra. Given a real functional $\Gamma : \mathcal{L}_{\infty}(\mathcal{A}) \to \mathbb{R}$, assume that for $f, g \in \mathcal{L}_{\infty}(\mathcal{A})$:*

(i) $\Gamma(cf) = c\Gamma(f)$, $\forall\ c > 0$,

(ii) $f \leq g$, implies $\Gamma(f) \leq \Gamma(g)$,

and

(iii) $\Gamma(f+g) = \Gamma(f) + \Gamma(g)$, for any comonotonic f, g.

Then $\gamma(A) := \Gamma(1_A)$, $\forall\ A \in \mathcal{A}$, defines a finite monotone set function on $\mathcal{A}$, and Γ is the Choquet integral with respect to γ, i.e.

$$\Gamma(f) = (C)\int_{\Omega} f(t)\, d\gamma(t), \quad \forall\ f \in \mathcal{L}_{\infty}(\mathcal{A}). \tag{46}$$

Above 1_A denotes the characteristic function on A.

Next we give nice interpretations of Theorems 9, 11, 16, 17 involving Choquet integrals and based on Theorem 34.

We make

Remark 35 *Consider here $[a,b] \subset \mathbb{R}_+$, $\mathcal{B} = \mathcal{B}([a,b])$ is the Borel σ-algebra on $[a,b]$, and $\mathcal{L}_{\infty}(\mathcal{B})$ is the vector space of $\mathcal{B}$-measurable bounded real valued functions on $[a,b]$. Let $(L_N)_{N\in\mathbb{N}}$ be a sequence of positive sublinear operators from $\mathcal{L}_{\infty}(\mathcal{B})$ into $C_+([a,b])$, and $x \in [a,b]$. That is here L_N fulfills the positive homogenuity, monotonicity and subadditivity properties, see (8)-(10).*

Assume $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. Clearly here $\mathcal{L}_{\infty}(\mathcal{B}) \supset C_+([a,b])$, where $[a,b] \subset [0,\infty)$. In particular we treat $L_N|_{C_+([a,b])}$, just denoted for simplicity by L_N, $\forall\ N \in \mathbb{N}$.

It is clear that $L_N(\cdot)(x) : \mathcal{L}_{\infty}(\mathcal{B}) \to \mathbb{R}$ is a functional, $\forall\ N \in \mathbb{N}$. It has the properties:

(i)

$$L_N(cf)(x) = cL_N(f)(x),\ \forall\ c > 0,\ \forall\ f \in \mathcal{L}_{\infty}(\mathcal{B}), \tag{47}$$

(ii)

$$f \leq g, \text{ implies } L_N(f)(x) \leq L_N(g)(x), \quad \text{where } f, g \in \mathcal{L}_{\infty}(\mathcal{B}), \tag{48}$$

and

(iii)

$$L_N(f+g)(x) \leq L_N(f)(x) + L_N(g)(x), \quad \forall\ f, g \in \mathcal{L}_{\infty}(\mathcal{B}). \tag{49}$$

For comonotonic $f, g \in \mathcal{L}_{\infty}(\mathcal{B})$, we further assume that

$$L_N(f+g)(x) = L_N(f)(x) + L_N(g)(x). \tag{50}$$

In that case L_N is called comonotonic.

16

By Theorem 34 we get that:

$$\gamma_{N,x}(A) := L_N(1_A)(x), \ \forall A \in \mathcal{B}, \forall N \in \mathbb{N}, \tag{51}$$

defines a finite monotone set function on $\mathcal{B}$, and

$$L_N(f)(x) = (C)\int_a^b f(t)\, d\gamma_{N,x}(t), \tag{52}$$

$\forall f \in \mathcal{L}_\infty(\mathcal{B})$, $\forall N \in \mathbb{N}$.

In particular (52) is valid for any $f \in C_+([a,b])$. Furthermore $\gamma_{N,x}$ is normalized, that is $\gamma_{N,x}([a,b]) = 1$, $\forall N \in \mathbb{N}$.

We give

Theorem 36 *Let $\alpha \in (0,1]$, $[a,b] \subseteq [0,\infty)$. Suppose f is $\mathbb{R}_+$ valued and is α-conformable fractional differentiable on $[a,b]$, with $D_\alpha f$ being continuous on $[a,b]$. Let $x \in [a,b]$ such that $D_\alpha f(x) = 0$, and $(L_N)_{N\in\mathbb{N}}$ be a sequence of positive sublinear comonotonic operators from $\mathcal{L}_\infty(\mathcal{B})$ into $C_+([a,b])$. We assume that $L_N(1) = 1$, and $(C)\int_a^b |t-x|^{\alpha+1} d\gamma_{N,x}(t) > 0$, $(C)\int_a^b (t-x)^{2(\alpha+1)} d\gamma_{N,x}(t) > 0$, $\forall N \in \mathbb{N}$. Then*

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1^\alpha\left(D_\alpha f, \left((C)\int_a^b (t-x)^{2(\alpha+1)} d\gamma_{N,x}(t)\right)^{\frac{\alpha}{2(\alpha+1)}}\right)}{\alpha} \cdot$$

$$\left[\left((C)\int_a^b |t-x|^{\alpha+1} d\gamma_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{2}\left((C)\int_a^b (t-x)^{2(\alpha+1)} d\gamma_{N,x}(t)\right)^{\frac{\alpha}{2(\alpha+1)}}\right], \tag{53}$$

$\forall N \in \mathbb{N}$.

As $(C)\int_a^b (t-x)^{2(\alpha+1)} d\gamma_{N,x}(t) \to 0$, $N \to \infty$, we get that $\lim_{N\to+\infty} L_N(f)(x) = f(x)$.

Proof. By Theorems 9, 34. ■

Theorem 37 *Let $\alpha \in (0,1]$, $n \in \mathbb{N}$. Suppose f is $\mathbb{R}_+$ valued and is n times conformable α-fractional differentiable on $[a,b] \subseteq [0,\infty)$, and $D_\alpha^n f$ is continuous on $[a,b]$. For a fixed $x \in [a,b]$ we have $D_\alpha^k f(x) = 0$, $k = 1,...,n$. Let positive sublinear comonotonic operators $\{L_N\}_{N\in\mathbb{N}}$ from $\mathcal{L}_\infty(\mathcal{B})$ into $C_+([a,b])$, such that $(C)\int_a^b |t-x|^{n(\alpha+1)} d\gamma_{N,x}(t)$, $(C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\gamma_{N,x}(t) > 0$, $\forall N \in \mathbb{N}$. Then*

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1^\alpha\left(D_\alpha^n f, \left((C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\gamma_{N,x}(t)\right)^{\frac{\alpha}{(n+1)(\alpha+1)}}\right)}{\alpha^n n!} \cdot \tag{54}$$

$$\left[\left((C)\int_a^b |t-x|^{n(\alpha+1)} d\gamma_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}} + \right.$$

$$\left. \frac{1}{(n+1)}\left((C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\gamma_{N,x}(t)\right)^{\frac{n\alpha}{(n+1)(\alpha+1)}}\right],$$

$\forall\ N \in \mathbb{N}$.

As $(C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\gamma_{N,x}(t) \to 0$, *when* $N \to \infty$, *we get that* $\lim_{N\to+\infty} L_N(f)(x) = f(x)$.

Proof. By Theorems 11, 34. ■

We continue with

Theorem 38 *Let* $\{L_N\}_{N\in\mathbb{N}}$ *from* $\mathcal{L}_\infty(\mathcal{B})$ *into* $C_+([a,b])$ *positive sublinear comonotonic operators, such that* $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. *Additionally assume that* $(C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\gamma_{N,x}(t) > 0$, $\forall\ N \in \mathbb{N}$; $x \in (a,b)$. *Here* $\alpha \in (0,1]$, *and* $n \in \mathbb{N}$. *Suppose* $f \in C_+([a,b])$ *is* n *times conformable* α*-fractional differentiable on* $[a,b] \subseteq [0,\infty)$, *and* $D_\alpha^n f$ *is continuous on* $[a,b]$. *Assume here* $0 < \left((C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\gamma_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}} \leq \min(x-a, b-x)$, $\forall\ N \in \mathbb{N}$: $N \geq N^* \in \mathbb{N}$, *and assume* $|D_\alpha^n f|$ *is convex over* $[a,b]$. *Furthermore assume that* $D_\alpha^k f(x) = 0$, $k = 1, ..., n$. *Then*

$$|L_N(f)(x) - f(x)| \leq \frac{b^{1-\alpha}\omega_1\left(D_\alpha^n f, \left((C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\gamma_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}}\right)}{(n+1)!\alpha^{n+1}}, \tag{55}$$

$\forall\ N \geq N^*$; $N, N^* \in \mathbb{N}$.

If $(C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\gamma_{N,x}(t) \to 0$, *then* $L_N(f)(x) \to f(x)$ *as* $N \to \infty$.

Proof. By Theorems 16, 34. ■

Theorem 39 *Let* $\{L_N\}_{N\in\mathbb{N}}$ *from* $\mathcal{L}_\infty(\mathcal{B})$ *into* $C_+([a,b])$ *positive sublinear comonotonic operators, such that* $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. *Additionally assume that* $(C)\int_a^b |t-x|^{(n+1)\alpha} d\gamma_{N,x}(t) > 0$, $\forall\ N \in \mathbb{N}$; $x \in (a,b)$. *Here* $\alpha \in (0,1]$, *and* $n \in \mathbb{N}$. *Suppose* $f \in C_+([a,b])$ *is* n *times conformable* α*-fractional differentiable on* $[a,b] \subseteq [0,\infty)$, *and* $D_\alpha^n f$ *is continuous on* $[a,b]$. *Assume here* $0 < \left((C)\int_a^b |t-x|^{(n+1)\alpha} d\gamma_{N,x}(t)\right) \leq \min(x-a, b-x)$, $\forall\ N \in \mathbb{N} : N \geq N^* \in \mathbb{N}$, *and assume* $|D_\alpha^n f|$ *is convex over* $[a,b]$. *Furthermore assume that* $D_\alpha^k f(x) = 0$, $k = 1, ..., n$. *Then*

$$|L_N(f)(x) - f(x)| \leq \frac{b^{1-\alpha}\omega_1\left(D_\alpha^n f, (C)\int_a^b |t-x|^{(n+1)\alpha} d\gamma_{N,x}(t)\right)}{(n+1)!\alpha^{n+1}}, \tag{56}$$

18

$\forall\ N \geq N^*$, *where* $N, N^* \in \mathbb{N}$.

If $(C)\int_a^b |t-x|^{(n+1)\alpha} d\gamma_{N,x}(t) \to 0$, *then* $L_N(f)(x) \to f(x)$ *as* $N \to \infty$.

Proof. By Theorems 17, 34. ■

We make

Remark 40 *Consider again* $[a,b] \subset \mathbb{R}_+$, $\mathcal{B} = \mathcal{B}([a,b])$ *the Borel* σ*-algebra on* $[a,b]$. *For each* $N \in \mathbb{N}$ *and each* $x \in [a,b]$ *consider the monotone set functions* $\mu_{N,x}; \mathcal{B} \to \mathbb{R}_+$. *We assume that all* $\mu_{N,x}$ *are normalized, that is* $\mu_{N,x}([a,b]) = 1$, *and submodular. Here we consider the operators* $T_N : C_+([a,b]) \to C_+([a,b])$ *given by the formula*

$$T_N(f)(x) = (C)\int_a^b f(t)\, d\mu_{N,x}(t), \tag{57}$$

$\forall\ N \in \mathbb{N}$, $\forall\ x \in [a,b]$.

Infact here $\mu_{N,x}$ *are chosen so that* $T_N(C_+([a,b])) \subseteq C_+([a,b])$.

We notice here that hold:

(i)

$$T_N(\alpha f)(x) = \alpha T_N(f)(x),\ \forall\ \alpha \geq 0, \tag{58}$$

(ii)

$$f \leq g, \text{ implies } T_N(f)(x) \leq T_N(g)(x), \tag{59}$$

and

(iii)

$$T_N(f+g)(x) \leq T_N(f)(x) + T_N(g)(x), \tag{60}$$

$\forall\ N \in \mathbb{N}$, $\forall\ x \in [a,b]$, $\forall\ f, g \in C_+([a,b])$.

Clearly T_N *are positive sublinear operators, compare to (8)-(10). We also have that* $T_N(1) = 1$, $\forall\ N \in \mathbb{N}$.

We give

Theorem 41 *Let* $\alpha \in (0,1]$, $[a,b] \subseteq [0,\infty)$. *Suppose* f *is* α*-conformable fractional differentiable on* $[a,b]$. $D_\alpha f$ *is continuous on* $[a,b]$. *Let an* $x \in [a,b]$ *such that* $D_\alpha f(x) = 0$. *Assume* $(C)\int_a^b |t-x|^{\alpha+1} d\mu_{N,x}(t)$, $(C)\int_a^b (t-x)^{2(\alpha+1)} d\mu_{N,x}(t) > 0$, $\forall\ N \in \mathbb{N}$. *Then*

$$|T_N(f)(x) - f(x)| \leq \frac{\omega_1^\alpha\left(D_\alpha f, \left((C)\int_a^b (t-x)^{2(\alpha+1)} d\mu_{N,x}(t)\right)^{\frac{\alpha}{2(\alpha+1)}}\right)}{\alpha} \cdot$$

$$\left[\left((C)\int_a^b |t-x|^{\alpha+1} d\mu_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}} + \frac{1}{2}\left((C)\int_a^b (t-x)^{2(\alpha+1)} d\mu_{N,x}(t)\right)^{\frac{\alpha}{2(\alpha+1)}}\right], \tag{61}$$

19

$\forall\ N \in \mathbb{N}$.

As $N \to \infty$, *and* $(C)\int_a^b (t-x)^{2(\alpha+1)} d\mu_{N,x}(t) \to 0$, *we obtain* $T_N(f)(x) \to f(x)$.

Proof. By Theorem 9. ■

Theorem 42 *Let* $\alpha \in (0,1]$, $n \in \mathbb{N}$. *Suppose* f *is* n *times conformable* α-*fractional differentiable on* $[a,b] \subseteq [0,\infty)$ *and takes values on* $\mathbb{R}_+$. $D_\alpha^n f$ *is continuous on* $[a,b]$. *For a fixed* $x \in [a,b]$ *we have* $D_\alpha^k f(x) = 0$, $k = 1,...,n$. *Assume that* $(C)\int_a^b |t-x|^{n(\alpha+1)} d\mu_{N,x}(t)$, $(C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\mu_{N,x}(t) > 0$, $\forall\ N \in \mathbb{N}$. *Then*

$$|T_N(f)(x) - f(x)| \leq \frac{\omega_1^\alpha\left(D_\alpha^n f, \left((C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\mu_{N,x}(t)\right)^{\frac{\alpha}{(n+1)(\alpha+1)}}\right)}{\alpha^n n!} \cdot$$

$$\left[\left((C)\int_a^b |t-x|^{n(\alpha+1)} d\mu_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}} + \right.$$

$$\left.\frac{1}{(n+1)}\left((C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\mu_{N,x}(t)\right)^{\frac{n\alpha}{(n+1)(\alpha+1)}}\right], \tag{62}$$

$\forall\ N \in \mathbb{N}$.

As $N \to \infty$, *and* $(C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\mu_{N,x}(t) \to 0$, *we get* $T_N(f)(x) \to f(x)$.

Proof. By Theorem 11. ■

We continue with

Theorem 43 *Assume* $(C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\mu_{N,x}(t) > 0$, $\forall\ N \in \mathbb{N}$; $x \in (a,b)$. *Here* $\alpha \in (0,1]$, *and* $n \in \mathbb{N}$. *Suppose* $f \in C_+([a,b])$ *is* n *times conformable* α-*fractional differentiable on* $[a,b] \subseteq [0,\infty)$, *and* $D_\alpha^n f$ *is continuous on* $[a,b]$. *Assume here that* $0 < \left((C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\mu_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}} \leq \min(x-a, b-x)$, $\forall\ N \in \mathbb{N} : N \geq N^* \in \mathbb{N}$, *and assume* $|D_\alpha^n f|$ *is convex over* $[a,b]$. *Furthermore assume that* $D_\alpha^k f(x) = 0$, $k = 1,...,n$. *Then*

$$|T_N(f)(x) - f(x)| \leq \frac{b^{1-\alpha}\omega_1\left(D_\alpha^n f, \left((C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\mu_{N,x}(t)\right)^{\frac{\alpha}{\alpha+1}}\right)}{(n+1)!\alpha^{n+1}}, \tag{63}$$

$\forall\ N \in \mathbb{N} : N \geq N^* \in \mathbb{N}$.

If $(C)\int_a^b |t-x|^{(n+1)(\alpha+1)} d\mu_{N,x}(t) \to 0$, *then* $T_N(f)(x) \to f(x)$ *as* $N \to \infty$.

Proof. By Theorem 16. ■

Theorem 44 *Assume* $(C)\int_a^b |t-x|^{(n+1)\alpha} d\mu_{N,x}(t) > 0$, $\forall\ N \in \mathbb{N}$. *Here* $\alpha \in (0,1]$, $n \in \mathbb{N}$ *and* $x \in (a,b)$; $[a,b] \subseteq [0,\infty)$. *Suppose* $f \in C_+([a,b])$ *is* n *times conformable* α*-fractional differentiable on* $[a,b]$, *and* $D_\alpha^n f$ *is continuous on* $[a,b]$. *Let* $0 < (C)\int_a^b |t-x|^{(n+1)\alpha} d\mu_{N,x}(t) \leq \min(x-a, b-x)$, $\forall\ N \geq N^*$; $N, N^* \in \mathbb{N}$, *and assume* $|D_\alpha^n f|$ *is convex over* $[a,b]$. *Furthermore assume that* $D_\alpha^k f(x) = 0$, $k = 1, ..., n$. *Then*

$$|T_N(f)(x) - f(x)| \leq \frac{b^{1-\alpha}\omega_1\left(D_\alpha^n f, (C)\int_a^b |t-x|^{(n+1)\alpha} d\mu_{N,x}(t)\right)}{(n+1)!\alpha^{n+1}}, \qquad (64)$$

$\forall\ N \geq N^*$, *where* $N, N^* \in \mathbb{N}$.

If $(C)\int_a^b |t-x|^{(n+1)\alpha} d\mu_{N,x}(t) \to 0$, *then* $T_N(f)(x) \to f(x)$ *as* $N \to \infty$.

Proof. By Theorem 17. ■

References

[1] G. Anastassiou, *Moments in probability and approximation theory*, Pitman Research Notes in Mathematics Series, Longman Group UK, New York, NY, 1993.

[2] G. Anastassiou, *Conformable Fractional Approximation by Max-Product Operators*, Studia Mathematica Babes Bolyai, 63 (1) (2018), 3-22.

[3] G. Anastassiou, *Conformable Fractional Approximations by Max-Product Operators using Convexity*, Arabian Journal of Mathematics, accepted for publication, 2018.

[4] G. Choquet, *Theory of capacities*, Ann. Inst. Fourier (Grenoble), 5 (1954), 131-295.

[5] D. Denneberg, *Non-additive Measure and Integral*, Kluwer, Dordrecht, 1994.

[6] D. Dubois and H. Prade, *Possibility Theory*, Plenum Press, New York, 1988.

[7] S. Gal, *Uniform and Pointwise Quantitative Approximation by Kantorovich-Choquet type integral Operators with respect to monotone and submodular set functions*, Mediterranean Journal of Mathematics, 14 (2017), no. 5, Art. 205, 12 pp.

[8] S. Gal and S. Trifa, *Quantitative estimates in uniform and pointwise approximation by Bernstein-Durrmeyer-Choquet operators*, Carpathian J. Math., 33 (2017) (1), 49-58.

[9] M. Abu Hammad and R. Khalil, *Abel's formula and Wronskian for conformable fractional differential equations*, International J. Differential Equations Appl., 13, No. 3 (2014), 177-183.

[10] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, *A new definition of fractional derivative*, J. Computational Appl. Math., 264 (2014), 65-70.

[11] D. Schmeidler, *Integral representation without additivity*, Proceedings of the American Mathematical Society, 97 (1986), 255-261.

[12] D. Schmeidler, *Subjective probability and expected utility without additivity*, Econometrica, 57 (1989), 571-587.

[13] Lloyd S. Shapley, *A Value for n-person Games*, in H.W. Kuhn, A.W. Tucker, *Contributions to the Theory of Games*, Annals of Mathematical Studies 28, Princeton University Press, (1953), 307-317.

[14] Z. Wang, G.J. Klir, *Generalized Measure Theory*, Springer, New York, 2009.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

The Minkowski Inequality and the Brunn-Minkowski Inequality for Dual Orlicz Mixed Affine Quermassintegrals

Tongyi Ma

(College of Mathematics and Statistics, Hexi University,

Zhangye, Gansu 734000, P.R.China)

Abstract

In this paper, the Orlicz version of the classical dual Cauchy-Kubota formula is given and the concept of dual affine quermassintegrals is extended to dual Orlicz mixed affine quermassintegrals in the framework of Orlicz Brunn-Minkowski theory. Some inequalities for dual Orlicz mixed affine quermassintegrals are obtained, such as dual Orlicz-Minkowski inequality and dual Orlicz-Brunn-Minkowski inequality.

Keywords: Orlicz Brunn-Minkowski theory, integral geometry, dual affine quermassintegral.

1 Introduction

We work in Euclidean space $\mathbb{R}^n$, and use $\mathrm{vol}_i(\cdot)$ to denote the i-dimensional volume. The unit sphere in $\mathbb{R}^n$ is written by S^{n-1}. In the projection of convex body K, quermassintegrals are important geometric invariants and have different definitions in many areas of mathematics. In the theory of mixed volumes quermassintegrals are usually called simple mixed volumes. The reader should refer to [24] and [26] for details. Lutwak [21] introduced the dual quermassintegrals, $\widetilde{W}_{n-i}$, of a star body K. Suppose $\widetilde{W}_0 = \mathrm{vol}_n(K)$ and $\widetilde{W}_n = \omega_n$. If $0 < i < n$, then

$$\widetilde{W}_{n-i}(K) = \frac{\omega_n}{\omega_i} \int_{G(n,i)} \mathrm{vol}_i(K \cap \xi) \mathrm{d}\mu_i(\xi), \tag{1.1}$$

where the Grassmann manifold $G(n,i)$ is endowed with the probability Haar measure μ_i, $\mathrm{vol}_i(K \cap \xi)$ is the i-dimensional volume of slice of K by an i-dimensional subspace $\xi \subset \mathbb{R}^n$ and $\omega_i = \pi^{i/2}/\Gamma(1 + i/2)$ denotes the i-dimensional volume of the unit ball in $\mathbb{R}^i$.

The quermassintegrals are connected with the projections of convex bodies, while the dual quermassintegrals are closely related to the cross sections of star bodies, which is proved in [11] that they are the only rotation invariant continuous star valuations with the corresponding homogeneity. Zhang [28] showed that the dual quermassintegrals have the same kind of kinematic formulas as the quermassintegrals.

Affine quermassintegrals [16] is an important geometric invariants in the projection of convex body. Lutwak [15] introduced the dual affine quermassintegrals, $\widetilde{\Phi}_{n-i}(K)$, of a star body K containing the

AMS Subject Classification: 52A39, 52A40, 52A22.

This work was Supported by the National Natural Science Foundation of China(Grant No. 11561020) and was partly supported by the National Natural Science Foundation of China (Grant No. 11371224).

E-mail: matongyi@126.com, matongyi_123@163.com (Tongyi Ma).

origin in its interior. Suppose $\widetilde{\Phi}_0(K) = \mathrm{vol}_n(K)$ and $\widetilde{\Phi}_n(K) = \omega_n$. If $0 < i < n$, then

$$\widetilde{\Phi}_{n-i}(K) = \frac{\omega_n}{\omega_i}\left(\int_{G(n,i)} \mathrm{vol}_i(K \cap \xi)^n \mathrm{d}\mu_i(\xi)\right)^{\frac{1}{n}}. \tag{1.2}$$

Grinberg [6] showed that both the affine quermassintegrals and the dual affine quermassintegrals are invariant under volume-preserving affine transformations. However, the dual affine quermassintegrals of star bodies received more considerable attention, see [6, 2, 16, 26, 27]. The aim of this paper is to study them further.

Some opened articles [9, 13, 17, 18, 23, 25], Gardner' work [3] and the classical Brunn-Minkowski theory of convex bodies (see, e.g., [4, 26]) were generalized to the Orlicz space, which is called the Orlicz Brunn-Minkowski theory and further extend the L_p-Brunn-Minkowski theory (see, e.g., [19, 20, 12]). We considers a non-zero convex function $\phi : [0, \infty) \to [0, \infty)$ in this paper. It is strictly increasing with $\phi(0) = 0$. Suppose that $\mathcal{C}$ is the class of convex and strictly increasing functions $\phi : [0, \infty) \to [0, \infty)$, where $\lim_{t\to\infty} \phi(t) = +\infty$, and $\phi(0) = 0$. Note that $\mathcal{S}^n_o$ denotes the set of star bodies in $\mathbb{R}^n$ containing the origin in their interiors.

The dual Orlicz mixed volume, $\widetilde{V}_{-\phi}(K, L)$, of $K, L \in \mathcal{S}^n_o$ is defined by

$$\widetilde{V}_{-\phi}(K, L) = \frac{-\phi'_r(1)}{n} \lim_{\varepsilon\to0^+} \frac{\mathrm{vol}_n(K \widetilde{+}_{-\phi} \varepsilon \diamond L) - \mathrm{vol}_n(K)}{\varepsilon}, \tag{1.3}$$

where $\phi'_r(1)$ is the right derivative of a real-valued function ϕ at 1 and $K \widetilde{+}_{-\phi} \varepsilon \diamond L$ denotes the Orlicz radial harmonic combination of K and L. It follows from (1.3) that the dual Orlicz mixed volume $\widetilde{V}_{-\phi}$ has the following integral representation:

$$\widetilde{V}_{-\phi}(K, L) = \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{\rho_K}{\rho_L}\right)\rho_K(u)^n \mathrm{d}S(u), \tag{1.4}$$

In [5, 10, 22, 31, 20], the dual mixed volume is extended to the dual L_p-mixed volume. If $\phi(t) = t^p, 1 \le p < \infty$, then

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n}\int_{S^{n-1}} \left(\frac{\rho_K(u)}{\rho_L(u)}\right)^p \rho^n_K(u)\mathrm{d}S(u). \tag{1.5}$$

Recently, Zhao [30] introduced the notion of dual Orlicz mixed quermassintegrals for $0 \le i \le n$ and established its integral representation. If $K, L \in \mathcal{S}^n_o$ and $\phi \in \mathcal{C}$, then

$$\widetilde{W}_{-\phi,i}(K, L) = \frac{-\phi'_r(1)}{n-i} \lim_{\varepsilon\to0^+} \frac{\widetilde{W}_i(K \widetilde{+}_{-\phi} \varepsilon \diamond L) - \widetilde{W}_i(K)}{\varepsilon}, \text{ and} \tag{1.6}$$

$$\widetilde{W}_{-\phi,i}(K, L) = \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{\rho_K}{\rho_L}\right)\rho_K(u)^{n-i}\mathrm{d}S(u), \quad i = 0, 1, \cdots, n. \tag{1.7}$$

In this paper, we first established the Orlicz version of the classical dual Cauchy-Kubota formula (1.1)

$$\widetilde{W}_{-\phi,n-i}(K, L) = \frac{\omega_n}{\omega_i}\int_{G(n,i)} \widetilde{V}^{(i)}_{-\phi}(K \cap \xi, L \cap \xi)\mathrm{d}\mu_i(\xi), \tag{1.8}$$

where $\widetilde{V}^{(i)}_{-\phi}(K \cap \xi, L \cap \xi)$ is the dual Orlicz mixed volume of the (i)-dimensional star bodies $K \cap \xi$ and $L \cap \xi$ in the subspace $\xi \in G(n, i)$.

For $i = 1, 2, \cdots, n$, we further consider the following formula.

$$\begin{aligned}\widetilde{\Phi}_{\phi,n-i}(K, L) &= \frac{\omega_n}{\omega_i}\left[\mathbb{E}(\widetilde{V}^{(i)}_{-\phi}(K \cap \xi, L \cap \xi)^n)\right]^{1/n} \\ &= \frac{\omega_n}{\omega_i}\left[\int_{G(n,i)} \widetilde{V}^{(i)}_{-\phi}(K \cap \xi, L \cap \xi)^n \mathrm{d}\mu_i(\xi)\right]^{\frac{1}{n}},\end{aligned} \tag{1.9}$$

and $\widetilde{\Phi}_{\phi,n-i}(K,L)$ is known as the dual Orlicz mixed affine quermassintegrals.

Let $\phi(t)=t^p$ with $p\geq 1$. Then

$$\widetilde{\Phi}_{p,n-i}(K,L)=\frac{\omega_n}{\omega_i}\left[\int\limits_{G(n,i)}\widetilde{V}^{(i)}_{-p}(K\cap\xi,L\cap\xi)^n\mathrm{d}\mu_i(\xi)\right]^{\frac{1}{n}},\tag{1.10}$$

where $\widetilde{V}^{(i)}_{-p}(K\cap\xi,L\cap\xi)$ denotes the dual L_p-mixed volume of $K\cap\xi$ and $L\cap\xi$ in the subspace $\xi\in G(n,i)$.

Taking $L=K$ in (1.9), $\widetilde{\Phi}_{\phi,n-i}(K,K)/\phi(1)=\widetilde{\Phi}_{n-i}(K)$ is just the classical dual affine quermassintegrals of K.

On the basis of the above concepts, one aim of this paper is to establish the following dual Orlicz-Minkowski inequality for dual Orlicz mixed affine quermassintegrals.

Theorem 1.1. *Suppose $K,L\in\mathcal{S}^n_o, n\geq 3$ and $\phi\in\mathcal{C}$. Then for $2\leq i\leq n$,*

$$\widetilde{\Phi}_{\phi,n-i}(K,L)\geq\widetilde{\Phi}_{n-i}(K)\phi\bigg(\bigg(\frac{\widetilde{\Phi}_{n-i}(K)}{\widetilde{\Phi}_{n-i}(L)}\bigg)^{\frac{1}{i}}\bigg).\tag{1.11}$$

If K and L are convex bodies containing the origin in their interiors, then equality holds in the inequality (1.11) *if and only if K and L are dilations.*

As an application of Theorem 1.1, we prove a uniqueness theorem of convex bodies.

The other aim of this paper is to prove Orlicz radial sum versions of the dual Brunn-Minkowski inequality for dual Orlicz mixed affine quermassintegrals.

Theorem 1.2. *Suppose $K,L\in\mathcal{S}^n_o$ and $\phi\in\mathcal{C}$. Then for $2\leq i\leq n$,*

$$\phi\bigg(\bigg(\frac{\widetilde{\Phi}_{n-i}(K\widetilde{+}_{-\phi}L)}{\widetilde{\Phi}_{n-i}(K)}\bigg)^{\frac{1}{i}}\bigg)+\phi\bigg(\bigg(\frac{\widetilde{\Phi}_{n-i}(K\widetilde{+}_{-\phi}L)}{\widetilde{\Phi}_{n-i}(L)}\bigg)^{\frac{1}{i}}\bigg)\leq\phi(1).\tag{1.12}$$

If K and L are convex bodies containing the origin in their interiors, then equality holds in the inequality (1.12) *if and only if K and L are dilations.*

In order to prove Theorems 1.1 and 1.2, we use the integral-geometric technique, motivated by Furstenberg and Tzkoni [1], Grinberg [7], Ma [22], Gardner and Hug, et al. [5] and Zhu et al. [31].

2 Preliminaries

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. We write $\mathcal{K}^n_o$ for the set of convex bodies containing the origin in their interiors. The support function of $K\in\mathcal{K}^n_o$, $h_K=h(K,\cdot):\mathbb{R}^n\backslash\{o\}\to[0,\infty)$, is defined by $h(K,x)=\max\{\langle x,y\rangle:y\in K\}$, where $x\in\mathbb{R}^n\backslash\{o\}$.

For $K\in\mathcal{K}^n_o$, its polar body, $K^*\in\mathcal{K}^n_o$, is defined by $K^*=\{x\in\mathbb{R}^n:\langle x,y\rangle\leq 1,\text{for any } y\in K\}$. It is easily known that $(K^*)^*=K$ for $K\in\mathcal{K}^n_o$, and for $c>0$ we have $(cK)^*=c^{-1}K^*$.

If K is a compact set in $\mathbb{R}^n$, then the radial function ρ_K of K is defined by $\rho_K(x)=\max\{\lambda\geq 0:\lambda x\in K\}$ for $x\in\mathbb{R}^n\backslash\{o\}$. If ρ_K is continuous then we call K a star body (about the origin).

Two star bodies K and L are dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u\in S^{n-1}$. It is easy to see that for $K,L\in\mathcal{S}^n_o$, $K\subseteq L$ if and only if $\rho_K\leq\rho_L$ and for $c>0$ and $x\in\mathbb{R}^n\backslash\{o\}$, $\rho(cK,x)=c\rho(K,x)$. More generally, for $T\in\mathrm{GL}(n)$ the radial function of the image $TK=\{Ty:y\in K\}$ of K is given by (see [26])

$$\rho(TK,x)=\rho(K,T^{-1}x),\quad\text{for } x\in\mathbb{R}^n\backslash\{o\},\tag{2.1}$$

where $\mathrm{GL}(n)$ denotes the linear transformation group on $\mathbb{R}^n$, and T^{-1} is the inverse of T.

For $K, L \in \mathcal{S}_o^n$, $\alpha, \beta \geq 0$ (not both zero) and $\phi \in \mathcal{C}$, the Orlicz radial combination $\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L$ of K and L is defined by (see [22])

$$\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u)^{-1} = \inf\left\{\lambda > 0 : \alpha\phi\Big(\frac{1}{\lambda\rho_K(u)}\Big) + \beta\phi\Big(\frac{1}{\lambda\rho_L(u)}\Big) \leq \phi(1)\right\}, \quad u \in S^{n-1}. \tag{2.2}$$

Note that for all $u \in S^{n-1}$, $\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u)$ is defined by

$$\alpha\phi\Big(\frac{\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u)}{\rho_K(u)}\Big) + \beta\phi\Big(\frac{\rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, u)}{\rho_L(u)}\Big) = \phi(1).$$

If $\phi(t) = t^p$ with $1 \leq p < \infty$, then $\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L$ is the L_p-radial harmonic combination $\alpha \diamond K \widetilde{+}_{-p} \beta \diamond L$, and correspondingly $\widetilde{V}_{-\phi}(K, L)$ is the dual L_p-mixed volume $\widetilde{V}_{-p}(K, L)$. See [20] for more details.

Lemma 2.1. *Let $K, L \in \mathcal{S}_o^n$ and $\alpha, \beta \geq 0$. If $\phi \in \mathcal{C}$, then for $T \in \mathrm{GL}(n)$,*

$$T(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L) = \alpha \diamond TK \widetilde{+}_{-\phi} \beta \diamond TL.$$

Proof. From (2.2) and (2.1), we have for $u \in S^{n-1}$,

$$\begin{aligned} \rho(\alpha \diamond TK \widetilde{+}_{-\phi} \beta \diamond TL, u)^{-1} &= \inf\left\{\lambda > 0 : \alpha\phi\Big(\frac{1}{\lambda\rho_{TK}(u)}\Big) + \beta\phi\Big(\frac{1}{\lambda\rho_{TL}(u)}\Big) \leq \phi(1)\right\} \\ &= \inf\left\{\lambda > 0 : \alpha\phi\Big(\frac{1}{\lambda\rho_K(T^{-1}u)}\Big) + \beta\phi\Big(\frac{1}{\lambda\rho_L(T^{-1}u)}\Big) \leq \phi(1)\right\} \\ &= \rho(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L, T^{-1}u)^{-1} \\ &= \rho(T(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L), u)^{-1}. \end{aligned}$$

Thus

$$T(\alpha \diamond K \widetilde{+}_{-\phi} \beta \diamond L) = \alpha \diamond TK \widetilde{+}_{-\phi} \beta \diamond TL.$$

□

Lemma 2.2. *Let $K, L \in \mathcal{S}_o^n, \phi \in \mathcal{C}$. Then for each $\xi \in G(n, i), i = 1, \cdots, n-1$ and $\varepsilon > 0$,*

$$(K \widetilde{+}_{-\phi} \varepsilon \diamond L) \cap \xi = (K \cap \xi) \widetilde{+}_{-\phi} \varepsilon \diamond (L \cap \xi).$$

Proof. Fixed $\xi \in G(n, i)$, and let $S^{i-1} = S^{n-1} \cap \xi$. For any $u \in S^{i-1}$ and $Q \in \mathcal{S}_o^n$, we get $\rho_Q(u) = \rho_{Q\cap\xi}(u)$. Applying the definition of $K \widetilde{+}_{-\phi} \varepsilon \diamond L$ to $u \in S^{i-1}$, it follows that

$$\phi\Big(\frac{\rho((K \widetilde{+}_{-\phi} \varepsilon \diamond L) \cap \xi, u)}{\rho_{K\cap\xi}(u)}\Big) + \varepsilon\phi\Big(\frac{\rho((K \widetilde{+}_{-\phi} \varepsilon \diamond L) \cap \xi, u)}{\rho_{L\cap\xi}(u)}\Big) = \phi(1).$$

On the other hand, from $(K \cap \xi) \widetilde{+}_{-\phi} \varepsilon \diamond (L \cap \xi)$ defined in ξ, we have

$$\phi\Big(\frac{\rho((K \cap \xi) \widetilde{+}_{-\phi} \varepsilon \diamond (L \cap \xi), u)}{\rho_{K\cap\xi}(u)}\Big) + \varepsilon\phi\Big(\frac{\rho((K \cap \xi) \widetilde{+}_{-\phi} \varepsilon \diamond (L \cap \xi), u)}{\rho_{L\cap\xi}(u)}\Big) = \phi(1).$$

Thus, $(K \widetilde{+}_{-\phi} \varepsilon \diamond L) \cap \xi$ and $(K \cap \xi) \widetilde{+}_{-\phi} \varepsilon \diamond (L \cap \xi)$ is a same star body in ξ. □

Lemma 2.3. (see [22]) *Suppose $K, L \in \mathcal{S}_o^n$ and $\phi \in \mathcal{C}$. Then*

$$\widetilde{V}_{-\phi}(K, L) \geq \mathrm{vol}_n(K)\phi\bigg(\Big(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\Big)^{\frac{1}{n}}\bigg), \tag{2.3}$$

with equality if and only if K and L are dilates of each other.

Taking $\phi(t)=t^p$ with $p\geq 1$. The above dual Orlicz-Minkowski inequality is Lutwak's L_p-dual Minkowski inequality (see [20]):

$$\widetilde{V}_{-p}(K,L)\geq \mathrm{vol}_n(K)^{\frac{n+p}{n}}\mathrm{vol}_n(L)^{-\frac{p}{n}}, \tag{2.4}$$

with equality holds if and only if K and L are dilations.

Lemma 2.4. (see [8]) *Suppose that μ is a probability measure on a space X and $f:X\to I\subset\mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. Jensen's inequality states that if $\phi:I\to\mathbb{R}$ is a convex function, then*

$$\int_X \phi\big(f(x)\big)\mathrm{d}\mu(x)\geq \phi\bigg(\int_X f(x)\mathrm{d}\mu(x)\bigg). \tag{2.5}$$

If ϕ is strictly convex, the equality holds in every inequality if and only if $f(x)$ is constant for μ-almost all $x\in X$.

3 The generalized dual Cauchy-Kubota formula

In this section, we prove the probabilistic essence of dual Orlicz mixed quermassintegrals. We first see the dual Cauchy-Kubota formula. For $K\in\mathcal{S}_o^n$,

$$\widetilde{W}_{n-i}(K)=\frac{\omega_n}{\omega_i}\int_{G(n,i)}\mathrm{vol}_i(K\cap\xi)\mathrm{d}\mu_i(\xi),\quad i=1,\cdots,n-1. \tag{3.1}$$

Theorem 3.1. *Suppose $K,L\in\mathcal{S}_o^n$ and $\phi\in\mathcal{C}$. Then for each $i=1,\cdots,n-1$,*

$$\widetilde{W}_{-\phi,n-i}(K,L)=\frac{\omega_n}{\omega_i}\int_{G(n,i)}\widetilde{V}_{-\phi}^{(i)}(K\cap\xi,L\cap\xi)\mathrm{d}\mu_i(\xi).$$

Proof. By (1.6), (3.1) and Lemma 2.2, we have

$$\begin{aligned}\widetilde{W}_{-\phi,n-i}(K,L)&=\frac{-\phi_r'(1)}{i}\lim_{\varepsilon\to0^+}\frac{\widetilde{W}_{n-i}(K\widetilde{+}_{-\phi}\varepsilon\diamond L)-\widetilde{W}_{n-i}(K)}{\varepsilon}\\&=\frac{-\phi_r'(1)}{i}\cdot\frac{\omega_n}{\omega_i}\int_{G(n,i)}\lim_{\varepsilon\to0^+}\frac{\mathrm{vol}_i((K\widetilde{+}_{-\phi}\varepsilon\diamond L)\cap\xi)-\mathrm{vol}_i(K\cap\xi)}{\varepsilon}\mathrm{d}\mu_i(u)\\&=\frac{-\phi_r'(1)}{i}\cdot\frac{\omega_n}{\omega_i}\int_{G(n,i)}\lim_{\varepsilon\to0^+}\frac{\mathrm{vol}_i((K\cap\xi\widetilde{+}_{-\phi}\varepsilon\diamond L\cap\xi))-\mathrm{vol}_i(K\cap\xi)}{\varepsilon}\mathrm{d}\mu_i(u).\end{aligned}$$

From (1.3), we have

$$\widetilde{W}_{-\phi,n-i}(K,L)=\frac{\omega_n}{\omega_i}\int_{G(n,i)}\widetilde{V}_{-\phi}^{(i)}(K\cap\xi,L\cap\xi)\mathrm{d}\mu_i(\xi).$$

□

Up to a constant, the quantity $\widetilde{W}_{-\phi,i}(K,L)$ is the expectation of the random variable

$$\widetilde{V}_{-\phi}^{(i)}(K\cap\cdot,L\cap\cdot):G(n,i)\to(0,\infty),\quad \xi\mapsto\widetilde{V}_{-\phi}^{(i)}(K\cap\xi,L\cap\xi),$$

which is defined on the probability space $(G(n,i),\mathcal{B},\mu_i)$ (where $\mathcal{B}$ is the Borel sigma-algebra on $G(n,i)$).

Taking $\phi(t)=t^p$ with $p>0$ in Theorem 3.1, we have the formula

$$\widetilde{W}_{-p,n-i}(K,L)=\frac{\omega_n}{\omega_i}\int_{G(n,i)}\widetilde{V}_{-p}^{(i)}(K\cap\xi,L\cap\xi)\mathrm{d}\mu_i(\xi).$$

For $K \in \mathcal{S}_o^n$, we extend the dual Cauchy-Kubota formula to $1 \le q \le i < n$,

$$\widetilde{W}_i(K) = \frac{\omega_n}{\omega_{n-q}} \int_{G(n,n-q)} \widetilde{W}_{i-q}^{(n-q)}(K \cap \xi) \mathrm{d}\mu_{n-q}(\xi), \tag{3.2}$$

where $\widetilde{W}_{i-q}^{(n-q)}$ denotes the (i-q)th dual harmonic quermassintegral in the subspace ξ.

It follows from (3.2) and (1.6) that we have the following theorem.

Theorem 3.2. *Suppose $K, L \in \mathcal{S}_o^n$ and $\phi \in \Phi_1$ or $\phi \in \Phi_2$. Then for $1 \le q \le i < n$,*

$$\widetilde{W}_{-\phi,i}(K,L) = \frac{\omega_n}{\omega_{n-q}} \int_{G(n,n-q)} \widetilde{W}_{-\phi,i-q}^{(n-q)}(K \cap \xi, L \cap \xi) \mathrm{d}\mu_{n-q}(\xi),$$

where $\widetilde{W}_{-\phi,i-q}^{(n-q)}(K \cap \xi, L \cap \xi)$ denotes the dual Orlicz harmonic mixed quermassintegral of the (n-q)-dimensional star bodies $K \cap \xi$ and $L \cap \xi$ in the subspace ξ.

4 Inequalities of dual Orlicz mixed affine quermassintegrals

In this section, we first show that the quantities $\widetilde{\Phi}_{\phi,1}(K,L), \cdots, \widetilde{\Phi}_{\phi,n}(K,L)$ are $\mathrm{SL}(n)$-invariant. Here, $\mathbb{E}(\widetilde{V}_{-\phi}^{(i)}(K \cap \cdot, L \cap \cdot)^n)$ is the expectation of $\widetilde{V}_{-\phi}^{(i)}(K \cap \cdot, L \cap \cdot)^n$.

Theorem 4.1. *Suppose $K, L \in \mathcal{S}_o^n$ and $\phi \in \mathcal{C}$. Then for $T \in \mathrm{SL}(n)$, there holds*

$$\widetilde{\Phi}_{\phi,i}(TK,TL) = \widetilde{\Phi}_{\phi,i}(K,L), \ \ i = 1, 2, \cdots, n.$$

Proof. Suppose $\xi \in G(n, n-i)$. For $S^{n-i-1} = S^{n-1} \cap \xi$, if

$$T \in \mathrm{SL}(n) = \{T \in \mathrm{GL}(n) : \det T = 1\},$$

then for $u \in S^{n-i-1}$ and $Q \in \mathcal{S}_o^n$, we get $\rho_{TQ}(u) = \rho_{TQ\cap\xi}(u)$. For $x \in \mathbb{R}^n \backslash \{o\}$, let $\langle x \rangle = x/||x||$. From (1.4) and (2.1), we obtain

$$\begin{aligned}
\widetilde{V}_{-\phi}^{(n-i)}(TK \cap \xi, TL \cap \xi) &= \frac{1}{n-i} \int_{S^{n-1}\cap\xi} \phi\left(\frac{\rho_{TK\cap\xi}(u)}{\rho_{TL\cap\xi}(u)}\right) \rho_{TK\cap\xi}^{n-i}(u) \mathrm{d}S_{n-i-1}(u) \\
&= \frac{1}{n-i} \int_{S^{n-1}\cap\xi} \phi\left(\frac{\rho_{TK}(u)}{\rho_{TL}(u)}\right) \rho_{TK}^{n-i}(u) \mathrm{d}S_{n-i-1}(u) \\
&= \frac{1}{n-i} \int_{S^{n-1}\cap\xi} \phi\left(\frac{\rho_K(\langle T^{-1}u \rangle)}{\rho_L(\langle T^{-1}u \rangle)}\right) \rho_K^{n-i}(\langle T^{-1}u \rangle) \mathrm{d}S_{n-i-1}(\langle T^{-1}u \rangle) \\
&= \frac{1}{n-i} \int_{S^{n-1}\cap\xi} \phi\left(\frac{\rho_{K\cap\xi}(v)}{\rho_{L\cap\xi}(v)}\right) \rho_{K\cap\xi}^{n-i}(v) \mathrm{d}S_{n-i-1}(v) \\
&= \widetilde{V}_{-\phi}^{(n-i)}(K \cap \xi, L \cap \xi),
\end{aligned}$$

where S_{n-i-1} denotes $n-i-1$-dimensional spherical Lebesgue measure. Thus, from (1.9), it follows that

$$\begin{aligned}
\widetilde{\Phi}_{\phi,i}(TK,TL) &= \frac{\omega_n}{\omega_{n-i}} \left(\int_{G(n,n-i)} \left[\widetilde{V}_{-\phi}^{(n-i)}(TK \cap \xi, TL \cap \xi) \right]^n \mathrm{d}\mu_{n-i}(\xi) \right)^{\frac{1}{n}} \\
&= \frac{\omega_n}{\omega_{n-i}} \left(\int_{G(n,n-i)} \left[\widetilde{V}_{-\phi}^{(n-i)}(K \cap \xi, L \cap \xi) \right]^n \mathrm{d}\mu_{n-i}(\xi) \right)^{\frac{1}{n}} \\
&= \widetilde{\Phi}_{\phi,i}(K,L).
\end{aligned}$$

□

6

To prove Theorem 1.1 and Theorem 1.2, the next three lemmas are needed.

Lemma 4.2. (see [14]) *Suppose $K \in \mathcal{K}_o^n$ and $\xi \in G(n,i)$, then $K^* \cap \xi = (K|\xi)^*$.*

Lemma 4.3. (see [12]) *Suppose $K_1, K_2 \in \mathcal{K}_o^n$ and $2 \le k \le n-1$. If $K_1|\xi$ and $K_1|\xi$ are dilations for each $\xi \in G(n,k)$, then K_1 and K_2 are dilations.*

Lemma 4.4. *Suppose $K_1, K_2 \in \mathcal{K}_o^n$ and $2 \le k \le n-1$. If $K_1 \cap \xi$ and $K_1 \cap \xi$ are dilations for each $\xi \in G(n,k)$, then K_1 and K_2 are dilations.*

Proof. If both $K_1 \cap \xi$ and $K_2 \cap \xi$ are dilations for each $\xi \in G(n,k)$, then $K_1 \cap \xi = a(K_2 \cap \xi)$ for $a > 0$. If follows from Lemma 4.2 that $(K_1^*|\xi)^* = a(K_2^*|\xi)^* = (a^{-1}K_2^*|\xi)^*$. Thus, $K_1^*|\xi = a^{-1}K_2^*|\xi$. From Lemma 4.3, we know $K_1^* = \frac{b}{a}K_2^*$. Therefore, $K_1 = cK_2$ for some $c > 0$. □

The normalized dual affine quermassintegrals measure of K are defined by

$$d\widetilde{\Phi}_i^*(K,\cdot) = \left(\frac{\omega_n}{\omega_i \widetilde{\Phi}_{n-i}(K)}\right)^n [\mathrm{vol}_i(K \cap \cdot)]^n \, d\mu_i, \tag{4.1}$$

where $d\mu_i$ is the normalized Haar measure on $G(n,i)$. Obviously, $\widetilde{\Phi}_i^*(K,\cdot)$ is a probability measure on $G(n,i)$.

Proof of Theorem 1.1. Note that $\widetilde{\Phi}_{\phi,0} = \widetilde{V}_{-\phi}(K,L), \widetilde{\Phi}_0(K) = \mathrm{vol}_n(K)$, and $\widetilde{\Phi}_0(L) = \mathrm{vol}_n(L)$. It follows directly from Lemma 2.3 that the case when $i = n$.

Now, we consider the case when $2 \le i \le n-1$. By (1.9), (2.3), (4.1), (2.5) and Hölder's inequality, it follows that

$$\begin{aligned}
&\frac{\widetilde{\Phi}_{\phi,n-i}(K,L)}{\widetilde{\Phi}_{n-i}(K)} \\
= \;& \frac{\omega_n}{\omega_i \widetilde{\Phi}_{n-i}(K)} \left[\int_{G(n,i)} \left(\widetilde{V}_{-\phi}^{(i)}(K\cap\xi, L\cap\xi)\right)^n d\mu_i(\xi)\right]^{\frac{1}{n}} \\
\ge \;& \frac{\omega_n}{\omega_i \widetilde{\Phi}_{n-i}(K)} \left[\int_{G(n,i)} (\mathrm{vol}_i(K\cap\xi))^n \phi^n\left(\left(\frac{\mathrm{vol}_i(K\cap\xi)}{\mathrm{vol}_i(L\cap\xi)}\right)^{\frac{1}{i}}\right) d\mu_i(\xi)\right]^{\frac{1}{n}} \\
= \;& \left[\int_{G(n,i)} \phi^n\left(\left(\frac{\mathrm{vol}_i(L\cap\xi)}{\mathrm{vol}_i(K\cap\xi)}\right)^{\frac{1}{i}}\right) d\widetilde{\Phi}_i^*(K,\xi)\right]^{\frac{1}{n}} \\
\ge \;& \phi\left[\int_{G(n,i)} \left(\frac{\mathrm{vol}_i(K\cap\xi)}{\mathrm{vol}_i(L\cap\xi)}\right)^{\frac{1}{i}} d\widetilde{\Phi}_i^*(K,\xi)\right] \\
= \;& \phi\left[\left(\frac{\omega_n}{\omega_i \widetilde{\Phi}_{n-i}(K)}\right)^n \int_{G(n,i)} (\mathrm{vol}_i(K\cap\xi))^{n\left(\frac{ni+1}{ni}\right)} (\mathrm{vol}_i(L\cap\xi))^{n\left(-\frac{1}{ni}\right)} d\mu_i(\xi)\right] \\
\ge \;& \phi\left(\frac{\widetilde{\Phi}_{n-i}(K)^{\frac{ni+1}{i}} \widetilde{\Phi}_{n-i}(L)^{-\frac{1}{i}}}{\widetilde{\Phi}_{n-i}(K)^n}\right) \\
= \;& \phi\left(\left(\frac{\widetilde{\Phi}_{n-i}(K)}{\widetilde{\Phi}_{n-i}(L)}\right)^{\frac{1}{i}}\right).
\end{aligned}$$

If K and L are dilations, then the equality holds in (1.11) is obvious. Conversely, let $K, L \in \mathcal{K}_o^n$. Together the equality conditions of the dual Brunn-Minkowski inequality (2.3), Jensen's inequality (2.5) with Hölder's inequality, we know equality holds in inequality (1.11) if and only if $K \cap \xi$ and $L \cap \xi$ are dilations for each $\xi \in G(n, n-i)$. Therefore, Lemma 4.4 can reduce that K and L are dilations. □

Let $\phi(t) = t^p$ with $p \ge 1$. An immediate consequence of Theorem 1.1 is:

Corolloary 4.5. *Suppose $K, L \in \mathcal{S}_o^n$. Then for $p \geq 1$ and $2 \leq i \leq n$,*

$$\widetilde{\Phi}_{p,n-i}(K,L) \geq \widetilde{\Phi}_{n-i}(K)^{1+\frac{p}{i}} \widetilde{\Phi}_{n-i}(L)^{-\frac{p}{i}}. \tag{4.2}$$

If $K, L \in \mathcal{K}_o^n$, then equality holds in the inequality (4.2) if and only if K and L are dilations.

A direct consequence of the dual Orlicz-Minkowski inequality is the following uniqueness.

Corolloary 4.6. *Suppose $\phi \in \mathcal{C}$ with $\phi(1) = 1$, and $\mathfrak{U} \subset \mathcal{K}_o^n$ $(n \geq 3)$ such that $K, L \in \mathfrak{U}$. If for $2 \leq i \leq n$, there holds*

$$\widetilde{\Phi}_{\phi,n-i}(M,K) = \widetilde{\Phi}_{\phi,n-i}(M,L), \quad \textit{for all } M \in \mathfrak{U}, \tag{4.3}$$

or

$$\frac{\widetilde{\Phi}_{\phi,n-i}(K,M)}{\widetilde{\Phi}_{n-i}(K)} = \frac{\widetilde{\Phi}_{\phi,n-i}(L,M)}{\widetilde{\Phi}_{n-i}(L)}, \quad \textit{for all } M \in \mathfrak{U}, \tag{4.4}$$

then $K = L$.

Proof. Suppose (4.3) holds. If we take K for M, then by (1.9), (1.2), and $\phi(1) = 1$, we have

$$\widetilde{\Phi}_{n-i}(K) = \phi(1)\widetilde{\Phi}_{n-i}(K) = \widetilde{\Phi}_{\phi,n-i}(K,K) = \widetilde{\Phi}_{\phi,n-i}(K,L).$$

Thus

$$1 = \phi(1) \geq \phi\left(\left(\frac{\widetilde{\Phi}_{n-i}(K)}{\widetilde{\Phi}_{n-i}(L)}\right)^{\frac{1}{i}}\right),$$

with equality if and only if K and L are dilates of each other. Since ϕ is strictly increasing on $(0,\infty)$, we have $\widetilde{\Phi}_{n-i}(K) \leq \widetilde{\Phi}_{n-i}(L)$, with equality if and only if K and L are dilates of each other.

If let L for M we similarly get $\widetilde{\Phi}_{n-i}(K) \geq \widetilde{\Phi}_{n-i}(L)$. Therefore, $\widetilde{\Phi}_{n-i}(K) = \widetilde{\Phi}_{n-i}(L)$, this obtains $\mathrm{vol}_i(K \cap \xi) = \mathrm{vol}_i(L \cap \xi)$, and from the equality conditions of the dual Orlicz-Minkowski inequality we obtain that K and L are dilates of each other. Since $K \cap \xi$ and $L \cap \xi$ have the same volume, this implies $K = L$.

Further, suppose that (4.4) holds. Similarly, we get

$$1 = \phi(1) = \frac{\widetilde{\Phi}_{\phi,n-i}(K,K)}{\widetilde{\Phi}_{n-i}(K)} = \frac{\widetilde{\Phi}_{\phi,n-i}(L,K)}{\widetilde{\Phi}_{n-i}(L)}.$$

Therefore,

$$1 = \phi(1) \leq \phi\left(\left(\frac{\widetilde{\Phi}_{n-i}(L)}{\widetilde{\Phi}_{n-i}(K)}\right)^{\frac{1}{i}}\right),$$

with equality if and only if K and L are dilates of each other. Since ϕ is strictly increasing on $(0,\infty)$, we have $\widetilde{\Phi}_{n-i}(L) \geq \widetilde{\Phi}_{n-i}(K)$, with equality if and only if K and L are dilates of each other.

Taking L for M, obviously $\widetilde{\Phi}_{n-i}(L) \leq \widetilde{\Phi}_{n-i}(K)$. Therefore, $\widetilde{\Phi}_{n-i}(L) = \widetilde{\Phi}_{n-i}(K)$ can obtain that K and L are dilates of each other. Since $K \cap \xi$ and $L \cap \xi$ have the same volume, this gets $K = L$. $\square$

Proof of Theorem 1.2. For the convenience, define $K_\phi = K \widetilde{+}_{-\phi} L$. From Lemma 2.2, we have for $\xi \in G(n, n-i)$, $K_\phi \cap \xi = (K \widetilde{+}_{-\phi} L) \cap \xi = (K \cap \xi) \widetilde{+}_{-\phi} (L \cap \xi)$. Note that $K_\phi \cap \xi \in \mathcal{S}_o^n$ implies that for $u \in S^{n-i-1}$,

$$\phi\left(\frac{\rho_{K_\phi \cap \xi}(u)}{\rho_{K \cap \xi}(u)}\right) + \phi\left(\frac{\rho_{K_\phi \cap \xi}(u)}{\rho_{L \cap \xi}(u)}\right) = \phi(1). \tag{4.5}$$

Suppose $\lambda_\phi = \omega_n/\big[\omega_i\widetilde{\Phi}_{n-i}(K_\phi)\big]$. By (1.2), (4.5), (1.4), (2.3), (4.1) and (2.5) we obtain

$$\begin{aligned}
&\phi(1)\\
=\ &\lambda_\phi\left[\int_{G(n,i)}(\phi(1)\mathrm{vol}_i(K_\phi\cap\xi))^n\,\mathrm{d}\mu_i(\xi)\right]^{\frac{1}{n}}\\
=\ &\lambda_\phi\left[\int_{G(n,i)}\left(\widetilde{V}^{(i)}_{-\phi}(K_\phi\cap\xi,K\cap\xi)+\widetilde{V}^{(i)}_{-\phi}(K_\phi\cap\xi,L\cap\xi)\right)^n\mathrm{d}\mu_i(\xi)\right]^{\frac{1}{n}}\\
\geq\ &\lambda_\phi\left\{\int_{G(n,i)}\mathrm{vol}_i(K_\phi\cap\xi)^n\left[\phi\left(\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(K\cap\xi)}\right)^{\frac{1}{i}}\right)+\phi\left(\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(L\cap\xi)}\right)^{\frac{1}{i}}\right)\right]^n\mathrm{d}\mu_i(\xi)\right\}^{\frac{1}{n}}\\
=\ &\left\{\int_{G(n,i)}\left[\phi\left(\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(K\cap\xi)}\right)^{\frac{1}{i}}\right)+\phi\left(\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(L\cap\xi)}\right)^{\frac{1}{i}}\right)\right]^n\mathrm{d}\widetilde{\Phi}^*_i(K_\phi,\xi)\right\}^{\frac{1}{n}}\\
\geq\ &\int_{G(n,i)}\left[\phi\left(\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(K\cap\xi)}\right)^{\frac{1}{i}}\right)+\phi\left(\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(L\cap\xi)}\right)^{\frac{1}{i}}\right)\right]\mathrm{d}\widetilde{\Phi}^*_i(K_\phi,\xi)\\
=\ &\int_{G(n,i)}\phi\left(\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(K\cap\xi)}\right)^{\frac{1}{i}}\right)\mathrm{d}\widetilde{\Phi}^*_i(K_\phi,\xi)+\int_{G(n,i)}\phi\left(\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(L\cap\xi)}\right)^{\frac{1}{i}}\right)\mathrm{d}\widetilde{\Phi}^*_i(K_\phi,\xi)\\
\geq\ &\phi\left(\int\limits_{G(n,i)}\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(K\cap\xi)}\right)^{\frac{1}{i}}\mathrm{d}\widetilde{\Phi}^*_i(K_\phi,\xi)\right)+\phi\left(\int\limits_{G(n,i)}\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(L\cap\xi)}\right)^{\frac{1}{i}}\mathrm{d}\widetilde{\Phi}^*_i(K_\phi,\xi)\right).
\end{aligned}\tag{4.6}$$

From Hölder inequality and (1.2), we get

$$\begin{aligned}
&\phi\left(\int\limits_{G(n,i)}\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(K\cap\xi)}\right)^{\frac{1}{i}}\mathrm{d}\widetilde{\Phi}^*_i(K_\phi,\xi)\right)\\
=\ &\phi\left[\left(\frac{\omega_n}{\omega_i\widetilde{\Phi}_{n-i}(K_\phi)}\right)^n\int_{G(n,i)}(\mathrm{vol}_i(K_\phi\cap\xi))^{n\left(\frac{ni+1}{ni}\right)}(\mathrm{vol}_i(K\cap\xi))^{n\left(-\frac{1}{ni}\right)}\mathrm{d}\mu_i(\xi)\right]\\
\geq\ &\phi\Bigg[\left(\frac{\omega_n}{\omega_i\widetilde{\Phi}_{n-i}(K_\phi)}\right)^n\left(\int_{G(n,i)}(\mathrm{vol}_i(K_\phi\cap\xi))^n\,\mathrm{d}\mu_i(\xi)\right)^{\frac{ni+1}{ni}}\\
&\times\left(\int_{G(n,i)}(\mathrm{vol}_i(K\cap\xi))^n\,\mathrm{d}\mu_i(\xi)\right)^{-\frac{1}{ni}}\Bigg]\\
=\ &\phi\left(\left(\frac{\widetilde{\Phi}_{n-i}(K_\phi)}{\widetilde{\Phi}_{n-i}(K)}\right)^{\frac{1}{i}}\right).
\end{aligned}\tag{4.7}$$

Similarly,

$$\phi\left(\int\limits_{G(n,i)}\left(\frac{\mathrm{vol}_i(K_\phi\cap\xi)}{\mathrm{vol}_i(L\cap\xi)}\right)^{\frac{1}{i}}\mathrm{d}\widetilde{\Phi}^*_i(K_\phi,\xi)\right)\geq\phi\left(\left(\frac{\widetilde{\Phi}_{n-i}(K_\phi)}{\widetilde{\Phi}_{n-i}(L)}\right)^{\frac{1}{i}}\right).\tag{4.8}$$

Together (4.6), (4.7) with (4.8), this yields

$$\phi\left(\left(\frac{\widetilde{\Phi}_{n-i}(K_\phi)}{\widetilde{\Phi}_{n-i}(K)}\right)^{\frac{1}{i}}\right)+\phi\left(\left(\frac{\widetilde{\Phi}_{n-i}(K_\phi)}{\widetilde{\Phi}_{n-i}(L)}\right)^{\frac{1}{i}}\right)\leq\phi(1).$$

Finally, we give the equality conditions. Suppose that K and L are dilations. with equality in (1.12) is obvious.

Conversely, Let $K, L \in \mathcal{K}_o^n$. From the equality conditions of the dual Orlicz-Minkowski inequality of star bodies, Jensen's inequality (2.5) and Hölder's inequality, we obtain that equality holds in inequality (1.12) if and only if $K \cap \xi$ and $L \cap \xi$ are dilations for each $\xi \in G(n, n-i)$. Therefore, Lemma 4.4 can get that K and L are dilations. $\square$

If $\phi(t) = t^p$ with $p \geq 1$, then we get:

Corolloary 4.7. *Let $K, L \in \mathcal{S}_o^n$. If $p \geq 1$ and $2 \leq i \leq n$, then*

$$\widetilde{\Phi}_{n-i}(K \widetilde{+}_{-p} L)^{-\frac{p}{i}} \geq \widetilde{\Phi}_{n-i}(K)^{-\frac{p}{i}} + \widetilde{\Phi}_{n-i}(L)^{-\frac{p}{i}}. \tag{4.9}$$

If $K, L \in \mathcal{K}_o^n$, then equality holds in the inequality (4.9) *if and only if K and L are dilations.*

An immediate consequence of the inequality (4.9) is:

Corolloary 4.8. *Let $K, L \in \mathcal{S}_o^n$. If $p \geq 1$ and $2 \leq i \leq n$, then*

$$2\widetilde{\Phi}_{n-i}(K \widetilde{+}_{-p} L)^{\frac{p}{i}} \leq \left(\widetilde{\Phi}_{n-i}(K)\widetilde{\Phi}_{n-i}(L)\right)^{\frac{p}{2i}} \leq \frac{1}{2}\left(\widetilde{\Phi}_{n-i}(K)^{\frac{p}{i}} + \widetilde{\Phi}_{n-i}(L)^{\frac{p}{i}}\right). \tag{4.10}$$

If $K, L \in \mathcal{K}_o^n$, with equality in (4.10) *if and only if $K = L$.*

Proof. By (4.9) and the arithmetic-geometric-harmonic mean inequality, we have

$$\begin{aligned}
2\widetilde{\Phi}_{n-i}(K \widetilde{+}_{-p} L)^{\frac{p}{i}} &\leq \frac{2}{\frac{1}{\widetilde{\Phi}_{n-i}(K)^{\frac{p}{i}}} + \frac{1}{\widetilde{\Phi}_{n-i}(L)^{\frac{p}{i}}}} \\
&\leq \left(\widetilde{\Phi}_{n-i}(K)\widetilde{\Phi}_{n-i}(L)\right)^{\frac{p}{2i}} \\
&\leq \frac{1}{2}\left(\widetilde{\Phi}_{n-i}(K)^{\frac{p}{i}} + \widetilde{\Phi}_{n-i}(L)^{\frac{p}{i}}\right).
\end{aligned}$$

We see easily that equality holds in the inequality (4.10) if and only if $K = L$. $\square$

The next result is a relationship between $\widetilde{\Phi}_{\phi,i}(K, L)$ and $\widetilde{W}_{-\phi,i}(K, L)$.

Theorem 4.9. *Suppose $K, L \in \mathcal{S}_o^n$ and $i = 1, 2, \cdots, n-1$. Then*

$$\widetilde{\Phi}_{\phi,i}(K, L) \geq \widetilde{W}_{-\phi,i}(K, L), \tag{4.11}$$

with equality if and only if $\widetilde{V}_{-\phi}^{(n-i)}(K \cap \xi, L \cap \xi)$ is constant for all $\xi \in G(n, n-i)$.

Proof. Notice that $\widetilde{V}_{-\phi}^{(n-i)}(K \cap \cdot, L \cap \cdot)$ is positive on $G(n, n-i)$ and that μ_{n-i} is a probability measure on $G(n, n-i)$. Hence, it follows from Jensen's inequality (2.5) that

$$\left(\int_{G(n,n-i)} \widetilde{V}_{-\phi}^{(n-i)}(K \cap \xi, L \cap \xi)^n \mathrm{d}\mu_{n-i}(\xi)\right)^{\frac{1}{n}} \geq \int_{G(n,n-i)} \widetilde{V}_{-\phi}^{(n-i)}(K \cap \xi, L \cap \xi) \mathrm{d}\mu_{n-i}(\xi),$$

with equality if and only if $\widetilde{V}_{-\phi}^{(n-i)}(K \cap \xi, L \cap \xi)$ is constant for all $\xi \in G(n, n-i)$. This inequality and the definitions of $\widetilde{\Phi}_{\phi,i}(K, L)$ and $\widetilde{W}_{-\phi,i}(K, L)$ can easily yield the desired inequality. $\square$

Conflict of Interests

The author declare that they have no competing interests.

Authors' Contribution

All authors contributed equally to the paper and read and approved its final version.

Acknowledgment

This work is supported by the National Natural Science Foundations of China (Grant No.11561020) and is partly supported by the National Natural Science Foundations of China (Grant No.11371224).

References

[1] H. Furstenberg, I. Tzkoni, *Spherical harmonics and integral geometry.* Israel J Math, 1971, **10**: 327-338.

[2] R. J. Gardner, Geometric Tomography, Encyclopedia of Mathematics and Its Applications, vol. 58, Cambridge University Press, Cambridge, 1995.

[3] R. J. Gardner, D. Hug, W. Weil, *The Orlicz-Brunn-Minkowski theory: A general framework, additions, and inequalities.* J Differential Geom, 2014, **97**: 427-476.

[4] R. J. Gardner, Geometric Tomography. Cambridge: Cambridge University Press, 2006.

[5] R. J. Gardner, D. Hug, W. Weil, et al., *The dual Orlicz-Brunn-Minkowski theory.* Journal of Mathematical Analysis and Applications, 2015, **430**(2): 810-829.

[6] E. L. Grinberg, *Isoperimetric inequalities and identities for k-dimensional cross-sections of a convex bodies.* London Mathematical Society, 1990, **22**: 478-484.

[7] E. L. Grinberg, *Isoperimetric inequalities and identities for k-dimensional cross-sections of convex bodies.* Math. Ann., 1991, **291**: 75-86.

[8] G. H. Hardy, J. E. Littlewood, G. Pólya. Inequalities, Cambridge Univ. Press, London, 1934.

[9] C. Haberl, E. Lutwak, D. Yang, G. Zhang, *The even Orlicz Minkowski problem.* Adv. Math., 2010, **224**: 2485-2510.

[10] H. L. Jin, S. F. Yuan and G. S. Leng, *On the dual Orlicz mixed volumes*, Chin. Ann. Math., 2015, **36B(6)**: 1019-1026.

[11] D. Klain, *Star valuations and dual mixed volumes.* Adv. Math., 1996, **121**(1): 80-101.

[12] D. Y. Li, D. Zou, G. Xiong, *Orlicz mixed affine quermassintegrals.* Science China Mathematics, 2015, **58**(8): 1715-1722.

[13] M. Ludwig, *General affine surface areas.* Adv. Math., 2010, **224**: 2346-2360.

[14] E. Lutwak, *Inequalities for Hadwiger's harmonic quermassintegrals.* Math. Ann., 1988, **280**: 165-175.

[15] E. Lutwak, *Dual mixed volumes.* Pacific Journal of Mathematics, 1975, **58**(2): 531-538.

[16] E. Lutwak, *A general isepiphanic inequality.* Proceedings of the AmericanMathematical Society, 1984, **90**(3): 415-421.

[17] E. Lutwak, D. Yang, G. Zhang, *Orlicz projection bodies.* Adv. Math., 2010, **223**: 220-242.

[18] E. Lutwak, D. Yang, G. Zhang, *Orlicz centroid bodies.* J. Differential Geom., 2010, **84**: 365-387.

[19] E. Lutwak, *The Brunn-Minkowski-Firey theory, I: Mixed volumes and the Minkowski problem.* J Differential Geom, 1993, **38**: 131-150.

[20] E. Lutwak, *The Brunn-Minkowski-Firey theory, II: Affine and geominimal surface areas.* Adv. Math., 1996, **118**: 244-294.

[21] E. Lutwak, *Intersection bodies and dual mixed volumes.* Adv. Math., 1988, **71**: 232-261.

[22] T. Y. Ma, *The minimal dual Orlicz surface area.* Tawanese Journal of Mathematics, 2016, **20**(2): 287-309.

[23] D. L. Ren, Topics in Integral Geometry. Singapore: World Scientific, 1994.

[24] L. A. Santaló, Integral geometry and geometric probability. Reading, MA: Addison-Wesley, 1976.

[25] L. A. Santaló, Integral Geometry and Geometric Probability. Cambridge: Cambridge University Press, 2004.

[26] R. Schneider, Convex Bodies: The Brunn-Minkowski Theory. Cambridge: Cambridge University Press, Second Expanded Edition, 2014.

[27] J. Yuan, G. S. Leng, *Inequalities for dual affine quermassintegrals.* Journal of Inequalities and Applications, 2006, **2006**(1): 1-7.

[28] G. Y. Zhang, *Dual kinematic formulas.* Tran. Amer. Math., 1999, **351**(3): 985-995.

[29] C. J. Zhao, *Reverse L_p-dual Minkowski's inequality.* Differential Geometry and its Applications, 2015, **40**: 243-251.

[30] C. J. Zhao, *Orlicz dual mixed volumes*, Results. Math., 2015, **68**: 93-104.

[31] B. Zhu, J. Zhou, W. Xu, *Dual Orlicz-Brunn-Minkowski theory.* Adv. Math., 2014, **264**: 700-725.

Existence and convergence for fixed points of a strict pseudo-contraction in CAT(0) spaces

Narongrit Puturong[a], Kasamsuk Ungchittrakool[a,b,*]

[a]*Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand*
[b]*Research Center for Academic Excellence in Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand*

Abstract

The purposes of this paper are to introduce and study some existence and convergence theorems for fixed points of a strict pseudo-contraction in the framework of complete CAT(0) spaces. By using available properties in the spaces together with some appropriate conditions of the mapping and under certain assumptions, we can create some suitable sets to be used to construct an iterative projection algorithm to guarantee the existence fixed points for a strict pseudo-contraction. The method allows us to obtain a strong convergence iteration for finding some fixed points of a strict pseudo-contraction in the framework of complete CAT(0) spaces.

Keywords: Strict pseudo-contraction; Iterative projection technique; CAT(0) space

1. Introduction

Let (X, d) be a metric space, and $x, y \in X$ with $l = d(x, y)$. A geodesic path from x to y is an isometry $\gamma : [0, l] \to X$ such that $\gamma(0) = x$ and $\gamma(l) = y$. The image of a geodesic path is called a geodesic segment. When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic space X consists of three points x_1, x_2, x_3 of X and three geodesic segments joining each pair of vertices. A comparison triangle of a geodesic triangle $\triangle(x_1, x_2, x_3)$ is the triangle $\bar{\triangle}(x_1, x_2, x_3) = \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean space $\mathbb{E}^2$ such that $d(x_i, x_j) = d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j)$ for all $i, j = 1, 2, 3$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.
CAT(0) : Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$,

$$d(x, y) \leq \ d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies that

$$d^2(x, y_0) \leq \ \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{1.1}$$

*Corresponding author. Tel.:+66 55963250; fax:+66 55963201.
Email addresses: `kasamsuku@nu.ac.th (Kasamsuk Ungchittrakool)`, `naputurong@gmail.com (Narongrit Puturong)`

This is the (CN) - inequality of Bruhat and Tits [5]. In fact ([3] p.163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) - inequality.

The study of CAT(0) spaces, Kirk [15, 16] first studied the fixed point theory in CAT(0) spaces. Since then, many authors have developed the fixed point theory for single-valued and set-valued mappings in the setting of CAT(0) spaces. Dhompongsa et al. [7] proved that a nonexpansive mapping from a nonempty bounded closed convex subset of a CAT(0) space to the family of nonempty compact subsets of the CAT(0) space has a fixed point under suitable conditions. In 2008, Berg and Nikolaev [2] introduced the concept of quasilinearization. In 2010, Kakavandi and Amini [13] introduced the concept of dual space for CAT(0) spaces. In 2012, Dehghan and Rooin [6] presented a characterization of metric projection in CAT(0) spaces. In 2014, Lu et al. [19] establish generalized CAT(0) versions of the Fan-Browder fixed point theorem. In the same year, Ungchittrakool [22] has discovered some significant inequalities for a strict pseudo-contraction in the framework of Hilbert spaces that has resulted in creating the important sets and the iterative shrinking projection technique to ensure the existence for fixed points of a strict pseudo-contraction in the terminology of Browder and Petryshyn [4].

Inspired and motivated by the significance of the problems mentioned above, we will pay attention to investigate and establish the existence theorem for fixed points of the mapping called strict pseudo-contraction mappings and some related mappings in complete CAT(0) spaces by employing suitable structure of certain sets based on the shrinking projection technique.

2. Preliminaries

Recall that a metric space (X, d) is said to be a geodesic space if every two points of X are joining by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. We write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z, x) = td(x, y) \quad and \quad d(z, y) = (1-t)d(x, y).$$

We also denote by $[x, y]$ the geodesic segment joining from x to y, that is $[x, y] = \{(1-t)x \oplus ty : t \in [0, 1]\}$. A subset C of a CAT(0) space is convex if $[x, y] \subseteq C$ for all $x, y \in C$. In 1976, Lim in [18] introduced the concept of $\triangle$-convergence, and Kirk and Panyanak [17] has obtained some results in CAT(0) spaces which is every similar for weak convergence in Banach space setting. Next , we present the concept of $\triangle$-convergence and collect some basic properties.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set
$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$.
The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by
$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$,
the asymptotic radius $r_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is given by
$r_C(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}$,
the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set
$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$
and the asymptotic center $A_C(\{x_n\})$ of $\{x_n\}$ with respect to $C \subset X$ is the set
$A_C(\{x_n\}) = \{x \in C : r(x, \{x_n\}) = r_C(\{x_n\})\}$.

It is known from Proposition 7 of [8] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

A subset of a CAT(0) space equipped with the induced metric, is a CAT(0) space if and only if it is convex ([3], p.167).

Definition 2.1 ([17], Definition 3.1)**.** A sequence $\{x_n\}$ in a CAT(0) space X is said to be $\triangle$-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle - \lim\limits_{n\to\infty} x_n = x$ and x is called the $\triangle$-limit of $\{x_n\}$.

Lemma 2.2 ([17], Opial's property)**.** *Let X be a complete CAT(0) space and a sequence $\{x_n\}$ in X such that $\{x_n\}$ $\triangle$-converge to x and given $y \in X$ with $y \neq x$. Then we have*
$\limsup\limits_{n\to\infty} d(x_n, x) < \limsup\limits_{n\to\infty} d(x_n, y)$.

It is known from [17] that, the uniqueness of asymtotic center implies that CAT(0) space X satisfies Opial's property.

Let X be a complete CAT(0) space. Bijan Ahmadi Kakavandi [12] introduced the properties of $\triangle$-convergence, i.e., every closed convex subset of X is $\triangle$-closed in the sense that it contains all $\triangle$-limit point of every $\triangle$-convergent sequence.

Lemma 2.3 ([20], Lemma 3.5)**.** *Every bounded closed convex set in a complete CAT(0) space always has a $\triangle$-convergent subsequence.*

Lemma 2.4 ([9], Proposition 2.1)**.** *If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Recall that a subset K of a metric space X is said to be $\triangle$- compact if every sequence in K has a $\triangle$- convergent subsequence.

Lemma 2.5 ([17], Proposition 3.6)**.** *Every bounded closed convex set in a complete CAT(0) space is $\triangle$- compact.*

Let C be a closed convex subset of a CAT(0) space X and $\{x_n\}$ be a bounded sequence in C. We use the following notation
$\{x_n\} \rightharpoonup w \iff \Phi(w) = \inf_{x\in C} \Phi(x)$ where $\Phi(x) = \limsup\limits_{n\to\infty} d(x_n, x)$.
Also, we have $\{x_n\} \rightharpoonup w \iff A_C(\{x_n\}) = \{w\}$.

Lemma 2.6 ([20], Proposition 3.12)**.** *Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X and let C be a closed convex subset of X which contain $\{x_n\}$. Then $\triangle - \lim\limits_{n\to\infty} x_n = x$ implies that $\{x_n\} \rightharpoonup x$.*

Berg and Nikolaev [2] have introduced the concept of quasilinearization as follows. Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then quasilinearization is the map
$\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$
defined by

$$\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle = \frac{1}{2}\{d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\} \text{ for all } a, b, c, d \in X. \tag{2.1}$$

It is easily seen that $\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle = \left\langle \overrightarrow{cd}, \overrightarrow{ab} \right\rangle$, $\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle = -\left\langle \overrightarrow{ba}, \overrightarrow{cd} \right\rangle$ and $\left\langle \overrightarrow{ax}, \overrightarrow{cd} \right\rangle + \left\langle \overrightarrow{xb}, \overrightarrow{cd} \right\rangle = \left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle$ for all $a, b, c, d, x \in X$.

We say that X satisfies the Cauchy - Schwarz inequality if $\left\langle \overrightarrow{ab}, \overrightarrow{cd} \right\rangle \leq d(a, b)d(c, d)$
for all $a, b, c, d \in X$.

It is known ([2], Corollary 3) that a geodesically connected metric space is CAT(0) space if and only if it satisfies the Cauchy- Schwarz inequality.

Definition 2.7 ([3], Proposition 2.4)**.** Let (X, d) be a metric space and $C \subseteq X$. The distance function $d(x, C) : X \to C$ is defined by $d(x, C) = \inf\limits_{c\in C} d(x, c)$ for any $x \in X$.

Lemma 2.8 ([3], Proposition 2.4). *Let C be a closed convex subset of a complete CAT(0) space X and $x \in X$. Then there exists a unique point $p \in C$ such that $d(x,C) = d(x,p) = \inf_{y \in C} d(x,y)$.*

Definition 2.9 ([3], Proposition 2.4). Let C be a closed convex subset of a complete CAT(0) space X and $P_C : X \to C$ is defined by $P_C x = p$ such that p satisfies Lemma 2.8. P_C is said to be the metric projection from X onto C.

Dehghan and Rooin [6] presented monotone and a characterization of metric projection in CAT(0) spaces as follows:

A self-mapping T of C where C is a subset of CAT(0) space (X,d) is said to be monotone if $\left\langle \overrightarrow{xy}, \overrightarrow{TxTy} \right\rangle \geq 0$ for all $x,y \in C$. Aslo, it is nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in C$.

Lemma 2.10 ([6], Lemma 2.1). *Let X be a CAT(0) space, $x,y \in X$, $\lambda \in [0,1]$ and $z = \lambda x \oplus (1-\lambda)y$. Then $\langle \overrightarrow{zy}, \overrightarrow{zw} \rangle \leq \lambda \langle \overrightarrow{xy}, \overrightarrow{zw} \rangle$ for all $w \in X$.*

Lemma 2.11 ([6], Theorem 2.2). *Let C be a nonempty convex subset of a CAT(0) space X, $x \in X$ and $u \in C$. Then $u = P_C x$ if and only if $\langle \overrightarrow{xu}, \overrightarrow{uy} \rangle \geq 0$ for all $y \in C$.*

Lemma 2.12 ([6], Proposition 2.4). *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Then $P_C : X \to C$ is monotone and nonexpansive.*

Lemma 2.13 ([23], Lemma 2.10). *Let X be a CAT(0) space. For any $u,v \in X$ and $t \in [0,1]$, let $u_t = tu \oplus (1-t)v$. Then, for all $x,y \in X$,*

(1) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \leq t \langle \overrightarrow{ux}, \overrightarrow{u_t y} \rangle + (1-t) \langle \overrightarrow{vx}, \overrightarrow{u_t y} \rangle$;

(2) $\langle \overrightarrow{u_t x}, \overrightarrow{uy} \rangle \leq t \langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t) \langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$ *and* $\langle \overrightarrow{u_t x}, \overrightarrow{vy} \rangle \leq t \langle \overrightarrow{ux}, \overrightarrow{vy} \rangle + (1-t) \langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$.

Lemma 2.14 ([3], Proposition 2.2). *Let X be a CAT(0) space, $p,q,r,s \in X$ and $\lambda \in [0,1]$. Then $d[\lambda p \oplus (1-\lambda)q, \lambda r \oplus (1-\lambda)s] \leq \lambda d(p,r) + (1-\lambda)d(q,s)$.*

Lemma 2.15 ([10], Lemma 2.5). *Let X be a CAT(0) space, $x,y,z \in X$ and $\lambda \in [0,1]$. Then $d^2(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d^2(x,z) + (1-\lambda)d^2(y,z) - \lambda(1-\lambda)d^2(x,y)$.*

Definition 2.16 ([1], Definition 3.2.2). Let X be a complete CAT(0) space and let f be a function of X into $(-\infty,\infty]$. Then f is said to be weakly lower semicontinuous on X if and only if for any $x_0 \in X$, $\{x_n\} \rightharpoonup x_0$ implies that $f(x_0) \leq \liminf_{n\to\infty} f(x_n)$.

Lemma 2.17 ([1], Corollary 3.2.4). *Let C be a nonempty closed convex subset of a complete CAT(0) space X. The distance function $d(x,C)$ as well as its square $d^2(x,C)$ are weakly lower semicontinuous.*

We first introduce the definition of k-strict pseudo-contraction in CAT(0) spaces.

Definition 2.18. Let (X,d) be a CAT(0) space and C be a nonempty subset of X. The mapping $T : C \to C$ is said to be a k-strict pseudo-contraction in the terminology of Browder and Petryshyn [4] if for all $x,y \in C$ there exists $k \in (-\infty,1)$ such that

$$d^2(Tx,Ty) \leq d^2(x,y) + k\{d^2(x,Tx) - 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle + d^2(y,Ty)\}.$$

Lemma 2.19. *Let C be a nonempty closed convex subset of a CAT(0) space X,and $T : C \to C$ be a k-strict pseudo-contraction, then T satisfies the Lipschitz condition with Lipschitz constant $L = \max\{\frac{1+k}{1-k}, 1\}$ for all $x,y \in C$. That is*

$$d(Tx,Ty) \leq \max\{\tfrac{1+k}{1-k}, 1\} d(x,y) \text{ for all } x,y \in C.$$

Proof. Let C be a nonempty closed convex subset of a CAT(0) space X. For $T : C \to C$ be a k-strict pseudo-contraction, we have

$$\begin{aligned} d^2(Tx,Ty) &\le d^2(x,y) + k\{d^2(x,Tx) - 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle + d^2(y,Ty)\} \\ &= d^2(x,y) + k\{d^2(x,Tx) - d^2(x,Ty) - d^2(y,Tx) + d^2(x,y) + d^2(Tx,Ty) + d^2(y,Ty)\} \end{aligned} \tag{2.2}$$

By simple calculation from (2.2) we have that

$$\begin{aligned} (1-k)d^2(Tx,Ty) &\le (1+k)d^2(x,y) + k\{d^2(x,Tx) - d^2(x,Ty) - d^2(y,Tx) + d^2(y,Ty)\} \\ &= (1+k)d^2(x,y) + 2k\left\langle \overrightarrow{xy}, \overrightarrow{TyTx} \right\rangle. \end{aligned} \tag{2.3}$$

Since X satisfies the Cauchy-Schwarz inequality, it follows from (2.3), we get that

$$(1-k)d^2(Tx,Ty) - 2kd(x,y)d(Tx,Ty) - (1+k)d^2(x,y) \le 0. \tag{2.4}$$

Next, we will divide the proof into two cases.

Case 1. $k \le 0$.
Notice that $k \le 0 \Leftrightarrow 2k \le 0 \Leftrightarrow 1+k \le 1-k \Leftrightarrow \frac{1+k}{1-k} \le 1 \Leftrightarrow \max\{\frac{1+k}{1-k}, 1\} = 1$. Since $k \le 0$, from (2.4), we have

$$\begin{aligned} &(1-k)d^2(Tx,Ty) + 2kd(x,y)d(Tx,Ty) - (1+k)d^2(x,y) \\ &\qquad \le (1-k)d^2(Tx,Ty) - 2kd(x,y)d(Tx,Ty) - (1+k)d^2(x,y) \le 0. \end{aligned}$$

Thus $(1-k)d^2(Tx,Ty) + 2kd(x,y)d(Tx,Ty) - (1+k)d^2(x,y) \le 0$.
Solving this quadratic inequality, we obtain
$d(Tx,Ty) \le d(x,y)$ or $d(Tx,Ty) \le \{\frac{1+k}{1-k}\}d(x,y)$ for all $x, y \in C$. It implies that
$d(Tx,Ty) \le d(x,y) = \max\{\frac{1+k}{1-k}, 1\}d(x,y)$ for all $x, y \in C$.

Case 2. $0 \le k < 1$.
We have $1-k > 0$ and then $k \ge 0 \Leftrightarrow 2k \ge 0 \Leftrightarrow 1+k \ge 1-k \Leftrightarrow \frac{1+k}{1-k} \ge 1 \Leftrightarrow \max\{\frac{1+k}{1-k}, 1\} = \frac{1+k}{1-k}$.
Similarly case 1, we have $(1-k)d^2(Tx,Ty) - 2kd(x,y)d(Tx,Ty) - (1+k)d^2(x,y) \le 0$.
It implies that $d(Tx,Ty) \le \{\frac{1+k}{1-k}\}d(x,y) = \max\{\frac{1+k}{1-k}, 1\}d(x,y)$ for all $x, y \in C$.
Therefore, the desired result. □

In this paper, we denote that Fix(T) is the set of fixed point of T such that Fix$(T) = \{x \in C : Tx = x\}$.

Lemma 2.20 ([11], Theorem 2.3). *Let C be a closed convex subset of a CAT(0) space X and $T : C \to C$ be a k - strict pseudo-contraction mapping. If Fix$(T) \ne \varnothing$, then Fix(T) is closed and convex so that the projection $P_{Fix(T)}$ is well defined.*

Lemma 2.21 ([19], Lemma 2.2). *Let (E,d) be a complete metric space. Then E is a geodesic space if and only if for every $x, y \in E$, there exists $z \in E$ such that $d(x,z) = d(z,y) = \frac{1}{2}d(x,y)$.*

Lemma 2.22 ([3], p.163). *A geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.*

Let (E,d) be a CAT(0) space and $D \subseteq E$. Niculescu and Roventa [21] introduced the notion of a convex hull of D as follows :

co$(D) = \bigcup_{n=0}^{\infty} D_n$ where $D_0 = D$ and for $n \ge 1$, the set D_n consists of all points in E which lie on geodesics which start and end in D_{n-1}.

$\bar{co}(D)$ denote the closure of the convex hull. It is easy to see that in a CAT(0) space, the closure of the convex hull will be convex and hence it is the smallest closed convex set containing D ([1], p.31).

Definition 2.23 ([19], Definition 2.2)**.** Let D be a nonempty subset of a CAT(0) space (E, d). A set-valued mapping $G : D \to 2^E$ is called to be a KKM mapping if $\mathrm{co}(F) \subset \bigcup_{x\in F} G(x)$ for every $F \in \langle D\rangle$ where $\langle D\rangle$ denotes the class of all nonempty finite subsets of D.

Lemma 2.24 ([14], Lemma 1.8)**.** *Suppose X is a complete CAT(0) space and K is a nonempty subset of X. Let $G : K \to 2^K$ be a mapping such that for each $x \in K$, $G(x)$ be $\triangle$-closed. Suppose that*

(1) *each $x_1, ..., x_m \in K$, $co(\{x_1, ..., x_m\}) \subset \bigcup_{i=1}^{m} G(x_i)$,*

(2) *there exists $x_0 \in K$ such that $G(x_0)$ is $\triangle$- compact.*

Then $\bigcap_{x\in K} G(x) \neq \varnothing$.

Lemma 2.25. *Let (E, d) be a complete CAT(0) space, K be a nonempty $\triangle$-compact subset of E, and $F, G : E \to 2^E$ be two set-valued mappings such that*
(1) *for every $y \in E$, $F(y) \subseteq G(y)$ and $G(y)$ is convex;*
(2) *for every $x \in E$, $F^{-1}(x)$ is open in E;*
(3) *for every $y \in K$, $F(y) \neq \varnothing$;*
(4) *there exists a point $x_0 \in E$ such that $\bar{co}(E\backslash G^{-1}(x_0)) \subseteq K$.*

Then, there exists $\widehat{y} \in E$ such that $\widehat{y} \in G(\widehat{y})$.

Proof. Suppose the contrary. Then, for every $y \in E, y \notin G(y)$. Now let us define two set-valued mappings $\widetilde{G}, \widetilde{F} : E \to 2^E$ by

$$\widetilde{G}(x) = \bar{\mathrm{co}}(E\backslash G^{-1}(x)) \text{ and } \widetilde{F}(x) = \mathrm{co}(E\backslash F^{-1}(x)) \text{ for all } x \in E.$$

By using (1) and $F(y) \subseteq G(y)$ for every $y \in E$, we have

$$F^{-1}(x) = \{y \in E : x \in F(y)\} \subseteq \{y \in E : x \in G(y)\} = G^{-1}(x).$$

Then, $E\backslash G^{-1}(x) \subseteq E\backslash F^{-1}(x)$ for every $x \in E$. It implies that $\mathrm{co}(E\backslash G^{-1}(x)) \subseteq \mathrm{co}(E\backslash F^{-1}(x))$. By using (2), we have $\mathrm{co}(E\backslash F^{-1}(x))$ is closed in E. Since $\bar{\mathrm{co}}(E\backslash G^{-1}(x))$ is the smallest closed set containing $\mathrm{co}(E\backslash G^{-1}(x))$. Then $\bar{\mathrm{co}}(E\backslash G^{-1}(x)) \subseteq \mathrm{co}(E\backslash F^{-1}(x))$. Therefore $\widetilde{G}(x) \subseteq \widetilde{F}(x)$ for every $x \in E$.

We next show that $\widetilde{G}$ is a KKM mapping. That is, for every $A \in \langle E\rangle$, $\mathrm{co}(A) \subseteq \bigcup_{x\in A} \widetilde{G}(x)$. Otherwise, there exist $A \in \langle E\rangle$ and a point $y \in \mathrm{co}(A)$ such that $y \notin \bigcup_{x\in A} \widetilde{G}(x) = \bigcup_{x\in A} (\bar{\mathrm{co}}(E\backslash G^{-1}(x)))$. For $(E\backslash G^{-1}(x)) \subseteq E$, we have $\mathrm{co}(E\backslash G^{-1}(x)) = \bigcup_{n=0}^{\infty} (E\backslash G^{-1}(x))_n$ where $(E\backslash G^{-1}(x))_0 = E\backslash G^{-1}(x)$ and for $n \geq 1$, $(E\backslash G^{-1}(x))_n$ consists of all points in E which lie on geodesics which start and end in $(E\backslash G^{-1}(x))_{n-1}$.

Let us consider, $y \notin \bigcup_{x\in A} \widetilde{G}(x) = \bigcup_{x\in A} (\bar{\mathrm{co}}(E\backslash G^{-1}(x))) = \bigcup_{x\in A} \mathrm{cl}_E\{\bigcup_{n=0}^{\infty} (E\backslash G^{-1}(x))_n\}$. Since $\bigcup_{x\in A} \mathrm{cl}_E(E\backslash G^{-1}(x)) \subseteq \bigcup_{x\in A} \mathrm{cl}_E\{\bigcup_{n=0}^{\infty} (E\backslash G^{-1}(x))_n\}$. It implies that

$$y \notin \bigcup_{x\in A} \mathrm{cl}_E(E\backslash G^{-1}(x)) = E\backslash \bigcap_{x\in A} \mathrm{int}_E G^{-1}(x).$$

It follows that $y \in \bigcap_{x\in A} G^{-1}(x)$. Therefore $A \subseteq G(y)$. Since $G(y)$ is convex by (1), and $\mathrm{co}(A)$ is the smallest convex set containing A. We get that $y \in \mathrm{co}(A) \subseteq G(y)$, which is a contradiction. Hence $\widetilde{G}$ is a KKM mapping. By the definition of $\widetilde{G}$, $\widetilde{G}(x)$ is $\triangle$-closed in E for every $x \in E$. By using (4), there exists a point $x_0 \in E$ such that $\widetilde{G}(x_0) = \bar{\mathrm{co}}(E\backslash G^{-1}(x_0)) \subseteq K$, it implies that $\widetilde{G}(x_0)$ is $\triangle$-compact. Then, by Lemma 2.24, we get that $\varnothing \neq \bigcap_{x\in E} \widetilde{G}(x) \subseteq \widetilde{G}(x_0) \subseteq K$. Therefore, we have

$$\varnothing \neq K \cap (\bigcap_{x\in E} \widetilde{G}(x)) \subseteq K \cap (\bigcap_{x\in E} \widetilde{F}(x)).$$

Taking $y_0 \in K \cap (\bigcap_{x\in E} \widetilde{F}(x))$, we have $y_0 \in K$ and $x \notin F(y_0)$ for every $x \in E$. Hence, we have $F(y_0) = \varnothing$ which contradicts (3). Therefore, there exists $\widehat{y} \in E$ such that $\widehat{y} \in G(\widehat{y})$. This completes our proof. □

Remark 2.26. If $F = G$, then (4) of Lemma 2.25 can be replaced by the following equivalent condition:

$(4)^*$ there exists a point $x_0 \in E$ such that $\mathrm{co}(E\backslash F^{-1}(x_0)) \subseteq K$.

3. Main Results

In this section, motivated by Ungchittrakool [22]. We discuss the existence and convergence for fixed point of a strict pseudo-contraction in the terminology of Browder and Petryshyn in the framework of complete CAT(0) spaces.

Lemma 3.1. *Let C be a bounded closed convex subset of a complete CAT(0) space (X, d). Then (C, d) is a complete CAT(0) space.*

Proof. Let C be a bounded closed convex subset of complete CAT(0) space (X, d). Notice that, a subset of a CAT(0) space equipped with the induced metric, is a CAT(0) space if and only if it is convex. This implies that (C, d) is a CAT(0) space. Since C is closed subset of complete metric space (X, d), then (C, d) is complete metric space. Therefore, we have (C, d) is a complete CAT(0) space. □

Lemma 3.2. *Let C be a bounded closed convex subset of a complete CAT(0) space X. Let T be a k- strict pseudo-contraction in the terminology of Browder and Petryshyn. Then, there exists an element $x_0 \in C$ such that $\left\langle \overrightarrow{xx_0}, \overrightarrow{xTx_0} \right\rangle \geq 0$ for all $x \in C$.*

Proof. Let C be a bounded closed convex subset of a complete CAT(0) space (X, d). We claim that there exists an element $x_0 \in C$ such that $\left\langle \overrightarrow{xx_0}, \overrightarrow{xTx_0} \right\rangle \geq 0$ for all $x \in C$. For any $y \in C$, we assume that the set $\{x \in C : \left\langle \overrightarrow{xy}, \overrightarrow{xTy} \right\rangle < 0\}$ is nonempty. We also define two set-valued mappings $F, G : C \to 2^C$ by $F(y) = G(y) = \{x \in C : \left\langle \overrightarrow{xy}, \overrightarrow{xTy} \right\rangle < 0\}$.

We first show that $G(y)$ is convex and $F^{-1}(x)$ is an open set.
Step1. To show that $G(y)$ is convex.
Let $x_1, x_2 \in G(y)$ and $u_t = tx_1 \oplus (1-t)x_2$ such that $t \in [0, 1]$. So, we have $x_1, x_2 \in C$, that $u_t \in C$.

Let us consider $\left\langle \overrightarrow{u_t y}, \overrightarrow{u_t Ty} \right\rangle$, by Lemma 2.13, we get that

$$\begin{aligned}
&\left\langle \overrightarrow{u_t y}, \overrightarrow{u_t Ty} \right\rangle \\
&\leq t \left\langle \overrightarrow{x_1 y}, \overrightarrow{u_t Ty} \right\rangle + (1-t) \left\langle \overrightarrow{x_2 y}, \overrightarrow{u_t Ty} \right\rangle = t \left\langle \overrightarrow{u_t Ty}, \overrightarrow{x_1 y} \right\rangle + (1-t) \left\langle \overrightarrow{u_t Ty}, \overrightarrow{x_2 y} \right\rangle \\
&\leq t\{t \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_1 y} \right\rangle + (1-t) \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_1 y} \right\rangle\} + (1-t)\{t \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_2 y} \right\rangle + (1-t) \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_2 y} \right\rangle\} \\
&= t^2 \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_1 y} \right\rangle + t(1-t) \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_1 y} \right\rangle + (1-t)^2 \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_2 y} \right\rangle + t(1-t) \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_2 y} \right\rangle \\
&= t^2 \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_1 y} \right\rangle + t(1-t)\{\langle \overrightarrow{x_2 x_1}, \overrightarrow{x_1 y} \rangle + \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_1 y} \right\rangle\} \\
&\quad + (1-t)^2 \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_2 y} \right\rangle + t(1-t)\{\langle \overrightarrow{x_1 x_2}, \overrightarrow{x_2 y} \rangle + \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_2 y} \right\rangle\} \\
&= t^2 \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_1 y} \right\rangle + t(1-t) \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_1 y} \right\rangle + (1-t)^2 \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_2 y} \right\rangle + t(1-t) \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_2 y} \right\rangle \\
&\quad + t(1-t)\{\langle \overrightarrow{x_2 x_1}, \overrightarrow{x_1 x_2} \rangle + \langle \overrightarrow{x_2 x_1}, \overrightarrow{x_2 y} \rangle\} + t(1-t) \langle \overrightarrow{x_1 x_2}, \overrightarrow{x_2 y} \rangle \\
&= t \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_1 y} \right\rangle + (1-t) \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_2 y} \right\rangle - t(1-t) \langle \overrightarrow{x_1 x_2}, \overrightarrow{x_1 x_2} \rangle \\
&\quad - t(1-t) \langle \overrightarrow{x_1 x_2}, \overrightarrow{x_2 y} \rangle + t(1-t) \langle \overrightarrow{x_1 x_2}, \overrightarrow{x_2 y} \rangle \\
&\leq t \left\langle \overrightarrow{x_1 Ty}, \overrightarrow{x_1 y} \right\rangle + (1-t) \left\langle \overrightarrow{x_2 Ty}, \overrightarrow{x_2 y} \right\rangle \\
&< 0.
\end{aligned}$$

Therefore $u_t \in G(y)$, that is $G(y)$ is convex.
Step 2. To show that $F^{-1}(x)$ is an open set.
For $F^{-1}(x) = \{y \in C : \left\langle \overrightarrow{xy}, \overrightarrow{xTy} \right\rangle < 0\}$, we show that $C \backslash F^{-1}(x) = \{y \in C : \left\langle \overrightarrow{xy}, \overrightarrow{xTy} \right\rangle \geq 0\}$ is a closed set. Let $\{y_n\} \subseteq C \backslash F^{-1}(x)$ such that $y_n \to y_0$. Then $\left\langle \overrightarrow{xy_n}, \overrightarrow{xTy_n} \right\rangle \geq 0$. We will show that $y_0 \in C \backslash F^{-1}(x)$. By Lemma 2.19, T is a Lipschitzian map. Inparticular, T is continuous. It follows that

$$\begin{aligned}
0 &\leq \left\langle \overrightarrow{xy_n}, \overrightarrow{xTy_n} \right\rangle = \left\langle \overrightarrow{xy_0}, \overrightarrow{xTy_n} \right\rangle + \left\langle \overrightarrow{y_0 y_n}, \overrightarrow{xTy_n} \right\rangle = \left\langle \overrightarrow{xy_0}, \overrightarrow{xTy_0} \right\rangle + \left\langle \overrightarrow{xy_0}, \overrightarrow{Ty_0 Ty_n} \right\rangle + \left\langle \overrightarrow{y_0 y_n}, \overrightarrow{xTy_n} \right\rangle \\
&\leq \left\langle \overrightarrow{xy_0}, \overrightarrow{xTy_0} \right\rangle + d(x, y_0) d(Ty_0, Ty_n) + d(y_0, y_n) d(x, Ty_n) \\
&\leq \left\langle \overrightarrow{xy_0}, \overrightarrow{xTy_0} \right\rangle + \max\{\frac{1+k}{1-k}, 1\} d(x, y_0) d(y_0, y_n) + d(y_0, y_n) d(x, Ty_n),
\end{aligned}$$

for all $n \in \mathbb{N}$. Taking the limit in both sides, we get that $\left\langle \overrightarrow{xy_0}, \overrightarrow{xTy_0} \right\rangle \geq 0$.
That is $y_0 \in C \backslash F^{-1}(x)$. Hence $C \backslash F^{-1}(x)$ is a closed set in C, therefore $F^{-1}(x)$ is an open set.

We next show that there exists an element $x_0 \in C$ such that $\left\langle \overrightarrow{xx_0}, \overrightarrow{xTx_0} \right\rangle \geq 0$ for all $x \in C$. By assumption, we have $F(y) \neq \varnothing$ for every $y \in C$, and by Lemma 2.5, we have C is $\triangle$- compact. Notice that there exists a point $z \in C$ such that $C \backslash F^{-1}(z) \subseteq C$. Also, $co(C \backslash F^{-1}(z))$ is the smallest convex set containing $C \backslash F^{-1}(z)$. Then, we get that there exists a point $z \in C$ such that $co(C \backslash F^{-1}(z)) \subseteq C$ where C is a nonempty $\triangle$-compact subset of C. Also, by Lemma 3.1, we have (C, d) is a complete CAT(0) space. By Lemma 2.25 and Remark 2.26, we have $x_0 \in C$ such that $x_0 \in G(x_0)$. This implies that $0 = \left\langle \overrightarrow{x_0 x_0}, \overrightarrow{x_0 Tx_0} \right\rangle < 0$. This is a contradiction. We obtain that $\{x \in C : \left\langle \overrightarrow{xx_0}, \overrightarrow{xTx_0} \right\rangle < 0\} = \varnothing$. Therefore $\left\langle \overrightarrow{xx_0}, \overrightarrow{xTx_0} \right\rangle \geq 0$ for all $x \in C$. □

Lemma 3.3. *Let C be a bounded closed convex subset of a complete CAT(0) space X. Let T be a k- strict pseudo-contraction in the terminology of Browder and Petryshyn. Then, there exists an element $x_0 \in C$ such that $\left\langle \overrightarrow{xx_0}, \overrightarrow{x_0 Tx_0} \right\rangle \geq 0$ for all $x \in C$.*

Proof. Let C be a bounded closed convex subset of complete CAT(0) space (X, d). By Lemma 3.1, we have (C, d) is a complete CAT(0) space. By Lemma 3.2, we have

$$x_0 \in C \text{ such that } \left\langle \overrightarrow{xx_0}, \overrightarrow{xTx_0} \right\rangle \geq 0 \text{ for all } x \in C. \tag{3.1}$$

Also, for any $u, z \in C$ and $0 < t < 1$ and since C is convex, we have $y_t = (1-t)u \oplus tz \in C$. Then, for $x_0 \in C$ we have

$$y_t = (1-t)x_0 \oplus tz \in C. \tag{3.2}$$

By using (3.1) and (3.2), we have $0 \leq \left\langle \overrightarrow{\{(1-t)x_0 \oplus tz\}x_0}, \overrightarrow{y_tTx_0} \right\rangle$. By Lemma 2.10, we have $0 \leq \left\langle \overrightarrow{\{(1-t)x_0 \oplus tz\}x_0}, \overrightarrow{y_tTx_0} \right\rangle \leq t \left\langle \overrightarrow{zx_0}, \overrightarrow{y_tTx_0} \right\rangle$. Since $t > 0$, it follows that $0 \leq \left\langle \overrightarrow{zx_0}, \overrightarrow{y_tTx_0} \right\rangle$. By Lemma 2.19, T is a Lipschitzian map. Inparticular, T is continuous. Also $y_t \to x_0$ as $t \to 0$. It follows that

$$0 \leq \left\langle \overrightarrow{zx_0}, \overrightarrow{y_tTx_0} \right\rangle = \langle \overrightarrow{zx_0}, \overrightarrow{y_tx_0} \rangle + \left\langle \overrightarrow{zx_0}, \overrightarrow{x_0Tx_0} \right\rangle \leq d(z, x_0)d(y_t, x_0) + \left\langle \overrightarrow{zx_0}, \overrightarrow{x_0Tx_0} \right\rangle,$$

for $0 < t < 1$. Taking the limit in both sides, we get that $\left\langle \overrightarrow{zx_0}, \overrightarrow{x_0Tx_0} \right\rangle \geq 0$ as $t \to 0$.

Therefore, $\left\langle \overrightarrow{xx_0}, \overrightarrow{x_0Tx_0} \right\rangle \geq 0$ for all $x \in C$. □

Lemma 3.4. *Let (X, d) be a complete CAT(0) space and T be a k-strict pseudo-contraction in the terminology of Browder and Petryshyn with domain $D(T)$ and range $R(T)$. Then for all $x, y \in D(T)$ the following inequalities hold and are equivalent :*

(1) $d^2(x, Tx) + d^2(y, Ty) \leq \frac{2}{1-k}\left\langle \overrightarrow{xy}, \overrightarrow{xTx} \right\rangle - \frac{2k}{1-k}\left\langle \overrightarrow{xy}, \overrightarrow{yTy} \right\rangle - 2\left\langle \overrightarrow{Txy}, \overrightarrow{yTy} \right\rangle$;

(2) $d^2(x, Tx) + d^2(y, Ty) \leq \frac{2}{1-k}\left\langle \overrightarrow{xy}, \overrightarrow{xTx} \right\rangle - \frac{2}{1-k}\left\langle \overrightarrow{xy}, \overrightarrow{yTy} \right\rangle + 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle$;

(3) $d^2(x, Tx) + d^2(y, Ty) \leq \frac{2}{1-k}\left\langle \overrightarrow{xTy}, \overrightarrow{xTx} \right\rangle - \frac{2}{1-k}\left\langle \overrightarrow{Txy}, \overrightarrow{yTy} \right\rangle - 2(\frac{1+k}{1-k})\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle$;

(4) $d^2(x, Tx) + d^2(y, Ty)$
$\leq \left\langle \overrightarrow{xTy}, \overrightarrow{xTx} \right\rangle - \left\langle \overrightarrow{Txy}, \overrightarrow{yTy} \right\rangle + \frac{1+k}{2}\{d^2(x, Tx) - 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle + d^2(y, Ty)\}$.

Proof. We first show that (2) holds.

$$\begin{aligned}
d^2(x, Tx) + d^2(y, Ty) &= d^2(x, Tx) - 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle + d^2(y, Ty) + 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle \\
&= d^2(x, Tx) - d^2(x, Ty) - d^2(Tx, y) + d^2(x, y) + d^2(Tx, Ty) \\
&\quad + d^2(y, Ty) + 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle \\
&\leq d^2(x, Tx) - d^2(x, Ty) - d^2(Tx, y) + d^2(x, y) + d^2(x, y) \\
&\quad + kd^2(x, Tx) - 2k\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle + kd^2(y, Ty) + d^2(y, Ty) + 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle.
\end{aligned}$$

By simple calculation from the inequality above we get that

$$\begin{aligned}
(1-k)\{d^2(x, Tx) + d^2(y, Ty)\} &\leq d^2(x, Tx) + d^2(y, Ty) + d^2(y, x) + d^2(x, y) - d^2(y, Tx) \\
&\quad - d^2(x, Ty) + 2(1-k)\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle.
\end{aligned}$$

Dividing throughout with $(1-k)$ we have that

$$\begin{aligned}
d^2(x, Tx) + d^2(y, Ty) &\leq \frac{1}{1-k}\{d^2(x, Tx) + d^2(y, Ty) + d^2(y, x) + d^2(x, y) \\
&\quad - d^2(y, Tx) - d^2(x, Ty)\} + 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle \\
&= \frac{2}{1-k}\left\langle \overrightarrow{xy}, \overrightarrow{xTx} \right\rangle - \frac{2}{1-k}\left\langle \overrightarrow{xy}, \overrightarrow{yTy} \right\rangle + 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy} \right\rangle.
\end{aligned} \tag{3.3}$$

Then, (2) is true. Next, we observe that

$$\begin{aligned} -\frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle+2\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle &= -\frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle+2\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle+2\left\langle\overrightarrow{yTx},\overrightarrow{yTy}\right\rangle \\ &= \{2-\frac{2}{1-k}\}\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle-2\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle \\ &= \frac{-2k}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle-2\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle. \end{aligned} \tag{3.4}$$

Substituting (3.4) in (3.3), we get that (1) holds; that is
$d^2(x,Tx)+d^2(y,Ty)\le \frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{xTx}\right\rangle-\frac{2k}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle-2\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle$,
and hence (1) and (2) are equivalent.

We next show that (3) is true. Let us consider

$$\frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{xTx}\right\rangle=\frac{2}{1-k}\{\left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle+\left\langle\overrightarrow{Tyy},\overrightarrow{xTx}\right\rangle\}=\frac{2}{1-k}\left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle-\frac{2}{1-k}\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle. \tag{3.5}$$

$$-\frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle=-\frac{2}{1-k}\{\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle+\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle\}=-\frac{2}{1-k}\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle-\frac{2}{1-k}\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle. \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$\frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{xTx}\right\rangle-\frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle=\frac{2}{1-k}\left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle-\frac{2}{1-k}\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle-\frac{4}{1-k}\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle.$$

We get that

$$\begin{aligned} &\frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{xTx}\right\rangle-\frac{2}{1-k}\left\langle\overrightarrow{xy},\overrightarrow{yTy}\right\rangle+2\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle \\ &=\frac{2}{1-k}\left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle-\frac{2}{1-k}\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle-\frac{4}{1-k}\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle+2\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle \\ &=\frac{2}{1-k}\left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle-\frac{2}{1-k}\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle-2(\frac{1+k}{1-k})\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle. \end{aligned}$$

This shows that (3) is true. We get that (2) and (3) are equivalent. Next, we will show that (3) and (4) are equivalent. We will show that (3) implies (4). Since $\frac{1-k}{2}>0$, for (3) is true, we get that

$$\begin{aligned} (\frac{1-k}{2})\{d^2(x,Tx)+d^2(y,Ty)\} &\le \left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle-\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle-(1+k)\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle \\ [1-(\frac{1+k}{2})]\{d^2(x,Tx)+d^2(y,Ty)\} &\le \left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle-\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle-(1+k)\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle \\ d^2(x,Tx)+d^2(y,Ty) &\le \left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle-\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle-(1+k)\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle \\ &\quad +(\frac{1+k}{2})\{d^2(x,Tx)+d^2(y,Ty)\} \\ &= \left\langle\overrightarrow{xTy},\overrightarrow{xTx}\right\rangle-\left\langle\overrightarrow{Txy},\overrightarrow{yTy}\right\rangle \\ &\quad +(\frac{1+k}{2})\{d^2(x,Tx)+d^2(y,Ty)-2\left\langle\overrightarrow{xTx},\overrightarrow{yTy}\right\rangle\}. \end{aligned}$$

Then, we get that (4) holds. By using a similar method, we get that (4) implis (3). That is, we get that (3) and (4) are equivalent. This completes our proof. ☐

Lemma 3.5. *Let (X,d) be a complete CAT(0) space and T be a k-strict pseudo-contraction in the terminology of Browder and Petryshyn with domain $D(T)$ and range $R(T)$. If, there exists $u \in D(T)$ such that $\left\langle \overrightarrow{xu}, \overrightarrow{uTu} \right\rangle \geq 0$ and $\left\langle \overrightarrow{Txu}, \overrightarrow{uTu} \right\rangle \geq 0$ for some $x \in D(T)$, the following inequalities hold :*

$$d^2(x,Tx) \leq \begin{cases} \frac{2}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{xTx} \right\rangle, & \begin{cases} \text{if } k \in [0,1); \\ \text{or if } \left\langle \overrightarrow{xTx}, \overrightarrow{uTu} \right\rangle \leq 0; \end{cases} \\ \begin{cases} \frac{2}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{xTx} \right\rangle \\ \frac{2}{1-k}\left\langle \overrightarrow{xTu}, \overrightarrow{xTx} \right\rangle \end{cases}, & \text{if } \left\langle \overrightarrow{xTx}, \overrightarrow{uTu} \right\rangle \geq 0 \text{ and } k \in [0,1); \\ \frac{2}{1-k}\left\langle \overrightarrow{xTu}, \overrightarrow{xTx} \right\rangle, & \text{if } \left\langle \overrightarrow{xTx}, \overrightarrow{uTu} \right\rangle \geq 0 \text{ and } k \in [-1,0); \\ \left\langle \overrightarrow{xTu}, \overrightarrow{xTx} \right\rangle, & \text{if } k \in (-\infty,-1]. \end{cases}$$

Proof. If $k \in [0,1)$, then $k<1 \Leftrightarrow 0<1-k$ and note that $0 \leq 2k$, so we have $\frac{2k}{1-k} \geq 0 \Leftrightarrow -\frac{2k}{1-k} \leq 0$. By Lemma 3.4(1), $\left\langle \overrightarrow{xu}, \overrightarrow{uTu} \right\rangle \geq 0$ and $\left\langle \overrightarrow{Txu}, \overrightarrow{uTu} \right\rangle \geq 0$, we get that

$$\begin{aligned} d^2(x,Tx) &\leq d^2(x,Tx)+d^2(u,Tu) \leq \frac{2}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{xTx} \right\rangle - \frac{2k}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{uTu} \right\rangle - 2\left\langle \overrightarrow{Txu}, \overrightarrow{uTu} \right\rangle \\ &\leq \frac{2}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{xTx} \right\rangle. \end{aligned} \tag{3.7}$$

If $\left\langle \overrightarrow{xTx}, \overrightarrow{uTu} \right\rangle \leq 0$, then by Lemma 3.4(2) and $\left\langle \overrightarrow{xu}, \overrightarrow{uTu} \right\rangle \geq 0$, we get that

$$\begin{aligned} d^2(x,Tx) &\leq d^2(x,Tx)+d^2(u,Tu) \leq \frac{2}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{xTx} \right\rangle - \frac{2}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{uTu} \right\rangle + 2\left\langle \overrightarrow{xTx}, \overrightarrow{uTu} \right\rangle \\ &\leq \frac{2}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{xTx} \right\rangle. \end{aligned}$$

Before we prove the next case, let us consider the following
$k \in [-1,1) \Leftrightarrow -1 \leq k < 1 \begin{cases} \Leftrightarrow 0 \leq 1+k < 2. \\ \Leftrightarrow 1 \geq -k > -1 \Leftrightarrow 2 \geq 1-k > 0 \Leftrightarrow \frac{1}{1-k} \geq \frac{1}{2}. \end{cases}$
Therefore, we have $2(\frac{1+k}{1-k}) \geq 1+k \geq 0$ and then

$$-2(\frac{1+k}{1-k}) \leq 0 \quad \text{whenever} \quad k \in [-1,1). \tag{3.8}$$

If $\left\langle \overrightarrow{xTx}, \overrightarrow{uTu} \right\rangle \geq 0$ and $k \in [0,1)$, then it follows from (3.8), Lemma 3.4(3) and $\left\langle \overrightarrow{Txu}, \overrightarrow{uTu} \right\rangle \geq 0$, we get that

$$\begin{aligned} d^2(x,Tx) &\leq d^2(x,Tx)+d^2(u,Tu) \leq \frac{2}{1-k}\left\langle \overrightarrow{xTu}, \overrightarrow{xTx} \right\rangle - \frac{2}{1-k}\left\langle \overrightarrow{Txu}, \overrightarrow{uTu} \right\rangle - 2(\frac{1+k}{1-k})\left\langle \overrightarrow{xTx}, \overrightarrow{uTu} \right\rangle \\ &\leq \frac{2}{1-k}\left\langle \overrightarrow{xTu}, \overrightarrow{xTx} \right\rangle. \end{aligned}$$

From (3.7), we can conclude in this case that $d^2(x,Tx) \leq \begin{cases} \frac{2}{1-k}\left\langle \overrightarrow{xu}, \overrightarrow{xTx} \right\rangle \\ \frac{2}{1-k}\left\langle \overrightarrow{xTu}, \overrightarrow{xTx} \right\rangle \end{cases}$. If $\left\langle \overrightarrow{xTx}, \overrightarrow{uTu} \right\rangle \geq 0$ and $k \in [-1,0)$, then by (3.8), Lemma 3.4(3) and $\left\langle \overrightarrow{Txu}, \overrightarrow{uTu} \right\rangle \geq 0$, we get that $d^2(x,Tx) \leq$

$\frac{2}{1-k}\left\langle \overrightarrow{xTu}, \overrightarrow{xTx}\right\rangle$. Finally, if $k \in (-\infty, -1]$, then by using lemma 3.4(4) and $\left\langle \overrightarrow{Txu}, \overrightarrow{uTu}\right\rangle \geq 0$, we get that

$$\begin{aligned} d^2(x,Tx) &\leq d^2(x,Tx)+d^2(u,Tu) \leq \left\langle \overrightarrow{xTu}, \overrightarrow{xTx}\right\rangle - \left\langle \overrightarrow{Txu}, \overrightarrow{uTu}\right\rangle + \frac{1+k}{2}d^2(xTx,uTu) \\ &\leq \left\langle \overrightarrow{xTu}, \overrightarrow{xTx}\right\rangle. \end{aligned}$$

This completes our proof. □

Every iteration process generated by the shrinking projection method for a k-strict pseudo-contraction T in the terminology of Browder and Petryshn is well defined even if T is fixed point free.

Lemma 3.6. *Let (X, d) be a complete CAT(0) space and C be nonempty closed and convex subset of X. Let $T : C \to C$ be a k-strict pseudo-contraction in the terminology of Browder and Petryshyn, that is for all $x, y \in X$ there exists an element $k \in (-\infty, 1)$ such that*

$$d^2(Tx,Ty) \leq d^2(x,y) + k[d^2(x,Tx) - 2\left\langle \overrightarrow{xTx}, \overrightarrow{yTy}\right\rangle + d^2(y,Ty)].$$

Let $x_0 \in X$, $C_1 = C$ and $\{x_n\}$ be a sequence in C generated by

$$\begin{cases} x_n &= P_{C_n}(x_0), \\ C_{n+1} &= \left\{ z \in C_n : d^2(x_n,Tx_n) \leq max\,\{\frac{2}{1-k},1\}\left\langle \overrightarrow{x_nz}, \overrightarrow{x_nTx_n}\right\rangle \right\}, \end{cases} \tag{3.9}$$

for all $n \in \mathbb{N}$. Then, C_n is nonempty closed convex subsets of X and consequently, $\{x_n\}$ is well defined for every $n \in \mathbb{N}$.

Proof. Clearly, C_1 is nonempty. Suppose that C_m is nonempty for some $m \in N$. We wish to show that C_{m+1} is nonempty. Since $C_m \subset C_{m-1} \subset ... \subset C_1$, we have that $C_1, C_2, ..., C_m$ are nonempty. Next, we will show that $C_1, C_2, ..., C_m$ are closed and convex. It is sufficient to show that C_m is closed and convex. It is not hard to show that for any $\{z_k\} \subseteq C_m$ such that $z_k \to z_0$, we have $z_0 \in C_m$. We get that C_m is closed.

We next show that C_m is convex. Notice that a subset of a CAT(0) space, equipped with the induced metric, is a CAT(0) space if and only if it is convex. Thus, we will show that (C_m, d) is a complete CAT(0) space. Let each $x, y \in C_m$, we have $x, y \in C$. By Lemma 3.1, we have (C, d) is a complete CAT(0) space and thus, it is a geodesic space, hence x, y are joined by a geodesic. Since x, y are arbitrary, thus we have $x, y \in C_m$ are joined by a geodesic. Hence C_m is a geodesic space. Since C_m is closed subset of complete metric space (C, d), then (C_m, d) is a complete metric space. It follows from Lemma 2.21 that, for every $y, z \in C_m$, there exists $p \in C_m$ such that $d(y, p) = d(p, z) = \frac{1}{2}d(y, z)$. Now, we claim that C_m satisfies the (CN) inequality. In fact, let $x, y, z \in C_m$ and $p \in C_m$ with $d(y, p) = d(p, z) = \frac{1}{2}d(y, z)$. Let α and β be two numbers satisfy $\alpha + \beta \geq 1$. Then $\alpha^2 + \beta^2 \geq \frac{1}{2}(\alpha + \beta)^2 \geq \frac{1}{2}$ with equality if and only if $\alpha = \beta = \frac{1}{2}$. By this fact and by the triangle inequality, we get that $(\frac{d(y,p)}{d(y,z)})^2 + (\frac{d(p,z)}{d(y,z)})^2 \geq \frac{1}{2}$. That is, $\frac{1}{2}d^2(y, z) \leq d^2(y, p) + d^2(p, z)$. It follows from the above inequality that, setting $x = p$, we get that $d^2(x, y) + d^2(x, z) \geq 2d^2(x, p) + \frac{1}{2}d^2(y, z)$, this implies that C_m satisfies the (CN) inequality. By Lemma 2.22, we know that (C_m, d) is a CAT(0) space. By above, we have (C_m, d) is a complete metric space. Then (C_m, d) is a complete CAT(0) space. This implies that C_m is convex subset of X. Thus, we have C_m is closed and convex. Finally, put $r = \max\{d(x_0, x_i), d(x_0, Tx_i) : i = 1, 2, ..., m\}$ and $B_r = \{z \in X : d(x_0, z) \leq r\}$. Obviously $C \cap B_r$ is a nonempty bounded closed convex subset of X. It follows from Lemma 3.3 that there exists an element $u \in C \cap B_r$ such that $\left\langle \overrightarrow{yu}, \overrightarrow{uTu}\right\rangle \geq 0$ for all $y \in C \cap B_r$. In particular, we have

$$\left\langle \overrightarrow{x_iu}, \overrightarrow{uTu}\right\rangle \geq 0 \quad \text{and} \quad \left\langle \overrightarrow{Tx_iu}, \overrightarrow{uTu}\right\rangle \geq 0 \tag{3.10}$$

for every $i = 1, 2, ..., m$.
Case I. $\max\{\frac{2}{1-k}, 1\} = \frac{2}{1-k}$.
Notice that $\max\{\frac{2}{1-k}, 1\} = \frac{2}{1-k} \Leftrightarrow 1 \le \frac{2}{1-k} \Leftrightarrow 1-k \le 2 \Leftrightarrow -1 \le k \Leftrightarrow k \in [-1, 1)$, it follows from (3.10) and Lemma 3.5 that

$$d^2(x_i, Tx_i) \le \begin{cases} \frac{2}{1-k}\left\langle \overrightarrow{x_iu}, \overrightarrow{x_iTx_i}\right\rangle, & \begin{cases} \text{if } k \in [0,1); \\ \text{or if } \left\langle \overrightarrow{x_iTx_i}, \overrightarrow{uTu}\right\rangle \le 0; \end{cases} \\ \begin{cases} \frac{2}{1-k}\left\langle \overrightarrow{x_iu}, \overrightarrow{x_iTx_i}\right\rangle \\ \frac{2}{1-k}\left\langle \overrightarrow{x_iTu}, \overrightarrow{x_iTx_i}\right\rangle \end{cases}, & \text{if } \left\langle \overrightarrow{x_iTx_i}, \overrightarrow{uTu}\right\rangle \ge 0 \text{ and } k \in [0,1); \\ \frac{2}{1-k}\left\langle \overrightarrow{x_iTu}, \overrightarrow{x_iTx_i}\right\rangle, & \text{if } \left\langle \overrightarrow{x_iTx_i}, \overrightarrow{uTu}\right\rangle \ge 0 \text{ and } k \in [-1,0) \end{cases}$$

for every $i = 1, 2, ..., m$. This shows that $u \vee Tu \in C_{m+1}$.
Case II. $\max\{\frac{2}{1-k}, 1\} = 1$.
Notice that $\max\{\frac{2}{1-k}, 1\} = \frac{2}{1-k} \le 1 \Leftrightarrow 2 \le 1-k \Leftrightarrow k \le -1 \Leftrightarrow k \in (-\infty, -1]$, it follows from (3.9) and Lemma 3.5 that $d^2(x_i, Tx_i) \le \left\langle \overrightarrow{x_iTu}, \overrightarrow{x_iTx_i}\right\rangle$, if $k \in (-\infty, -1]$ for every $i = 1, 2, ..., m$. This shows that $Tu \in C_{m+1}$. By Case I and Case II, we can conclude that $u \vee Tu \in C_{m+1}$. Hence C_{m+1} is nonempty. By induction on n, therefore the desired result. □

Theorem 3.7. *Let all the assumptions be the same as in Lemma 3.6. Then, the following are equivalent :*

(1) $\bigcap_{n=1}^{\infty} C_n$ *is nonempty;*
(2) $\{x_n\}$ *is bounded;*
(3) $Fix(T)$ *is nonempty.*

Proof. [(1) ⇒ (2)] Let $u \in \bigcap_{n=1}^{\infty} C_n$. By Lemma 2.12, it follows from the nonexpansiveness of P_{C_n} that $d(x_n, u) = d(P_{C_n}x_0, P_{C_n}u) \le d(x_0, u)$. This shows that x_n is bounded.

[(2) ⇒ (3)] Suppose that x_n is bounded, we first claim that $0 \le d^2(x_{n+1}, x_n) \le d^2(x_{n+1}, x_0) - d^2(x_n, x_0)$. Since $x_n = P_{C_n}x_0$, by Lemma 2.11, we have $\langle \overrightarrow{x_0x_n}, \overrightarrow{x_nx_{n+1}}\rangle \ge 0$ for all $x_{n+1} \in C_n$. So, we have $\langle \overrightarrow{x_0x_n}, \overrightarrow{x_0x_{n+1}}\rangle - \langle \overrightarrow{x_0x_n}, \overrightarrow{x_0x_n}\rangle = \langle \overrightarrow{x_0x_n}, \overrightarrow{x_nx_{n+1}}\rangle \ge 0$ and hence $d^2(x_0, x_n) = \langle \overrightarrow{x_0x_n}, \overrightarrow{x_0x_n}\rangle \le \langle \overrightarrow{x_0x_n}, \overrightarrow{x_0x_{n+1}}\rangle$. By Lemma 2.14, 2.15 and using (2.1), we have

$$\begin{aligned} d^2(x_n, x_{n+1}) &\le 2d^2(x_n, x_{n+1}) = 4d^2(\frac{1}{2}x_0 \oplus \frac{1}{2}x_n, \frac{1}{2}x_0 \oplus \frac{1}{2}x_{n+1}) \\ &\le d^2(x_n, x_0) + d^2(x_{n+1}, x_0) + d^2(x_0, x_0) + d^2(x_n, x_{n+1}) - d^2(x_0, x_{n+1}) - d^2(x_0, x_n) \\ &= d^2(x_n, x_0) + d^2(x_{n+1}, x_0) + 2\langle \overrightarrow{x_{n+1}x_0}, \overrightarrow{x_0x_n}\rangle \\ &= d^2(x_n, x_0) + d^2(x_{n+1}, x_0) - 2\langle \overrightarrow{x_0x_{n+1}}, \overrightarrow{x_0x_n}\rangle \\ &\le d^2(x_n, x_0) + d^2(x_{n+1}, x_0) - 2d^2(x_0, x_n). \end{aligned} \tag{3.11}$$

This shows that $\{d(x_n, x_0)\}$ is nondecreasing and with is the bounded of $\{x_n\}$, we have $\lim_{n\to\infty} d(x_n, x_0)$ exists. From (3.11), we get that $d^2(x_{n+1}, x_n) \to 0$ as $n \to \infty$. Thus $\overrightarrow{x_nx_{n+1}} = 0$. Since $x_{n+1} \in C_{n+1}$, we have $d^2(x_n, Tx_n) \le \max\{\frac{2}{1-k}, 1\}\left\langle \overrightarrow{x_nx_{n+1}}, \overrightarrow{x_nTx_n}\right\rangle \le \max\{\frac{2}{1-k}, 1\}d(x_n, x_{n+1})d(x_n, Tx_n)$. Thus $d(x_n, Tx_n) \to 0$ as $n \to \infty$. Since $\{x_n\}$ is bounded and by Lemma 2.3, we have $\triangle - \lim_{j\to\infty} x_{n_j} = w$. Since $d(x_{n_j}, Tx_{n_j}) \to 0$ as $j \to \infty$, then we get that

$$\Phi(x) = \limsup_{j\to\infty} d(x_{n_j}, x) = \limsup_{j\to\infty} d(Tx_{n_j}, x) \text{ for all } x \in C. \tag{3.12}$$

By taking $x = Tw$ in (3.12), we have

$$\begin{aligned}
\Phi(Tw)^2 &= \limsup_{j\to\infty}\ d^2(Tx_{n_j}, Tw) \\
&\le \limsup_{j\to\infty}\{d^2(x_{n_j}, w) + k[d^2(x_{n_j}, Tx_{n_j}) - 2\left\langle \overrightarrow{x_{n_j}Tx_{n_j}}, \overrightarrow{wTw}\right\rangle + d^2(w, Tw)]\} \\
&= \limsup_{j\to\infty}\ d^2(x_{n_j}, w) + k\limsup_{j\to\infty}[d^2(x_{n_j}, Tx_{n_j}) + 2\left\langle \overrightarrow{Tx_{n_j}x_{n_j}}, \overrightarrow{wTw}\right\rangle + d^2(w, Tw)] \\
&\le \limsup_{j\to\infty} d^2(x_{n_j}, w) + k\limsup_{j\to\infty} d^2(x_{n_j}, Tx_{n_j}) \\
&\quad + 2k\limsup_{j\to\infty}[d(Tx_{n_j}, x_{n_j})d(w, Tw)] + k\limsup_{j\to\infty} d^2(w, Tw) \\
&= \Phi(w)^2 + k\ d^2(w, Tw). \qquad (3.13)
\end{aligned}$$

Since $-\infty < k < 1$, we can choose a real number $\lambda \in [0, 1]$ be such that $\max\{0, k\} < \lambda < 1$. By Lemma 2.15, we have

$$d^2\left(x_{n_j}, \lambda w \oplus (1-\lambda)Tw\right) \le \lambda d^2\left(x_{n_j}, w\right) + (1-\lambda)d^2\left(x_{n_j}, Tw\right) - \lambda(1-\lambda)d^2\left(w, Tw\right)$$

Taking the superior limit on both sides of the above inequality, we get that

$$\Phi(\lambda w \oplus (1-\lambda)Tw)^2 \le \lambda\Phi(w)^2 + (1-\lambda)\Phi(Tw)^2 - \lambda(1-\lambda)d^2\left(w, Tw\right)$$

Since $\triangle - \lim_{j\to\infty} x_{n_j} = w$. By using (3.13) and Lemma 2.2, we have

$$\begin{aligned}
\Phi(w)^2 &\le \Phi(\lambda w \oplus (1-\lambda)Tw)^2 \le \lambda\Phi(w)^2 + (1-\lambda)\Phi(Tw)^2 - \lambda(1-\lambda)d^2\left(w, Tw\right) \\
&\le \lambda\Phi(w)^2 + (1-\lambda)\left(\Phi(w)^2 + kd^2\left(w, Tw\right)\right) - \lambda(1-\lambda)d^2\left(w, Tw\right) \\
&= \lambda\Phi(w)^2 + (1-\lambda)\Phi(w)^2 + (1-\lambda)kd^2\left(w, Tw\right) - \lambda(1-\lambda)d^2\left(w, Tw\right) \\
&= \Phi(w)^2 + (1-\lambda)(k-\lambda)d^2\left(w, Tw\right).
\end{aligned}$$

This implies that $(1-\lambda)(\lambda-k)d^2\left(w, Tw\right) \le 0$. Since $\max\{0, k\} < \lambda < 1$, we have $(1-\lambda)(\lambda-k) > 0$. This implies that $Tw = w$, that is $w \in Fix(T) \neq \varnothing$.

[(3) $\Rightarrow$ (1)] Suppose that $Fix(T) \neq \varnothing$. We claim that $Fix(T) \subset C_n$ for all $n \in N$. If $w \in Fix(T)$, then we have $\left\langle \overrightarrow{ab}, \overrightarrow{wTw}\right\rangle = 0$ for all $a, b \in X$. Taking $u = w$ in the proof of Lemma 3.6, it is not hard to observe that all inequalities are satisfied. This implies that $w \in C_n$ for all $n \in N$. Therefore $Fix(T) \subset \bigcap_{n=1}^{\infty} C_n \neq \varnothing$. ☐

Theorem 3.8. *Let all the assumptions be the same as in Theorem 3.7 Then, if* $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$ *(*$\Leftrightarrow \{x_n\}$ *is bounded* $\Leftrightarrow Fix(T) \neq \varnothing$*),then the sequence* $\{x_n\}$ *generated by* (3.9) *converges strongly to some points of* C *and its strong limit point is a member of* $Fix(T)$*, that is* $\lim_{n\to\infty} x_n = P_{Fix(T)}x_0 \in Fix(T)$.

Proof. If $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$, then Theorem 3.7 ensures that $\{x_n\}$ is bounded sequence in C. By Lemma 2.3, we have $\{x_{n_j}\} \subseteq \{x_n\}$ such that $\triangle - \lim_{j\to\infty} x_{n_j} = u$. By the proof of Theorem 3.7[(2) $\Rightarrow$ (3)], we have $u \in Fix(T)$. By Lemma 2.20, we have $P_{Fix(T)}$ is well defined. Also, $P_{Fix(T)}x_0 \in Fix(T) \subset C_n$, we observe that

$$d(x_n, x_0) = d(P_{C_n}x_0, x_0) \le d(P_{Fix(T)}x_0, x_0) \qquad (3.14)$$

for all $n \in N$. Since $\{d(x_n, x_0)\}$ is nondecreasing, we get that $\lim_{n\to\infty} d(x_n, x_0)$ exists. Since $\triangle - \lim_{j\to\infty} x_{n_j} = u$, by using Lemma 2.6, we have $\{x_{n_j}\} \rightharpoonup u$. By Definition 2.16 and Lemma 2.17, we get that $d(u, \{x_0\}) \le \liminf_{j\to\infty} d(x_{n_j}, \{x_0\})$. By Lemma 2.8, there exists an $x_0 \in \{x_0\}$ such that

$$d(u, x_0) = d(u, \{x_0\}) \le \liminf_{j\to\infty} d(x_{n_j}, x_0) \tag{3.15}$$

By using (3.14) and (3.15), we get that

$$d(u, x_0) \le \liminf_{j\to\infty} d(x_{n_j}, x_0) = \lim_{n\to\infty} d(x_n, x_0) \le d(P_{Fix(T)}x_0, x_0). \tag{3.16}$$

Taking into account $u \in Fix(T)$, from (3.16), we have $d(u, x_0) \le d(P_{Fix(T)}x_0, x_0) \le d(u, x_0)$. This implies that $d(u, x_0) = d(P_{Fix(T)}x_0, x_0)$. By Lemma 2.8, we obtain that $u = P_{Fix(T)}x_0$. Therefore $\{x_n\} \rightharpoonup P_{Fix(T)}x_0$ and $d(x_n, x_0) \to d(P_{Fix(T)}x_0, x_0)$. Consequently, from (3.11), we get that $d^2(x_n, P_{Fix(T)}x_0) \le d^2(P_{Fix(T)}x_0, x_0) - d^2(x_n, x_0) \to 0$ as $n \to \infty$. This completes our proof. $\square$

Remark 3.9. The results in this section extend and improve the corresponding Theorem 3.4 and 3.5 in [22] in the case of an iterative projection technique in a Hilbert space.

4. Conclusion

In the present paper, we study some existence and convergence theorems for fixed points of a strict pseudo-contraction by using an iterative projection technique with some suitable conditions. We obtain the sufficient conditions for the existence and convergence theorem for the fixed points of strict pseudo-contraction mappings in complete CAT(0) spaces.

Acknowledgements

The authors would like to thank, Naresuan University, Phitsanulok 65000, Thailand.

References

[1] M. Bacak, Convex Analysis and Optimization in Hadamard spaces, Walter de Gruyter GmbH, Berlin (2014).

[2] I. D. Berg, I. G. Nikolaev, Quasilinearization and curvature of Alexandrov spaces, Geom Dedicata, 133, 195-218 (2008).

[3] M. Bridson, A. Haefliger, Metric Spaces of Nonpositive Curvature, Springer, Berlin (1999).

[4] F. E. Browder, W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20, 197-228 (1967).

[5] F. Bruhat, J. Tits, Groupes re ductifs sur un corps local. I. Donees radicielles valuees, Inst., Hautes Etudes Sci. Publ. Math., 41, 5-251 (1972).

[6] H. Dehghan, J. Rooin, A characterization of metric projection in CAT(0) spaces, International Conference on Functional Equation, Geometric Functions and Applications(ICFGA 2012), Payame Noor University, Tabriz, Iran, 10-12 May 2012.

[7] S. Dhompongsa, A. Kaewkhao, B. Panyanak, Lim's theorems for multivalued mappings in CAT(0) spaces, J. Math. Anal. Appl., 312, 478-487 (2005).

[8] S. Dhompongsa, W. A. Kirk, B. Sims, Fixed points of uniformly lipschitzian mappings, Nonlinear Anal., 65, 762-772 (2006).

[9] S. Dhompongsa, W. A. Kirk, B. Panyanak, Nonexpansive set-valued mappings in metric and Banach spaces, J. Nonlinear Convex Anal., 8, 35-45 (2007).

[10] S. Dhompongsa, B. Panyanak, On $\triangle$-convergence theorems in CAT(0) spaces, Comput. Math. Appl., 56, 2572-2579 (2008).

[11] A. Gharajelo, H. Dehghan, Convergence theorems for strict pseudo-contractions in CAT(0) metric spaces, Filomat 31(7), 1967-1971 (2017). doi 10.2298/FIL1707967G.

[12] B. A. Kakavandi, Weak topologies in complete CAT(0) metric spaces, Proc. Amer. Math. Soc. 141, 1029 - 1039 (2012).

[13] B. A. Kakavandi, M. Amini, Duality and subdifferential for convex functions on complete CAT(0) metric spaces, Nonlinear Anal., 73, 3450-3455 (2010).

[14] H. Khatibzadeh, V. Mohebbi, Monotone and pseudo-monotone equilibrium problems in Hadamard spaces, https://arxiv.org/abs/1611.01829.

[15] W. A. Kirk, Geodesic geometry and fixed point theory, in Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), Colecc. Abierta, vol. 64, pp. 195-225. Univ. Sevilla Secr. Publ.,Seville (2003).

[16] W. A. Kirk, Geodesic geometry and fixed point theory II, in International Conference on Fixed Point Theory and Application, pp.113-142. Yokohama Publ., Yokohama (2004).

[17] W. A. Kirk, B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Anal., 68, 3689-3696 (2008).

[18] T. C. Lim, Remarks on some fixed point theorems, Proc. Amer. Soc., 60, 179-182 (1976).

[19] H. Lu, D. Lan, Q. Hu, G. Yuan, Fixed point theorems in CAT(0) spaces with applications, J. Inequa. Appl., 320, 1-26 (2014).

[20] B. Nanjaras, B. Panyanak, Demiclosed principle for asymptotically nonexpansive mappings in CAT(0) spaces. Fixed Point Theory Appl., 2010:doi:10.1155/2010/268780.

[21] C. P. Niculescu, I. Roventa, Fan's inequality in geodesic spaces, Appl. Math. Lett. 22, 1529-1533 (2009).

[22] K. Ungchittrakool, Existence and convergence of fixed points for a strict pseudo-contraction via an iterative shrinking projection technique, J. Nonlinear Convex Anal., 15, 693-710 (2014).

[23] R. Wangkeeree, P. Preechasilp, Viscosity approximation method, for nonexpansive mappings in CAT(0) spaces, J. Inequal. Appl., 93, 1-15 (2013).

ON THE CEU-DEGREE OF SIMILARITY IN INTERNATIONAL TRADE BY USING THE CHOQUET INTEGRAL EXPECTED UTILITY

LEE-CHAE JANG, JACOB WOOD

Graduate School of Education,
Konkuk University, Seoul 143-701, Republic of Korea
E-mail : lcjang@konkuk.ac.kr

Department of Business
James Cook University Singapore, 149 Sims Road, Singapore 387380
E-mail : jacobwood0528@gmail.com

Abstract. Recently, we considered the Choquet integrals with respected to a fuzzy measure and the Choquet-expected utility(CEU)which was represented by preference functionals. We note that the CEU provides a useful tool to calculate the subjective capacity of trade values between Korea and some countries in Wood-Jang [4,10].

In this paper, by using the Choquet-expected utility in Wood-Jang [4] and the degree of similarity in Biswas [1], we define the CEU-degree of the similarity related with the CEU of trade values between Korea and some countries. In particular, we investigate some applications of the CEU-degree of similarity related with the CEU of trade values.

1. Introduction

By using fuzzy sets and Choquet integrals in [1,2,4,5,6,10], many researchers have studied the concept of Choquet intgeral expected utility and its related areas(see[3,4,8.9.11,12]). Recently, Wood-Jang [6,7] studied some applications of the Choquet integral as imprecise market premium functionals with respect to an imprecise set function which was an interval-valued measure of risk and the Choquet integral with respect to a fuzzy measure of a utility function. In 1995, Biswas [1] investigated a student's evaluation on the space of fuzzy sets which include data information for the students respective classes.

In this paper, by using the degree of similarity in Biswas [1], we define the CEU-degree of the similarity which is related to the CEU for the trade values that exist between Korea and some of its important trading partners (such as Korea-USA, Korea-New Zealand, Korea-India, and Korea-Turkey). In particular, we investigate the evaluation of the CEU-degree of similarity which is related with the CEU of trade values $CEU(u(a))$ of a utility u from an act a on S for specified HS product codes for animal product exports between Korea and selected trading partners for years 2010-2013. We note that we include the dates used in our previous studies [4,10].

In particular, we investigate the following applications:

1991 *Mathematics Subject Classification.* 28E10, 28E20, 03E72, 26E50 11B68.

Key words and phrases. Choquet integral, Choquet expected utility, fuzzy neasure, the degree of similarity.

(1) we calculate CEU-degree of contribution from an economic value perspective, for animal exports with HS product code $i = 1, 2, 3, 4, 5$ between Korea and selected trading partners for years 2010-2013 and

(2) we compare these values with the USA and other trading partners in terms of CEU-degrees (13), (14), and (15) of the similarity which is related to the relationships and characterizations involved in the value of international trade between Korea and each of the four countries analyzed in this study(see[14]).

2. Preliminaries and definitions

Let S be a finite set of states of nature and $F(S)$ be the set of all fuzzy sets $A = \{(s, m_A(s)) \mid s \in S,\ m_A \longrightarrow [0,1] \text{ is a function}\}$. Recall that m_A is called a membership function of A.

Definition 2.1. ([4-7,9,10,11,13])

(1) A real-valued function μ on S the subsets of is called a fuzzy measure if it satisfies

$$\begin{array}{ll} \text{(i)} & \mu(\emptyset) = 0,\ \mu(S) = 1, \\ \text{(ii)} & A \subset B \Rightarrow \mu(A) \leq \mu(B). \end{array} \tag{1}$$

(2) The Choquet integrals with respect to a fuzzy measure μ of $A \in F(S)$ is defined by

$$(C)\int f_A d\mu = \int_0^1 \mu(\{s \in S | f_A(s) \geq \alpha\}) d\alpha, \tag{2}$$

Definition 2.2. ([4-7,9,10,11,13]) (1) Let $A \in F(S)$. The Choquet integrals with respect to a fuzzy measure μ of a fuzzy set $A = (S, f_A)$ is defined by

where the integral on the right-hand side is an ordinary one.

(2) Let $S = \{s_1, s_2, \cdots, s_n\}$ be a finite set. The discrete Choquet integral with respect to a fuzzy measure μ is defined by

$$(C)\int m_A d\mu = \sum_{i=1}^{n} f_A(s^{(i)}) \left[\mu(E^{(i)}) - \mu(E^{(i+1)})\right], \tag{3}$$

where $E^{(i)} = \{s \in S | m_A(s) \geq m_A(s^i)\}$ for $i = 1, 2, \cdots, n$. By convention, let $E^{n+1} = \emptyset$.

By using the Choquet integral, we consider the Choquet expected utility(CEU) of a utility u from an act a as follows.

Definition 2.3. ([4]) Let $u : X \longrightarrow [0,1]$ be a utility and a be an act from S to X. The Choquet expected utility(CEU) with respect to a fuzzy measure μ of utility u from act a is defined by

$$CEU(u(a)) = (C)\int u(a(s)) d\mu(s). \tag{4}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

We note that if $m_A(s) = u(a(s))$ and $A = (S, m_A)$, then $A \in F(S)$, that is, A is a fuzzy set. From Definition 2.1(3) and Definition 2.2 with a finite set S, we get $CEU(u(a))$ as follows:

$$CEU(u(a)) = \sum_{i=1}^{n} u(a(s^{(i)})) \left[\mu(E^{(i)}) - \mu(E^{(i+1)})\right]. \tag{5}$$

where $E^{(i)} = \{s \in S | u(a(s)) \geq u(a(s^{(i)})\}$ for all $i = 1, 2, \cdots, n$.

3. CEU-fuzzy marks and CEU-degree of similarity

In this section, we consider the CEU of a utility on a set of trade values (in USD) that represent the trading relationship that Korea shares with selected trading partners(i.e. Korea-USA, Korea-New Zealand, Korea-India, and Korea-Turkey). We also examine these respective trading relationships by incorporating a clearly defined set of Harmonized System (HS) product code product categories (i.e. HS Codes $i = 1, 2, 3, 4, 5$) for each individual year that is under review (i.e. $2010, 2011, 2012, 2013$). We note that the product code definitions have been provided by the UN Comtrade's online database and the relevant categories are defined as follows(see[14]):

1. Live animals; animal products.
2. Meat and edible meat offal.
3. Fish and crustaceans, mollusks and other aquatic invertebrates.
4. Dairy produce; birds' eggs; natural honey; edible products of animal origin, not elsewhere specified or included.
5. Products of animal origin, not elsewhere specified or included.

Firstly, we denote that HSPC=HS Product Code, s=Year, $a(s)$=Trade Value, $u(a(s))$=the utility of $a(s)$, $CEU(u, a)$=the Choquet Expected Utility of u from a. By using the trade values in tables A1 A4, we can calculate the Choquet integral of an utility on the set of trade values (in USD) that represent Korea's trading relationship with a particular country for years $2010, 2012, 2012, 2013$. Let $s_1 = 2010, s_2 = 2011, s_3 = 2012, s_4 = 2013$. If we define a fuzzy measure μ on S as follows(see[4]):

$$\begin{aligned}&\mu(E^{(4)}) = \mu_1(\{s^{(4)}\}) = 0.1, \ \ \mu(E^{(3)}) = \mu_1(\{s^{(3)}, s^{(4)}\}) = 0.3,\\ &\mu(E^{(2)}) = \mu_1(\{s^{(2)}, s^{(3)}, s^{(4)}\}) = 0.6, \ \ \mu(E^{(1)}) = \mu_1(\{s^{(4)}, s^{(3)}, s^{(2)}, s^{(1)}\}) = 1,\end{aligned} \tag{6}$$

and if $a(s)$ is the trade value of s and $u(a) = \sqrt{\frac{a}{100141401}}$, then we obtain the following $CEU(u(a))$ as follows:

$$\begin{aligned} CEU(u(a)) &= \sum_{i=1}^{4} u(a(s^{(i)})) \left(\mu(E^{(i)}) - (\mu(E^{(i+1)})\right) \\ &= 0.4u(a(s^{(1)})) + 0.3u(a(s^{(2)})) + 0.2u(a(s^{(3)})) + 0.1u(a(s^{(4)})). \end{aligned} \tag{7}$$

By using (5), we calculate the four tables $A_1 \sim A_4$ as follows(see [4]): By using four tables, we get the four X-fuzzy sets $X : \{1, 2, 3, 4, 5\} \to [0, 1]$ by $X = \{(i, m_X(i)) | i = 1, 2, 3, 4, 5\}$ (i.e., USA-fuzzy set U, NZ-fuzzy set N, IN-fuzzy set I, TR-fuzzy set T) defined by

$$U = \{(1, 0.05664), \ (2, 0.04483), \ (3, 0.93879), \ (4, 0.20821), \ (5, 0.04858)\} \tag{8}$$

$$N = \{(1, 0.00533), \ (2, 0.00000), \ (3, 0.78873), \ (4, 0.15976), \ (5, 0.01557)\} \tag{9}$$

$$I = \{(1, 0.00154),\ (2, 0.00000),\ (3, 0.04570),\ (4, 0.00000),\ (5, 0.00000)\} \tag{10}$$

$$T = \{(1, 0.00264),\ (2, 0.00887),\ (3, 0.00368),\ (4, 0.00470),\ (5, 0.00000)\} \tag{11}$$

Definition 3.1. ([1]) The degree of similarity between X-fuzzy set and Y-fuzzy set is defined by

$$S(X,Y) = \frac{\hat{X} \cdot \hat{Y}}{\max\{\hat{X} \cdot \hat{X}, \hat{Y} \cdot \hat{Y}\}} \tag{12}$$

where

$$\begin{aligned} \hat{X} &=< m_X(1), m_X(2), m_X(3), m_X(4), m_X(5) >, \\ \hat{Y} &=< m_Y(1), m_Y(2), m_Y(3), m_Y(4), m_Y(5) > \end{aligned} \tag{13}$$

are vectors and

$$\begin{aligned} &\hat{X} \cdot \hat{Y} \\ &= m_X(1) \cdot m_Y(1) + m_X(2) \cdot m_Y(2) + m_X(3) \cdot m_Y(3) + m_X(4) \cdot m_Y(4) + m_X(5) \cdot m_Y(5). \end{aligned} \tag{14}$$

By using Definition 3.1, we define the degree of similarity between X-fuzzy set and Y-fuzzy set is called the CEU-degree of similarity as follows.

Definition 3.2. If X and Y are elements of $\{U, N, I, T\}$, then the degree of similarity between X-fuzzy set and Y-fuzzy set is called the CEU-degree of similarity.

From Definition 3.1 and Definition 3.2, we get the CEU-degree of similarity between X-fuzzy set and Y-fuzzy set where X and Y are elements of $\{U, N, I, T\}$.

Example 3.1. (1) From Definition 3.1 and Definition 3.2, we get the CEU-degree of similarity between U-fuzzy set and N-fuzzy set as follows:

$$S(U,N) = \frac{\hat{U} \cdot \hat{N}}{\max\{\hat{U} \cdot \hat{U}, \hat{N} \cdot \hat{N}\}} = \frac{0.7747737841}{\max\{0.932255903, 0.6478891043\}} = 0.83174152. \tag{15}$$

(2) From (8) and (10), we get the CEU-degree of similarity between U-fuzzy set and I-fuzzy set as follows:

$$S(U,I) = \frac{\hat{U} \cdot \hat{I}}{\max\{\hat{U} \cdot \hat{U}, \hat{I} \cdot \hat{I}\}} = \frac{0.0429899286}{\max\{0.932255903, 0.0020908616\}} = 0.0461138712. \tag{16}$$

(3) From (8) and (11), we get the CEU-degree of similarity between U-fuzzy set and T-fuzzy set as follows:

$$S(U,T) = \frac{\hat{U} \cdot \hat{T}}{\max\{\hat{U} \cdot \hat{U}, \hat{T} \cdot \hat{T}\}} = \frac{0.004979328}{\max\{0.932255903, 0.0001219413\}} = 0.0053411601. \tag{17}$$

By using four information with those of the CEU-degrees of similarity (13), (14), and (15), we understand the exact difference of similarity between the USA and each of the other three trading partners. By using the CEU-degrees of similarity between USA and another country, we are able to provide a useful plan to find a more effective method of improving the value of international trade between Korea and each of the four countries analyzed in this study. We provide information that may well be of interest to international business practitioners that want a clearer understanding of the relationship and characterizations related to the value of international trade between Korea and each of the four countries measured.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Table A1: The CEU for animal product exports between Korea and the USA for years 2010-2013

HSPC	s	$a(s)$(USD)	$u(a(s))$	$CEU_{(i,USA)}(u(a))$
1	s_1	$286892 = a(s^{(1)})$	0.05352	0.05664
	s_2	$330299 = a(s^{(2)})$	0.05743	
	s_3	$358496 = a(s^{(3)})$	0.05983	
	s_4	$364918 = a(s^{(4)})$	0.06037	
2	s_1	$997539 = a(s^{(4)})$	0.09981	0.04483
	s_2	$376805 = a(s^{(3)})$	0.06034	
	s_3	$30005 = a(s^{(1)})$	0.01731	
	s_4	$272884 = a(s^{(2)})$	0.05220	
3	s_1	$74866073 = a(s^{(1)})$	0.86464	0.93879
	s_2	$95654573 = a(s^{(2)})$	0.97734	
	s_3	$100141401 = a(s^{(4)})$	1.00000	
	s_4	$99871717 = a(s^{(3)})$	0.99865	
4	s_1	$3722326 = a(s^{(1)})$	0.19280	0.20821
	s_2	$4323214 = a(s^{(2)})$	0.20778	
	s_3	$5016833 = a(s^{(4)})$	0.22382	
	s_4	$4910771 = a(s^{(3)})$	0.22145	
5	s_1	$235669 = a(s^{(2)})$	0.04851	0.04858
	s_2	$359747 = a(s^{(3)})$	0.05994	
	s_3	$101795 = a(s^{(1)})$	0.05994	
	s_4	$863858 = a(s^{(4)})$	0.09088	

Remark 3.1. *As demonstrated in (13) (14) and (15) this study compares the similarities that exist between Korea and its respective trading partners. As such, our study details the following information:*

$$\text{Korea} - \text{USA} : \text{Korea} - \text{NZ} : \text{Korea} - \text{India} : \text{Korea} - \text{Turkey} = 1 : 0.832 : 0.046 : 0.005 \tag{18}$$

(2) Given a situation whereby Korea spends 10 *million USD as a means of developing a strong trading relationship between itself and its US trading partner, we are able to also ascertain the level of support that is needed to develop effective trading ties with other countries, for example:*

$$\begin{aligned} &\text{NewZealand} : 8,320,000\text{USD} \\ &\text{India} : 460,000\text{USD} \\ &\text{Turkey}50,000\text{USD}. \end{aligned} \tag{19}$$

Table A2: The CEU for animal product exports between Korea and New Zealand for years 2010-2013

HSPC	s	$a(s)$(USD)	$u(a(s))$	$CEU_{(i,NZ)}(u(a))$
1	s_1	$6650 = a(s^{(4)})$	0.00815	0.00533
	s_2	$4497 = a(s^{(3)})$	0.00670	
	s_3	$1589 = a(s^{(1)})$	0.00398	
	s_4	$2779 = a(s^{(2)})$	0.00527	
2	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	
3	s_1	$70759196 = a(s^{(2)})$	0.84059	0.78873
	s_2	$91263506 = a(s^{(4)})$	0.95464	
	s_3	$70763937 = a(s^{(3)})$	0.84062	
	s_4	$46632301 = a(s^{(1)})$	0.68240	
4	s_1	$165773 = a(s^{(3)})$	0.04069	0.15976
	s_2	$113751 = a(s^{(1)})$	0.03370	
	s_3	$148756 = a(s^{(2)})$	0.03854	
	s_4	$277350 = a(s^{(4)})$	0.05263	
5	s_1	$0 = a(s^{(1)})$	0.00000	0.01557
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$218022 = a(s^{(3)})$	0.04666	
	s_4	$393025 = a(s^{(4)})$	0.00265	

Table A3: the CEU for Animal product expert between Korea and India for years 2010-2013

HSPC	s	$a(s)$(USD)	$u(a(s))$	$CEU(u(a))$
1	s_1	$1050 = a(s^{(3)})$	0.00324	0.00264
	s_2	$1300 = a(s^{(4)})$	0.00360	
	s_3	$450 = a(s^{(1)})$	0.00212	
	s_4	$700 = a(s^{(2)})$	0.00264	
2	s_1	$35432 = a(s^{(3)})$	0.01881	0.00887
	s_2	$50639 = a(s^{(4)})$	0.02249	
	s_3	$2656 = a(s^{(1)})$	0.00515	
	s_4	$8230 = a(s^{(2)})$	0.00907	
3	s_1	$8695 = a(s^{(4)})$	0.009318	0.00368
	s_2	$5247 = a(s^{(3)})$	0.00724	
	s_3	$0 = a(s^{(1)})$	0.00000	
	s_4	$1865 = a(s^{(2)})$	0.00432	
4	s_1	$0 = a(s^{(1)})$	0.00000	0.00470
	s_2	$21614 = a(s^{(3)})$	0.01469	
	s_3	$30938 = a(s^{(4)})$	0.01758	
	s_4	$0 = a(s^{(2)})$	0.00000	
5	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Table A4: The CEU for animal product exports between Korea and Turkey for years 2010-2013

HSPC	s	$a(s)$(USD)	$u(a(s))$	$CEU(u(a))$
1	s_1	$0 = a(s^{(1)})$	0.00000	0.00154
	s_2	$6900 = a(s^{(4)})$	0.00830	
	s_3	$150 = a(s^{(2)})$	0.00122	
	S_4	$300 = a(s^{(3)})$	0.00173	
2	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	
3	s_1	$0 = a(s^{(1)})$	0.00000	0.04570
	s_2	$672952 = a(s^{(3)})$	0.08198	
	s_3	$2532837 = a(s^{(4)})$	0.15904	
	s_4	$199874 = a(s^{(2)})$	0.04468	
4	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	
5	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	

4. Conclusions

The Choquet expected utility(see Definition 2.3) is a useful tool which can be used to calculate the evaluation of the contribution of animal exports between Korea and selected trading partners. By using the Choquet expected utility, we obtained Tables $A1 \sim A4$ in [10]. From these Tables $A1 \sim A4$, we gave four X- fuzzy sets (8),(9),(10),(11) which are representations of the evaluation of contribution to animal exports for HP product codes $i = 1, 2, 3, 4, 5$ between Korea and selected trading partners for years 2010-2013.

By using these X-fuzzy sets, we obtained three CEU-degrees (13), (14), and (15) of similarity. From three CEU-degrees (13), (14), and (15) of similarity, we can clearly understand the difference of similarity that exists between the USA and each of three countries measured in the study. By using CEU-degrees of similarity between the USA and a respective trading partner, we are able to provide a more effective method of improving the value of international trade between Korea and its trading partners. We also provide valuable information that can be used to compare the USA and another countries as was the case with the three CEU-degrees (13), (14), and (15) of similarity that is related with the relationship and characterizations of the international trade values that exist between Korea and its respective trading partner.

References

[1] R. Biswas, *An apllication of fuzzy sets in students' evaluation*, *Fuzzy Sets Syst.* **74** (1995) 187-194.
[2] G. Choquet, *Theory of capacities*, *Ann. Inst. Fourier* **5** (1953) 131-295.

[3] I. Gilboa, D. Schmeidler, *Maximin expected utility with non-unique prior*, *J. Math. Econ.* **18** (1989) 141-153.
[4] L.C. Jang, J. Wood, *The application of the Choquet integral expected utility in international trade*, *Advan. Stud. Contem. Math.*, **27(2)** (2017) 159-173.
[5] L.C. Jang, *A note on convergence properties of interval-valued capacity functionals and Choquet integrals*, *Information Sciences* **183** (2012) 151-158.
[6] L.C. Jang, *A note on the interval-valued fuzzy integral by means of an interval-representable pseudo-multiplication and their convergence properties*, *Fuzzy Sets Syst.* **137** (2003) 11-26.
[7] L. Mangelsdorff, M. Weber, *Testing Choquet expected utility J. Econ. Behavior and Organization* **25** (1994) 437-457.
[8] D. Schmeidler, *Subjective probability and expected utility without additivity*, *Econometrica* **57(3)** (1989) 571-587.
[9] J. Wood, L.C. Jang, *A note on Choquet integrals and imprecise market premium functionals*, *Proc. Jangjeon Math.Soc.*, **18(4)** (2015) 601-608.
[10] J. Wood, L.C. Jang, *A study on the Choquet integral with respect to a capacity and its applications*, *Global J. Pure Applied Math.*, **12(2)** (2016) 1593-1599.
[11] L. Xuechang, *Entropy, distance maesure and similarity measure of fuzzy sets and their relations*, *Fuzzy Sets Syst.*, **52** (1992) 201-227.
[12] W. Zeng, H. Li, *Relationship btween similarity maesure and entropy of interval-valued fuzzy sets* , *Fuzzy Sets Syst.*, **57** (2006) 1477-1484.
[13] D. Zhang, *Subjectiv ambiguity, expected utility and Choquet expected utility*, *Econ. Theory* **20** (2002) 159-181.
[14] WTO (2016). WTO Regional Trade Database.

The general solution of a mixed cubic-quartic functional equation and the Ulam stability of matrix fuzzy normed spaces

Yali Ding[1,*], Tian-Zhou Xu[1], John Michael Rassias[2]

(*1. School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, P. R. China*)

(*2. Pedagogical Department E.E., Section of Mathematics and Informatics, National and Capodistrian University of Athens, 4, Agamemnonos Str., Aghia Paraskevi, Athens 15342, Greece*)

Abstract In this paper, we consider the following new type cubic-quartic (CQ) functional equation

$$\begin{aligned} f(\lambda x+y)+f(\lambda x-y) &= \frac{\lambda^2+\lambda}{2}[f(x+y)+f(x-y)]+\frac{\lambda^2-\lambda}{2}[f(-x-y)+f(y-x)] \\ &+(\lambda^4+\lambda^3-\lambda^2-\lambda)f(x)+(\lambda^4-\lambda^3-\lambda^2+\lambda)f(-x)+(1-\lambda^2)[f(y)+f(-y)], \end{aligned}$$

where $\lambda \geq 2$ is a fixed integer. We investigate the general solution of the functional equation, and then, using the fixed point method, we prove some stability results for this functional equation in matrix fuzzy normed spaces.

Keywords Ulam stability; Cubic-quartic mapping; Cubic-quartic functional equation; Matrix fuzzy normed spaces.

Mathematics Subject Classification(2010) 39B82; 39B52; 46H25.

1 Introduction

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{R}$ and $\mathbb{C}$ stand for the sets of reals and complex numbers, respectively. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$, and $\mathbb{N}_{m_0}$ denotes the set of all positive integers greater than or equal to a given $m_0 \in \mathbb{N}$.

The study of stability problems for functional equations is related to a question of Ulam [14] concerning the stability of group homomorphisms. Subsequently, the partial result of Ulam's problem was proved by Hyers [8], The solution of Hyers was generalized by Rassias [13] for approximate linear mappings by allowing the Cauchy difference $||f(x+y)-f(x)-f(y)||$ to be controlled by $\epsilon\,(\|x\|^p+\|y\|^p)$. In 1994, a further generalization was obtained by Găvruţa [10], who replaced $\epsilon(\|x\|^p+\|y\|^p)$ by a general control function $\varphi(x,y)$. This new idea is known as the Hyers-Ulam-Rassias stability of functional equations.

Park [7] considered the following cubic-quartic functional equation

$$f(2x+y)+f(2x-y)=3f(x+y)+3f(x-y)+f(-x-y)+f(y-x)+18f(x)+6f(-x)-3f(y)-3f(-y), \quad (1.1)$$

and investigated the orthogonally stability of (1.1). Very recently, Song [5] proved Ulam stability of this equation (1.1) in matrix intuitionistic fuzzy normed spaces. For more interesting discussions and generalizations of the original problem of Ulam have been investigated, see for instance [1, 2, 9, 11, 12, 15] and the references therein.

*Corresponding author.

E-mail addresses: dingyaliding@126.com (Y. Ding), xutianzhou@bit.edu.cn (T.Z. Xu), jrassias@primedu.uoa.gr (J.M. Rassias).

In the present paper, we introduce a new mixed type cubic and quartic functional equation:

$$\begin{aligned} f(\lambda x+y)+f(\lambda x-y)=&\frac{\lambda^2+\lambda}{2}[f(x+y)+f(x-y)]+\frac{\lambda^2-\lambda}{2}[f(-x-y)+f(y-x)]\\ &+(\lambda^4+\lambda^3-\lambda^2-\lambda)f(x)+(\lambda^4-\lambda^3-\lambda^2+\lambda)f(-x)+(1-\lambda^2)[f(y)+f(-y)], \end{aligned} \tag{1.2}$$

where $\lambda \geq 2$ is a fixed integer. One can see that the functional equation (1.1) is a special case of (1.2) when we take the integer $\lambda = 2$. Every solution of the functional equation (1.2) is said to be a cubic-quartic mapping.

The aim of this paper is to discuss the general solution and then establish the Ulam stability of (1.2). More precisely, we discuss the Ulam stability of (1.2) in matrix fuzzy normed spaces by applying the fixed point method.

2 Preliminaries

In this section, we recall some basic facts concerning fuzzy normed spaces, matrix fuzzy normed spaces and some useful results.

Definition 2.1 ([4]) Let X be a real vector space. A function $N : X\times\mathbb{R}\to[0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t\in\mathbb{R}$:
(1) $N(x,t) = 0$ for $t \leq 0$; (2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$; (3) $N(cx,t) = N(x,\frac{t}{|c|})$ if $c\neq 0$; (4) $N(x+y,s+t)\geq\min\{N(x,s),N(y,t)\}$; (5) $N(x,\cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t) = 1$; (6) $N(x,\cdot)$ is continuous on $\mathbb{R}$ for $x\neq 0$.

In this case (X,N) is called a fuzzy normed vector space.

Definition 2.2 ([4]) Let (X,N) be a fuzzy normed space. A sequence x_n in X is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty}N(x_n-x,t) = 1(t>0)$. A sequence x_n in X is called Cauchy if for each $\epsilon > 0$ and $t > 0$, there exists $n_0\in\mathbb{N}$ such that $N(x_m-x_n,t) > 1-\epsilon$ $(m,n\geq n_0)$. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

We will use the following notations: $M_{m,n}(X)$ is the set of all $m\times n$ matrices in X; When $m = n$, the matrix $M_{m,n}(X)$ will be written as $M_n(X)$; $e_j\in M_{1,n}(\mathbb{R})$ denote the row vector whose jth component is 1 and the other components are zero; $E_{ij}\in M_n(\mathbb{R})$ is that (i,j)-component is 1 and the other components are zero; $E_{ij}\otimes x\in M_n(X)$ is that (i,j)-component is x and the other components are zero.

Let $(X,\|\cdot\|)$ be a normed space. Note that $(X,\{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X),\|\cdot\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k\leq\|A\|\,\|B\|\,\|x\|_n$ holds for $A\in M_{k,n}(\mathbb{R}), x=[x_{ij}]\in M_n(X)$ and $B\in M_{n,k}(\mathbb{R})$, and that $(X,\{\|\cdot\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X,\{\|\cdot\|_n\})$ is a matrix normed space.

For $x\in M_n(X), y\in M_k(X), x\oplus y := \begin{pmatrix} x & 0\\ 0 & y\end{pmatrix}$, we introduce the concept of matrix fuzzy normed spaces.

Let X, Y be vector space. For a given mapping $h : X\to Y$ and a given positive integer n, define $h_n : M_n(X)\to M_n(Y)$ by $h_n([x_{ij}]) := [h(x_{ij})]$ for all $[x_{ij}]\in M_n(X)$.

Definition 2.3 ([6, 15]) Let (X,N) be a fuzzy normed space.

(1) $(X,\{N_n\})$ is called a matrix fuzzy normed space if for each positive integer n, $(M_n(X),N_n)$ is a fuzzy normed space and $N_k(AxB,t)\geq N_n(x,\frac{t}{\|A\|\cdot\|B\|})$ for all $t>0, A\in M_{k,n}(\mathbb{R}), x=[x_{ij}]\in M_n(X)$ and $B\in M_{n,k}(\mathbb{R})$ with $||A||\cdot||B||\neq 0$.

(2) $(X,\{N_n\})$ is called a matrix fuzzy Banach space if (X,N) is a fuzzy Banach space and $(X,\{N_n\})$ is a matrix fuzzy normed space.

Lemma 2.1 ([6]) *Let $(X, \{N_n\})$ be a matrix fuzzy normed space. Then*

(1) $N_n(E_{kl} \otimes x, t) = N(x, t)$ *for all* $t > 0, x \in X$,

(2) *For all* $[x_{ij}] \in M_n(X)$ *and* $t = \sum_{i,j=1}^{n} t_{ij}$,

$$N(x_{kl}, t) \geq N_n([x_{ij}], t) \geq \min\{N(x_{ij}, t_{ij}) : i, j = 1, 2, \ldots, n\},$$
$$N(x_{kl}, t) \geq N_n([x_{ij}], t) \geq \min\left\{N(x_{ij}, \frac{t}{n^2}) : i, j = 1, 2, \ldots, n\right\}.$$

(3) $\lim_{n\to\infty} x_n = x$ *if and only if* $\lim_{n\to\infty} x_{ijn} = x_{ij}$ *for* $x_n = [x_{ijn}], x = [x_{ij}] \in M_k(X)$.

Theorem 2.1 ([3]) *Let (E, d) be a complete generalized metric space and $J : E \to E$ be a strictly contractive mapping, that is*

$$d(Jx, Jy) \leq Ld(x, y), \forall x, y \in E$$

for some $0 < L < 1$. Then, for each given element $x \in E$, either $d(J^n x, J^{n+1}x) = +\infty, \forall n \geq 0$ or $d(J^n x, J^{n+1}x) < \infty, \forall n \geq n_0$, for some natural number n_0. Moreover, if the second alternative holds, then

(1) *The sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;*

(2) *y^* is the unique fixed point of J in the set $E' = \{y \in E | d(J^{n_0}x, y) < +\infty\}$ and $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in E'$.*

3 General solution of the functional equation (1.2)

In this section, we investigate the general solution of the mixed cubic-quartic functional equation (1.2). Throughout this section, let X be a vector space over $\mathbb{Q}$, Y be a vector space, and $\lambda \in \mathbb{N}_2$. Some basic facts on n-additive symmetric mappings can be found in [12].

Lemma 3.1 *If an odd mapping $f : X \to Y$ satisfies (1.2), then f is of the form $f(x) = A^3(x)$ for all $x \in X$, where $A^3(x)$ is the diagonal of the 3-additive symmetric map $A_3 : X^3 \to Y$.*

Proof. Using the oddness of f, we have $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$. (1.2) with $y = 0$ yields

$$f(\lambda x) = \lambda^3 f(x). \tag{3.1}$$

Applying (3.1) to (1.2), we obtain

$$f(\lambda x + y) + f(\lambda x - y) = \lambda[f(x + y) + f(x - y)] + 2\lambda(\lambda^2 - 1)f(x), \tag{3.2}$$

From (3.2), by Theorems 3.4 and 3.5 in [12], f is a generalized polynomial function of degree at most 3:

$$f(x) = A^3(x) + A^2(x) + A^1(x) + A^0(x), \tag{3.3}$$

where $A^0(x) = A^0$ is an arbitrary element of Y, and A^i is the diagonal of the i-additive symmetric map $A_i : X^i \to Y$ for $i = 1, 2, 3$. By $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$, we get $A^0(x) = A^0 = 0$ and $A^2(x) = 0$ for all $x \in X$. By $f(\lambda x) = \lambda^3 f(x)$ and $A^i(rx) = r^i A^i(x)$ whenever $x \in X$ and $r \in \mathbb{Q}$, we obtain $A^1(x) = 0$ for all $x \in X$. Therefore, $f(x) = A^3(x)$ for all $x \in X$. □

Lemma 3.2 *If an even mapping $f : X \to Y$ satisfies (1.2), then f is of the form $f(x) = A^4(x)$ for all $x \in X$, where $A^4(x)$ is the diagonal of the 4-additive symmetric map $A_4 : X^4 \to Y$.*

Proof. In view of the evenness of f, we have $f(-x) = f(x)$ for all $x \in X$. Let $y = 0$ in (1.2), we obtain

$$f(\lambda x) = \lambda^4 f(x). \tag{3.4}$$

The rest of the proof is similar to the proof of Lemma 3.1.

□

Theorem 3.1 *A mapping $f : X \to Y$ satisfies* (1.2) *for all $x, y \in X$ if and only if f is the form*

$$f(x) = A^4(x) + A^3(x), \tag{3.5}$$

where A^i is the diagonal of the i-additive symmetric map $A_i : X^i \to Y$ for $i = 3, 4$.

Proof. Assume that f satisfies the functional equation (1.2), we decompose f into the odd part and the even part by putting

$$f_o(x) = \frac{f(x) - f(-x)}{2}, f_e(x) = \frac{f(x) + f(-x)}{2}, \tag{3.6}$$

then, $f(x) = f_o(x) + f_e(x)$ for all $x \in X$. It is easy to show that the mapping f_o and f_e satisfy (1.2). Therefore our assertion follows immediately from Lemmas 3.1 and 3.2. Conversely, assume that $f(x) = A^4(x) + A^3(x)$ for all $x \in X$, where $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i : X^i \to Y$ for $i = 3, 4$. Using

$$\begin{aligned}
&A^4(x+y) + A^4(x-y) = 2A^4(x) + 2A^4(y) + 12A^{2,2}(x, y),\\
&A^3(x+y) + A^3(x-y) = 2A^3(x) + 6A^{1,2}(x, y),\\
&A^i(rx) = r^i A^i(x), i \in \{3, 4\}, r \in \mathbb{Q},\\
&A^{i,j}(rx, sy) = r^i s^j A^{i,j}(x, y), i \in \{1, 2\}, r, s \in \mathbb{Q},
\end{aligned} \tag{3.7}$$

by a simple computation, one can see that f satisfies (1.2), which complete the proof of Theorem 3.1. □

4 Stability of the functional equation (1.2)

Throughout this section, let $(X, \{N_n\})$ be a matrix fuzzy normed space, $(Y, \{N_n\})$ be a matrix fuzzy Banach space, $\lambda \in \mathbb{N}_2$ and $n \in \mathbb{N}$. Using the fixed point method, we prove the Ulam stability of the CQ-functional equation (1.2) in matrix fuzzy normed spaces.

Now before taking up the main subject, for a given mapping $f : X \to Y$, we define the difference operator $Df : X^2 \to Y$, and $Df_n : M_n(X^2) \to M_n(Y)$.

$$\begin{aligned}
(Df)(a, b) :=& f(\lambda a + b) + f(\lambda a - b) - \frac{\lambda^2 + \lambda}{2}[f(a + b) + f(a - b)] - \frac{\lambda^2 - \lambda}{2}[f(-a - b) + f(b - a)]\\
&- (\lambda^4 + \lambda^3 - \lambda^2 - \lambda)f(a) - (\lambda^4 - \lambda^3 - \lambda^2 + \lambda)f(-a) - (1 - \lambda^2)[f(b) + f(-b)],
\end{aligned}$$

$$\begin{aligned}
(Df_n)([x_{ij}], [y_{ij}]) :=& f_n(\lambda[x_{ij}] + [y_{ij}]) + f_n(\lambda[x_{ij}] - [y_{ij}]) - \frac{\lambda^2 + \lambda}{2}[f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}])]\\
&- \frac{\lambda^2 - \lambda}{2}[f_n(-[x_{ij}] - [y_{ij}]) + f_n([y_{ij}] - [x_{ij}])] - (\lambda^4 + \lambda^3 - \lambda^2 - \lambda)f_n([x_{ij}])\\
&- (\lambda^4 - \lambda^3 - \lambda^2 + \lambda)f_n(-[x_{ij}]) - (1 - \lambda^2)[f_n([y_{ij}]) + f_n(-[y_{ij}])]
\end{aligned}$$

for all $a, b \in X, x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 4.1 *Let $\varphi_1 : X^2 \to [0, \infty)$ be a function such that for some real number α with $0 < \alpha < 1$,*

$$\varphi_1(\frac{a}{\lambda}, \frac{b}{\lambda}) \le \frac{\alpha}{\lambda^4}\varphi_1(a, b), \qquad a, b \in X. \tag{4.1}$$

Suppose that $f : X \to Y$ is an even function with $f(0) = 0$ and such that

$$N_n(Df_n([x_{ij}],[y_{ij}]),t) \geq \frac{t}{t+\sum_{i,j=1}^{n}\varphi_1(x_{ij},y_{ij})}, \qquad t>0, x=[x_{ij}], y=[y_{ij}] \in M_n(X). \tag{4.2}$$

Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$N_n\left(f_n([x_{ij}])-Q_n([x_{ij}]),t\right) \geq \frac{2\lambda^4(1-\alpha)t}{2\lambda^4(1-\alpha)t+\alpha n^2\sum_{i,j=1}^{n}\varphi_1(x_{ij},0)}, \qquad t>0, x=[x_{ij}] \in M_n(X).$$

Proof. When $n = 1$, (4.2) is equivalent to

$$N(Df(a,b),t) \geq \frac{t}{t+\varphi_1(a,b)}, \qquad t>0, a,b \in X. \tag{4.3}$$

Putting $b = 0$ in (4.3), we obtain that

$$N(2f(\lambda a)-2\lambda^4 f(a),t) \geq \frac{t}{t+\varphi_1(a,0)}, \qquad t>0, a \in X. \tag{4.4}$$

Hence

$$N(f(a)-\lambda^4 f(\frac{a}{\lambda}),\frac{t}{2}) \geq \frac{t}{t+\varphi_1(\frac{a}{\lambda},0)}, \qquad t>0, a \in X. \tag{4.5}$$

Using (4.1) we get

$$N(f(a)-\lambda^4 f(\frac{a}{\lambda}),t) \geq \frac{t}{t+\frac{\alpha}{2\lambda^4}\varphi_1(a,0)}, \qquad t>0, a \in X. \tag{4.6}$$

Consider the set $E_1 = \{g : X \to Y, g(0) = 0\}$, and introduce the generalized metric d_1:

$$d_1(g,h) := \inf\left\{\epsilon \in \mathbb{R}_+ : N(g(a)-h(a),\epsilon t) \geq \frac{t}{t+\varphi_1(a,0)}, \qquad t>0, a \in X\right\},$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to prove that (E_1, d_1) is a complete generalized metric space.

Now, let us consider the linear mapping $J_1 : E_1 \to E_1$ such that

$$J_1 g(a) = \lambda^4 g(\frac{a}{\lambda}), \qquad g \in E_1, a \in X.$$

It is easy to see that J_1 is a strictly contractive self-mapping of E_1 with the Lipschitz constant $L = \alpha$. Indeed, given $g, h \in E_1$, let $\epsilon \in (0,\infty)$ be an arbitrary constant with $d_1(g,h) = \epsilon$. From the definition of d_1, it follows that

$$N(g(a)-h(a),\epsilon t) \geq \frac{t}{t+\varphi_1(a,0)}, \qquad t>0, a \in X.$$

Hence

$$\begin{aligned} N(J_1 g(a)-J_1 h(a),\alpha\epsilon t) &= N(\lambda^4 g(\frac{a}{\lambda})-\lambda^4 h(\frac{a}{\lambda}),\alpha\epsilon t) = N(g(\frac{a}{\lambda})-h(\frac{a}{\lambda}),\frac{\alpha\epsilon t}{\lambda^4}) \\ &\geq \frac{\frac{\alpha}{\lambda^4}t}{\frac{\alpha}{\lambda^4}t+\varphi_1(\frac{a}{\lambda},0)} \geq \frac{t}{t+\varphi_1(a,0)}, \qquad t>0, a \in X. \end{aligned}$$

So, $d_1(g,h) = \epsilon$ implies that $d(J_1 g, J_1 h) \leq \alpha\epsilon$. This means that $d_1(J_1 g, J_1 h) \leq \alpha d_1(g,h)$ for all $g, h \in E_1$, thus J_1 is a strictly contractive self-mapping, and the Lipschitz constant $L = \alpha$.

It follows from (4.6) that

$$N(f(a)-J_1 f(a),t) \geq \frac{t}{t+\frac{\alpha}{2\lambda^4}\varphi_1(a,0)}, \qquad t>0, a \in X,$$

thus we have that $d_1(f, J_1 f) \leq \frac{\alpha}{2\lambda^4} < +\infty$.

According to Theorem 2.1, we deduce the existence of a fixed point of J_1, that is, the existence of a mapping $Q : X \to Y$ such that $Q(a) = J_1Q(a) = \lambda^4 Q(\frac{a}{\lambda})$, i.e., $Q(\frac{a}{\lambda}) = \frac{1}{\lambda^4}Q(a)$ for each $a \in X$. Moreover, we have $d_1(J_1^l f, Q) \to 0 (l \to +\infty)$, which implies

$$\lim_{l\to+\infty} N(J_1^l f(a) - Q(a), t) = 1, \qquad t > 0, a \in X. \tag{4.7}$$

Also, $d_1(f, Q) \leq \frac{1}{1-L} d_1(J_1 f, f)$ implies the inequality $d_1(f, Q) \leq \frac{\alpha}{2\lambda^4(1-\alpha)}$, which means that

$$N(f(a) - Q(a), t) \geq \frac{2\lambda^4(1-\alpha)t}{2\lambda^4(1-\alpha)t + \alpha\varphi_1(a, 0)}, \qquad t > 0, a \in X. \tag{4.8}$$

Replacing a and b by $\frac{a}{\lambda^l}$ and $\frac{b}{\lambda^l}$ in (4.3), respectively, we have

$$N\left(\lambda^{4l} Df(\frac{a}{\lambda^l}, \frac{b}{\lambda^l}), t\right) = N\left(Df(\frac{a}{\lambda^l}, \frac{b}{\lambda^l}), \frac{t}{\lambda^{4l}}\right) \geq \frac{\frac{t}{\lambda^{4l}}}{\frac{t}{\lambda^{4l}} + \varphi_1(\frac{a}{\lambda^l}, \frac{b}{\lambda^l})}, \qquad t > 0, a, b \in X. \tag{4.9}$$

It follows from (4.1) that

$$\varphi_1(\frac{a}{\lambda^l}, \frac{b}{\lambda^l}) \leq \frac{\alpha^l}{\lambda^{4l}}\varphi_1(a, b), \qquad a, b \in X,$$

thus

$$N\left(\lambda^{4l} Df(\frac{a}{\lambda^l}, \frac{b}{\lambda^l}), t\right) \geq \frac{t}{t + \alpha^l \varphi_1(a, b)}, \qquad t > 0, a, b \in X. \tag{4.10}$$

Letting $l \to +\infty$ in (4.10), we obtain

$$N\left(\lambda^{4l} Df(\frac{a}{\lambda^l}, \frac{b}{\lambda^l}), t\right) \to 1, \qquad t > 0, a, b \in X, \tag{4.11}$$

which means

$$N\left(DQ(a, b), t\right) = 1, \qquad t > 0, a, b \in X. \tag{4.12}$$

Thus, $DQ(a, b) = 0$ for all $a, b \in X$. By the definition of Q, it is clear that $Q(-a) = Q(a)$ for all $a \in X$. Then by Lemma 3.1, the mapping Q is quartic.

Assume that there exists another quartic function $F : X \to Y$ which satisfies (4.8). Then it is clear that $F(\frac{a}{\lambda}) = \frac{1}{\lambda^4}F(a)$, and while $a = 0$, we have $F(a) = 0$, thus $J_1F(a) = \lambda^4 F(\frac{a}{\lambda}) = F(a)$ for all $a \in X$, i.e., F is a fixed point of J_1. By (4.8) we get

$$N(f(a) - F(a), t) \geq \frac{2\lambda^4(1-\alpha)t}{2\lambda^4(1-\alpha)t + \alpha\varphi_1(a, 0)}, \qquad t > 0, a \in X.$$

Hence, $d_1(f, F) \leq \frac{\alpha}{2\lambda^4(1-\alpha)}$. So, $F \in E_1^{'} = \{g \in E_1, d_1(f, g) < \infty\}$. By Theorem 2.1, Q is the unique fixed point in E_1, which means that $Q = F$.

By Lemma 2.1 and (4.8), we have

$$\begin{aligned} N\left(f_n([x_{ij}]) - Q_n([x_{ij}]), t\right) &\geq \min\left\{N(f(x_{ij}) - Q(x_{ij}), \frac{t}{n^2}) : \quad i, j = 1, 2, ..., n\right\} \\ &\geq \min\left\{\frac{2\lambda^4(1-\alpha)t}{2\lambda^4(1-\alpha)t + \alpha n^2\varphi_1(x_{ij}, 0)} : \quad i, j = 1, 2, ..., n\right\} \\ &\geq \frac{2\lambda^4(1-\alpha)t}{2\lambda^4(1-\alpha)t + \alpha n^2 \sum_{i,j=1}^n \varphi_1(x_{ij}, 0)} \end{aligned}$$

for all $x = [x_{ij}] \in M_n(X), t > 0$. This completes the proof. □

Theorem 4.2 *Let $\varphi_2 : X^2 \to [0, \infty)$ be a function such that for some real number α with $0 < \alpha < \lambda$,*

$$\varphi_2(\frac{a}{\lambda}, \frac{b}{\lambda}) \leq \frac{\alpha}{\lambda^4}\varphi_2(a, b), \qquad a, b \in X. \tag{4.13}$$

Suppose that $f : X \to Y$ is an odd function such that

$$N_n(Df_n([x_{ij}],[y_{ij}]),t) \geq \frac{t}{t+\sum_{i,j=1}^{n}\varphi_2(x_{ij},y_{ij})}, \qquad t>0, x=[x_{ij}], y=[y_{ij}] \in M_n(X). \tag{4.14}$$

Then there exists a unique cubic mapping $C : X \to Y$ such that

$$N_n\left(f_n([x_{ij}])-C_n([x_{ij}]),t\right) \geq \frac{2\lambda^3(\lambda-\alpha)t}{2\lambda^3(\lambda-\alpha)t+\alpha n^2\sum_{i,j=1}^{n}\varphi_2(x_{ij},0)}, \qquad t>0, x=[x_{ij}] \in M_n(X).$$

Proof. The proof is similar to the proof of Theorem 4.1. $\square$

Theorem 4.3 *Let $\varphi : X^2 \to [0,\infty)$ be a function such that for some real number α with $0<\alpha<1$,*

$$\varphi(\frac{a}{\lambda},\frac{b}{\lambda}) \leq \frac{\alpha}{\lambda^4}\varphi(a,b), \qquad a,b \in X. \tag{4.15}$$

Suppose that $f : X \to Y$ is a function such that $f(0)=0$, and for all $x=[x_{ij}], y=[y_{ij}] \in M_n(X)$, satisfying

$$N_n(Df_n([x_{ij}],[y_{ij}]),t) \geq \frac{t}{t+\sum_{i,j=1}^{n}\varphi(x_{ij},y_{ij})}, \qquad t>0. \tag{4.16}$$

Then there exist a unique cubic mapping $C : X \to Y$ and a unique quartic mapping $Q : X \to Y$ such that

$$N_n\left(f_n([x_{ij}])-Q_n([x_{ij}])-C_n([x_{ij}]),t\right) \geq \frac{\lambda^4(1-\alpha)t}{\lambda^4(1-\alpha)t+\alpha n^2\sum_{i,j=1}^{n}\psi(x_{ij},0)},$$

where $\psi(a,b) := \varphi(a,b)+\varphi(-a,-b)$ for all $a,b \in X$.

Proof. Let $f_e(a)=\frac{1}{2}(f(a)+f(-a))$, it is easy to see that $f_e(0)=0, f_e(-a)=f_e(a)$.

$$\begin{aligned} N(Df_e(a,b),t) &= N(\frac{1}{2}Df(a,b)+\frac{1}{2}Df(-a,-b),t) = N(Df(a,b)+Df(-a,-b),2t) \\ &\geq \min\{N(Df(a,b),t),N(Df(-a,-b),t)\} \geq \frac{t}{t+\psi(a,b)}. \end{aligned}$$

Let $f_o(a)=\frac{1}{2}(f(a)-f(-a))$, we can get $N(Df_o(a,b),t) \geq \frac{t}{t+\psi(a,b)}$. From (4.15), it follows that $\psi(\frac{a}{\lambda},\frac{b}{\lambda}) \leq \frac{\alpha}{\lambda^4}\psi(a,b)$. It is easy to check that all conditions of Theorems 4.1 and 4.2 hold, by the proofs of Theorems 4.1 and 4.2, we know that there exist a quartic mapping $Q : X \to Y$ and a cubic mapping $C : X \to Y$ such that

$$N\left(f_e(a)-Q(a),t\right) \geq \frac{2\lambda^4(1-\alpha)t}{2\lambda^4(1-\alpha)t+\alpha\psi(a,0)}, \qquad t>0, a \in X,$$

and

$$N\left(f_o(a)-C(a),t\right) \geq \frac{2\lambda^3(\lambda-\alpha)t}{2\lambda^3(\lambda-\alpha)t+\alpha\psi(a,0)}, \qquad t>0, a \in X.$$

Therefore

$$\begin{aligned} N\left(f(a)-C(a)-Q(a),t\right) &= N(f_e(a)-Q(a)+f_o(a)-C(a),t) \\ &\geq \min\{N(f_e(a)-Q(a),\frac{t}{2}),N(f_o(a)-C(a),\frac{t}{2})\} \\ &\geq \min\{\frac{\lambda^4(1-\alpha)t}{\lambda^4(1-\alpha)t+\alpha\psi(a,0)},\frac{\lambda^3(\lambda-\alpha)t}{\lambda^3(\lambda-\alpha)t+\alpha\psi(a,0)}\} \\ &= \frac{\lambda^4(1-\alpha)t}{\lambda^4(1-\alpha)t+\alpha\psi(a,0)}, \qquad t>0, a \in X. \end{aligned} \tag{4.17}$$

Using a proof method similar to Theorem 3.10 in [11], we can prove the uniqueness of C and Q. By Lemma 2.1 and (4.17), we have

$$\begin{aligned}N\left(f_n([x_{ij}]) - Q_n([x_{ij}] - C_n([x_{ij}]), t\right) &\geq \min\left\{N(f(x_{ij}) - Q(x_{ij}) - C(x_{ij}), \frac{t}{n^2}) : \quad i,j = 1,2,...,n\right\} \\ &\geq \min\left\{\frac{\lambda^4(1-\alpha)t}{\lambda^4(1-\alpha)t + \alpha n^2\psi(x_{ij},0)} : \quad i,j = 1,2,...,n\right\} \\ &\geq \frac{\lambda^4(1-\alpha)t}{\lambda^4(1-\alpha)t + \alpha n^2\sum_{i,j=1}^n \psi(x_{ij},0)}\end{aligned}$$

for all $x = [x_{ij}] \in M_n(X), t > 0$. This completes the proof. □

References

[1] A. Bahyrycz, J. Brzdęk, E. Jabłońska, R. Malejki, Ulam's stability of a generalization of the Fréchet functional equation, J. Math. Anal. Appl., 442(2016), 537-553.

[2] A. Bahyrycz, K. Ciepliński, On an equation characterizing multi-Jensen-quadratic mappings and its Hyers-Ulam stability via a fixed point method, J. Fixed Point Theory Appl., 18(2016), 737-751.

[3] J.B. Diaz, B. Margolis, A fixed point theorem of the alternative for contractions on a generalized complete metric space, Bull. Amer. Math. Soc., 74(1968), 305-309.

[4] A.K. Mirmostafaee, M.S. Moslehian, Fuzzy versions of Hyers-Ulam-Rassias theorem, Fuzzy Sets and Systems, 159(2008), 720-729.

[5] A.M. Song, The Ulam stability of matrix intuitionistic fuzzy normed spaces, J. Intell. Fuzzy Syst., 32(2017), 629-641.

[6] C. Park, D. Shin, J.R. Lee, Fuzzy stability of functional inequalities in matrix fuzzy normed spaces, J. Ineq. Appl., 2013(2013), Artical ID 547, 13 pages.

[7] C. Park, Orthogonal stability of a cubic-quartic functional equation, J. Nonlinear Sci. Appl., 5(2012), 28-36.

[8] D. H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. USA, 27(1941), 222-224.

[9] M.E. Gordji, M.B. Savadkouhi, Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces, Applied Mathematics Letters, 23(10)(2010), 1198-1202.

[10] P. Găvruţa, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl., 184(1994), 431-436.

[11] T.Z. Xu, J.M. Rassias, W.X. Xu, Intuitionistic fuzzy stability of a genereal mixed additive-cubic equation, J. Math. Phys., 51(2010), 063519, 21 pages.

[12] T.Z. Xu, J.M. Rassias, W.X. Xu, A generalized mixed quadratic-quartic functional equation, Bull. Malays. Math. Sci. Soc., 35(3)(2012), 633-649.

[13] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Am. Math. Soc., 72(1978), 297-300.

[14] S.M. Ulam, A Collection of the Mathematical Problems, Interscience, New York, 1960.

[15] Z.H. Wang, P.K. Sahoo, Stability of ACQ-functional equation in various matrix normed spaces, J. Nonlinear Sci. Appl., 8(2015), 64-85.

A High-Accuracy Collocation Method for Solving Mixed Boundary Value Problems on Nonsmooth Boundaries*

Xin Luo †   Chuan-Long Wang‡

Abstract

By potential theory, the mixed Dirichlet-Neumann boundary value problem for the Laplacian is converted into the boundary integral equations (BIEs) with logarithmic singularity. Then the resulting system of the integral equations is solved by the Sidi-Israeli quadrature method (SIQM) with a Sigmoidal transformation. The convergence of numerical solutions by SIQM is proved based on Anselone's collective compact theory. Furthermore, a convergence estimate of the solution error is presented, which possesses high accuracy order $O\ (h_{\max}^3)$, where $h_{\max}$ is the mesh size. Finally, The efficiency of the method is illustrated by examples.

Keyword: Boundary value problem, collective compact theory, singularity, integral equations

1 Introduction

Consider the following mixed Dirichlet-Neumann boundary value problem for the Laplacian

$$\begin{cases} \Delta u = 0, \quad \text{in } \ \Omega, \\ u|_{\Gamma_{D_i}} = f_i, \quad i = 1, 2, \cdots, p, \\ \dfrac{\partial u}{\partial n}|_{\Gamma_{N_j}} = g_j, \quad j = 1, 2, \cdots, q, \end{cases} \tag{1.1}$$

where Ω is a simply connected region with the piecewise-smooth boundary $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, and $\Gamma_D = \cup_{i=1}^{p} \Gamma_{D_i}$ and $\Gamma_N = \cup_{j=1}^{q} \Gamma_{N_j}$. Here, f_i and g_j are given on Γ_{D_i} and

*This work is supported by Project (NO. KYTZ201505) Supported by the Scientific Research Foundation of CUIT

†College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610225, P.R. China, corresponding author: luoxin919@163.com

‡Department of Mathematics, Taiyuan Normal University, Taiyuan 030012, P.R. China

Γ_{N_j} respectively, and $\partial u/\partial n$ denotes the derivative of u with respect to the outward normal vector n.

By the potential theory [17], the solution of Eq. (1.1) can be represented as a single-layer potential of the form

$$u(P) = -\frac{1}{\pi}\int_\Gamma \ln|P-Q|z(Q)\mathrm{d}S_Q, \quad P \in \Omega, \tag{1.2}$$

where z is an unknown function called the "the single layer" density. From the jump condition for the normal derivative of the single layer potential at the boundary, we then have the following boundary integral equations (BIEs)

$$\begin{cases} -\dfrac{1}{\pi}\displaystyle\sum_{j=1}^{p}\int_{\Gamma_{D_j}} \ln|P-Q|z_{D_j}(Q)\mathrm{d}S_Q \\ -\dfrac{1}{\pi}\displaystyle\sum_{j=1}^{q}\int_{\Gamma_{N_j}} \ln|P-Q|z_{N_j}(Q)\mathrm{d}S_Q = f_i, \quad P\in\Gamma_{D_i}, \quad i=1,2,\cdots,p, \\ z_{N_i}(P) - \dfrac{1}{\pi}\displaystyle\sum_{j=1}^{p}\int_{\Gamma_{D_j}} \frac{\partial \ln|P-Q|}{\partial n_P} z_{D_j}(Q)\mathrm{d}S_Q \\ -\dfrac{1}{\pi}\displaystyle\sum_{j=1}^{q}\int_{\Gamma_{N_j}} \frac{\partial \ln|P-Q|}{\partial n_P} z_{N_j}(Q)\mathrm{d}S_Q = g_i, \quad P\in\Gamma_{N_i}, \quad i=1,2,\cdots,q, \end{cases} \tag{1.3}$$

where $z_{D_j} := z|_{\Gamma_{D_j}}$ and $z_{N_j} := z|_{\Gamma_{N_j}}$ are sought on Γ_{D_i} and Γ_{N_j}, respectively. Once z_{D_j} and z_{N_j} are solved from the Eq. (1.3), the solution $u(P)$ can be computed by

$$u(P) = -\frac{1}{\pi}\sum_{j=1}^{p}\int_{\Gamma_{D_j}} \ln|P-Q|z_{D_j}(Q)\mathrm{d}S_Q - \frac{1}{\pi}\sum_{j=1}^{q}\int_{\Gamma_{N_j}} \ln|P-Q|z_{N_j}(Q)\mathrm{d}S_Q, \quad P\in\Omega. \tag{1.4}$$

Even for the boundary data f_i and g_i are smooth, the solutions z_{D_j} and z_{N_j} may not be smooth. We denote by $P_i,\ i=0,1$ of the two interface points of the boundary Γ and by β_i with $0<\beta_i<2\pi,\ i=0,1$ the interior angle of Γ at P_i. In fact, from [1, 2] it follows that around P_i we have

$$u(P) = c(\Theta)r^{\pi/(2\beta_i)} + \text{smoother terms}, \quad P\in\Omega, \tag{1.5}$$

where (r,Θ) are the polar coordinates centered at P_i. Then , using (1.2) to define a potential not only in Ω but also in $R^2\backslash\bar{\Omega}$, the single z is the difference between the normal derivatives of u on Γ from inside and outside Γ. Therefore, near $P_i,\ i=0,1$, we get

$$z(P) = cr^{\min\{\pi/(2\beta_i),\ \pi/(4\pi-2\beta_i)\}-1} + \text{smoother terms}, \quad P\in\Omega. \tag{1.6}$$

Hence, z_{D_j} and z_{N_j} have this behavior near the corners P_i. To smooth these irregularities, in the next section we will introduce a smoothing parameterization $\psi_\gamma(t)$,

which improves the behavior of the unknown function z by incorporating the Jacobian of the transformation. In fact, the new unknown function will be $z(\psi_\gamma(t))|\psi'_\gamma(t)|$, whose smoothness degree at the corner depends upon a smoothing parameter: the larger its value, the smoother the transformed density. There exist numerical methods for approximately solving mixed value problems on polygonal domains by means of boundary integral equations (see [18, 19]). They are based on the collocation method, and in general no error estimates are available [20]. After that, the proof of asymptotic error estimates for the finite element Galerkin approximation of the boundary integral equations for a mixed Dirichlet-Neumann boundary value problem for the Laplacian in a plane polygonal domain is given in [21]. This was a generalization of [22], where the case of a domain with a smooth boundary was treated. In [6], the trigonometric collocation method which uses a mesh grading transformation and a cosine approximating space is proposed for solving the mixed boundary value problems on domains with curved polygonal boundaries, the complete stability and solvability analysis of the transformed integral equations is given by use of a Mellin transform technique, in which each arc of the polygon has associated with it a periodic Sobolev space. Inspired by the technique developed in [6], A collocation method using Chebyshev polynomial expansions as approximants and the zeros of Chebyshev polynomials as collocation nodes is applied to solved (1.3) [2]. From [5], we know that the Sidi transformation [3] is the important one of "integral" sigmoidal transformations, which can yield fast convergence of the collocation solution by smoothing the singularities of the exact solution. Hence, we apply Sidi-Israeli quadrature method [4] and trapezoidal rule with Sidi transformation [3, 16] to calculate the integrals with weakly singular kernels and continuous kernels in (1.3) respectively.

This paper is organized as follows: in Section 2, the convergence analysis is carried out based on the theory of collectively compact operators [7, 8, 9, 10] for closed curved polygons. in Section 3, a convergence estimate of the solution error is given. Numerical examples are provided to verify the theoretical results in Section 4, and conclusions are made in Section 5.

2 Collocation method for the boundary integral equations

2.1 Discretization for integral operators

In [4], high-accuracy numerical quadrature methods based on the appropriate Euler-Maclaurin expansions of trapezoidal rule approximations are proposed for the singular and weakly singular Fredholm integral equations. These integral equations are used in the solution of planar elliptic boundary value problems such as those that arise in free surface flows, elasticity, potential theory, conformal mapping, etc. Let the functions $G(x,t) = \log|x-t|g(x) + \tilde{g}(x)$ are periodic with period $T = b-a$, and that they are $2m$ times differentiable on $R\backslash\{t+kT\}_{k=-\infty}^{\infty}$. Then the Sidi-Israeli quadrature

formula [4] for integrals with kernel $G(x,t)$ can be described by

$$Q_n[G(x,t)] = h\Big\{\sum_{\substack{j=1\\ x_j\neq t}}^{n} G(x_j,t)+\tilde{g}(t)+\log(\frac{h}{2\pi})g(t)\Big\},\ h=(b-a)/n,\ x_j=a+jh,\tag{2.1}$$

and

$$\int_a^b G(x,t)dx - Q_n[G(x,t)] = 2\sum_{\mu=1}^{m-1}\frac{\zeta'(-2\mu)}{(2\mu)!}g^{(2\mu)}h^{2\mu+1}+O\,(h^{2m}),\quad \text{as}\quad h\to 0,$$

where $\zeta(z)$ is a Riemann function.

Define the following boundary integral operators on Γ_{D_j} and Γ_{N_j}

$$\begin{aligned}
U_{ij}z_j(x) &= -\frac{1}{\pi}\int_{\Gamma_{D_j}}\ln|x-y|z_j(y)\mathrm{d}s_y,\ x\in\Gamma_{D_i},\ i,j=1,\cdots,p,\\
M_{ij}z_j(x) &= -\frac{1}{\pi}\int_{\Gamma_{N_j}}\frac{\partial\ln|x-y|}{\partial n_x}z_j(y)\mathrm{d}s_y,\ x\in\Gamma_{N_i},\ i,\ j=1,\cdots,q,\\
V_{ij}z_j(x) &= -\frac{1}{\pi}\int_{\Gamma_{N_j}}\ln|x-y|z_j(y)\mathrm{d}s_y,\ x\in\Gamma_{D_i},\ i=1,\cdots,p,\ j=1,\cdots,q,\\
W_{ij}z_j(x) &= -\frac{1}{\pi}\int_{\Gamma_{D_j}}\frac{\partial\ln|x-y|}{\partial n_x}z_j(y)\mathrm{d}s_y,\ x\in\Gamma_{N_i},\ i=1,\cdots,q,\ j=1,\cdots,p.
\end{aligned}$$

Assume that Γ_{D_j} or Γ_{N_j} can be described by the parameter mapping: $x_j(t) = (x_{j1}(t),\ x_{j2}(t)) : [0,1]\to\Gamma_{D_j}(\text{or }\Gamma_{N_j})$ with $|x_j'(t)| = [|x_{j1}'(t)|^2+\ |x_{j2}'(t)|^2]^{1/2} > 0$. In order to degrade the singularities at corners, we apply the Sidi transformation [3, 16] to the parameter mapping, which is defined by

$$\psi_\gamma(t) = \frac{\int_0^t(\sin\pi\tau)^\gamma\mathrm{d}\tau}{\int_0^1(\sin\pi\tau)^\gamma\mathrm{d}\tau} : [0,1]\to[0,1],\quad \gamma\geq 1.\tag{2.2}$$

Define the following "smoothing parameterization"

$$\alpha(t) = \begin{cases}\alpha_i^{(1)}(t) = x_i(\psi_\gamma(t))\in\Gamma_{D_i} & t\in[-1,1],\\ \alpha_i^{(2)}(t) = x_i(\psi_\gamma(t))\in\Gamma_{N_i} & t\in[-1,1],\end{cases}\tag{2.3}$$

Thus, we can rewrite equations (1.3) as a $p\times q$ matrix integral equation system

$$\begin{cases}\displaystyle\sum_{j=1}^{p}\int_0^1 u(t,s)\bar{z}_j^{(1)}(s)\mathrm{d}s+\sum_{j=1}^{q}\int_0^1 v(t,s)\bar{z}_j^{(2)}(s)\mathrm{d}s = f_i(t),\ t\in[0,1],\ i=1,2,\cdots,p,\\ \displaystyle z_i(t)+\sum_{j=1}^{p}\int_0^1 w(t,s)\bar{z}_j^{(1)}(s)\mathrm{d}s+\sum_{j=1}^{q}\int_0^1 m(t,s)\bar{z}_j^{(2)}(s)\mathrm{d}s = g_i(t),\ i=1,2,\cdots,q,\end{cases}\tag{2.4}$$

4

where

$$u(t,s) = -\frac{1}{\pi}\ln|\alpha_i^{(1)}(t) - \alpha_j^{(1)}(s)|, \tag{2.5}$$

$$v(t,s) = -\frac{1}{\pi}\ln|\alpha_i^{(1)}(t) - \alpha_j^{(2)}(s)|, \tag{2.6}$$

$$w(t,s) = -\frac{1}{\pi}\frac{\alpha_{i2}^{(2)'}(t)[\alpha_{i1}^{(2)}(t) - \alpha_{j1}^{(1)}(s)] - \alpha_{i1}^{(2)'}(t)[\alpha_{i2}^{(2)}(t) - \alpha_{j2}^{(1)}(s)]}{[\alpha_{i1}^{(2)}(t) - \alpha_{j1}^{(1)}(s)]^2 + [\alpha_{i2}^{(2)}(t) - \alpha_{j2}^{(1)}(s)]^2}, \tag{2.7}$$

$$m(t,s) = \begin{cases} -\frac{1}{\pi}\frac{\alpha_{i2}^{(2)'}(t)[\alpha_{i1}^{(2)}(t)-\alpha_{j1}^{(2)}(s)]-\alpha_{i1}^{(2)'}(t)[\alpha_{i2}^{(2)}(t)-\alpha_{j2}^{(2)}(s)]}{[\alpha_{i1}^{(2)}(t)-\alpha_{j1}^{(2)}(s)]^2+[\alpha_{i2}^{(2)}(t)-\alpha_{j2}^{(2)}(s)]^2}, & \text{as } t \neq s, \\ -\frac{1}{2\pi}\frac{\alpha_{i2}^{(2)''}(t)\alpha_{i1}^{(2)'}(t)-\alpha_{i1}^{(2)''}(t)\alpha_{i2}^{(2)'}(t)}{(\alpha_{i1}^{(2)'}(t))^2+(\alpha_{i2}^{(2)'}(t))^2}, & \text{as } t = s. \end{cases} \tag{2.8}$$

and

$$\begin{aligned} &\bar{z}_j^{(1)}(s) = z_{D_j}(x_j(\psi_\gamma(s)))|x_j'(\psi_\gamma(s))|\psi_\gamma'(s), \\ &\bar{z}_j^{(2)}(s) = z_{N_j}(x_j(\psi_\gamma(s)))|x_j'(\psi_\gamma(s))|\psi_\gamma'(s), \\ &f_i(t) = f(\alpha_i^{(1)}(t)),\ \ g_i(t) = g(\alpha_i^{(2)}(t)),\ \ \alpha_i^{(k)}(t) = (\alpha_{i1}^{(k)}(t), \alpha_{i2}^{(k)}(t)),\ k = 1,2. \end{aligned}$$

Lemma 2.1. Although $z_{D_j}(s)$ and $z_{N_j}(s)$ have singularities at endpoints $s = 0$ and $s = 1$, $\bar{z}_j^{(1)}(s)$ and $\bar{z}_j^{(2)}(s)$ have no singularities by Sidi transformation at $s = 0$ and $s = 1$.

Proof. Let $d_j = \min\{\pi/(2\beta_j), \pi/(4\pi - 2\beta_j)\} - 1$ in (1.6), then we have $-1/2 \leq d_j < 0$. Suppose that $z_{DN}(s) = s^{d_j}\varphi_j(s)$ near $s = 0$, where $z_{DN} = z_{D_j}$ or z_{N_j}, and the function $\varphi_j(s)$ is differentiable enough on $[0,1]$ with $\varphi_j(0) \neq 0$. Using Taylor's formula, we can obtain

$$z_{DN}(s) = \sum_{i=0}^{l}\frac{\varphi_j^{(i)}(0)}{i!}s^{i+d_j} + O(s^{l+d_j+1}), \quad \text{as} \quad s \to 0^+. \tag{2.9}$$

From [3], we have

$$\psi_\gamma(s) \sim \sum_{i=0}^{\infty}\epsilon_i s^{\gamma+2i+1}, \quad \psi_\gamma'(s) \sim \sum_{i=0}^{\infty}\delta_i s^{\gamma+2i}, \quad \text{as} \quad s \to 0^+, \quad \epsilon_0,\ \delta_0 > 0. \tag{2.10}$$

By substituting (2.9) and (2.10) into the expression of $\bar{z}_j^{(k)}(s)$, $k = 1,2$, we have

$$\bar{z}_j^{(k)}(s) = c_1\varphi_j(0)s^{(\gamma+1)d_j+\gamma}(1 + O(s^2)) \quad \text{as} \quad s \to 0^+, \tag{2.11}$$

where c_1 is a constant. Also assume that $z_{DN}(s) = (1-s)^{d_j}\varphi_j(s)$ near $s = 1$. Similarly, we have

$$\bar{z}_j^{(k)}(s) = c_2\varphi_j(0)(1-s)^{(\gamma+1)d_j+\gamma}(1 + O((1-s)^2)) \quad \text{as} \quad s \to 1^-, \tag{2.12}$$

5

where c_2 is a constant independent of s. By (2.11), (2.12) and $d_j \geq -\frac{1}{2}$, we can obtain $(\gamma+1)(d_j+1)-1 \geq 0$ for $\gamma \geq 1$. The proof is completed. $\square$

Now we can rewrite the Eqs. (2.4) as follows

$$\begin{bmatrix} U & V \\ W & I+M \end{bmatrix} \begin{bmatrix} z^{(1)} \\ z^{(2)} \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{2.13}$$

where

$$\begin{aligned} &f=(f_1,f_2,\cdots,f_p),\ g=(g_1,g_2,\cdots,g_q),\\ &z^{(1)}=(\bar{z}_1^{(1)},\bar{z}_2^{(1)},\cdots,\bar{z}_p^{(1)})^T,\ z^{(2)}=(\bar{z}_1^{(2)},\bar{z}_2^{(2)},\cdots,\bar{z}_q^{(2)})^T,\\ &U=[U_{ij}]_{i,j=1}^{p,p},\ V=[V_{ij}]_{i,j=1}^{p,q},\ W=[W_{ij}]_{i,j=1}^{q,p},\ M=[M_{ij}]_{i,j=1}^{q,q}. \end{aligned}$$

Let $U=A+B$, where $A=\mathrm{diag}(A_{11},A_{22},\cdots,A_{pp})$ and $B=[B_{ij}]_{i,j=1}^{p,p}$, where

$$A_{ii}\bar{z}_i^{(1)}(t)=\int_0^1 a(t,s)\bar{z}_i^{(1)}(s)\mathrm{d}s,$$

with the kernel

$$a_{ii}(t,s)=-\frac{1}{\pi}\ln|2e^{-1/2}\sin(\pi(t-s))|,$$

and

$$B_{ij}\bar{z}_j^{(1)}(t)=\int_0^1 b(t,s)\bar{z}_j^{(1)}(s)\mathrm{d}s,$$

with the kernel

$$b_{ij}(t,s)=\begin{cases} -\frac{1}{\pi}\ln|\frac{\alpha_i^{(1)}(t)-\alpha_j^{(1)}(s)}{2e^{-1/2}\sin(\pi(t-s))}|, & \text{as } i=j,\\ \\ -\frac{1}{\pi}\ln|\alpha_i^{(1)}(t)-\alpha_j^{(1)}(s)|, & \text{as } i\neq j. \end{cases} \tag{2.14}$$

In the subsequent analysis we will focus on the singularity of the kernels $b_{ij}(t,s)$. Obviously, if $\Gamma_{D_i}\cap\Gamma_{D_j}=\emptyset$, $b_{ij}(t,s)$ are continuous in $[0,1]^2$, and if $\Gamma_{D_i}\cap\Gamma_{D_j}\neq\emptyset$, $b_{ij}(t,s)$ have singularities at the points $(t,s)=(0,1)$ and $(t,s)=(1,0)$. For convenience of analysis, we only discuss the case in which $(t,s)=(1,0)$. Defining the following function

$$\tilde{b}_{ij}(t,s)=b_{ij}(t,s)\sin^\gamma(\pi t),\quad \gamma\geq 1,\quad \Gamma_{D_i}\cap\Gamma_{D_j}\neq\emptyset. \tag{2.15}$$

Lemma 2.2. Let $\tilde{b}_{ij}(t,s)$ be defined by (2.15), then $\tilde{b}_{ij}(t,s)$ and $\frac{\partial^k\tilde{b}_{ij}(t,s)}{\partial t^k}$ $(k=1,2)$ are smooth on $[0,1]^2$.

Proof. By the continuity of $\tilde{b}_{ii}(t,s)$ in (2.14) and the boundness of $\sin^\gamma(\pi t)$, we can immediately complete the proof for the case $i=j$. Hence, we only consider the case in which $j-i=1$. Let $\Gamma_{D_{i-1}}\cap\Gamma_{D_i}=P_i=(0,0)$ and $\theta_i\in(0,2\pi)$ be the corresponding interior angle. Then we have

$$\ln|\alpha_i^{(1)}(t)-\alpha_{i-1}^{(1)}(s)|=\frac{1}{2}\ln[(|\alpha_i^{(1)}(t)|-|\alpha_{i-1}^{(1)}(s)|)^2+4|\alpha_i^{(1)}(t)||\alpha_{i-1}^{(1)}(s)|\sin^2(\theta_i/2)] \tag{2.16}$$

6

which shows the kernel $b_{i-1,i}(t,s)$ has a logarithmic singularity at $(t,s) = (1,0)$. Suppose that $a_0(t) = |\alpha_{i-1}^{(1)}(t)|$ and $a_1(s) = |\alpha_i^{(1)}(s)|$, we have $a_0(0) = a_1(0) = 0$. If $\theta_i \in (0,\pi) \cup (\pi, 2\pi)$, then $b_{i-1,i}(1,0) = 0$. If $(t,s) \neq (1,0)$, then we can obtain

$$\begin{aligned}
&\tilde{b}_{i-1,i}(t,s) = -\frac{1}{2\pi}\sin^\gamma(\pi t)\ln[a_0^2(t) + a_1^2(s) - 2a_0(t)a_1(s)\cos\theta_{i-1}] \qquad (2.17)\\
&= -\frac{1}{2\pi}\sin^\gamma(\pi t)\ln[a_0^2(t) + a_1^2(s)]\\
&- \frac{1}{2\pi}\sin^\gamma(\pi t)\ln[1 - 2a_0(t)a_1(s)\cos\theta_{i-1}/(a_0^2(t) + a_1^2(s))]\\
&= \varpi_1(t,s) + \varpi_2(t,s).
\end{aligned}$$

Since

$$|2a_0(t)a_1(s)\cos\theta_{i-1}/(a_0^2(t) + a_1^2(s))| \leq |\cos\theta_{i-1}| < 1,$$

the function $\varpi_2(t,s)$ and its second derivative are bounded. Noting that

$$\psi_\gamma^{(k)}(0) = \psi_\gamma^{(k)}(1) = 0, \quad k = 0, \cdots, \gamma,$$

we have

$$a_{\bar{i}}^{(k)}(0) = a_{\bar{i}}^{(k)}(1) = 0, \quad \bar{i} = i-1 \text{ or } i, \quad k = 1, \cdots, \gamma.$$

Let $(t,s) \in [\varepsilon/2, \varepsilon] \times [1-\varepsilon, 1-\varepsilon/2]$ for all $\varepsilon > 0$, we have $|\varpi_1(t,s)| = O(\varepsilon^\gamma |\ln\varepsilon|)$, so $\varpi_1(t,s)$ is also bounded. In addition, we have

$$\begin{aligned}
|\frac{\partial}{\partial t}\varpi_1(t,s)| &\leq \frac{1}{2\pi}|\sin^\gamma(\pi t)\frac{2a_0(s)|\alpha_{i-1}^{(1)'}(\psi_\gamma(s))|\psi_\gamma'(s)}{a_0^2(t) + a_1^2(s)}|\\
&= O(\varepsilon^\gamma)O(\varepsilon^{2\gamma})/O(\varepsilon^{2\gamma}) = O(\varepsilon^\gamma)
\end{aligned}$$

and

$$|\frac{\partial^2}{\partial t^2}\varpi_1(t,s)| = O(\varepsilon^{\gamma-1}).$$

This shows $\frac{\partial^k \tilde{b}_{ij}(t,s)}{\partial t^k}$ $(k = 0,1,2)$ are also continuous in $[0,1]^2$. At last, if $\theta_{i-1} = \pi$, then

$$\tilde{b}_{i-1,i}(t,s) = -\frac{1}{\pi}\sin^\gamma(\pi t)\ln(a_0(t) + a_1(s)), \qquad (2.18)$$

we can use the same method mentioned above to prove $\tilde{b}_{i-1,i}(t,s)$ and its second derivative are bounded. The proof of Lemma 2.2 is completed. □

Let $h_j = 1/n_j$ $(n_j \in N)$ and $t_j = s_j = (j-1/2)h_j$ $(j = 1, ..., n_j)$ be the mesh sizes and nodes respectively. By the trapezoidal or the midpoint rule [11] we construct the Nyström's approximation operator $B_{ij}^{h_j}$ of the integral operator B_{ij}, defined by

$$(B_{ij}^{h_j}\bar{z}_j^{(1)})(t) = h_j\sum_{j=1}^{n_j} b_{ij}(t,s_j)\bar{z}_j^{(1)}(s_j), \quad t \in [0,1], \quad i = 1, ..., p, \qquad (2.19)$$

7

which has the error bounds [3, 11]

$$(B_{ij}\bar{z}_j^{(1)})(t) - (B_{ij}^{h_j}\bar{z}_j^{(1)})(t) = O\,(h_j^{2\ell}), \quad \text{for} \;\; \Gamma_{D_i} \cap \Gamma_{D_j} = \emptyset, \quad \ell \in N, \tag{2.20}$$

and

$$(B_{ij}\bar{z}_j^{(1)})(t) - (B_{ij}^{h_j}\bar{z}_j^{(1)})(t) = O\,(h_j^{\omega}), \quad \text{for} \;\; \Gamma_{D_i} \cap \Gamma_{D_j} \in \{P_j\}, \tag{2.21}$$

where (see [3])

$$\omega = \begin{cases} \min\{(\gamma+1)(d_j+1), \gamma+1\}, & \gamma \;\; \text{odd}, \\ \\ \min\{(\gamma+1)(d_j+1), 2(\gamma+1)\}, & \gamma \;\; \text{even}. \end{cases} \tag{2.22}$$

For the logarithmically singular operators A_{ii}, by the Sidi-Israeli quadrature formula, we can also construct the approximate operator $A_{ii}^{h_i}$,

$$\begin{aligned} (A_{ii}^{h_i}\bar{z}_j^{(1)})(t) = &-\frac{h_i}{\pi}\left\{\sum_{\substack{j=1 \\ s_j \neq t}}^{n_j} \ln\left|2e^{-1/2}\sin\pi(t-s_j)\right|\bar{z}_j^{(1)}(s_j)\right\} \\ &- \frac{h_i}{\pi}\left\{\ln\left(2\pi e^{-1/2}h_i/(2\pi)\right)\bar{z}_j^{(1)}(t)\right\} \quad (i = 1, ..., n_i), \end{aligned} \tag{2.23}$$

which has the error bounds [4]

$$(A_{ii}^{h_i}\bar{z}_j^{(1)})(t) - (A_{ii}\bar{z}_j^{(1)})(t) = -\frac{2}{\pi}\sum_{\mu=1}^{2\ell-1}\frac{\zeta'(-2\mu)}{(2\mu)!}[\bar{z}_j^{(1)}]^{(2\mu)}h_i^{2\mu+1} + O(h_i^{2\ell}), \quad t \in \{t_i\},$$

where $\zeta'(t)$ is the derivative of the Riemann zeta function.

By the trapezoidal or the midpoint rule, we can also construct the Nyström's approximation operators $V_{ij}^{h_j}$, $W_{ij}^{h_j}$ and $M_{ij}^{h_j}$ for the continuous operators V_{ij}, W_{ij} and M_{ij}, that is,

$$(\Xi_{ij}^{h_j}\bar{z}_j)(t) = h_j\sum_{j=1}^{n_j}\chi_{ij}(t, s_j)\bar{z}_j(s_j), \quad t \in [0, 1], \tag{2.24}$$

which have the error bounds $O(h_j^{2\ell})$ or $O(h_j^{\omega})$. Here, $\Xi_{ij} = V_{ij}, W_{ij}$ or M_{ij}, $\chi_{ij}(t, s) = v_{ij}(t, s), w_{ij}(t, s)$ or $m_{ij}(t, s)$.

Now we write the discrete equations for (2.13) are

$$\begin{bmatrix} U^h & V^h \\ W^h & I^h + M^h \end{bmatrix}\begin{bmatrix} z^{(1)h} \\ z^{(2)h} \end{bmatrix} = \begin{bmatrix} f^h \\ g^h \end{bmatrix}, \tag{2.25}$$

where

$$
\begin{aligned}
&U^h = A^h + B^h,\ A^h = \mathrm{diag}(A_{11}^{h_1}, ..., A_{pp}^{h_p}), \quad A_{ii}^{h_i} = [a(t_i, s_j)]_{i,j=1}^{n_p},\\
&B^h = [B_{ij}^{h_j}]_{i,j=1}^{p}, \quad B_{ij}^{h_j} = [b_{ij}(t_i, s_j)]_{i,j=1}^{n_p,n_p}, \quad V^h = [V_{ij}^{h_j}]_{i,j=1}^{p,q},\\
&V_{ij}^{h_j} = [v_{ij}(t_i, s_j)]_{i,j=1}^{n_p,n_q}, \quad W^h = [W_{ij}^{h_j}]_{i,j=1}^{q,p}, \quad W_{ij}^{h_j} = [w_{ij}(t_i, s_j)]_{i,j=1}^{n_q,n_p},\\
&M^h = [M_{ij}^{h_j}]_{i,j=1}^{p,q}, \quad M_{ij}^{h_j} = [m_{ij}(t_i, s_j)]_{i,j=1}^{n_q,n_q},\\
&z^{(1)h} = (z_1^{(1)h_1}(t_1), ..., z_1^{(1)h_1}(t_{n_1}), ..., z_p^{(1)h_p}(t_1), ..., z_p^{(1)h_p}(t_{n_p}))^T,\\
&z^{(2)h} = (z_1^{(2)h_1}(t_1), ..., z_1^{(2)h_1}(t_{n_1}), ..., z_q^{(2)h_q}(t_1), ..., z_q^{(2)h_q}(t_{n_q}))^T,\\
&f^h = (f_1^{h_1}(t_1), ..., f_1^{h_1}(t_{n_1}), ..., f_p^{h_p}(t_1), ..., f_p^{h_p}(t_{n_p}))^T,\\
&g^h = (g_1^{h_1}(t_1), ..., g_1^{h_1}(t_{n_1}), ..., g_q^{h_q}(t_1), ..., g_q^{h_q}(t_{n_q}))^T.
\end{aligned}
$$

Let

$$
\begin{bmatrix} U^h & V^h \\ W^h & I^h + M^h \end{bmatrix} = \begin{bmatrix} A^h & 0 \\ 0 & I^h \end{bmatrix} + \begin{bmatrix} B^h & V^h \\ W^h & M^h \end{bmatrix}, \tag{2.26}
$$

then (2.25) is equivalent to

$$
\left(\begin{bmatrix} E_1^h & 0 \\ 0 & E_2^h \end{bmatrix} + \begin{bmatrix} (A^h)^{-1}B^h & (A^h)^{-1}V^h \\ W^h & M^h \end{bmatrix} \right) \begin{bmatrix} z^{(1)h} \\ z^{(2)h} \end{bmatrix} = \begin{bmatrix} (A^h)^{-1}f^h \\ g^h \end{bmatrix}. \tag{2.27}
$$

2.2 The collectively compact convergence

For the convenience of the analysis of the existence and convergence of numerical solutions, we first introduce the subspaces and some special operators to be used. Define the subspace $C_0[0,1] = \{v(t) \in C[0,1] : v(t)(\sin^\gamma(\pi t))^{-1} \in C[0,1]\}$ of the space $C[0,1]$ with the norm $||v||^* = \max_{0\le t\le 1} |v(t)(\sin^\gamma(\pi t)))^{-1}|$. Let $S^{h_j} = \mathrm{span}\{e_j(t), j = 1, ..., n_j\} \subset C_0[0,1]$ be a piecewise linear function subspace with the basis nodes $\{t_i\}_{i=1}^{n_j}$, where $e_j(t)$ are the basis functions satisfying $e_j(t_i) = \delta_{ji}$. Also define a prolongation operator $\Pi_1^{h_j} : \Re^{n_j} \to S^{h_j}$ satisfying $\Pi_1^{h_j} v = \sum_{j=1}^{n_j} v_j e_j(t), \forall v = (v_1, ..., v_{n_j}) \in \Re^{n_j}$, and a restricted operator $\Pi_2^{h_j} : C_0[0,1] \to \Re^{n_j}$ satisfying $\Pi_2^{h_j} v = (v(t_1), ..., v(t_{n_j})) \in \Re^{n_j}, \forall v \in C_0[0,1]$.

Lemma 2.3. Let $\Gamma = \Gamma_D \cup \Gamma_N$ satisfy $C_\Gamma \neq 1$, and also let

$$
\bar{B}_{ij}^{h_j} = \begin{cases} B_{ij}^{h_j}, & \Gamma_{D_i} = \Gamma_{D_j} \text{ or } \Gamma_{D_i} \cap \Gamma_{D_j} = \emptyset, \\ \tilde{B}_{ij}^{h_j}, & \Gamma_{D_i} \cap \Gamma_{D_j} \in \{P_j\}, \end{cases}
$$

where the kernel $\tilde{b}_{ij}(t,s)$ of $\tilde{B}_{ij}$ is defined by (2.15). Then under the transformation (2.2), we have

$$
\|(A_{ij})^{-1} \bar{B}_{ij}^{h_j}\|_{2,0} \le M \tag{2.28}
$$

and

$$
\Pi_1^{h_i}(A_{ii}^{h_i})^{-1}\Pi_2^{h_j}\bar{B}_{ij}^{h_j} \xrightarrow{c.c} (A_{ij})^{-1}B_{ij}, \quad \text{in } \ C[0,1] \to C[0,1], \tag{2.29}
$$

where M is a constant and $\xrightarrow{c.c}$ denotes the collectively compact convergence.

Proof. From [14] and by Lemma 2.2, $b_{ij}(t,s)$ and $\tilde{b}_{ij}(t,s)$ are continuous on $[0,1]^2$,

9

and then we have (2.28). Using the following result [14],

$$\Pi_1^{h_i}(A_{ii}^{h_i})^{-1}\Pi_2^{h_i}A_{ii} \xrightarrow{p} I, \quad \text{in} \quad C^2[0,1] \to C[0,1],$$

where I is the embedding operator and $\xrightarrow{p}$ denotes the pointwisely convergence, and by

$$\begin{aligned}\|\Pi_1^{h_i}(A_{ii}^{h_i})^{-1}\Pi_2^{h_i}\bar{B}_{ij}^{h_j}\|_{0,0} &= \|(\Pi_1^{h_i}(A_{ij}^{h_i})^{-1}\Pi_2^{h_i}A_{ij})((A_{ij})^{-1}\bar{B}_{ij}^{h_j})\|_{0,0}\\ &\le \|\Pi_1^{h_i}(A_{ij}^{h_i})^{-1}\Pi_2^{h_i}A_{ij}\|_{0,2}\|(A_{ij})^{-1}\bar{B}_{ij}^{h_j}\|_{2,0}\\ &\le C,\end{aligned}$$

where C is a constant. Thus, we complete the proof of Lemma 2.3. $\square$

Replacing $(A_{ii}^{h_i})^{-1}$, $B_{ij}^{h_j}$ $(i,j{=}1,...,p)$, $V_{ij}^{h_j}$ $(i = 1,...,p, j = 1,...,q)$, $W_{ij}^{h_j}$ $(i = 1,...,q, j = 1,...,p)$ and $M_{ij}^{h_j}$ $(i,j = 1,...,q)$ by $\Pi_1^{h_i}(A_{ii}^{h_i})^{-1}\Pi_2^{h_i}$, $\Pi_1^{h_i}B_{ij}^{h_j}\Pi_2^{h_j}$, $\Pi_1^{h_i}V_{ij}^{h_j}\Pi_2^{h_j}$, $\Pi_1^{h_i}W_{ij}^{h_j}\Pi_2^{h_j}$, and $\Pi_1^{h_i}M_{ij}^{h_j}\Pi_2^{h_j}$, respectively.

Define the following operators

$$(\hat{A}^h)^{-1}\hat{B}^h,\ \hat{W}^h : (C_0[0,1])^p \to \cup_{j=1}^p S^{h_j}, \quad (\hat{A}^h)^{-1}\hat{V}^h,\ \hat{M}^h : (C[0,1])^q \to \cup_{j=1}^q S^{h_j},$$

where

$$\begin{aligned}&(\hat{A}^h)^{-1}\hat{B}^h = \Pi_{11}^h(A^h)^{-1}\Pi_{21}^hB^h, \quad (\hat{A}^h)^{-1}\hat{V}^h = \Pi_{11}^h(A^h)^{-1}\Pi_{22}^hV^h,\\ &\hat{W}^h = \Pi_{11}^h(A^h)^{-1}\Pi_{21}^hW^h, \quad \hat{M}^h = \Pi_{12}^h(A^h)^{-1}\Pi_{22}^hM^h,\\ &\Pi_{11}^h = \operatorname{diag}(\Pi_1^{h_1},...,\Pi_1^{h_p}), \quad \Pi_{12}^h = \operatorname{diag}(\Pi_1^{h_1},...,\Pi_1^{h_q}),\\ &\Pi_{21}^h = \operatorname{diag}(\Pi_2^{h_1},...,\Pi_2^{h_p}), \quad \Pi_{22}^h = \operatorname{diag}(\Pi_2^{h_1},...,\Pi_2^{h_q}).\end{aligned}$$

Hence, we can write (2.27) as the following operator equation

$$\left(\begin{bmatrix} E_1^h & 0 \\ 0 & E_2^h \end{bmatrix} + \begin{bmatrix} (\hat{A}^h)^{-1}\hat{B}^h & (\hat{A}^h)^{-1}\hat{V}^h \\ \hat{W}^h & \hat{M}^h \end{bmatrix}\right)\begin{bmatrix} \hat{z}^{(1)h} \\ \hat{z}^{(2)h} \end{bmatrix} = \begin{bmatrix} (\hat{A}^h)^{-1}\hat{f}^{(1)h} \\ \hat{f}^{(2)h} \end{bmatrix}.$$

Theorem 2.4. (see [12, 15]) Assume $\Gamma_D = \cup_{j=1}^p\Gamma_{D_j}$ satisfy $C_{\Gamma_D} \neq 1$, and Γ_{D_j} $(j = 1,\ ...,\ p)$ are smooth curves. Then the operator sequence$\{(\hat{A}^h)^{-1}\hat{B}^h\}$ is collectively compact convergent to $A^{-1}B$ in $V = (C_0[0,1])^p$. That is, we have

$$(\hat{A}^h)^{-1}\hat{B}^h \xrightarrow{c.c} A^{-1}B. \tag{2.30}$$

Consider the integral

$$Q(g) = \int_\Omega g(x)\mathrm{d}x,$$

where Ω is the bounded domain. Supposed that the quadrature formulae for $Q(g)$ is

$$Q_n(g) = \sum_{j=1}^n \omega_j^{(n)}g(x_j^{(n)}),$$

10

where the weights $\omega_j^{(n)}$ satisfy to the following condition

$$\sum_{j=1}^{n} |\omega_j^{(n)}| \leq C, \tag{2.31}$$

where C is a constant.

Theorem 2.5. [8, 13] Assume that the kernel $k(x, y)$ of K is continuous on $\Omega \times \Omega$ and the K_n is the Nyström's approximation operator for K, and the condition (2.31) holds. Then we have

$$K_n \stackrel{c.c}{\to} K$$

By the Theorem 2.5, we can immediately obtain the following theorem.

Theorem 2.6. Let $\Gamma = \Gamma_D \cup \Gamma_N$ satisfy $C_\Gamma \neq 1$, Γ_{D_j} $(j = 1, ..., p)$ and Γ_{N_j} $(j = 1, ..., q)$are smooth curves, then we have

$$(\hat{A}^h)^{-1}\hat{V}^h \stackrel{c.c}{\to} A^{-1}V, \quad \hat{W}^h \stackrel{c.c}{\to} W, \quad \hat{M}^h \stackrel{c.c}{\to} M. \tag{2.32}$$

3 Errors analysis

In this section, we give the following theorem, which provides a convergence estimate of the solution error.

Theorem 3.1. Assume $\Gamma = \Gamma_D \cup \Gamma_N$ satisfy $C_\Gamma \neq 1$, $f_j = f|_{\Gamma_{D_j}} \in C^6(\Gamma_{D_j})$ and $g_j = g|_{\Gamma_{N_j}} \in C^6(\Gamma_{N_j})$, then when we choose an appropriate number γ in (2.22) such that $\omega > 3$, the following estimate hold

$$\|\hat{z}^{(1)h} - z^{(1)}\|_\infty = O(h_{\max}^3), \quad \|\hat{z}^{(2)h} - z^{(2)}\|_\infty = O(h_{\max}^3) \tag{3.1}$$

where $h_{\max} = \max_{1\leq j\leq \max\{p,q\}} h_j$.

Proof. By the trapezoidal rule, the asymptotic expansion holds

$$\begin{bmatrix} \hat{U}^h & \hat{V}^h \\ \hat{W}^h & I^h + \hat{M}^h \end{bmatrix} \left(\begin{bmatrix} \hat{z}^{(1)h} \\ \hat{z}^{(2)h} \end{bmatrix} - \begin{bmatrix} z^{(1)} \\ z^{(2)} \end{bmatrix} \right)$$

$$= \begin{bmatrix} \Pi_{11}^h U \Pi_{21}^h & \Pi_{12}^h V \Pi_{22}^h \\ \Pi_{11}^h W \Pi_{21}^h & \Pi_{21}^h I \Pi_{22}^h + \Pi_{21}^h M \Pi_{22}^h \end{bmatrix} \begin{bmatrix} z^{(1)} \\ z^{(2)} \end{bmatrix} - \begin{bmatrix} U^h & V^h \\ W^h & I^h + M^h \end{bmatrix} \begin{bmatrix} z^{(1)} \\ z^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} \Pi_{11}^h \Pi_{21}^h \Sigma_1 \psi_1 + \Pi_{12}^h \Pi_{22}^h \Sigma_2 \psi_2 \\ \Pi_{11}^h \Pi_{21}^h \Sigma_1 \psi_3 + \Pi_{12}^h \Pi_{22}^h \Sigma_2 \psi_4 \end{bmatrix} + O(h^{3.5}) \cdot I_{p+q}$$

where $\Sigma_1 = \text{diag}(h_1^3, \cdots, h_p^3)$, $\Sigma_2 = \text{diag}(h_1^3, \cdots, h_q^3)$, $\psi_1 = (\psi_{11}, \psi_{12}, \cdots, \psi_{1p})^T$, $\psi_2 = (\psi_{21}, \psi_{22}, \cdots, \psi_{2q})^T$, $\psi_3 = (\psi_{31}, \psi_{32}, \cdots, \psi_{3p})^T$ and $\psi_4 = (\psi_{41}, \psi_{42}, \cdots, \psi_{4q})^T$. Hence, we have

$$\left(\begin{bmatrix} E_1^h & 0 \\ 0 & E_2^h \end{bmatrix} + \begin{bmatrix} (\hat{A}^h)^{-1}\hat{B}^h & (\hat{A}^h)^{-1}\hat{V}^h \\ \hat{W}^h & \hat{M}^h \end{bmatrix} \right) \begin{bmatrix} \hat{z}^{(1)h} - z^{(1)} \\ \hat{z}^{(2)h} - z^{(2)} \end{bmatrix}$$

11

$$= \begin{bmatrix} \Pi_{11}^h \Pi_{21}^h (\hat{A}^h)^{-1} \Sigma_1 \psi_1 + \Pi_{12}^h \Pi_{22}^h (\hat{A}^h)^{-1} \Sigma_2 \psi_2 \\ \Pi_{11}^h \Pi_{21}^h \Sigma_1 \psi_3 + \Pi_{12}^h \Pi_{22}^h \Sigma_2 \psi_4 \end{bmatrix} + o(h^5) \cdot I_{p+q} \tag{3.2}$$

Define the auxiliary equation

$$\left(\begin{bmatrix} E_1^h & 0 \\ 0 & E_2^h \end{bmatrix} + \begin{bmatrix} (A^h)^{-1} B^h & (A^h)^{-1} V^h \\ W^h & M^h \end{bmatrix} \right) \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} A^{-1}\psi_1 + A^{-1}\psi_2 \\ \psi_3 + \psi_4 \end{bmatrix} \tag{3.3}$$

and its approximate equation

$$\begin{aligned} &\left(\begin{bmatrix} E_1^h & 0 \\ 0 & E_2^h \end{bmatrix} + \begin{bmatrix} (\hat{A}^h)^{-1} \hat{B}^h & (\hat{A}^h)^{-1} \hat{V}^h \\ \hat{W}^h & \hat{M}^h \end{bmatrix} \right) \begin{bmatrix} \Phi_1^h \\ \Phi_2^h \end{bmatrix} \\ &= \begin{bmatrix} \Pi_{11}^h \Pi_{21}^h (\hat{A}^h)^{-1} \Sigma_1 \psi_1 + \Pi_{12}^h \Pi_{22}^h (\hat{A}^h)^{-1} \Sigma_2 \psi_2 \\ \Pi_{11}^h \Pi_{21}^h \Sigma_1 \psi_3 + \Pi_{21}^h \Pi_{22}^h \Sigma_2 \psi_4 \end{bmatrix} \end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.2), we can obtain

$$\begin{aligned} &\left(\begin{bmatrix} E_1^h & 0 \\ 0 & E_2^h \end{bmatrix} + \begin{bmatrix} (\hat{A}^h)^{-1} \hat{B}^h & (\hat{A}^h)^{-1} \hat{V}^h \\ \hat{W}^h & \hat{M}^h \end{bmatrix} \right) \\ &\times \left(\begin{bmatrix} \hat{z}^{(1)h} - z^{(1)} \\ \hat{z}^{(2)h} - z^{(2)} \end{bmatrix} - \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_3 & \Sigma_4 \end{bmatrix} \begin{bmatrix} \Phi_1^h \\ \Phi_2^h \end{bmatrix} \right) = o(h^5). \end{aligned} \tag{3.5}$$

Since

$$\left(\begin{bmatrix} E_1^h & 0 \\ 0 & E_2^h \end{bmatrix} + \begin{bmatrix} (\hat{A}^h)^{-1} \hat{B}^h & (\hat{A}^h)^{-1} \hat{V}^h \\ \hat{W}^h & \hat{M}^h \end{bmatrix} \right)^{-1}$$

is bounded. we have

$$\begin{bmatrix} \hat{z}^{(1)h} - z^{(1)} \\ \hat{z}^{(2)h} - z^{(2)} \end{bmatrix} - \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_3 & \Sigma_4 \end{bmatrix} \begin{bmatrix} \tilde{\Phi}_1^h \\ \tilde{\Phi}_2^h \end{bmatrix} = o(h^5)$$

that is

$$\|\hat{z}^{(1)h} - z^{(1)}\|_\infty = O(h_{\max}^3), \quad \|\hat{z}^{(2)h} - z^{(2)}\|_\infty = O(h_{\max}^3) \tag{3.6}$$

where $h_{\max} = \max_{1 \le j \le \max\{p,q\}} h_j$. $\square$

4 Numerical experiments

In this section, we will test the SIQM proposed in this paper for the numerical solution of the mixed problem (1.1) via the boundary integral equations (1.3).

Let $err_n^u(P) = \big|u(P) - u_n(P)\big|$ be the errors by SIQM using n boundary nodes, and let $EOC = \log(err_n/err_{2n})/\log 2$ be the estimated order of convergence.

Example 1. [2] Ω is a domain with a re-entrant corner, enclosed by the curve:

$$\Gamma: \quad (-\frac{1}{2}\sin(\frac{3\pi}{2}x),\ -\sin(\pi x)), \quad 0 \le x \le 2,$$

and the Dirichlet and Neumman arcs Γ_D and Γ_N are parameterized by the interval $[0, 1]$ and $[1, 2]$, respectively. Setting $u(x_1, x_2) = x_1^2 - x_2^2$. Because u is the real part of an analytic function, u satisfies the Laplace equation in Ω. Let $u = \bar{f}_1$ on Γ_D and $\partial u/\partial n = \bar{f}_2$ on Γ_N. Let each boundary be divided into 2^k $(k = 3, \cdots, 8)$ segments. The errors and error ratio of the interior points $P_1 = (0.4,\ 0)$, $P_2 = (0,\ 0.6)$ and $P_3 = (0.1,\ 0.5)$ using n $(= 2 \times 2^k,\ k = 3, \cdots, 8)$ nodes by transformation $\psi_6(t)$ are listed in Table 1. In addition, the numerical solution u of the interior points along the line $x_2 = x_1 - 0.4$ are computed, where $x_1 = -0.4 : 0.01 : 0.4$. The plots of computed errors are shown in Figure 1 (b) to Figure 2.

Table 1: The Errors of u.

n	2×2^3	2×2^4	2×2^5	2×2^6	2×2^7	2×2^8
$err_n^u(P_1)$	5.007-04	3.872-03	2.040-04	2.581-05	3.222-06	4.026-07
$EOC(P_1)$	–	-2.951	4.246	2.982	3.002	3.000
$err_n^u(P_2)$	2.973-01	5.533-02	1.737-03	1.878-05	2.008-06	2.509-07
$EOC(P_2)$	–	2.426	4.993	6.531	3.226	3.000
$err_n^u(P_3)$	2.414-01	1.931-02	4.130-04	1.687-05	2.035-06	2.543-07
$EOC(P_3)$	–	3.644	5.547	4.613	3.052	3.000

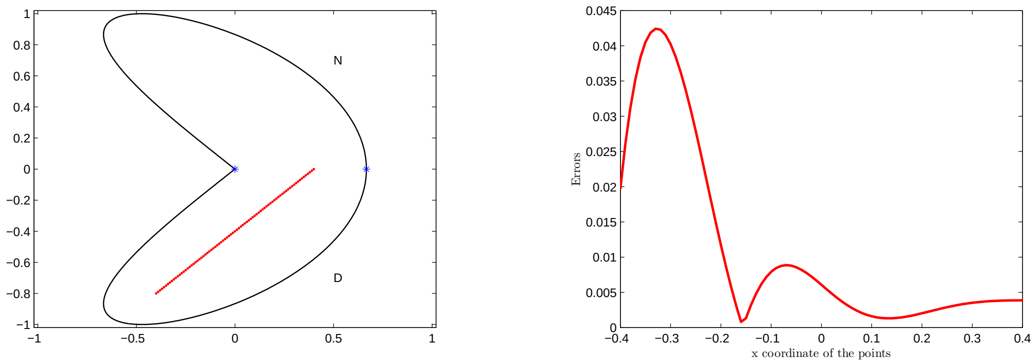
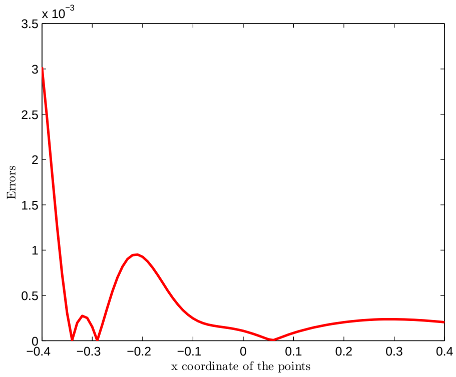


Figure 1: *Left: The contour* Γ *for Example 1; Right: Errors of* u *by* 2×2^4 *boundary nodes.*

Example 2. Consider the following problem where the domain is a quarter-circle.

$$\Delta u = 0 \quad \text{for } x_1 > 0,\ x_2 > 0,\ \text{and } x_1^2 + x_2^2 < 1,$$

13



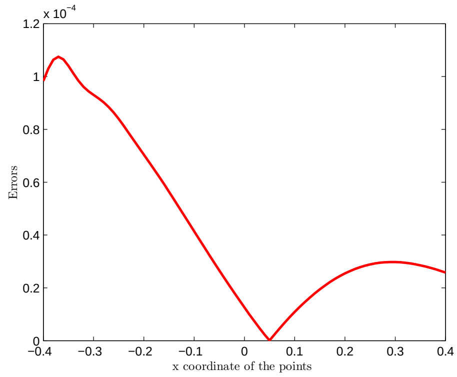


Figure 2: *Left: Errors of* u *by* 2×2^5 *boundary nodes; Right: Errors of* u *by* 2×2^6 *boundary nodes.*

subject to the boundary conditions

$$\Gamma_{D_1}: \ u = 1, \quad \text{on } x_1^2 + x_2^2 = 1, \quad \text{for } x_1 > 0, \ x_2 > 0;$$
$$\Gamma_{D_2}: \ u = 0, \quad x_2 = 0;$$
$$\Gamma_{N_1}: \ \frac{\partial u}{\partial n} = 0, \quad x_1 = 0.$$

The analytical solution of this problem is $u = \frac{2}{\pi}\arctan(\frac{2x_2}{1-x_1^2-x_2^2})$.

Let each boundary be divided into 2^k $(k = 3, \cdots, 8)$ segments. The errors and error ratio of the interior points $P_1 = (0.1, \ 0.1)$, $P_2 = (0.8, \ 0.1)$ and $P_3 = (0.1, \ 0.7)$ using n $(= 3 \times 2^k, \ k = 3, \cdots, 8)$ nodes by transformation $\psi_6(t)$ are listed in Table 2. In addition, the numerical solution u of the interior points along the curve segment $L: \ x_1 = 0.7\cos(\frac{\pi}{2}t), \ x_2 = 0.7\sin(\frac{\pi}{2}t)$ are computed, where $t = 0.05 : 0.01 : 0.95$. The plots of computed errors are shown in Figure 3 (b) to Figure 4. From the numerical results of Table 1 and Table 2 we can see that $EOC \approx 3$.

Table 2: The Errors of u .

n	3×2^3	3×2^4	3×2^5	3×2^6	3×2^7	3×2^8
$err_n^u(P_1)$	1.167-03	1.223-04	1.341-05	1.675-06	2.093-07	2.617-08
$EOC(P_1)$	–	3.255	3.188	3.001	3.000	3.000
$err_n^u(P_2)$	7.409-03	9.133-04	1.691-05	2.421-06	3.025-07	3.781-08
$EOC(P_2)$	–	3.020	5.755	2.805	3.000	3.000
$err_n^u(P_3)$	1.997-02	1.694-03	3.282-05	1.062-06	1.313-07	1.641-08
$EOC(P_3)$	–	3.559	5.689	4.950	3.016	3.000

14

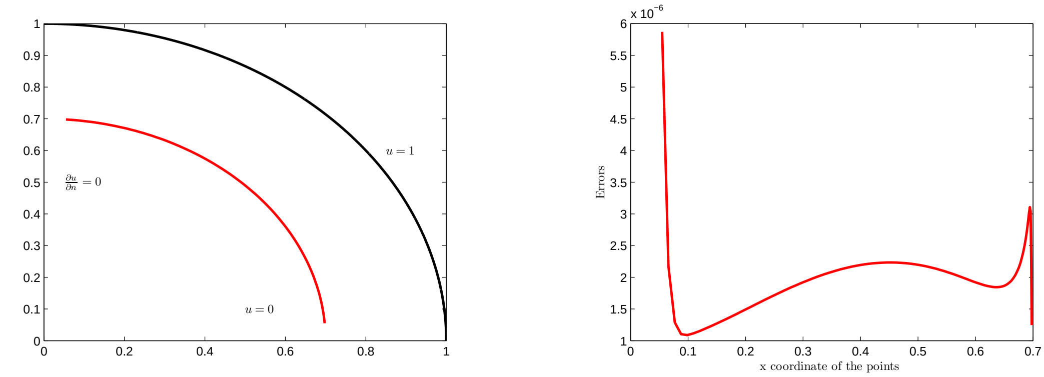


Figure 3: *Left: The contour* Γ *for Example 2; Right: Errors of* u *by* 3×2^6 *boundary nodes.*

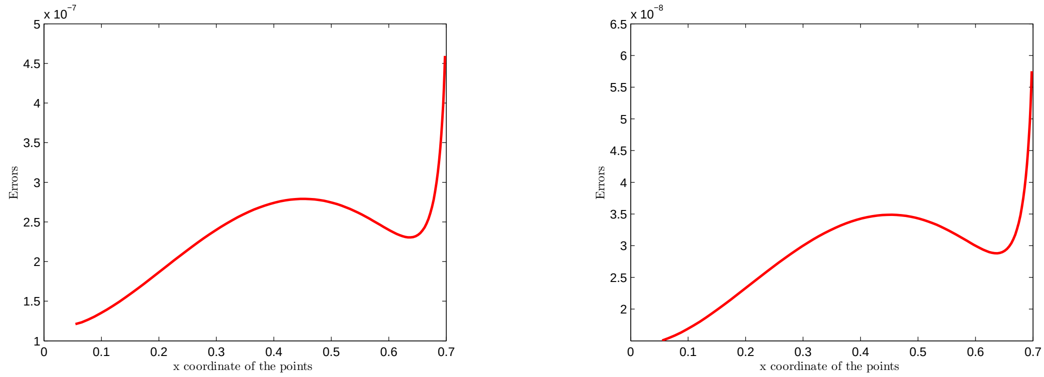


Figure 4: *Left: Errors of* u *by* 3×2^7 *boundary nodes; Right: Errors of* u *by* 3×2^8 *boundary nodes.*

5 Conclusions

In this paper, the convergence and error of SIQM for the boundary integral equations of the mixed Dirichlet-Neumann boundary value problem for the Laplacian are studied on nonsmooth boundaries. Especially, in order to provide a good accuracy in the solution near the singular points, the Sidi transformation is used for the boundary integral equations of problems (1.1). The numerical results show that the presented algorithm has a high accuracy of $O\ (h^3_{\max})$, which coincides with our theoretical analysis.

References

[1] M. Costabel and E.P. Stephan, *On the convergence of collocation methods for boundary integral equations on polygons*, Math. Comp. 50 (1987), pp. 461–478.

[2] L.Scuderi, *A chebyshev polynomial collocation BIEM for mixed boundary value problems on nonsmooth boundaries*, J. Integral Equations Appl. 14 (2002), pp. 179–221.

[3] A. Sidi, *A new variable transformation for numerical integration*, Int Ser Numer Math. 112 (1993), pp. 359–373.

[4] A. Sidi and M. Israeli, *Quadrature methods for periodic singular and weakly singular Fredholm integral equation* J. Sci. Comput. 3 (1988), pp. 201–231.

[5] David Elliott, *Sigmoidal Transformations and the Trapezoidal Rule*, J. Austral. Math. Soc. B 40 (1998) 77-137.

[6] J. Elschner, Y. Jeon, I.H. Sloan and E.P. Stephan, *The collocation method for mixed boundary value problem on domains with curved polygonal boundaries*, Numer. Math. 76 (1997), 335–381.

[7] P.M. Anselone, *Singularity subtraction in numerical solution of integral equations*, J. Austral Math. Soc. 22 (1981), pp. 408–418.

[8] P.M. Anselone, *Collectively Compact Operator Approximation Theory and Applications to Integral Equations*, Prentice-Hall, Englewood Cliffs, NJ. 1971.

[9] P.M. Anselone and T.W. Palmer, *Collectively compact sets of linear operators*, Pacific J. Math. 25 (1968), pp. 417–422.

[10] P.M. Anselone and M.L. Treuden, *Regular operator approximation theory*, Pacific J. Math. 120 (1985), pp. 257–268.

[11] P. Davis, *Methods of Numerical Integration* Second edition, Academic Press, New York, 1984.

[12] J. Huang and T. Lü, *The mechanical quadrature methods and their extrapolation for solving BIE of Steklov eigenvalue problems*, J. Comput. Math., 22 (2004), pp. 719-726.

[13] F. Chatelin, *Spectral approximation of linear operator, Academic Press*, New York, 1983.

[14] J. Huang and Z. Wang, *Extrapolation algorithms for solving mixed boundary integral equations of the Helmholtz equation by mechanical quadrature methods*, SIAM J. Sci. Comput. 31 (2009), pp. 4115–4129.

[15] J. Huang, G. Zeng, X.-M. He and Z.-C. Li,*Splitting extrapolation algorithm for first kind boundary integral equations with singularities by mechanical quadrature methods*, Adv. Comput. Math., 36 (2012), pp. 79-97.

[16] A. Sidi, *Exitension of a class of periodizing variable transformations for numerical integration*, Math. Comp. 75 (2005), pp. 327–343.

[17] K. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, Cambridge, UK, 1997.

[18] M. A. Jaswon, G. T. Symm, *Integral equation methods in potential theory and elastostatics*, Acaademic Press. Now York-San Francisco-London 1977.

[19] N. Papamicheal, G. T. Symm, *Numerical techniques for two-dimensional Laplacian problems*, Comp. Math. Appl. Mech. Eng. 6 (1975). 175-194.

[20] S. Prössdorf, G. Schmidt, *Notwendige und hinreichende Bedingungen für die Konvergenzdes Kollokationsverfahrens bei singulären Integralgleichungen*, Math. Nachr. 89 (1979), 203-215.

[21] M. Costabel and E. Stephan, *Boundary integral equations for mixed boundary value problems in polygonal domains and Galerkin approximations*, Math. Models and Meth. Mech. 15 (1985), 175-251.

[22] W. L. Wendland, E. Stephan, G. C. Hsiao, *On the integal equation method for the plane mixed boundary value problem of the Laplacian*, Math. Meth. in the Appl. Sci. 1 (1979), 265-321.

Adaptive Modified Function Projective Synchronization of Chaotic Dynamical System with Different Order

M. M. El-Dessoky[1,2], Ebraheem Alzahrani[1] and N. A. Almohammadi[1]

[1]Mathematics Department, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia.
[2]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.
E-mail: dessokym@mans.edu.eg; eoalzahrani@kau.edu.sa; nalmohammadi0010@stu.kau.edu.sa

Abstract: This work present the adaptive modified function projective synchronization of two systems with different order, which is a further extension of many existing synchronization schemes, such as function projection synchronization, modified projective synchronization and so on. Based on Lyapunov direct method of stability, an adaptive control is proposed to realize the modified function projective synchronization. Finally, numerical results are provided to illustrate the effectiveness of the obtained result.

1 Introduction

In the last few years, control and synchronization of chaos have generate much interest according to its application in secure communications [2]. Synchronization of chaotic systems means that two or more systems adjust each other to a common dynamical behavior. Up to now, many different kind of synchronization were studied such as: complete and anti synchronization, generalized synchronization, projective synchronization [9]-[39]. Recently, projective synchronization has a lot of attention because it obtain faster communication. Modifief projective synchronization is one of the important projective synchronization methods. It means that the drive and response systems could be synchronized up to constant scaling matrix [28]-[31] .Later, a new projective synchronization method called function projective synchronization where the responses of the synchronized dynamical states synchronize up to a scaling function [32]-[37]. More recently, researcher introduces a new type of synchronization phenomenon, modified function projective synchronization ,where the drive and response systems could be synchronized up to a desired scaling function matrix [38]-[39]. In recent years, most of researches for the synchronization assumed that the drive and response are identical or different systems with the same order. But in the real systems, especially in biology and social systems the synchronization is applied even though the oscillators haven't the same order. Hence, studying the synchronization of two systems with different order plays significant role in application.

The rest of this paper is as the following: The Liu chaotic and hyperchaotic dynamical systems are introduced in Section 2. Section 3 gives the definition of MFPS. In Section 4, an adaptive modified projective synchronization of Liu chaotic and hyperchaotic systems is proposed based on Lyapunov direct method of stability. Section 5 gives the numerical result and the conclusion is obtained in the last Section.

2 The Liu (chaotic and hyperchaotic) systems

The Liu hyperchaotic system is defined by:

$$\begin{cases} \dot{x} = a(y-x), \\ \dot{y} = bx + kxz + ew, \\ \dot{z} = -cz - hx^2 + mw, \\ \dot{w} = -dy, \end{cases} \tag{1}$$

where $x,\ y,\ z$ and w are the state vectors, and $a,\ b,\ c,\ d,\ e,\ k,\ h$ and m are constant parameters.It can be generate a chaotic attractor for the parameters $a = 10,\ b = 40,\ c = 2.5,\ d = 2.5,\ e = 1,\ k = 1,\ h = 4$, and $m = 1$ in Figure 1 and the chaotic motions of Liu system are illustrated in Figure 2.

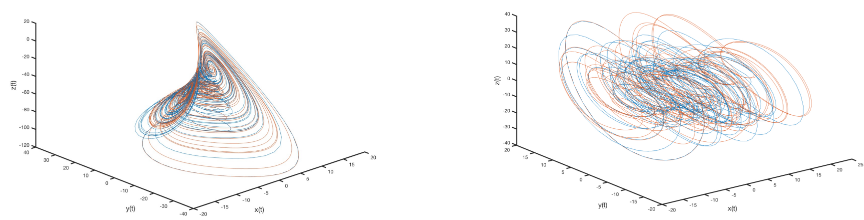


Figure 1: Liu hyperchaotic system at a=10, b=40, c=2.5, d=2.5, e=1, m=1, k=1 and h=4

The Liu chaotic system is given by:

$$\begin{cases} \dot{x} = a(y-x), \\ \dot{y} = bx - kxz, \\ \dot{z} = -cz + hx^2, \end{cases} \tag{2}$$

where $x,\ y$, and z are the state vectors, and the parameters a, b, c, h and k are positive real constants. A chaotic attractor for the parameters $a = 10,\ b = 40,\ c = 2.5,\ k = 1$ and $h = 4$ is shown in Figure 3, and the system states responses in time domain are shown in Figure 4.

3 The modified function projective synchronization scheme

We define the drive and the response systems as follows:

$$\dot{x} = \chi(x),$$
$$\dot{y} = \Psi(y) + U(t, x, y),$$

2

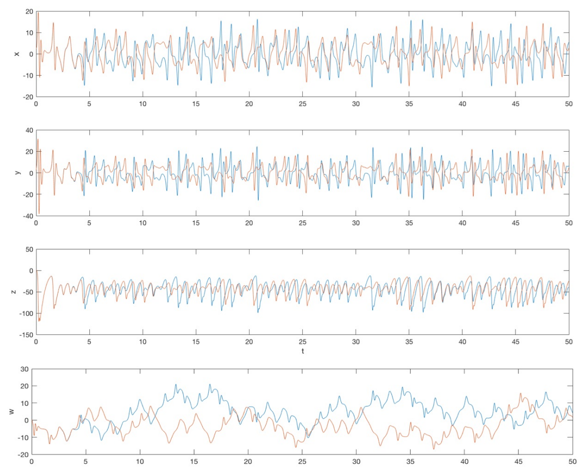


Figure 2: The behavior of the trajectories of the Liu hyper chaotic system

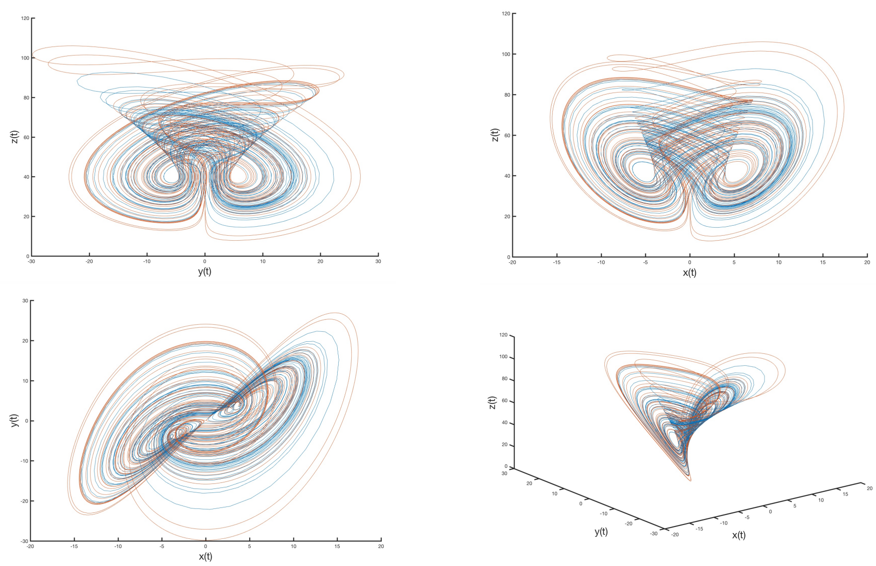


Figure 3: Phase portrait of Liu chaotic system at a=10, b=40, c=2.5, k=1 and h=4.

3

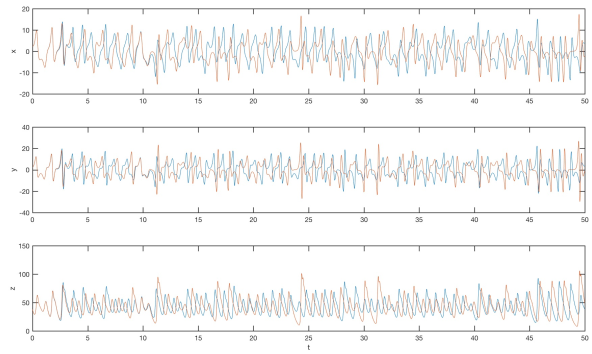


Figure 4: The behavior of the trajectories of the Liu chaotic system.

where x, y are the state variables, $\chi, \Psi : R^n \to R^n$ are continuous nonlinear functions and $U(t, x, y)$ is a control function.

Let the error state be $e = y - \Lambda(t)x$ where $\Lambda(t) = diag\{\beta_1(t), \beta_2(t), \ldots, \beta_n(t)\}$ is n-order diagonal matrix where $\beta_i = \eta_{i1}x + \eta_{i2},\ (i = 1, 2, \ldots, n),\ \eta \in R.$

Definition 1. (MFPS)
We say that the drive system and the response system are modified function projective synchronization (MFPS), if there is a scaling function $\Lambda(t)$, such that

$$\lim_{t\to+\infty} \|e\| = 0.$$

4 Modified function projective synchronization between Liu chaotic and hyperchaotic systems

Following the scheme of Zheng in [39], we apply this scheme to achieve the MFPS between Liu chaotic and hyperchaotic systems with different order. The Liu hyperchaotic system is defined below as a drive (or master) system:

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1), \\ \dot{y}_1 = bx_1 + kx_1z_1 + ew_1, \\ \dot{z}_1 = -cz_1 - hx_1^2 + mw_1, \\ \dot{w}_1 = -dy_1, \end{cases} \tag{3}$$

where $x_1,\ y_1,\ z_1$ and w_1 are the state vectors. Moreover, the Liu system as the response (or slave) system is given by::

$$\begin{cases} \dot{x}_2 = a(y_2 - x_2) + u_1, \\ \dot{y}_2 = bx_2 - kx_2z_2 + u_2, \\ \dot{z}_2 = -cz_2 + hx_2^2 + u_3, \end{cases} \tag{4}$$

where $x_2,\ y_2$ and z_2 are the state vectors, and $u_i,\ (i=1,2,3)$ are the controller to be determined later.

Since the order of the drive system is greater than the response system, we must increase the order of the response system by structure a state vector. Based on the method in [39], we structure a state variable $w_2=\frac{1}{2}x_2^2$, then the response system become:

$$\begin{cases} \dot{x}_2 = a(y_2-x_2)+u_1, \\ \dot{y}_2 = bx_2-kx_2z_2+u_2, \\ \dot{z}_2 = -cz_2+hx_2^2+u_3, \\ \dot{w}_2 = a(y_2-x_2)x_2+u_4. \end{cases} \tag{5}$$

Let the error state vector be expressed by:

$$\begin{cases} e_1 = x_2-(\eta_{11}x_1+\eta_{12})x_1, \\ e_2 = y_2-(\eta_{21}y_1+\eta_{22})y_1, \\ e_3 = z_2-(\eta_{31}z_1+\eta_{32})z_1, \\ e_4 = w_2-(\eta_{41}w_1+\eta_{42})w_1, \end{cases} \tag{6}$$

Moreover, the error dynamical system can be described by:

$$\begin{cases} \dot{e}_1 = ay_2-ax_2-2\eta_{11}ax_1y_1+2\eta_{11}ax_1^2-\eta_{21}ay_1+\eta_{12}ax_1+u_1, \\ \dot{e}_2 = bx_2-kx_2z_2-2\eta_{21}bx_1y_1-2\eta_{21}kx_1y_1z_1-2\eta_{21}ey_1w_1-\eta_{22}bx_1-\eta_{22}kx_1z_1-\eta_{22}ew_1+u_2, \\ \dot{e}_3 = -cz_2+hx_2^2+2\eta_{31}cz_1^2+2\eta_{31}hz_1x_1^2-2\eta_{31}mz_1w_1+\eta_{32}cz_1+\eta_{32}hx_1^2-\eta_{32}mw_1+u_3, \\ \dot{e}_4 = ay_2x_2-ax_2^2+2\eta_{41}dy_1w_1+\eta_{42}y_1+u_4. \end{cases} \tag{7}$$

Now, the aim is to design the control function $u_i(t),\ (i=1,2,3,4)$ to achieve the MFPS.

Consider the following Lyapunov function:

$$V=\frac{1}{2}(e_1^2+e_2^2+e_3^2+e_4^2),$$

which is a positive definite function, then the time derivative of the Lyapunov function is given as follows:

$$\dot{V}=e_1\dot{e_1}+e_2\dot{e_2}+e_3\dot{e_3}+e_4\dot{e_4}.$$

Moreover,

$$\begin{aligned} \dot{V} &= e_1(ay_2-ax_2-2\eta_{11}ax_1y_1+2\eta_{11}ax_1^2-\eta_{12}ay_1+\eta_{12}ax_1+u_1), \\ &+e_2(bx_2-kx_2z_2-2\eta_{21}bx_1y_1-2\eta_{21}kx_1y_1z_1-2\eta_{21}ey_1w_1-\eta_{22}bx_1-\eta_{22}kx_1z_1-\eta_{22}ew_1+u_2), \\ &+e_3(-cz_2+hx_2^2+2\eta_{31}cz_1^2+2\eta_{31}hz_1x_1^2-2\eta_{31}mz_1w_1+\eta_{32}cz_1+\eta_{32}hx_1^2-\eta_{32}mw_1+u_3), \\ &+e_4(ay_2x_2-ax_2^2+2\eta_{41}dy_1w_1+\eta_{42}y_1+u_4). \end{aligned} \tag{8}$$

Thus, we choose the controller as the following:

$$\begin{cases} u_1 = -ay_2 + 2\eta_{11}ax_1y_1 - \eta_{11}ax_1^2 + \eta_{12}ay_1, \\ u_2 = -bx_2 + kx_2z_2 + 2\eta_{21}bx_1y_1 + 2\eta_{21}kx_1y_1z_1 + 2\eta_{21}ey_1w_1 + \eta_{22}bx_1 + \eta_{22}kx_1z_1 \\ \quad + \eta_{22}ew_1 - by_2 + \eta_{21}by_1^2 + \eta_{22}by_1, \\ u_3 = -hx_2^2 - \eta_{31}cz_1^2 - 2\eta_{31}hz_1x_1^2 + 2\eta_{31}mz_1w_1 - \eta_{32}hx_1^2 + \eta_{32}mw_1, \\ u_4 = -ay_2x_2 + ax_2^2 - 2\eta_{41}dy_1w_1 - \eta_{42}y_1 - dw_2 + d\eta_{41}w_1^2 + d\eta_{42}w_1, \end{cases} \tag{9}$$

by this choice, the time derivative of Lyapunov function is:

$$\begin{aligned} \dot{V} &= e_1(-ax_2 + \eta_{11}ax_1^2 + \eta_{12}ax_1) + e_2(-by_2 + \eta_{21}by_1^2 + \eta_{22}by_1) \\ &+ e_3(-cz_2 + \eta_{31}cz_1^2 + \eta_{32}cz_1) + e_4(-dw_2 + \eta_{41}dw_1^2 + \eta_{42}dw_1), \\ &= -(ae_1^2 + be_2^2 + ce_3^2 + de_4^2), \\ &= -e^TPe, \end{aligned} \tag{10}$$

where $P = diag[a,\ b,\ c,\ d]$.

Obviously, the origin of the error dynamical system is asymptotically stable since $\dot{V}$ is negative definite. Thus, the drive and the response systems are achieving the MFPS.

5 Numerical results

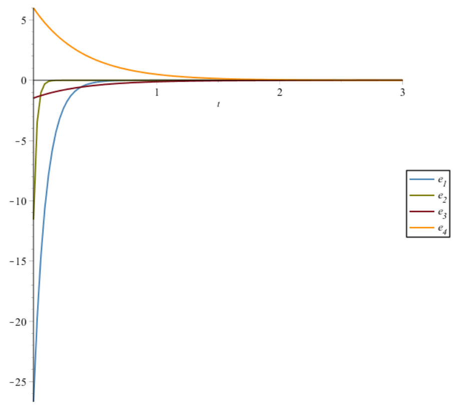
In this section, we show a numerical simulation to verify the influence of the synchronization controller (9). We assume that the initial states of the drive and the response systems are

$[x_1(0),\ y_1(0),\ z_1(0),\ w_1(0)]^T = [2.4,\ 2.2,\ 0.8,\ 0]^T$ and $[x_2(0),\ y_2(0),\ z_2(0),\ w_2(0)]^T = [0.2,\ 0.1,\ 3,\ 6]^T$.

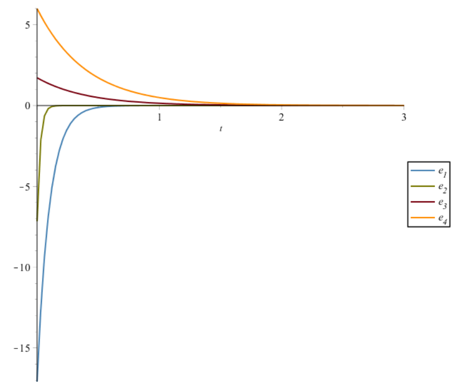
These numerical simulation are presented in Figure 5. Firstly, when the scaling functions are given by:

$$\beta_1 = 3x_1 + 4,\ \beta_2 = 1.5y_1 + 2,\ \beta_3 = 2z_1 + 4 \text{ and } \beta_4 = w_1 + 7,$$

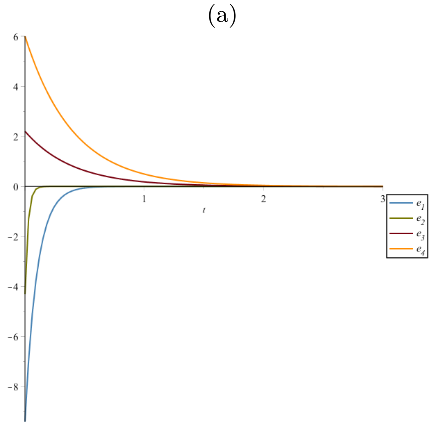
we get adaptive modified function projective synchronization (MFPS) in Figure 5 (a) . Furthermore, Figure 5 (b) shows the generalized function projective synchronization (GFPS) when the scaling functions are given by $\beta_1 = 3x_1,\ \beta_2 = 1.5y_1,\ \beta_3 = 2z_1$, and $\beta_4 = w_1$. Also, we get the modified projective synchronization (MPS) according to the constants $\beta_1 = 4,\ \beta_2 = 2,\ \beta_3 = 1$, and $\beta_4 = 7$ shown in Figure 5 (c) . The complete synchronization error of the drive and response systems are displayed in Figure 5 (d) when the scaling function is simplified to $\beta_i = +1,\ (i = 1, 2, 3, 4)$ with $\eta_{i1} = 0,\ \eta_{i2} = +1,\ (i = 1, 2, 3, 4)$. Finally, if we choose the scaling function $\beta_i = -1, (i = 1, 2, 3, 4)$ in which $\eta_{i1} = 0$, and $\eta_{i2} = -1,\ (i = 1, 2, 3, 4)$ we gained the anti-phase synchronization between the two systems in Figure 5 (e). From these results, they clearly show that the synchronization errors $e = [e_1, e_2, e_3, e_4]^T$ are converge to zero as time goes to infinity.



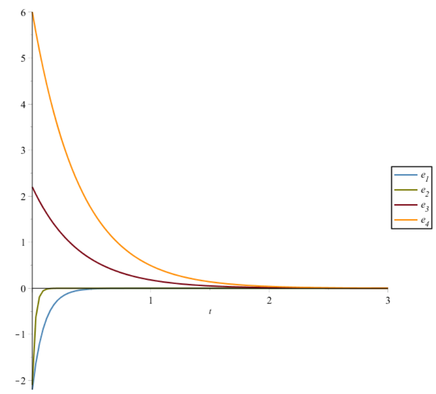
(a)



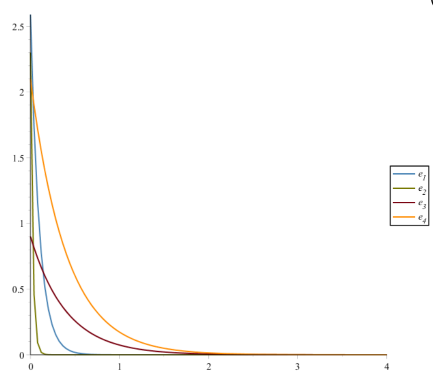
(b)



(c)



(d)



(e)

Figure 5: The errors between Liu (chaotic and hyperchaotic) systems for (a) MFPS (b) GFPS (c) MPS (d) Complete synchronization (e) Anti-phase synchronization.

7

6 Conclusion

In this paper, we have introduced a modified function projective synchronization between two chaotic systems with different dimensional. The Liu chaotic system (third order) and Liu hyperchaotic system (fourth order) are chosen to illustrate the proposed technique. The results show that we can apply the MFPS between the two systems if we increased the order. By using adaptive control method, some conditions are derived for the stability of the error proved according to Lyapunov direct method of stability. Finally, the graphical presentation of the numerical results with error states tending to zero as time becomes large, clearly exhibit that the applied adaptive control method is effective and convenient to achieve global synchronization among non identical chaotic systems with different order.

Acknowledgement

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

References

[1] L. Xin, *A new hyperchaotic dynamical system*, Chinese J. Phys., Vol 16 (11), (2007), 3279-3284.

[2] X. Xu, *Generalized function projective synchronization of chaotic systems for secure communication*, Advances in Signal Processing , Vol.2011 (1), (2011), 6180-6187.

[3] G. Chen, *Chaos on some controllability conditions for chaotic dynamics control*, Chaos,Solitons and Fractals, Vol.8 (9), (1997), 1461-1470.

[4] E. Ott, C. Grebogi and J. Yorke, *Controlling chaos*, Physical Review Letters, Vol.64 (11), (1999), 1179-1184.

[5] H. N. Agiza, *On the analysis of stability, bifurcation, chaos and chaos control of kopel map*, Chaos,Solitons and Fractals, Vol.10 (11), (1999), 1909-1916.

[6] A. Hegazi, H. N. Agiza and M. M. El-Dessoky, *Controlling chaotic behaviour for spin generator and Rossler dynamical systems with feedback control*, Chaos, Solitons and Fractals, Vol.12 (4), (2001), 631-658.

[7] S. Dadras and H. Momeni, *Control of a fractional-order economical system via sliding mode*, Physica A, Vol.389 (12), (2010), 2434-2442

[8] A. Singh and S. Gakkhar, *Controlling chaos in a food chain model*, Mathematics and Computers in Simulation, Vol.115 (C), (2015), 24-36.

[9] L. M. Pecora and T. L. Carroll, *Synchronization in chaotic systems*, Physical Review Letters,Vol. 64 (8), (1990), 821-824.

[10] T. L. Carroll and L. M. Perora, *Synchronizing a chaotic systems*, IEEE Transactions on circuits and systems,Vol.38 (4), (1991), 453-456.

[11] N. Rulkov, M. Sushchik, L. Tsimring and H. Abarbanel, *Generalized synchronization of chaos in directionally coupled chaotic systems*, Physical Review Letters, Vol.51 (2), (1995), 980- 994.

[12] S. Yang and C. Duan, *Generalized synchronization in chaotic systems*, Chaos, Solitons and Fractals, Vol.9 (10), (1998), 1703-1707.

[13] X. Yang, *A framework for synchronization theory* Chaos, Solitons and Fractals, Vol.11 (9), (2000), 1365-1368.

[14] E. Bai and K. Lonngren, *Sequential synchronization of two Lorenz system using active control*, Chaos, Solitons and Fractals, Vol.11 (7), (2000), 1041-1044.

[15] E. M. Elabbasy, H. N. Agiza and M. M. El-Dessoky, *Global chaos synchronization for four-scroll attractor by nonlinear control*, Scientific Research and Essay, Vol.1 (3), (2006), 65-71.

[16] E. M. Elabbasy and M. M. El-Dessoky, *Adaptive coupled synchronization of coupled chaotic dynamical systems*, Applied Sciences Research, Vol.2 (2), (2007), 88-102.

[17] E. M. Elabbasy and M. M. El-Dessoky, *Synchronization of Van Der Pol oscillator and chen chaotic dynamical system*, Chaos,Solitons and Fractals, Vol.36 (5), (2008), 1425-1435.

[18] J. Huang, *Adaptive synchronization between different hyperchaotic systems with fully uncertain parameters* Physics Letters A, Vol.372 (27-28), (2008), 4799-4804.

[19] G. Li, *Generalized synchronization of chaos based on suitable separation*, Chaos, Solitons and Fractals, Vol.39 (5), (2009), 2056-2062.

[20] A. Loria, *Master-slave synchronization of fourth order Lu chaotic oscillators via linear output feadback* IEEE Transactions on circuits and systems, Vol.57 (3), (2010), 213-217.

[21] M. M. El-Dessoky and M. T. Yassen, *Adaptive feedback control for chaos control and synchronization for new chaotic dynamical system*, Mathematical Problems in Engineering, Vol. 2012, (2012), Article ID 347210, 12 pages, doi:10.1155/2012/347210

[22] C. H. Yang and C. L. Wu, *Nonlinear dynamic analysis and synchronization of four-dimensional Lorenz-Stenflo system and its circuit experimental implementation*, Abstract and Applied Analysis, Vol. 2014, (2014), Article ID 213694, 17 pages.

[23] K. Vishal and S. Agrawal, *On the dynamics, existence of chaos, control and synchronization of a novel complex chaotic system*, Chinese Journal of Physics, Vol.55 (2), (2017), 519-532.

[24] J. Petereit and A. Pikovsky, *Chaos synchronization by nonlinear coupling*, Communications in Nonlinear Science and Numerical Simulation, Vol.44 (C), (2017), 344-351.

[25] K. Ojo, S. Ogunjo and A. Olagundoye, *Projective synchronization via active control of identical chaotic oscillators with parametric and external excitation*, International Journal of Nonlinear Science, Vol.24 (2), (2017), 76-83.

[26] M. M. El-Dessoky, *Synchronization and anti-synchronization of a hyperchaotic Chen system*, Chaos, Solitons and Fractals, Vol.39 (4), (2009), 1790-1797.

[27] M. M. El-Dessoky, *Anti-synchronization of four scroll attractor with fully unknown parameters*, Nonlinear Analysis: Real World Applications, Vol. 11 (2), (2010), 778-783.

[28] G. Li, *Modified projective synchronization of chaotic system*, Chaos, Solitons and Fractals, Vol.32 (5), (2007), 1786-1790.

[29] J. Park, *Adaptive modified projective synchronization of a unified chaotic system with an uncertain parameter*, Chaos, Solitons and Fractals, Vol.34 (5), (2007), 1552-1559.

[30] Y. Tang and J. Fang, *General method for modified projective synchronization of hyperchaotic systems with known or unknown parameter*, Physics Letters A, Vol.372 (11), (2008), 1816-1826.

[31] N. Cai, Y. Jing and S. Zhang, *Modified projective synchronization of chaotic systems with disturbances via active sliding mode control*, Communications in Nonlinear Science and Numerical Simulation, Vol.15 (6), (2010), 1613-1620.

[32] Y. Chen and X. Li, *Function projective synchronization between two identical chaotic systems*, International Journal of Modern Physics C, Vol.18 (5), (2007), 883-888.

[33] H. Du, Q. Zeng and C. Wang, *Function projective synchronization of different chaotic systems with uncertain parameters*, Physics Letters A, Vol.372 (33), (2008), 5402-5410.

[34] L. Runzi and W. Zhengmin, *Adaptive function projective synchronization of unified chaotic systems with uncertain parameters*, Chaos, Solitons and Fractals, Vol.42 (2), (2009), 1266-1272.

[35] Y. Yua and H. Li, *Adaptive generalized function projective synchronization of uncertain chaotic systems*, Nonlinear Analysis: Real World Applications, Vol.11 (4), (2010), 2456-2464.

[36] S. K. Agrawal and S. Das, *Function projective synchronization between four dimensional chaotic systems with uncertain parameters using modified adaptive control method*, Journal of process Control, Vol.24 (5), (2014), 517-530.

[37] M. M. El-Dessoky, E. O. Alzahrany, and N. A. Almohammadi. *Function Projective Synchronization for Four Scroll Attractor by Nonlinear Control*, Applied Mathematical Sciences, Vol.11 (26), (2017), 1247-1259.

[38] S. Zheng, G. Dong and Q. Bi, *Adaptive modified function projective synchronization of hyperchaotic systems with unknown parameters*, Communications in Nonlinear Science and Numerical Simulation, Vol.15 (11), (2010), 3547-3556.

[39] S. Zheng, *Adaptive modified function projective synchronization of unknown chaotic systems with different order*, Applied Mathematics and Computation, Vol.218 (10), (2011), 5891-5899.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.2, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Dual log-Minkowski inequality for star bodies

Tongyi Ma

(School of Mathematics and Statistics, Hexi University,

Zhangye, Gansu 734000, P.R.China)

Abstract

We validate a modified dual log-Minkowski inequality and prove some variants of the dual log-Minkowski inequality for star bodies in $\mathbb{R}^n$ containing the origin in their interior. In addition, we point out that the equivalence between the dual log-Minkowski inequality and the dual log-Brunn-Minkowski inequality.

Keywords: Dual cone-volume measure, L_0-Minkowski problem, dual log-Brunn-Minkowski inequality, dual log-Minkowski inequality.

1 Introduction

The classical Brunn-Minkowski theory of convex bodies was placed in a larger theory by Lutwak's L_p-Minkowski problem [13, 14]. Therefore, many classical results for convex bodies became a part of the extended L_p-Brunn-Minkowski-Firey theory, while many other results of the extended theory bring new and original insight in convex geometric analysis.

One such strikingly new behavior is due to the log-Brunn-Minkowski inequality [2]. That is, let K, L be convex bodies that contain the origin in their interiors and $0 \leq \lambda \leq 1$, the log-Minkowski combination which is defined by

$$(1-\lambda) \cdot K +_0 \lambda \cdot L = \cap_{u \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \leq h_K(u)^{1-\lambda} h_L(u)^{\lambda}\}, \tag{1.1}$$

where, $x \cdot u$ denotes the standard inner product of x and u in $\mathbb{R}^n$, $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$ and h_K denotes the support function of convex body. Böröczky, Lutwak, Yang and Zhang [2] conjectured that for origin-symmetric convex bodies K and L in $\mathbb{R}^n$ with $0 \leq \lambda \leq 1$,

$$\mathrm{vol}_n((1-\lambda) \cdot K +_0 \lambda \cdot L) \geq \mathrm{vol}_n(K)^{1-\lambda} \mathrm{vol}_n(L)^{\lambda}, \tag{1.2}$$

where $\mathrm{vol}_n(\cdot)$ denotes the n-dimensional volume of body in $\mathbb{R}^n$. They call (1.2) as the log-Brunn-Minkowski inequality. Note that while the inequality (1.2) is not true for general convex bodies, it implies the classical Brunn-Minkowski inequality for origin-symmetric convex bodies. In [2], Böröczky, et al. proved the inequality (1.2) when $n = 2$ and showed that (1.2) is equivalent to the logarithmic Minkowski inequality (log-Minkowski inequality) for all n, that is

$$\int_{\mathbb{S}^{n-1}} \ln\left(\frac{h_K(u)}{h_L(u)}\right) \mathrm{d}\bar{v}_L(u) \geq \frac{1}{n} \ln\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right), \tag{1.3}$$

AMS Subject Classification: 52A30, 52A40.

This work was Supported by the National Natural Science Foundation of China(Grant No.11561020) and was partly supported by the National Natural Science Foundation of China (Grant No.11371224).

Email: matongyi@126.com, matongyi_123@163.com (Tongyi Ma).

where $\mathrm{d}v_L(u) = \frac{1}{n}h_L(u)\mathrm{d}S_L(u)$ is the cone-volume measure of L, $\mathrm{d}\bar{v}_L(u) = \frac{1}{\mathrm{vol}_n(L)}\,\mathrm{d}v_L(u)$ and S_L is surface area measure of L on $\mathbb{S}^{n-1}$.

In [23], Stancu proved some variants of the log-Minkowski inequality for general convex bodies without the symmetry assumption.

The dual L_p-Brunn-Minkowski theory for star bodies developed by Lutwak [15, 16] and received considerable attention, see [1, 4, 6, 10, 11, 17, 20, 21, 22, 25]. Recently, Gardner, et al. [7] established dual log-Minkowski inequality as follows. If K and L be star bodies in $\mathbb{R}^n$ containing the origin in their interior, then

$$\int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_K(u) \leq \frac{1}{n}\ln\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right), \tag{1.4}$$

with equality if and only if K and L are dilatates, where $\mathrm{d}\overline{\widetilde{v}}_K$ is the dual cone-volume probability measure of K (see definition (2.11)). In the present paper, we prove a modified dual log-Minkowski inequality and obtain the double dual log-Minkowski inequality through the Gibbs' inequality. Secondly, we prove an analogue of the dual log-Minkowski inequality. In addition, we point out the equivalence between the dual log-Minkowski inequality and the dual log-Brunn-Minkowski inequality.

Our first result is the following dual log-Minkowski inequality:

Theorem 1.1. *Let K and L be star bodies in $\mathbb{R}^n$ containing the origin in their interior. Then*

$$\int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_{-1}(K,L;u) \geq \ln\left(\frac{\widetilde{V}_{-1}(K,L)}{\mathrm{vol}_n(K)}\right) \geq \frac{1}{n}\ln\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right), \tag{1.5}$$

with equality if and only if K and L are dilates, where $\mathrm{d}\widetilde{v}_{-1}(K,L;\cdot)$ is the dual mixed volume measure $\widetilde{V}_{-1}(K,L) = \int_{\mathbb{S}^{n-1}} \mathrm{d}\widetilde{v}_{-1}(K,L;u)$ and $\mathrm{d}\overline{\widetilde{v}}_{-1}(K,L;u) = \frac{1}{\widetilde{V}_{-1}(K,L)}\,\mathrm{d}\widetilde{v}_{-1}(K,L;u)$.

Secondly, we obtain the following double log-Minkowski inequality.

Theorem 1.2. *Let K and L be star bodies in $\mathbb{R}^n$ containing the origin in their interior. Then*

$$\int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_K(u) \leq \ln\left(\frac{\widetilde{V}_{-1}(K,L)}{\mathrm{vol}_n(K)}\right) \leq \int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_{-1}(K,L;u), \tag{1.6}$$

with equality in inequality if and only if K and L are dilates.

Further, we prove an analogue of the dual log-Minkowski inequality. In what follows, we will denote

$$\left(\frac{\rho_K}{\rho_L}\right)_{\text{average}} = \frac{\int_{\mathbb{S}^{n-1}} \frac{\rho_K(u)}{\rho_L(u)}\mathrm{d}\widetilde{v}_K(u)}{\int_{\mathbb{S}^{n-1}} \mathrm{d}\widetilde{v}_K(u)},$$

$$\left(\frac{\rho_K}{\rho_L}\right)_{\max} = \max_{u\in\mathbb{S}^{n-1}}\left(\frac{\rho_K(u)}{\rho_L(u)}\right) \text{ and } \left(\frac{\rho_K}{\rho_L}\right)_{\min} = \min_{u\in\mathbb{S}^{n-1}}\left(\frac{\rho_K(u)}{\rho_L(u)}\right).$$

Theorem 1.3. *Let K and L be star bodies in $\mathbb{R}^n$ containing the origin in their interior with $L \subseteq K$. Then*

$$\int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_K(u) \geq \frac{\left(\frac{\rho_K}{\rho_L}\right)_{\text{average}}}{\left(\frac{\rho_K}{\rho_L}\right)_{\max}} \cdot \frac{1}{n}\ln\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right), \tag{1.7}$$

with equality if and only if $K = \lambda L$, where $0 < \lambda \leq 1$.

In general, if $K, L \in \mathcal{S}_o^n$, then

$$\int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right) \mathrm{d}\overline{\widetilde{v}}_K(u) \geq \frac{\left(\frac{\rho_K}{\rho_L}\right)_{\text{average}}}{\left(\frac{\rho_K}{\rho_L}\right)_{\max}} \cdot \frac{1}{n} \ln\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right)$$
$$+ \ln\left[\left(\frac{\rho_K}{\rho_L}\right)_{\min}\right] \cdot \left[1 - \frac{\left(\frac{\rho_K}{\rho_L}\right)_{\text{average}}}{\left(\frac{\rho_K}{\rho_L}\right)_{\max}}\right], \tag{1.8}$$

with equality if and only if K is homothetic to L.

Finally, we point out the equivalence between the dual log-Minkowski inequality (1.4) and the dual log-Brunn-Minkowski inequality (2.8). We give a different proof with Wang and Liu [24].

2 Notation and preliminaries

The support function $h_K : \mathbb{R}^n \to \mathbb{R}$, of a compact, convex set $K \subset \mathbb{R}^n$ is defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max(x \cdot y : y \in K), \tag{2.1}$$

and uniquely determines the convex set. Let $\mathcal{K}_o^n$ be the set of convex bodies in $\mathbb{R}^n$ containing the origin in their interior.

If L is a compact star-shaped (about the origin) in $\mathbb{R}^n$, its radial function, $\rho_L = \rho(L, \cdot) : \mathbb{R}^n \backslash \{o\} \to [0, +\infty)$, is defined by

$$\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in L\}, \quad x \in \mathbb{R}^n \backslash \{o\}. \tag{2.2}$$

If ρ_L is positive and continuous, then L will be called a star body (about the origin). Let $\mathcal{S}_o^n$ denotes the set of star bodies in $\mathbb{R}^n$ containing the origin in their interior. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in \mathbb{S}^{n-1}$. Obviously, for a pair $K, L \in \mathcal{S}_o^n$, we have

$$\rho_K \leq \rho_L, \quad \text{if and only if,} \quad K \subseteq L. \tag{2.3}$$

If $K, L \in \mathcal{S}_o^n$ and $\lambda, \mu \geq 0$ (not both zero), then, for $p \geq 1$, the harmonic L_p-combination, $\lambda \diamond K \widehat{+}_p \mu \diamond L \in \mathcal{S}_o^n$ is defined by (see [14])

$$\rho(\lambda \diamond K \widehat{+}_p \mu \diamond L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}. \tag{2.4}$$

For $p \geq 1$ and $K, L \in \mathcal{S}_o^n$, the dual mixed volume, $\widetilde{V}_{-p}(K, L)$, is defined

$$-\frac{n}{p} \widetilde{V}_{-p}(K, L) = \lim_{\varepsilon \to 0} \frac{\mathrm{vol}_n(K \widehat{+}_p \varepsilon \diamond L) - \mathrm{vol}_n(K)}{\varepsilon}.$$

The following integral representation for the dual mixed volume $\widetilde{V}_{-p}$ is obtained (see [14]): If $p \geq 1$ and $K, L \in \mathcal{S}_o^n$, then

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho(K, u)^{n+p} \rho(L, u)^{-p} \mathrm{d}S(u),$$

where $\mathrm{d}S$ is the spherical Lebesgue measure on $\mathbb{S}^{n-1}$. This integral representation, together the Hölder inequality with the polar coordinate formula, immediately gives the dual L_p-Minkowski inequality: If $p \geq 1$ and $K, L \in \mathcal{S}_o^n$, then

$$\widetilde{V}_{-p}(K, L)^n \geq \mathrm{vol}_n(K)^{n+p} \mathrm{vol}_n(L)^{-p}, \tag{2.5}$$

with equality if and only if K and L are dilates.

Using the dual L_p-Minkowski inequality, we can obtain the following dual L_p-Brunn-Minkowski inequality (see [14]). Suppose $K, L \in \mathcal{S}_o^n$, $\lambda, \mu > 0$ and $p \geq 1$, then

$$\mathrm{vol}_n(\lambda \diamond K \widehat{+}_p \mu \diamond L)^{-p/n} \geq \lambda \mathrm{vol}_n(K)^{-p/n} + \mu \mathrm{vol}_n(L)^{-p/n}, \tag{2.6}$$

with equality if and only if K and L are dilates.

Note that definition (2.4) makes sense for all $p > 0$. The case $p = 0$ is the limiting case given by

$$\rho((1-\lambda) \diamond K \hat{+}_0 \lambda \diamond L, \cdot) = \rho(K, \cdot)^{1-\lambda} \rho(L, \cdot)^{\lambda}, \quad 0 \leq \lambda \leq 1, \tag{2.7}$$

it is called the radial log-Minkowski-combination.

Similarly, the inequality (2.6) makes sense for all $p > 0$. The case $p = 0$ is the limiting case given by an dual log-Brunn-Minkowski inequality. Namely, if $K, L \in \mathcal{S}_o^n$, then for all $\lambda \in [0, 1]$,

$$\mathrm{vol}_n((1-\lambda) \diamond K \hat{+}_0 \lambda \diamond L) \leq \mathrm{vol}_n(K)^{1-\lambda} \mathrm{vol}_n(L)^{\lambda}, \tag{2.8}$$

with equality if and only if K and L are dilates.

If $K \in \mathcal{S}_o^n$, then

$$\mathrm{d}\widetilde{v}_K(u) = \frac{1}{n} \rho_K^n(u) \mathrm{d}S(u) \tag{2.9}$$

is the dual cone-volume measure of K and

$$\mathrm{d}\widetilde{v}_{-1}(K, L; u) = \frac{1}{n} \rho_K^{n+1}(u) \rho_L^{-1}(u) \mathrm{d}S(u) \tag{2.10}$$

is the dual mixed volume measure with $(n+1)$ copies of K and (-1) copies of L. Note that we usually write $\widetilde{V}_{-1}(K, L) = \int_{\mathbb{S}^{n-1}} \mathrm{d}\widetilde{v}_{-1}(K, L; u)$. The dual cone-volume measure of a star body K in $\mathbb{R}^n$ with $\mathrm{vol}_n(K)$ is the Borel probability measure $\overline{\widetilde{v}}_K$ in $\mathbb{S}^{n-1}$ defined by

$$\mathrm{d}\overline{\widetilde{v}}_K = \frac{\rho_K^n(u)}{n \mathrm{vol}_n(K)} \mathrm{d}S(u). \tag{2.11}$$

And the normalized dual mixed cone measure of a star bodies K, L in $\mathbb{R}^n$ with $\widetilde{V}_{-1}(K, L)$ is the Borel probability measure $\overline{\widetilde{v}}_{-1}(K, L; \cdot)$ on $\mathbb{S}^{n-1}$ defined by

$$\mathrm{d}\overline{\widetilde{v}}_{-1}(K, L; u) = \frac{1}{\widetilde{V}_{-1}(K, L)} \mathrm{d}\widetilde{v}_{-1}(K, L; u). \tag{2.12}$$

3 Proofs of dual log-Minkowski type results

In this section, we will prove the theorems mentioned in Section 1.

Proof of Theorem 1.1. Consider the function $G_{K,L}(p) : [1, \infty] \to \mathbb{R}$ defined by

$$G_{K,L}(p) = \frac{1}{\widetilde{V}_{-1}(K, L)} \int_{\mathbb{S}^{n-1}} \left(\frac{\rho_K(u)}{\rho_L(u)} \right)^{\frac{p}{n+p}} \mathrm{d}\widetilde{v}_K(u).$$

Through using L'Hôpital's rule, we obtain

$$
\begin{aligned}
\lim_{p\to\infty} \ln(G_{K,L}(p))^{n+p} &= \lim_{p\to\infty} \ln\left(\frac{\int_{\mathbb{S}^{n-1}}\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^{\frac{p}{n+p}} \mathrm{d}\widetilde{v}_K(u)}{\widetilde{V}_{-1}(K,L)}\right)^{n+p} \\
&= \lim_{p\to\infty} \ln\left(\frac{\int_{\mathbb{S}^{n-1}}\rho_K(u)^{n+1}\rho_L(u)^{-1}\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^{-\frac{n}{n+p}} \mathrm{d}S(u)}{\int_{\mathbb{S}^{n-1}}\rho_K(u)^{n+1}\rho_L(u)^{-1}\mathrm{d}S(u)}\right)^{n+p} \\
&= \ln\exp\left(\frac{\int_{\mathbb{S}^{n-1}}\rho_K(u)^{n+1}\rho_L(u)^{-1}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^{-n} \mathrm{d}S(u)}{\int_{\mathbb{S}^{n-1}}\rho_K(u)^{n+1}\rho_L(u)^{-1}\mathrm{d}S(u)}\right) \\
&= \frac{\int_{\mathbb{S}^{n-1}}\rho_K(u)^{n+1}\rho_L(u)^{-1}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^{-n} \mathrm{d}S(u)}{\int_{\mathbb{S}^{n-1}}\rho_K(u)^{n+1}\rho_L(u)^{-1}\mathrm{d}S(u)} \\
&= -\frac{n}{\widetilde{V}_{-1}(K,L)}\int_{\mathbb{S}^{n-1}}\frac{\rho_K(u)}{\rho_L(u)}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\widetilde{v}_K(u).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\exp\left[-\frac{n}{\widetilde{V}_{-1}(K,L)}\int_{\mathbb{S}^{n-1}}\frac{\rho_K(u)}{\rho_L(u)}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\widetilde{v}_K(u)\right] \\
= &\lim_{p\to\infty}\left[\frac{1}{\widetilde{V}_{-1}(K,L)}\int_{\mathbb{S}^{n-1}}\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^{\frac{p}{p+n}}\mathrm{d}\widetilde{v}_K(u)\right]^{p+n}, \qquad (3.1)
\end{aligned}
$$

and it follows from Hölder's inequality that

$$
\begin{aligned}
&\left(\int_{\mathbb{S}^{n-1}}\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^{\frac{p}{p+n}}\mathrm{d}\widetilde{v}_K(u)\right)^{\frac{p+n}{p}}\left(\int_{\mathbb{S}^{n-1}}\mathrm{d}\widetilde{v}_K(u)\right)^{-\frac{n}{p}} \\
\leq &\int_{\mathbb{S}^{n-1}}\frac{\rho_K(u)}{\rho_L(u)}\mathrm{d}\widetilde{v}_K(u) = \widetilde{V}_{-1}(K,L). \qquad (3.2)
\end{aligned}
$$

Note that $\int_{\mathbb{S}^{n-1}}\mathrm{d}\widetilde{v}_K(u) = \mathrm{vol}_n(K)$, (2.10) and (2.12), together (3.1) with (3.2), we have

$$
\int_{\mathbb{S}^{n-1}}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_{-1}(K,L;u) \geq \ln\left(\frac{\widetilde{V}_{-1}(K,L)}{\mathrm{vol}_n(K)}\right).
$$

According to the condition of equality in Hölder's inequality, we easily see that with equality in the above inequality if and only if K and L are dilates.

Using dual Minkowski's inequality (2.5), we have the second inequality in the theorem, this is,

$$
\int_{\mathbb{S}^{n-1}}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_{-1}(K,L;u) \geq \ln\left(\frac{\widetilde{V}_{-1}(K,L)}{\mathrm{vol}_n(K)}\right) \geq \frac{1}{n}\ln\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right). \qquad (3.3)
$$

From the condition of equality in dual Minkowski's inequality, we know that with equality if and only if K and L are dilates. Which completes the proof of the theorem. □

Remark 3.1. *Our first inequality in* (1.5) *can be written as*

$$
\int_{\mathbb{S}^{n-1}}\frac{\rho_K(u)}{\rho_L(u)}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_K(u) \geq \frac{\widetilde{V}_{-1}(K,L)}{\mathrm{vol}_n(K)}\ln\left(\frac{\widetilde{V}_{-1}(K,L)}{\mathrm{vol}_n(K)}\right). \qquad (3.4)
$$

Use dual Minkowski's inequality in (3.4), *we have*

$$
\int_{\mathbb{S}^{n-1}}\frac{\rho_K(u)}{\rho_L(u)}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_K(u) \geq \frac{1}{n}\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right)^{1/n}\ln\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right). \qquad (3.5)
$$

Proof of Theorem 1.2. We consider Gibbs' inequality from information theory (see [3], (8.57), p. 252-253): If p and q are probability density functions on a measure space (X,ν), then

$$\int p\ln p\mathrm{d}\nu \geq \int p\ln q\mathrm{d}\nu, \tag{3.6}$$

with equality if and only if $p=q$ almost everywhere (a.e.).

By taking

$$p\mathrm{d}\nu=\frac{\rho_L(u)}{\rho_K(u)}\cdot\frac{1}{\mathrm{vol}_n(K)}\mathrm{d}\widetilde{v}_{-1}(K,L;u) \quad \text{and} \quad q\mathrm{d}\nu=\frac{1}{\widetilde{V}_{-1}(K,L)}\mathrm{d}\widetilde{v}_{-1}(K,L;u)$$

(and later reversing the two measures above so that the first is $q\mathrm{d}\nu$ and the second is $p\mathrm{d}\nu$), we obtain the double inequality as follows.

$$\int_{\mathbb{S}^{n-1}}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_K(u)\leq\ln\left(\frac{\widetilde{V}_{-1}(K,L)}{\mathrm{vol}_n(K)}\right)\leq\int_{\mathbb{S}^{n-1}}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_{-1}(K,L;u). \tag{3.7}$$

According to the condition of equality in Gibbs' inequality (3.6), we obtain that with equality in inequality (3.7) if and only if

$$\frac{\rho_L(u)}{\rho_K(u)}=\frac{\mathrm{vol}_n(K)}{\widetilde{V}_{-1}(K,L)}\Longleftrightarrow\frac{1}{n}\rho_L^n(u)=\frac{1}{n}\left(\frac{\mathrm{vol}_n(K)}{\widetilde{V}_{-1}(K,L)}\right)^n\rho_K^n(u)$$

almost everywhere (a.e.) on $\mathbb{S}^{n-1}$. Integrating both sides of the last equation over $\mathbb{S}^{n-1}$ with the sphere Lebesgue measure $\mathrm{d}S(u)$, we get

$$\frac{\mathrm{vol}_n(L)}{\mathrm{vol}_n(K)}=\left(\frac{\mathrm{vol}_n(K)}{\widetilde{V}_{-1}(K,L)}\right)^n.$$

From the condition of equality in the dual L_p-Minkowski inequality (2.5) ($p=1$), we see that with equality in inequality (3.7) if and only if K and L are dilates. □

Remark 3.2. *The proof of Theorem 1.2 can be seen that we provide a new proof for the dual Minkowski inequality itself. In fact, it is consistent with the idea of splitting mentioned by Gardner, Hug and Weil in [8] and [9].*

A natural idea is to give a proof of the dual log-Minkowski inequality similar to the proof of the Theorem 1.1. However, as such, we obtain again the left-hand side inequality of (1.6) due to the following lemma:

Lemma 3.3. *Let $K,L\in\mathcal{S}_o^n$, then*

$$\begin{aligned}&\exp\left(\int_{\mathbb{S}^{n-1}}\ln\frac{\rho_K(u)}{\rho_L(u)}\mathrm{d}\overline{\widetilde{v}}_K(u)\right)\\ &=\lim_{p\to\infty}\left(\frac{1}{\mathrm{vol}_n(K)}\int_{\mathbb{S}^{n-1}}\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^{\frac{1}{p+n}}\mathrm{d}\widetilde{v}_K(u)\right)^{p+n}.\end{aligned} \tag{3.8}$$

The proof follows the same idea used in deriving (3.1).

From Hölder's inequality, we have

$$\begin{aligned}&\left(\int_{\mathbb{S}^{n-1}}\left(\frac{\rho_K(u)}{\rho_L(u)}\right)^{\frac{1}{p+n}}\mathrm{d}\widetilde{v}_K(u)\right)^{p+n}\left(\int_{\mathbb{S}^{n-1}}\mathrm{d}\widetilde{v}_K(u)\right)^{1-(p+n)}\\ &\leq\int_{\mathbb{S}^{n-1}}\frac{\rho_K(u)}{\rho_L(u)}\mathrm{d}\widetilde{v}_K(u)=\widetilde{V}_{-1}(K,L).\end{aligned} \tag{3.9}$$

Lemma 3.3, together with (3.9), implies that

$$\int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right) \mathrm{d}\overline{\widetilde{v}}_K(u) \leq \ln\left(\frac{\widetilde{V}_{-1}(K,L)}{\mathrm{vol}_n(K)}\right).$$

It will be convenient to invoke the logarithmic mean $L(x,y)$ of two positive numbers x, y, which is given by

$$L(x,y) = \begin{cases} \frac{x-y}{lnx - \ln y}, & \text{for } x \neq y \\ x, & \text{for } x = y. \end{cases} \tag{3.10}$$

To prove Theorem 1.3, the following Hadamard type inequality for positive log-convex functions will be used [12].

Lemma 3.4. *Let f be a positive, integrable, log-convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \leq L\left(f(a), f(b)\right). \tag{3.11}$$

Suppose f has two derivative. The equality holds in the inequality (3.11) if and only if $f(t) = c$ almost everywhere (a.e.) or $\frac{f'(t)}{f(t)} = c$ almost everywhere (a.e.), where c is the constant.

The condition of the equality holds in the inequality (3.11) is that we supplements. Indeed, since f is log-convex function on $[a,b]$, and then $f(t)$ and $\frac{f'(t)}{f(t)}$ are monotonically increasing at the same time. So, we have

$$\begin{aligned} L(f(a), f(b)) &= \frac{f(b) - f(a)}{\ln f(b) - \ln f(b)} = \frac{\int_a^b f'(x)\mathrm{d}x}{\int_{f(a)}^{f(b)} \frac{1}{x}\mathrm{d}x} \\ &\overset{\text{x=f(t)}}{=} \frac{\int_a^b f'(x)\mathrm{d}x}{\int_a^b \frac{f'(t)}{f(t)}\mathrm{d}t} \geq \frac{1}{b-a}\int_a^b f(t)\mathrm{d}x \\ &= \frac{\int_a^b f(t)\mathrm{d}t}{\int_a^b 1\mathrm{d}t}. \end{aligned} \tag{3.12}$$

Thus, the inequality (3.11) is transformed into

$$\int_a^b f'(x)\mathrm{d}x \int_a^b 1\mathrm{d}x \geq \int_a^b f(t)\mathrm{d}t \int_a^b \frac{f'(t)}{f(t)}\mathrm{d}t. \tag{3.13}$$

Note that $f(t)$ and $\frac{f'(t)}{f(t)}$ are monotonically increasing at the same time, According to the condition of equality in Chebyshev's inequality, we see with equality in inequality (3.11) if and only if $f(t) = c$ or $\frac{f'(t)}{f(t)} = c$. Namely, $f(t) = c$ or $f(t) = e^{ct}$. □

Proof of Theorem 1.3. Consider the case $L \subseteq K$ and the function

$$F(q) : q \mapsto \int_{\mathbb{S}^{n-1}} \left(\frac{\rho_K(u)}{\rho_L(u)}\right)^q \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right) \mathrm{d}\widetilde{v}_K(u), \quad q \in \mathbb{R}.$$

Apparently, $F(q)$ is non-negative. If $u \mapsto \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)$ is zero on $\mathbb{S}^{n-1}$, then $F(q)$ is identically zero. Now, we assume that this is not the case, which also implies $F(1) \geq F(0) > 0$. If $F(1) = F(0)$, the conclusion is trivial (as using (3.7), K must be equal to L), and then, we assume $F(1) > F(0)$.

A simple verification shows that $F(q)$ is a log-convex function, this is because $\frac{\mathrm{d}^2}{\mathrm{d}q^2}\ln F(q) \geq 0$. By employing Hadamard type inequality (3.11) for positive log-convex functions [12], we have that

$$\frac{F(1) - F(0)}{\ln\big(F(1)/F(0)\big)} \geq \int_0^1 \left[\int_{\mathbb{S}^{n-1}} \left(\frac{\rho_K(u)}{\rho_L(u)}\right)^q \ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right) \mathrm{d}\widetilde{v}_K(u)\right] \mathrm{d}q. \tag{3.14}$$

Using Fubini-Tonelli's theorem, the following inequality

$$F(0) \geq F(1) \cdot \exp\left[- \frac{F(1) - F(0)}{\int_{\mathbb{S}^{n-1}} \left(\frac{\rho_K(u)}{\rho_L(u)} - 1 \right) \mathrm{d}\widetilde{v}_K(u)} \right] \tag{3.15}$$

is true. Note that

$$\begin{aligned} \frac{F(1) - F(0)}{\int_{\mathbb{S}^{n-1}} \left(\frac{\rho_K(u)}{\rho_L(u)} - 1 \right) \mathrm{d}\widetilde{v}_K(u)} &= \frac{\int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K(u)}{\rho_L(u)} \cdot \right) \cdot \left(\frac{\rho_K(u)}{\rho_L(u)} - 1 \right) \mathrm{d}\widetilde{v}_K(u)}{\int_{\mathbb{S}^{n-1}} \left(\frac{\rho_K(u)}{\rho_L(u)} - 1 \right) \mathrm{d}\widetilde{v}_K(u)} \\ &\leq \ln \left(\frac{\rho_K}{\rho_L} \right)_{\max}, \end{aligned} \tag{3.16}$$

then combining (3.15) and (3.16), we have

$$\begin{aligned} &\int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K(u)}{\rho_L(u)} \right) \mathrm{d}\widetilde{\widetilde{v}}_K(u) \\ \geq{}& \exp\left[- \ln \left(\frac{\rho_K}{\rho_L} \right)_{\max} \right] \cdot \frac{\widetilde{V}_{-1}(K,L)}{\mathrm{vol}_n(K)} \int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K(u)}{\rho_L(u)} \right) \mathrm{d}\widetilde{\widetilde{v}}_{-1}(K,L;u), \end{aligned} \tag{3.17}$$

it follows from (3.3) that

$$\int_{\mathbb{S}^{n-1}} \ln\left(\frac{\rho_K}{\rho_L} \right) \mathrm{d}\widetilde{\widetilde{v}}_K(u) \geq \frac{\left(\frac{\rho_K}{\rho_L} \right)_{\text{average}}}{\left(\frac{\rho_K}{\rho_L} \right)_{\max}} \frac{1}{n} \ln\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)} \right).$$

Now we discuss the conditions of equality in inequality (1.7), and the discussion is split into two cases. Assuming that $F(q)$ is identically zero, then $\rho_K(u) = \rho_L(u)$ for all u's with respect to $\mathbb{S}^{n-1}$ if and only if $K = L$.

Case 1. According to the conditions of equality in Hadamard type inequality (3.11) and inequality (3.3), we see with equality in inequality (1.7) if and only if

$$\begin{cases} F(q) = c \text{ for } q \geq 1, \\ K = \lambda L \quad \text{for } \lambda > 0. \end{cases} \tag{3.18}$$

From the definition of function $F(q)$, (3.18) is equivalent to

$$\begin{cases} \frac{\rho_K(u)}{\rho_L(u)} = 1 \text{ for any } u \in \mathbb{S}^{n-1}, \\ K = \lambda L \quad \text{for } \lambda > 0. \end{cases} \tag{3.19}$$

Namely, $K = L$.

Case 2. According to the conditions of equality in Hadamard type inequality (3.11) and inequality (3.3), we see with equality in inequality (1.7) if and only if

$$\begin{cases} \frac{F'(q)}{F(q)} = c \text{ for } q \geq 1, \\ K = \lambda L \quad \text{for } \lambda > 0. \end{cases} \tag{3.20}$$

Using mean value theorem for multiple integral [5, 19], there is a $u_0 \in \mathbb{S}^{n-1}$, such that (3.18) is equivalent to

$$\begin{cases} \frac{\rho_K(u_0)}{\rho_L(u_0)} = cq \text{ for a } u_0 \in \mathbb{S}^{n-1}, \\ K = \lambda L \quad \text{for } \lambda > 0. \end{cases} \tag{3.21}$$

Since $L \subseteq K$, $K = \lambda L$ with $0 < \lambda \leq 1$.

As mentioned above, we see with equality in inequality (1.7) if and only if $K = \lambda L$ with $0 < \lambda \leq 1$.

Assume now that K and L are arbitrary star bodies. If L is not included in K, there exists a λ, $0<\lambda<1$, such that $\tilde{L}:=\lambda L\subseteq K$. By using (1.7) for $\tilde{L}$ and K. Thus,

$$\int_{\mathbb{S}^{n-1}}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_K(u)-\ln\lambda\geq\frac{\left(\frac{\rho_K}{\rho_L}\right)_{\text{average}}}{\left(\frac{\rho_K}{\rho_L}\right)_{\max}}\cdot\frac{1}{n}\ln\left(\frac{\mathrm{vol}_n(K)}{\lambda^n\mathrm{vol}_n(L)}\right)\tag{3.22}$$

or

$$\begin{aligned}\int_{\mathbb{S}^{n-1}}\ln\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\mathrm{d}\overline{\widetilde{v}}_K(u)\geq&\frac{\left(\frac{\rho_K}{\rho_L}\right)_{\text{average}}}{\left(\frac{\rho_K}{\rho_L}\right)_{\max}}\cdot\frac{1}{n}\ln\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right)\\&+\ln\lambda\cdot\left(1-\frac{\left(\frac{\rho_K}{\rho_L}\right)_{\text{average}}}{\left(\frac{\rho_K}{\rho_L}\right)_{\max}}\right).\end{aligned}\tag{3.23}$$

Taking $\lambda=\min_{u\in\mathbb{S}^{n-1}}\left(\frac{\rho_K(u)}{\rho_L(u)}\right)$ will suffice, we now obtain the second inequality.

The claim that the homothety of K and L is the only case of equality follows from the first part. $\square$

Remark3.5. *Note that, if* $L\subseteq K$ *then* (1.8) *implies* (1.7)*. Also,* $\left(\frac{\rho_K}{\rho_L}\right)_{\text{average}}$, $\left(\frac{\rho_K}{\rho_L}\right)_{\max}$ *and* $\left(\frac{\rho_K}{\rho_L}\right)_{\min}$ *depend only on the values of the ratio* $\left(\frac{\rho_K(u)}{\rho_L(u)}\right)$ *on* $\mathbb{S}^{n-1}$.

We conclude this paper by pointing out that the equivalence between inequalities (1.4) and (2.8). We give a different proof with Wang and Liu [24]. For any $K\in\mathcal{S}_o^n$, define the real numbers R_K and r_K by

$$R_K=\max_{u\in\mathbb{S}^{n-1}}\rho_K(u),\qquad r_K=\min_{u\in\mathbb{S}^{n-1}}\rho_K(u).\tag{3.24}$$

Note that the definition of $\mathcal{S}_o^n$ is such that $0<r_K\leq R_K<\infty$, for all $K\in\mathcal{S}_o^n$.

Theorem 3.6. *For* $K,L\in\mathcal{S}_o^n$*, the dual log-Brunn-Minkowski inequality* (2.8) *and the dual log-Minkowski inequality* (1.4) *are equivalent.*

Proof. Suppose that K and L are fixed star bodies in $\mathcal{S}_o^n$. For $0\leq\lambda\leq 1$, let

$$Q_\lambda=(1-\lambda)\diamond K\hat{+}_0\lambda\diamond L,$$

i.e., the radial function of star body Q_λ is $q_\lambda:=\rho_{Q_\lambda}=\rho_K^{1-\lambda}\rho_L^\lambda$. Since q_0 and q_1 are the radial functions of star bodies, we have $Q_0=K$ and $Q_1=L$.

Suppose that we have the dual log-Minkowski inequality (1.4) for K and L. Now $\rho_{Q_\lambda}=\rho_K^{1-\lambda}\rho_L^\lambda$ a.e. with respect to $\mathcal{S}^{n-1}$, and thus

$$\begin{aligned}0&=\frac{1}{n\mathrm{vol}_n(Q_\lambda)}\int_{\mathbb{S}^{n-1}}\rho_{Q_\lambda}(u)^n\ln\frac{\rho_K(u)^{1-\lambda}\rho_L(u)^\lambda}{\rho_{Q_\lambda}(u)}\mathrm{d}S(u)\\&=(1-\lambda)\frac{1}{n\mathrm{vol}_n(Q_\lambda)}\int_{\mathbb{S}^{n-1}}\rho_{Q_\lambda}(u)^n\ln\frac{\rho_K(u)}{\rho_{Q_\lambda}(u)}\mathrm{d}S(u)\\&\quad+\lambda\frac{1}{n\mathrm{vol}_n(Q_\lambda)}\int_{\mathbb{S}^{n-1}}\rho_{Q_\lambda}(u)^n\ln\frac{\rho_L(u)}{\rho_{Q_\lambda}(u)}\mathrm{d}S(u)\\&=-(1-\lambda)\int_{\mathbb{S}^{n-1}}\ln\frac{\rho_{Q_\lambda}(u)}{\rho_K(u)}\mathrm{d}\overline{\widetilde{v}}_{Q_\lambda}(u)-\lambda\int_{\mathbb{S}^{n-1}}\ln\frac{\rho_{Q_\lambda}(u)}{\rho_L(u)}\mathrm{d}\overline{\widetilde{v}}_{Q_\lambda}(u)\\&\leq-(1-\lambda)\frac{1}{n}\ln\frac{\mathrm{vol}_n(Q_\lambda)}{\mathrm{vol}_n(K)}-\lambda\frac{1}{n}\ln\frac{\mathrm{vol}_n(Q_\lambda)}{\mathrm{vol}_n(L)}\\&=\frac{1}{n}\ln\frac{\mathrm{vol}_n(K)^{1-\lambda}\mathrm{vol}_n(L)^\lambda}{\mathrm{vol}_n(Q_\lambda)}.\end{aligned}\tag{3.25}$$

This gives the dual log-Brunn-Minkowski inequality (2.8).

Suppose now that we have the dual log-Brunn-Minkowski inequality (2.8) for K and L. Namely,

$$\mathrm{vol}_n((1-\lambda)\diamond K\hat{+}_0\lambda\diamond L)\leq \mathrm{vol}_n(K)\left(\frac{\mathrm{vol}_n(L)}{\mathrm{vol}_n(K)}\right)^{\lambda}. \tag{3.26}$$

Using the polar coordinates formula of volume, the radial log-Minkowski-combination (2.7) and the Borel probability measure (2.11), it follows from (3.26) that

$$\int_{\mathbb{S}^{n-1}}\left(\frac{\rho_L(u)}{\rho_K(u)}\right)^{n\lambda}\mathrm{d}\overline{\widetilde{v}}_K(u)\leq\left(\frac{\mathrm{vol}_n(L)}{\mathrm{vol}_n(K)}\right)^{\lambda}. \tag{3.27}$$

Therefore

$$\frac{\int_{\mathbb{S}^{n-1}}\left(\frac{\rho_L(u)}{\rho_K(u)}\right)^{n\lambda}\mathrm{d}\overline{\widetilde{v}}_K(u)-1}{\lambda}\leq\frac{\left(\frac{\mathrm{vol}_n(L)}{\mathrm{vol}_n(K)}\right)^{\lambda}-1}{\lambda}.$$

Taking the limit on both sides of the last inequality as $\lambda\to 0$, we get

$$\lim_{\lambda\to 0}\frac{\int_{\mathbb{S}^{n-1}}\left(\frac{\rho_L(u)}{\rho_K(u)}\right)^{n\lambda}\mathrm{d}\overline{\widetilde{v}}_K(u)-1}{\lambda}\leq\lim_{\lambda\to 0}\frac{\left(\frac{\mathrm{vol}_n(L)}{\mathrm{vol}_n(K)}\right)^{\lambda}-1}{\lambda}.$$

We are easy to prove the function $f(x)=\frac{a^x-1}{x}$ is uniformly continuous on $(0,\infty)$ for $0<a\leq 1$, and the Bernoulli's inequality leads to the function $f(x)=\frac{a^x-1}{x}$ is uniform boundness for $a>1$. From the definition (3.24), we have

$$\frac{\left(\frac{\rho_L}{\rho_K}\right)^{n\lambda}-1}{\lambda}\leq\frac{\left(\frac{R_L}{r_K}\right)^{n\lambda}-1}{\lambda}.$$

Using Lebesgue dominated convergence theorem we know that the order of the integral and the limit can be changed. Therefore, we can obtain

$$\int_{\mathbb{S}^{n-1}}\lim_{\lambda\to 0}\frac{\left(\frac{\rho_L(u)}{\rho_K(u)}\right)^{n\lambda}-1}{\lambda}\mathrm{d}\overline{\widetilde{v}}_K(u)\leq\lim_{\lambda\to 0}\frac{\left(\frac{\mathrm{vol}_n(L)}{\mathrm{vol}_n(K)}\right)^{\lambda}-1}{\lambda}.$$

Since $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$, then

$$\int_{\mathbb{S}^{n-1}}\ln\left(\frac{\rho_L(u)}{\rho_K(u)}\right)^{n}\mathrm{d}\overline{\widetilde{v}}_K(u)\leq\ln\left(\frac{\mathrm{vol}_n(L)}{\mathrm{vol}_n(K)}\right). \tag{3.28}$$

This is the dual log-Minkowski inequality (1.4), which completes the proof.

Acknowledgment

The authors are deeply indebted to Professor Li Yin, Professor Xiaoming Zhang, Doctor Xuezhong Wang, Doctor Denghui Wu and Doctor Yibin Feng for many invaluable suggestions in preparing the manuscript. The authors also would like to thank the referee for the many suggested improvements.

References

[1] A. Bernig, *The isoperimetrix in the dual Brunn-Minkowski theory*, Adv. Math. **254** (2014) 1-14.

[2] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang, *The log-Brunn-Minkowski inequality*, Adv. Math. **231** (2012) 1974-1997.

[3] T. M. Cover, J. A. Thomas, Elements of Information Theory, second edition, Wiley-Interscience, Hoboken, NJ, 2006.

[4] P. Dulio, R.J. Gardner and C. Peri, *Characterizing the dual mixed volume via additive functionals*, arXiv preprint arXiv: 1312.4072, 2013.

[5] J. Fan, B. Yang, *The mean value theorem for multiple integral*, Mathematics in Practice and Theory **37** (12) (2007) 197-200 (in Chinese).

[6] R. J. Gardner, D. Hug, W. Weil, *The Orlicz-Brunn-Minkowski theory: a general framework, addi-tions, and inequalities*, J. Differential Geom. **97** (2014) 427-476.

[7] R. J. Gardner, D. Hug, W. Weil, et al., *The dual Orlicz-Brunn-Minkowski theory*, J. Math. Anal. Appl. **430**(2)(2015) 810-829.

[8] R. J. Gardner, *The dual Brunn-Minkowski theory for bounded borel sets: Dual affine quermassintegrals and inequalities*, Adv. Math. **216** (2007) 358-386.

[9] R. J. Gardner and S. Vassallo, *Inequalities for dual isoperimetric deficits*, Mathematika. **45** (1998) 269-285.

[10] R. J. Gardner and S. Vassallo, *Stability of inequalities in the dual Brunn-Minkowski theory*, J. Math. Anal. Appl. **231** (1999) 568-587.

[11] R. J. Gardner and S. Vassallo, *The Brunn-Minkowski inequality, Minkowskis first inequality, and their duals'*, J. Math. Anal. Appl. **245** (2000) 502-512.

[12] P. M. Gill, C. E. M. Pearce, J. *Pečarić Hadamard's inequality for r-convex functions*, J. Math. Anal. Appl. **215** (1997) 461-470.

[13] E. Lutwak, *The Brunn-Minkowski-Firey theory. I: mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993) 131-150.

[14] E. Lutwak, *The Brunn-Minkowski-Firey theory II: affine and geominimal surface areas*, Adv. Math. **118** (1996) 244-294.

[15] E. Lutwak, *Dual mixed volumes*, Pacific J. Math. **58** (1975) 531-538.

[16] E. Lutwak, *Intersection bodies and dual mixed volume*, Adv. Math. **71** (1988) 232-261.

[17] E. Lutwak, *Centered bodies and dual mixed volumes*, Proc. London. Math. Soc. **60** (1990) 365-391.

[18] E. Lutwak, D. Yang, G. Zhang, *Orlicz centroid bodies*, J. Differential Geom. **84** (2010), 365-387.

[19] H. Liu, *The inverse proposition about mid-value theorem of multiple integral*, Journal of Huanggang teachers college (Natural Science Edition). **17**(1)(1997) 39-42 (in Chinese).

[20] T. Ma, *The minimal dual Orlicz surface area*, Taiwanese Journal of Mathematics **20**(2)(2016) 287-309.

[21] T. Ma, W. Wang, *Dual Orlicz geominimal surface area*, Journal of Inequalities and Applications **2016**(1)(2016) 1-13.

[22] E. Milman, *Dual mixed volumes and the slicing problem*, Adv. Math. **207** (2006) 566-598.

[23] A. Stancu, *The logarithmic Minkowski inequality fornon-symmetric convex bodies*, Adv. Appl. Math. **73**(2016) 43-58.

[24] W. Wang, L. Liu, *The dual Log-Brunn-Minkowski inequalities*, Taiwanese Jouranl of Mathematics **20**(4)(2016), 909-919.

[25] G. Zhang, *Centered bodies and dual mixed volumes*, Trans. Amer. Math. Soc. **345** (1994) 777-801.

Subalgebra and ideal-type hyper values in BCK/BCI-algebras

Young Bae Jun[1] **and Sun Shin Ahn**[2,*]

[1] *Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea*

[2] *Department of Mathematics Education, Dongguk University, Seoul 04620, Korea*

Abstract. The notions of subalgebra-type hyper value and ideal-type hyper value are introduced, and related properties are investigated. The relation between subalgebra-type hyper value and ideal-type hyper value is considered. Conditions for a pair (α, β) in $[0,1] \times [0,1]$ to be subalgebra-type hyper value and ideal-type hyper value are discussed. For a hyperfuzzy structure, conditions for its level sets to be S-energetic, I-energetic, right vanished and right stable are founded.

1. Introduction

Jun et al. [3] introduced the notion of energetic (resp. right vanish, right stable) subsets in BCK/BCI-algebras, and investigated several related properties. Ghosh et al. [1] introduced the concept of hyperfuzzy sets which is a generalization of fuzzy sets and interval-valued fuzzy sets. Jun et al. [4] and Song et al. [6] applied hyper structure to BCK/BCI-algebras, and discussed hyperfuzzy subalgebras and hyperfuzzy ideals in BCK/BCI-algebras.

In this article, we introduce the concepts of subalgebra-type hyper value and ideal-type hyper value, and investigate several properties. We discuss the relation between subalgebra-type hyper value and ideal-type hyper value. We provide an example to show that any subalgebra-type hyper value is not an ideal-type hyper value. We consider conditions for a pair (α, β) in $[0,1] \times [0,1]$ to be subalgebra-type hyper value and ideal-type hyper value. Given a hyperfuzzy structure, we find conditions for its level sets to be S-energetic, I-energetic, right vanished and right stable.

2. Preliminaries

By a *BCI-algebra* we mean a system $X := (X, *, 0)$ in which the following axioms hold:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = y * x = 0 \;\Rightarrow\; x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra*. We can define a partial ordering $\leq$ by

$$(\forall x, y \in X)\,(x \leq y \iff x * y = 0).$$

[0] **2010 Mathematics Subject Classification**: 08F35; 03G25; 03B52.

[0] **Keywords**: hyperfuzzy subalgebra; hyperfuzzy ideal; subalgebra-type hyper value; ideal-type hyper value; energetic subset; right vanished subset; right stable subset.

[*] Correspondence: Tel: +82 2 2260 3410, Fax: +82 2 2266 3409 (S. S. Ahn).

[0]**E-mail**: `skywine@gmail.com (Y. B. Jun); sunshine@dongguk.edu (S. S. Ahn).`

In a BCK/BCI-algebra X, the following hold:

$$(\forall x \in X)\ (x * 0 = x), \tag{2.1}$$

$$(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y). \tag{2.2}$$

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

A subset I of a BCK/BCI-algebra X is called an *ideal* of X if

$$0 \in I, \tag{2.3}$$

$$(\forall x \in X)(\forall y \in I)\,(x * y \in I\ \Rightarrow\ x \in I)\,. \tag{2.4}$$

We refer the reader to the books [2] and [5] for further information regarding BCK/BCI-algebras.

By a *fuzzy structure* over a nonempty set X we mean an ordered pair (X, ρ) of X and a fuzzy set ρ on X.

Let X be a nonempty set. A mapping $\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1])$ is called a *hyperfuzzy set* over X (see [1]), where $\tilde{\mathcal{P}}([0,1])$ is the family of all nonempty subsets of $[0,1]$. An ordered pair $(X, \tilde{\mu})$ is called a *hyper structure* over X.

Given a hyper structure $(X, \tilde{\mu})$ over a nonempty set X, we consider two fuzzy structures $(X, \tilde{\mu}_{\inf})$ and $(X, \tilde{\mu}_{\sup})$ over X in which

$$\tilde{\mu}_{\inf} : X \to [0,1],\ x \mapsto \inf\{\tilde{\mu}(x)\},$$

$$\tilde{\mu}_{\sup} : X \to [0,1],\ x \mapsto \sup\{\tilde{\mu}(x)\}.$$

Given a nonempty set X, let $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$ denote the collection of all BCK-algebras and all BCI-algebras, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$. In what follows, let $(X, *, 0) \in \mathcal{B}(X)$ unless otherwise specified.

Definition 2.1 ([4])**.** For any $(X, *, 0) \in \mathcal{B}(X)$, a fuzzy structure (X, μ) over $(X, *, 0)$ is called a

- *fuzzy subalgebra* of $(X, *, 0)$ with type 1 (briefly, 1-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\mu(x * y) \geq \min\{\mu(x), \mu(y)\})\,, \tag{2.5}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 2 (briefly, 2-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\mu(x * y) \leq \min\{\mu(x), \mu(y)\})\,, \tag{2.6}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 3 (briefly, 3-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\mu(x * y) \geq \max\{\mu(x), \mu(y)\})\,, \tag{2.7}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 4 (briefly, 4-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X)\,(\mu(x * y) \leq \max\{\mu(x), \mu(y)\})\,. \tag{2.8}$$

It is clear that every 3-fuzzy subalgebra is a 1-fuzzy subalgebra and every 2-fuzzy subalgebra is a 4-fuzzy subalgebra.

Definition 2.2 ([4])**.** For any $(X, *, 0) \in \mathcal{B}(X)$ and $i, j \in \{1, 2, 3, 4\}$, a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$ is called an (i, j)-*hyperfuzzy subalgebra* of $(X, *, 0)$ if $(X, \tilde{\mu}_{\inf})$ is an i-fuzzy subalgebra of $(X, *, 0)$ and $(X, \tilde{\mu}_{\sup})$ is a j-fuzzy subalgebra of $(X, *, 0)$.

Given a hyper structure $(X, \tilde{\mu})$ over X and $\alpha, \beta \in [0,1]$, we consider the following sets (see [6]):

$$U(\tilde{\mu}_{\inf}; \alpha) := \{x \in X \mid \tilde{\mu}_{\inf}(x) \geq \alpha\},$$
$$L(\tilde{\mu}_{\inf}; \alpha) := \{x \in X \mid \tilde{\mu}_{\inf}(x) \leq \alpha\},$$
$$U(\tilde{\mu}_{\sup}; \beta) := \{x \in X \mid \tilde{\mu}_{\sup}(x) \geq \beta\},$$
$$L(\tilde{\mu}_{\sup}; \beta) := \{x \in X \mid \tilde{\mu}_{\sup}(x) \leq \beta\}.$$

Definition 2.3 ([6]). A fuzzy structure (X, μ) over $(X, *, 0)$ is called a

- *fuzzy ideal* of $(X, *, 0)$ with type 1 (briefly, 1-*fuzzy ideal* of $(X, *, 0)$) if

$$(\forall x \in X)\,(\mu(0) \geq \mu(x))\,, \tag{2.9}$$
$$(\forall x, y \in X)\,(\mu(x) \geq \min\{\mu(x * y), \mu(y)\})\,, \tag{2.10}$$

- *fuzzy ideal* of $(X, *, 0)$ with type 2 (briefly, 2-*fuzzy ideal* of $(X, *, 0)$) if

$$(\forall x \in X)\,(\mu(0) \leq \mu(x))\,, \tag{2.11}$$
$$(\forall x, y \in X)\,(\mu(x) \leq \min\{\mu(x * y), \mu(y)\})\,, \tag{2.12}$$

- *fuzzy ideal* of $(X, *, 0)$ with type 3 (briefly, 3-*fuzzy ideal* of $(X, *, 0)$) if it satisfies (2.9) and

$$(\forall x, y \in X)\,(\mu(x) \geq \max\{\mu(x * y), \mu(y)\})\,, \tag{2.13}$$

- *fuzzy ideal* of $(X, *, 0)$ with type 4 (briefly, 4-*fuzzy ideal* of $(X, *, 0)$) if it satisfies (2.11) and

$$(\forall x, y \in X)\,(\mu(x) \leq \max\{\mu(x * y), \mu(y)\})\,. \tag{2.14}$$

It is clear that every 3-fuzzy ideal is a 1-fuzzy ideal and every 2-fuzzy ideal is a 4-fuzzy ideal.

Definition 2.4 ([6]). For any $i, j \in \{1, 2, 3, 4\}$, a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$ is called an (i, j)-*hyperfuzzy ideal* of $(X, *, 0)$ if $(X, \tilde{\mu}_{\inf})$ is an i-fuzzy ideal of $(X, *, 0)$ and $(X, \tilde{\mu}_{\sup})$ is a j-fuzzy ideal of $(X, *, 0)$.

3. **Subalgebra and ideal-type hyper values**

Definition 3.1 ([3]). A nonempty subset A of $(X, *, 0)$ is said to be S-*energetic* if it satisfies:

$$(\forall a, b \in X)\,(a * b \in A \;\Rightarrow\; \{a, b\} \cap A \neq \emptyset)\,.$$

Let A be a proper subset of X containing 0. Then there exists $a \in X \setminus A$, and so $a * a = 0 \in A$ but $\{a\}$ and A are disjoint. Thus every proper subset A of X containing 0 cannot be S-energetic.

Theorem 3.2. *Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, if it is a $(4, 1)$-hyperfuzzy subalgebra of $(X, *, 0)$, then its nonempty level subsets $U(\tilde{\mu}_{\inf}; \alpha)$ and $L(\tilde{\mu}_{\sup}; \beta)$ are S-energetic subsets of $(X, *, 0)$ for all $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0, 1] \times [0, 1]$.*

Proof. Assume that $U(\tilde{\mu}_{\inf};\alpha)$ and $L(\tilde{\mu}_{\sup};\beta)$ are nonempty for every $(\alpha,\beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0,1] \times [0,1]$. If $x * y \in U(\tilde{\mu}_{\inf};\alpha)$ and $a * b \in L(\tilde{\mu}_{\sup};\beta)$ for all $x, y, a, b \in X$, then

$$\alpha \leq \tilde{\mu}_{\inf}(x * y) \leq \max\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\}$$

and

$$\beta \geq \tilde{\mu}_{\sup}(a * b) \geq \min\{\tilde{\mu}_{\sup}(a), \tilde{\mu}_{\sup}(b)\}.$$

It follows that

$$\tilde{\mu}_{\inf}(x) \geq \alpha \text{ or } \tilde{\mu}_{\inf}(y) \geq \alpha, \text{ that is, } x \in U(\tilde{\mu}_{\inf};\alpha) \text{ or } y \in U(\tilde{\mu}_{\inf};\alpha)$$

and

$$\tilde{\mu}_{\sup}(a) \leq \beta \text{ or } \tilde{\mu}_{\sup}(b) \leq \beta, \text{ that is, } a \in L(\tilde{\mu}_{\sup};\beta) \text{ or } b \in L(\tilde{\mu}_{\sup};\beta).$$

Hence $\{x, y\} \cap U(\tilde{\mu}_{\inf};\alpha) \neq \emptyset$ and $\{a, b\} \cap L(\tilde{\mu}_{\sup};\beta) \neq \emptyset$. Therefore $U(\tilde{\mu}_{\inf};\alpha)$ and $L(\tilde{\mu}_{\sup};\beta)$ are S-energetic subsets of $(X, *, 0)$ for all $(\alpha,\beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0,1] \times [0,1]$. □

Corollary 3.3. *Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, if it is a $(2,1)$-hyperfuzzy (resp., $(2,3)$-hyperfuzzy and $(4,3)$-hyperfuzzy) subalgebra of $(X, *, 0)$, then its nonempty level subsets $U(\tilde{\mu}_{\inf};\alpha)$ and $L(\tilde{\mu}_{\sup};\beta)$ are S-energetic subsets of $(X, *, 0)$ for all $(\alpha,\beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0,1] \times [0,1]$.*

Proof. Straightforward. □

Definition 3.4 ([3])**.** A nonempty subset A of $(X, *, 0)$ is said to be *I-energetic* if it satisfies:

$$(\forall x, y \in X)\,(y \in A \;\Rightarrow\; \{x, y * x\} \cap A \neq \emptyset)\,.$$

Theorem 3.5. *Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, if it is a $(4,1)$-hyperfuzzy ideal of $(X, *, 0)$, then its nonempty level subsets $U(\tilde{\mu}_{\inf};\alpha)$ and $L(\tilde{\mu}_{\sup};\beta)$ are I-energetic subsets of $(X, *, 0)$ for all $(\alpha,\beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0,1] \times [0,1]$.*

Proof. Let $(\alpha,\beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0,1] \times [0,1]$ be such that $U(\tilde{\mu}_{\inf};\alpha)$ and $L(\tilde{\mu}_{\sup};\beta)$ are nonempty. Let $x, y, a, b \in X$ be such that $y \in U(\tilde{\mu}_{\inf};\alpha)$ and $b \in L(\tilde{\mu}_{\sup};\beta)$. Then

$$\alpha \leq \tilde{\mu}_{\inf}(y) \leq \max\{\tilde{\mu}_{\inf}(y * x), \tilde{\mu}_{\inf}(x)\}$$

and

$$\beta \geq \tilde{\mu}_{\sup}(b) \geq \min\{\tilde{\mu}_{\sup}(b * a), \tilde{\mu}_{\sup}(a)\}.$$

Hence

$$\tilde{\mu}_{\inf}(y * x) \geq \alpha \text{ or } \tilde{\mu}_{\inf}(x) \geq \alpha, \text{ i.e., } y * x \in U(\tilde{\mu}_{\inf};\alpha) \text{ or } x \in U(\tilde{\mu}_{\inf};\alpha)$$

and

$$\tilde{\mu}_{\sup}(b * a) \leq \beta \text{ or } \tilde{\mu}_{\sup}(a) \leq \beta, \text{ i.e., } b * a \in L(\tilde{\mu}_{\sup};\beta) \text{ or } a \in L(\tilde{\mu}_{\sup};\beta).$$

It follows that $\{x, y * x\} \cap U(\tilde{\mu}_{\inf};\alpha) \neq \emptyset$ and $\{a, b * a\} \cap L(\tilde{\mu}_{\sup};\beta) \neq \emptyset$. Therefore $U(\tilde{\mu}_{\inf};\alpha)$ and $L(\tilde{\mu}_{\sup};\beta)$ are I-energetic subsets of $(X, *, 0)$ for all $(\alpha,\beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0,1] \times [0,1]$. □

Corollary 3.6. *Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, if it is a $(2,1)$-hyperfuzzy (resp., $(2,3)$-hyperfuzzy and $(4,3)$-hyperfuzzy) subalgebra of $(X, *, 0)$, then its nonempty level subsets $U(\tilde{\mu}_{\inf}; \alpha)$ and $L(\tilde{\mu}_{\sup}; \beta)$ are I-energetic subsets of $(X, *, 0)$ for all $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0,1] \times [0,1]$.*

Proof. Straightforward. □

Definition 3.7. Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, let $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0,1] \times [0,1]$ be such that $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are nonempty. Then (α, β) is called a *subalgebra-type hyper value* for $(X, \tilde{\mu})$ if the following assertion is valid.

$$(\forall x, y \in X) \left(\begin{array}{l} \tilde{\mu}_{\inf}(x * y) \leq \alpha \;\Rightarrow\; \min\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\} \leq \alpha, \\ \tilde{\mu}_{\sup}(x * y) \geq \beta \;\Rightarrow\; \max\{\tilde{\mu}_{\sup}(x), \tilde{\mu}_{\sup}(y)\} \geq \beta \end{array} \right). \tag{3.1}$$

Example 3.8. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation $*$ which is given in Table 1.

Table 1. Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	1	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Then $(X, *, 0)$ is a BCK-algebra (see [5]). Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $\tilde{\mu}$ is given as follows:

$$\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1]),\; x \mapsto \begin{cases} [0.5, 0.53) & \text{if } x = 0, \\ (0.3, 0.58] & \text{if } x = 1, \\ [0.3, 0.44) \cup [0.45, 0.58) & \text{if } x = 2, \\ (0.4, 0.5] \cup [0.60, 0.68] & \text{if } x = 3, \\ [0.2, 0.63] & \text{if } x = 4. \end{cases}$$

It is routine to verify that every pair $(\alpha, \beta) \in [0.2, 0.5] \times [0.53, 0.68)$ is a subalgebra-type hyper value for $(X, \tilde{\mu})$.

Theorem 3.9. *For a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, let $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0,1] \times [0,1]$ be such that $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are nonempty. If $(X, \tilde{\mu})$ is a $(1,4)$-hyperfuzzy subalgebra of $(X, *, 0)$, then (α, β) is a subalgebra-type hyper value for $(X, \tilde{\mu})$.*

Proof. Let $x, y, a, b \in X$ be such that $\tilde{\mu}_{\inf}(x * y) \leq \alpha$ and $\tilde{\mu}_{\sup}(a * b) \geq \beta$. Since $(X, \tilde{\mu})$ is a $(1,4)$-hyperfuzzy subalgebra of $(X, *, 0)$, we have

$$\alpha \geq \tilde{\mu}_{\inf}(x * y) \geq \min\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\}$$

and

$$\beta \leq \tilde{\mu}_{\sup}(a * b) \leq \max\{\tilde{\mu}_{\sup}(a), \tilde{\mu}_{\sup}(b)\}.$$

Hence (α, β) is a subalgebra-type hyper value for $(X, \tilde{\mu})$. □

Corollary 3.10. *For a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, let $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0,1] \times [0,1]$ be such that $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are nonempty. If $(X, \tilde{\mu})$ is a $(1,2)$-hyperfuzzy (resp., $(3,2)$-hyperfuzzy and $(3,4)$-hyperfuzzy) subalgebra of $(X, *, 0)$, then (α, β) is a subalgebra-type hyper value for $(X, \tilde{\mu})$.*

Proof. Straightforward. □

Theorem 3.11. *Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$. If (α, β) is a subalgebra-type hyper value for $(X, \tilde{\mu})$, then $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are S-energetic subsets of $(X, *, 0)$.*

Proof. Let $x, y, a, b \in X$ be such that $x*y \in L(\tilde{\mu}_{\inf}; \alpha)$ and $a*b \in U(\tilde{\mu}_{\sup}; \beta)$. Then $\tilde{\mu}_{\inf}(x*y) \leq \alpha$ and $\tilde{\mu}_{\sup}(a*b) \geq \beta$. Since (α, β) is a subalgebra-type hyper value for $(X, \tilde{\mu})$, it follows from (3.1) that $\min\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\} \leq \alpha$ and $\max\{\tilde{\mu}_{\sup}(a), \tilde{\mu}_{\sup}(b)\} \geq \beta$. Hence

$$\tilde{\mu}_{\inf}(x) \leq \alpha \text{ or } \tilde{\mu}_{\inf}(y) \leq \alpha$$

and

$$\tilde{\mu}_{\sup}(a) \geq \beta \text{ or } \tilde{\mu}_{\sup}(b) \geq \beta,$$

that is,

$$x \in L(\tilde{\mu}_{\inf}; \alpha) \text{ or } y \in L(\tilde{\mu}_{\inf}; \alpha)$$

and

$$a \in U(\tilde{\mu}_{\sup}; \beta) \text{ or } b \in U(\tilde{\mu}_{\sup}; \beta).$$

Thus $\{x, y\} \cap L(\tilde{\mu}_{\inf}; \alpha) \neq \emptyset$ and $\{a, b\} \cap U(\tilde{\mu}_{\sup}; \beta) \neq \emptyset$, and therefore $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are S-energetic subsets of $(X, *, 0)$. □

Combining Theorems 3.9 and 3.11, we have the following corollary.

Corollary 3.12. *For a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, let $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0,1] \times [0,1]$ be such that $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are nonempty. If $(X, \tilde{\mu})$ is a $(1,4)$-hyperfuzzy subalgebra of $(X, *, 0)$, then $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are S-energetic subsets of $(X, *, 0)$.*

Definition 3.13. Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, let $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0,1] \times [0,1]$ be such that $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are nonempty. Then (α, β) is called an *ideal-type hyper value* for $(X, \tilde{\mu})$ if the following assertion is valid.

$$(\forall x, y \in X) \left(\begin{array}{l} \tilde{\mu}_{\inf}(y) \leq \alpha \;\Rightarrow\; \min\{\tilde{\mu}_{\inf}(y * x), \tilde{\mu}_{\inf}(x)\} \leq \alpha, \\ \tilde{\mu}_{\sup}(y) \geq \beta \;\Rightarrow\; \max\{\tilde{\mu}_{\sup}(y * x), \tilde{\mu}_{\sup}(x)\} \geq \beta \end{array} \right). \tag{3.2}$$

Example 3.14. In Example 3.8, the pair (α, β) is an ideal-type hyper value for $(X, \tilde{\mu})$.

Example 3.15. Let $X = \{0, 1, 2, a, b\}$ be a set with the binary operation $*$ which is given in Table 2.

Table 2. Cayley table for the binary operation "$*$"

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Then $(X, *, 0)$ is a BCI-algebra (see [5]). Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $\tilde{\mu}$ is given as follows:

$$\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1]),\ x \mapsto \begin{cases} [0.54, 0.72) & \text{if } x = 0, \\ (0.58, 0.64] & \text{if } x = 1, \\ [0.56, 0.72) & \text{if } x = 2, \\ (0.60, 0.68] & \text{if } x = a, \\ [0.60, 0.64) & \text{if } x = b. \end{cases}$$

If we take $(\alpha, \beta) \in (0.54, 0.60] \times [0.64, 0.72)$, then (α, β) is an ideal-type hyper value for $(X, \tilde{\mu})$.

We consider a relation between subalgebra-type hyper value and ideal-type hyper value.

Theorem 3.16. *Let $(X, \tilde{\mu})$ be a hyper structure over $(X, \tilde{\mu}) \in \mathcal{B}_K(X)$ such that*

$$(\forall x \in X)\left(\tilde{\mu}_{\inf}(0) \geq \tilde{\mu}_{\inf}(x),\ \tilde{\mu}_{\sup}(0) \leq \tilde{\mu}_{\sup}(x)\right). \tag{3.3}$$

Then every ideal-type hyper value for $(X, \tilde{\mu})$ is a subalgebra-type hyper value for $(X, \tilde{\mu})$.

Proof. Let (α, β) be an ideal-type hyper value for $(X, \tilde{\mu})$. Assume that $\tilde{\mu}_{\inf}(x * y) \leq \alpha$ and $\tilde{\mu}_{\sup}(a * b) \geq \beta$ for $x, y, a, b \in X$. Using (3.2), (2.2) and (3.3), we have

$$\begin{aligned} \alpha &\geq \min\{\tilde{\mu}_{\inf}((x * y) * x), \tilde{\mu}_{\inf}(x)\} \\ &= \min\{\tilde{\mu}_{\inf}((x * x) * y), \tilde{\mu}_{\inf}(x)\} \\ &= \min\{\tilde{\mu}_{\inf}(0 * y), \tilde{\mu}_{\inf}(x)\} \\ &= \min\{\tilde{\mu}_{\inf}(0), \tilde{\mu}_{\inf}(x)\} = \tilde{\mu}_{\inf}(x) \end{aligned}$$

and

$$\begin{aligned} \beta &\leq \max\{\tilde{\mu}_{\sup}((a * b) * a), \tilde{\mu}_{\sup}(a)\} \\ &= \max\{\tilde{\mu}_{\sup}((a * a) * b), \tilde{\mu}_{\sup}(a)\} \\ &= \max\{\tilde{\mu}_{\sup}(0 * b), \tilde{\mu}_{\sup}(a)\} \\ &= \max\{\tilde{\mu}_{\sup}(0), \tilde{\mu}_{\sup}(a)\} = \tilde{\mu}_{\sup}(a). \end{aligned}$$

It follows that

$$\min\{\tilde{\mu}_{\inf}(x), \tilde{\mu}_{\inf}(y)\} \leq \tilde{\mu}_{\inf}(x) \leq \alpha \text{ and } \max\{\tilde{\mu}_{\sup}(a), \tilde{\mu}_{\sup}(b)\} \geq \tilde{\mu}_{\sup}(a) \geq \beta.$$

Therefore (α, β) is a subalgebra-type hyper value for $(X, \tilde{\mu})$. □

The converse of Theorem 3.16 is not true in general as seen in the following example.

Example 3.17. Let $X = \{0, 1, a, b, c\}$ be a set with the binary operation $*$ which is given in Table 3.

Table 3. Cayley table for the binary operation "$*$"

$*$	0	1	a	b	c
0	0	0	a	a	a
1	1	0	a	a	a
a	a	a	0	0	0
b	b	a	1	0	0
c	c	a	1	1	0

Then $(X, *, 0)$ is a BCI-algebra (see [5]). Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$ in which $\tilde{\mu}$ is given as follows:

$$\tilde{\mu} : X \to \tilde{\mathcal{P}}([0,1]),\ x \mapsto \begin{cases} [0.51, 0.55) & \text{if } x = 0, \\ (0.48, 0.63] & \text{if } x = 1, \\ [0.45, 0.58) & \text{if } x = a, \\ (0.41, 0.5] \cup [0.60, 0.63] & \text{if } x = b, \\ [0.35, 0.65] & \text{if } x = c. \end{cases}$$

If we take $(\alpha, \beta) \in (0.41, 0.45) \times (0.63, 0.65]$, then (α, β) is a subalgebra-type hyper value for $(X, \tilde{\mu})$, but it is not an ideal-type hyper value for $(X, \tilde{\mu})$ since

$$\tilde{\mu}_{\inf}(b) = 0.41 \leq \alpha \text{ and } \min\{\tilde{\mu}_{\inf}(b * a), \tilde{\mu}_{\inf}(a)\} = 0.45 \nleq \alpha$$

and/or

$$\tilde{\mu}_{\sup}(c) = 0.65 \geq \beta \text{ and } \max\{\tilde{\mu}_{\sup}(c * a), \tilde{\mu}_{\sup}(a)\} = 0.63 \ngeq \beta.$$

We provide conditions for a pair (α, β) to be an ideal-type hyper value.

Theorem 3.18. *Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, let $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0,1] \times [0,1]$ be such that $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are nonempty. If $(X, \tilde{\mu})$ is a $(1,4)$-hyperfuzzy ideal of $(X, *, 0)$, then (α, β) is an ideal-type hyper value for $(X, \tilde{\mu})$.*

Proof. Let $x, y, a, b \in X$ be such that $\tilde{\mu}_{\inf}(y) \leq \alpha$ and $\tilde{\mu}_{\sup}(b) \geq \beta$. Since $(X, \tilde{\mu})$ is a $(1,4)$-hyperfuzzy ideal of $(X, *, 0)$, it follows that

$$\alpha \geq \tilde{\mu}_{\inf}(y) \geq \min\{\tilde{\mu}_{\inf}(y * x), \tilde{\mu}_{\inf}(x)\}$$

and

$$\beta \leq \tilde{\mu}_{\sup}(b) \leq \max\{\tilde{\mu}_{\sup}(b * a), \tilde{\mu}_{\sup}(a)\}.$$

Therefore (α, β) is an ideal-type hyper value for $(X, \tilde{\mu})$. □

Corollary 3.19. *Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, let $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0,1] \times [0,1]$ be such that $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are nonempty. If $(X, \tilde{\mu})$ is a $(1,2)$-hyperfuzzy (resp., $(3,2)$-hyperfuzzy and $(3,4)$-hyperfuzzy) ideal of $(X, *, 0)$, then (α, β) is an ideal-type hyper value for $(X, \tilde{\mu})$.*

Proof. Straightforward. □

Theorem 3.20. *Let $(X, \tilde{\mu})$ be a hyper structure over $(X, *, 0)$. If (α, β) is an ideal-type hyper value for $(X, \tilde{\mu})$, then $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are I-energetic subsets of $(X, *, 0)$.*

Proof. Let $x, y, a, b \in X$ be such that $y \in L(\tilde{\mu}_{\inf}; \alpha)$ and $b \in U(\tilde{\mu}_{\sup}; \beta)$. Then $\tilde{\mu}_{\inf}(y) \leq \alpha$ and $\tilde{\mu}_{\sup}(b) \geq \beta$. Since (α, β) is an ideal-type hyper value for $(X, \tilde{\mu})$, it follows from (3.2) that $\min\{\tilde{\mu}_{\inf}(y * x), \tilde{\mu}_{\inf}(x)\} \leq \alpha$ and $\max\{\tilde{\mu}_{\sup}(b * a), \tilde{\mu}_{\sup}(a)\} \geq \beta$. Hence

$$\tilde{\mu}_{\inf}(y * x) \leq \alpha \text{ or } \tilde{\mu}_{\inf}(x) \leq \alpha$$

and

$$\tilde{\mu}_{\sup}(b * a) \geq \beta \text{ or } \tilde{\mu}_{\sup}(a) \geq \beta,$$

that is,

$$y * x \in L(\tilde{\mu}_{\inf}; \alpha) \text{ or } x \in L(\tilde{\mu}_{\inf}; \alpha)$$

and

$$b * a \in U(\tilde{\mu}_{\sup}; \beta) \text{ or } a \in U(\tilde{\mu}_{\sup}; \beta).$$

Thus $\{y * x, x\} \cap L(\tilde{\mu}_{\inf}; \alpha) \neq \emptyset$ and $\{b * a, a\} \cap U(\tilde{\mu}_{\sup}; \beta) \neq \emptyset$, and therefore $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are I-energetic subsets of $(X, *, 0)$. □

Combining Theorems 3.18 and 3.20, we have the following corollary.

Corollary 3.21. *Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0)$, let $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta \subseteq [0,1] \times [0,1]$ be such that $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are nonempty. If $(X, \tilde{\mu})$ is a $(1,4)$-hyperfuzzy ideal of $(X, *, 0)$, then $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are I-energetic subsets of $(X, *, 0)$.*

Definition 3.22 ([3]). A nonempty subset A of $(X, *, 0)$ is said to be *right vanished* if it satisfies:

$$(\forall x, y \in X) \, (x * y \in A \;\Rightarrow\; x \in A) \,. \tag{3.4}$$

A is said to be *right stable* if $A * X := \{a * x \mid a \in A, \, x \in X\} \subseteq A$.

Lemma 3.23 ([6]). *If $(X, \tilde{\mu})$ is a $(4,1)$-hyperfuzzy ideal of $(X, *, 0)$, then*

$$(\forall x, y \in X) \, (x \leq y \;\Rightarrow\; \tilde{\mu}_{\inf}(x) \leq \tilde{\mu}_{\inf}(y), \, \tilde{\mu}_{\sup}(x) \geq \tilde{\mu}_{\sup}(y)) \,. \tag{3.5}$$

Theorem 3.24. *Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0) \in \mathcal{B}_K(X)$ and $(\alpha, \beta) \in [0,1] \times [0,1]$, if $(X, \tilde{\mu})$ is a $(4,1)$-hyperfuzzy ideal of $(X, *, 0)$, then $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are right stable subsets of $(X, *, 0)$ whenever they are nonempty.*

Proof. Let $(\alpha, \beta) \in [0,1] \times [0,1]$ be such that $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are nonempty. Let $x, a, b \in X$ be such that $a \in L(\tilde{\mu}_{\inf}; \alpha)$ and $b \in U(\tilde{\mu}_{\sup}; \beta)$. Then $\tilde{\mu}_{\inf}(a) \leq \alpha$ and $\tilde{\mu}_{\sup}(b) \geq \beta$. Since $a * x \leq a$ and $b * x \leq b$, it follows from Lemma 3.23 that $\tilde{\mu}_{\inf}(a * x) \leq \tilde{\mu}_{\inf}(a) \leq \alpha$ and $\tilde{\mu}_{\sup}(b * x) \geq \tilde{\mu}_{\sup}(b) \geq \beta$, that is, $a * x \in L(\tilde{\mu}_{\inf}; \alpha)$ and $b * x \in U(\tilde{\mu}_{\sup}; \beta)$. Hence $L(\tilde{\mu}_{\inf}; \alpha) * X \subseteq L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta) * X \subseteq U(\tilde{\mu}_{\sup}; \beta)$. Therefore $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are right stable subsets of $(X, *, 0)$. □

Corollary 3.25. *Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0) \in \mathcal{B}_K(X)$ and $(\alpha, \beta) \in [0,1] \times [0,1]$, if $(X, \tilde{\mu})$ is a $(2,1)$-hyperfuzzy (resp., $(2,3)$-hyperfuzzy and $(4,3)$-hyperfuzzy) ideal of $(X, *, 0)$, then $L(\tilde{\mu}_{\inf}; \alpha)$ and $U(\tilde{\mu}_{\sup}; \beta)$ are right stable subsets of $(X, *, 0)$ whenever they are nonempty.*

Proof. Straightforward. □

Theorem 3.26. *Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0) \in \mathcal{B}_K(X)$ and $(\alpha, \beta) \in [0,1] \times [0,1]$, if $(X, \tilde{\mu})$ is a $(4,1)$-hyperfuzzy ideal of $(X, *, 0)$, then $U(\tilde{\mu}_{\inf}; \alpha)$ and $L(\tilde{\mu}_{\sup}; \beta)$ are right vanished subsets of $(X, *, 0)$ whenever they are nonempty.*

Proof. Let $(\alpha, \beta) \in [0,1] \times [0,1]$ be such that $U(\tilde{\mu}_{\inf}; \alpha)$ and $L(\tilde{\mu}_{\sup}; \beta)$ are nonempty. Assume that $x * y \in U(\tilde{\mu}_{\inf}; \alpha)$ and $a * b \in L(\tilde{\mu}_{\sup}; \beta)$ for any $x, y, a, b \in X$. Using Lemma 3.23 implies that

$$\alpha \leq \tilde{\mu}_{\inf}(x * y) \leq \tilde{\mu}_{\inf}(x), \text{ that is, } x \in U(\tilde{\mu}_{\inf}; \alpha)$$

and

$$\beta \geq \tilde{\mu}_{\sup}(a * b) \geq \tilde{\mu}_{\sup}(a), \text{ that is, } a \in L(\tilde{\mu}_{\sup}; \beta).$$

Hence $U(\tilde{\mu}_{\inf}; \alpha)$ and $L(\tilde{\mu}_{\sup}; \beta)$ are right vanished subsets of $(X, *, 0)$. □

Corollary 3.27. *Given a hyper structure $(X, \tilde{\mu})$ over $(X, *, 0) \in \mathcal{B}_K(X)$ and $(\alpha, \beta) \in [0,1] \times [0,1]$, if $(X, \tilde{\mu})$ is a $(2,1)$-hyperfuzzy (resp., $(2,3)$-hyperfuzzy and $(4,3)$-hyperfuzzy) ideal of $(X, *, 0)$, then $U(\tilde{\mu}_{\inf}; \alpha)$ and $L(\tilde{\mu}_{\sup}; \beta)$ are right vanished subsets of $(X, *, 0)$ whenever they are nonempty.*

Proof. Straightforward. □

References

[1] J. Ghosh and T. K. Samanta, Hyperfuzzy sets and hyperfuzzy group, Int. J. Advanced Sci Tech. 41 (2012), 27–37.

[2] Y. S. Huang, BCI-algebra, Science Press, Beijing, 2006.

[3] Y. B. Jun, S. S. Ahn and E. H. Roh, Energetic subsets and permeable values with applications in BCK/BCI-algebras, Appl. Math. Sci. 7 (2013), no. 89, 4425–4438.

[4] Y. B. Jun, K. Hur and K. J. Lee, Hyperfuzzy subalgebras of BCK/BCI-algebras, Ann. Fuzzy Math. Inform. 15 (2018), no. 1, 17–28.

[5] J. Meng and Y. B. Jun, BCK-algebras, Kyungmoon Sa Co., Seoul, 1994.

[6] S. Z. Song, S. J. Kim and Y. B. Jun, Hyperfuzzy ideals in BCK/BCI-algebras, Mathematics **2017**, 5, 81.

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO. 2, 2020

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO. 2, 2020

(continued)

Volume 28, Number 3 May 2020
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of

Computational

Analysis and

Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(six times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, St.Martin Univ.,Olympia,WA,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Editorial Board
Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

Journal Article

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus homepage.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

Control problems for semilinear impulsive differential control systems

Ah-ran Park[1] and Jin-Mun Jeong[2,*]

[1,2]Department of Applied Mathematics, Pukyong National University
Busan 48513, Republic of Korea

Abstract

In this paper, we establish the approximate controllability for the semilinear impulsive differential equation in relation to the the corresponding linear control system based on the regularity for the equation under natural assumptions such as the local Lipschitz continuity of nonlinear term.

Keywords: approximate controllability, semilinear equation, ,impulsive differential equation, local lipschitz continuity, controller operator, reachable set

AMS Classification Primary 35B37; Secondary 93C20

1 Introduction

In this paper, we are concerned with the approximate controllability for the semilinear impulsive control system in Hilbert spaces:

$$\begin{cases} x^{'}(t) + Ax(t) = f(t, x(t)) + (Bu)(t), \quad t \in (0, T], \quad t = t_k, \\ k = 1, 2, \cdots, m, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, m, \\ x(0) = x_0. \end{cases} \tag{1.1}$$

Let H be identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections be continuous. Here, A is the operator associated with a sesquilinear form $a(\cdot, \cdot)$ defined on $V \times V$ satisfying Gårding's inequality:

$$(Au, v) = a(u, v), \quad u, \ v \in V$$

Email: [1]alanida@naver.com, [2,*]jmjeong@pknu.ac.kr(Corresponding author)
This work was supported by a Research Grant of Pukyong National University(2019Year).

where V is a Hilbert space such that $V \subset H \subset V^*$. Then $-A$ generates an analytic semigroup in both H and V^*(see [1, Theorem 3.6.1]) and so the equation (1.1) may be considered as an equation in H as well as in V^*. The nonlinear operator f from $[0,T]\times V$ to H is assumed to be locally Lipschitz continuous with respect to the second variable. Let U be a Banach space of control variables and the controller operator B be a bounded linear operator from the Banach space $L^2(0,T;U)$ to $L^2(0,T;H)$. The impulsive condition

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, m,$$

is a combination of traditional evolution systems. Let $x(t; f, u)$ be a solution of the equation (1.1) associated with a nonlinear term f and a control u. We will show the approximate controllability for the equation (1.1), namely that the reachable set $R_T(f) = \{x(T; f, u) : u \in L^2(0,T;U)\}$ is a dense subset of H. This kind of equations arise naturally in biology, in physics, control engineering problem, etc.

In the first part of this paper we establish the wellposedness and regularity property for the following equation:

$$\begin{cases} x^{'}(t) + Ax(t) = f(t, x(t)) + k(t), \quad t \in (0,T], \quad t = t_k, \\ k = 1, 2, \cdots, m, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, m, \\ x(0) = x_0. \end{cases} \tag{1.2}$$

The regularity for the semilinear heat equations has been developed as seen in Barbu [2] and [3, 4, 5, 6].

In this paper, based on the regularity for (1.2), we intend to establish the approximate controllability for (1.1). Approximate controllability for semilinear control systems can be founded in [7-15]. Similar considerations of linear and semilinear systems have been dealt with in many references, linear problems in the book [15] and Nakagiri [14], semilinear cases with the uniform bounded nonlinear term in [16], and with the uniform Lipschtz continuous nonlinear term in [3, 17, 18, 19]. However, there are few papers treating the systems with local Lipschipz continuity, we can just find a recent article Wang [20]. Among these literatures, in [17, 20], they assumed that the semigroup $S(t)$ generated by A is compact in order to guarantee the compactness of the solution mapping, and investigated the approximate controllability for the equation (1.1).

In this paper, in order to show that the main result of Naito [17] is extended to the nonlinear differential equation, we assume that the embedding $D(A) \subset V$ is compact instead of the compact property of semigroup used in [17, 21]. Then by virtue of the result in Aubin [22], we can take advantage of the fact that the solution mapping $u \in L^2(0,T;U) \mapsto x(T; f, u)$ is compact. Under natural assumptions such as the local Lipschtiz continuity of nonlinear term, we obtain the approximate controllability for the equation (1.1) when the corresponding linear system is approximately controllable.

The paper is organized as follows. In section 2, the results of general linear evolution equations besides notations and assumptions are stated. In section 3, we investigate the

approximate controllability for the problem (1.1). The approach used here is similar to that developed in [1, 3] on the general semilnear evolution equations, which is an important role to extend the theory of practical nonlinear partial differential equations.

2 Regularity for semilinear impulsive systems

The norm on V, H and V^* will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. We assume that V has a stronger topology than H and, for brevity, we may regard that

$$||u||_* \leq |u| \leq ||u||, \quad \forall u \in V. \tag{2.1}$$

Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u,u) \geq \omega_1 ||u||^2 - \omega_2 |u|^2, \tag{2.2}$$

where $\omega_1 > 0$ and ω_2 is a real number. Let A be the operator associated with this sesquilinear form:

$$(Au, v) = a(u,v), \quad u,\ v \in V.$$

Then $-A$ is a bounded linear operator from V to V^* by the Lax-Milgram Theorem. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by A. Then we consider the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{2.3}$$

where each space is dense in the next one which continuous injection. It is also well known that A generates an analytic semigroup $S(t)$ in both H and V^*. For the sake of simplicity, we assume that $\omega_2 = 0$ and hence the closed half plane $\{\lambda : \text{Re}\,\lambda \geq 0\}$ is contained in the resolvent set of A.

If X is a Banach space, $L^2(0,T;X)$ is the collection of all strongly measurable square integrable functions from $(0,T)$ into X and $W^{1,2}(0,T;X)$ is the set of all absolutely continuous functions on $[0,T]$ such that their derivative belongs to $L^2(0,T;X)$. $C([0,T];X)$ will denote the set of all continuously functions from $[0,T]$ into X with the supremum norm. Let the solution spaces $\mathcal{W}(T)$ and $\mathcal{W}_1(T)$ of strong solutions be defined by

$$\mathcal{W}(T) = L^2(0,T;D(A)) \cap W^{1,2}(0,T;H),$$
$$\mathcal{W}_1(T) = L^2(0,T;V) \cap W^{1,2}(0,T;V^*).$$

Here, we note that by using interpolation theory, we have

$$\mathcal{W}(T) \subset C([0,T];V), \quad \mathcal{W}_1(T) \subset C([0,T];H).$$

Thus, there exists a constant $M_0 > 0$ such that

$$||x||_{C([0,T];V)} \le M_0||x||_{\mathcal{W}(T)}, \quad ||x||_{C([0,T];H)} \le M_0||x||_{\mathcal{W}_1(T)}. \tag{2.4}$$

The semigroup generated by $-A$ is denoted by $S(t)$ and there exists a constant M such that

$$|S(t)| \le M, \quad ||s(t)||_* \le M.$$

Let f be a nonlinear mapping from V into H. We need to impose the following conditions on nonlinear term f.

Assumption (F). There exists a function $L : \mathbb{R}_+ \to \mathbb{R}$ such that $L(r_1) \le L(r_2)$ for $r_1 \le r_2$ and

$$|f(t,x)| \le L(r), \quad |f(t,x) - f(t,y)| \le L(r)||x-y||$$

hold for any $t \in [0,T]$, $||x|| \le r$ and $||y|| \le r$.

Assumption (I). The functions $I_k : V \to H$ are continuous and there exist positive constants $L(I_k)$ and $\beta \in (1/3, 1]$ such that

$$|A^\beta I_k(x)| \le L(I_k)||x||, \quad |A^\beta I_k(x) - I_k(y)| \le L(I_k)||x-y||, \quad k = 1, 2, \cdots, m$$

for each $x, y \in V$, and

$$||x(t_k^-)|| \le K, \quad k = 1, 2, \cdots, m.$$

From now on, we establish the following results on the local solvability of the following equation;

$$\begin{cases} x'(t) + Ax(t) = f(t, x(t)) + k(t), \quad t \in (0,T], \quad t \ne t_k, \\ \ \ k = 1, 2, \cdots, m, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \ k = 1, 2, \cdots, m, \\ x(0) = x_0. \end{cases} \tag{2.5}$$

Let us rewrite $(Fx)(t) = f(t, x(t))$ for each $x \in L^2(0,T;V)$. Then there is a constant, denoted again by $L(r)$, such that

$$||Fx||_{L^2(0,T;H)} \le L(r)\sqrt{T}, \quad ||Fx_1 - Fx_2||_{L^2(0,T;H)} \le L(r)||x_1 - x_2||_{L^2(0,T;V)}$$

hold for $x_1,\ x_2 \in B_r(T) = \{x \in L^2(0,T;V) : ||x||_{L^2(0,T;V)} \le r\}$. Here, we note that by using interpolation theory, we have that for any $t > 0$,

$$L^2(0,t;V) \cap W^{1,2}(0,t;V^*) \subset C([0,t];H).$$

Thus, for any $t > 0$, there exists a constant $c > 0$ such that

$$||x||_{C([0,t];H)} \le c||x||_{L^2(0,t;V) \cap W^{1,2}(0,t;V^*)}. \tag{2.6}$$

Let

$$0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m = T.$$

Then by Assumption (I) and (2.5), it is immediately seen that

$$x \in W^{1,2}(t_i, t_{i+1}; V^*), \quad i = 0, \cdots, m-1.$$

Thus by virtue of Assumption (I) and (2.6), we may consider that there exists a constant $C_3 > 0$ such that

$$\max_{0 \le t \le T} \{|x(t)| : x \text{ is a solution of } (2.5)\} \le C_3 ||x||_{L^2(0,T:V)}. \tag{2.6}$$

With the notations (2.2), (2.3), we have

$$(V, V^*)_{1/2,2} = H, \quad (D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between V and V^*(Section 1.3.3 of [23]). From now on, we establish the following results on the solvability of the equation (2.5).

Theorem 2.1. *1) Let Assumption (F) be satisfied. Assume that $x_0 \in H$, $k \in L^2(0,T;V^*)$. Then, there exists a time $T_0 \in (0,T)$ such that the equation (2.5) admits a solution*

$$x \in W_1(T_0) \subset C([0, T_0]; H). \tag{2.7}$$

2) Under Assumption (F) for the nonlinear mapping f, there exists a unique solution x of (2.5) such that

$$x \in \mathcal{W}_1(T) \equiv L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H), \quad T > 0.$$

for any $x_0 \in H$, $k \in L^2(0,T;V^*)$. Moreover, there exists a constant C_1 such that

$$||x||_{\mathcal{W}_1(T)} \le C_1(1 + |x_0| + ||k||_{L^2(0,T;V^*)}), \tag{2.8}$$

where C_1 is a constant depending on T.
3) Let Assumptions (F) and (I) be satisfied and $(x_0, k) \in H \times L^2(0,T;V)$. Then the solution x of the equation (2.5) belongs to $x \in \mathcal{W}_1 \equiv L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$ and the mapping

$$H \times L^2(0,T;V^*) \ni (x_0, k) \mapsto x \in \mathcal{W}_1(T) \tag{2.9}$$

is continuous.

Corollary 2.1. *Suppose that $k \in L^2(0,T;H)$ and $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \le t \le T$. Then there exists a constant C_2 such that*

$$||x||_{L^2(0,T;V)} \le C_2\sqrt{T}||k||_{L^2(0,T;H)}. \tag{2.10}$$

Proof. From Theorem 2.3 of [24], it follows that there exists a $C > 0$ such that

$$||x||_{L^2(0,T;D(A))} \leq C||k||_{L^2(0,T;H)}. \tag{2.11}$$

Moreover, we have

$$||x||^2_{L^2(0,T;H)} \leq M \int_0^T t \int_0^t |k(s)|^2 ds dt \leq M \frac{T^2}{2} \int_0^T |k(s)|^2 ds. \tag{2.12}$$

Since

$$(D(A), H)_{1/2,2} = V,$$

there exists a constant $C_0 > 0$ such that

$$||u|| \leq C_0 ||u||^{1/2}_{D(A)} |u|^{1/2}. \tag{2.13}$$

Thus, by (2.11), (2.12) and (2.13), if $C_2 = C_0\sqrt{CT}(M/2)^{1/4}$, then the inequality (2.10) holds. □

3 Approximate Controllability

Let U be a Banach space of control variables. Here B is a linear bounded operator from $L^2(0,T;U)$ to $L^2(0,T;H)$, which is called a controller. Consider the following nonlinear impulsive control systems.

$$\begin{cases} x^{'}(t) + Ax(t) = f(t, x(t)) + (Bu)(t), \quad t \in (0, T], \\ x(0) = x_0. \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \ k = 1, 2, \cdots, m. \end{cases} \tag{3.1}$$

Let $x(T; f, u)$ be a state value of the system (3.1) at time T corresponding to the nonlinear term f and the control u. Let $S(\cdot)$ be the analytic semigroup generated by $-A$. Then the solution $x(t; f, u)$ can be written as

$$x(t; f, u) = S(t)x_0 + \int_0^t S(t-s)\{f(s, x(s, f, u)) + (Bu)(s)\}ds + \sum_{0<t_k<t} S(t-s)I_k(x(t_k^-)),$$

and in view of Theorem 2.1

$$||x(\cdot; f, u)||_{\mathcal{W}_1(T)} \leq C_1(1 + |x_0| + ||B||||u||_{L^2(0,T;U)}). \tag{3.2}$$

We define the reachable sets for the system (3.1) as follows:

$$R_T(f) = \{x(T; f, u) : u \in L^2(0, T; U)\},$$
$$R_T(0) = \{x(T; 0, u) : u \in L^2(0, T; U)\}.$$

Definition 3.1. *The system (3.1) is said to be approximately controllable at time T if for every desired final state $x_1 \in H$ and $\epsilon > 0$ there exists a control function $u \in L^2(0,T;U)$ such that the solution $x(T;f,u)$ of (3.1) satisfies $|x(T;f,u) - x_1| < \epsilon$, that is, $\overline{R_T(f)} = H$ where $\overline{R_T(f)}$ is the closure of $R_T(f)$ in H.*

We define a linear bounded operator $\hat{S}$ from $L^2(0,T;H)$ to H by

$$\hat{S}p = \int_0^T S(T-t)p(t)dt,$$

for $p(\cdot) \in L^2(0,T;H)$.

Assumption (B) For any $\varepsilon > 0$, $p \in L^2(0,T;H)$ there exists a $u \in L^2(0,T;U)$ such that

$$\begin{cases} |\hat{S}p - \hat{S}Bu| \leq \varepsilon \\ ||Bu||_{L^2(0,t;H)} \leq q_1 ||p||_{L^2(0,t;H)}, \quad 0 \leq t \leq T \end{cases}$$

where q is a constant independent of p.

Assumption (F1) The nonlinear operator f is a nonlinear mapping of $[0,T] \times H$ into H satisfying the following. There exists a constant $L_1 = L_1(r) > 0$ such that

$$|f(t,x) - f(t,y)| \leq L_1 ||x - y||, \quad t \in [0,T],$$

hold for $||x|| \leq r$ and $||y|| \leq r$.

Assumption (H) We assume the following inequality condition:

$$max\{q, 1\}\{1 - M_2\}^{-1} C_2 L_1 \sqrt{T} < 1.$$

where C_2 is the constant in (2.10),

$$M_2 = C_2 \sqrt{T} L_1 + (3\beta)^{-1/2} 2(3\beta - 1)^{-1} C_{1-\beta} C_3 T^{3\beta/2} \sum_{0 \leq t_k \leq T} L(I_k).$$

Lemma 3.1. *Let u_1 and u_2 be in $L^2(0,T;U)$. Then under Assumption(B) and Assumption$(F1)$, one has that, for $0 \leq t \leq T$,*

$$||x(t:f,u_1) - x(t:f,u_2)||_{L_2(0,T;V)} \leq \{1 - M_2\}^{-1} C_2 \sqrt{t} ||Bu_1 - Bu_2||_{L^2(0,T;H)}. \quad (3.3)$$

Proof. Let $x_1(t) = x(t : f, u_1)$ and $x_2(t) = x(t : f, u_2)$. Then for $0 \le t \le T$,we have

$$\begin{aligned} x_1(t) - x_2(t) = &\int_0^t S(t-s)\{f(s, x_1(s)) - f(s, x_2(s))\}ds \\ &+ \int_0^t S(t-s)\{Bu_1 - Bu_2\}ds \\ &+ \sum_{0 \le t_k \le T} S(t-s)\{I_k(x_1(t_k^-)) - I_k(x_2(t_k^-))\}. \end{aligned} \tag{3.4}$$

By $Assumption(F1)$ and (2.10), we obtain

$$||\int_0^t S(t-s)\{f(s, x_1(s)) - f(s, x_2(s))\}ds||_{L^2(0,t;V)} \le C_2\sqrt{t}L_1||x_1 - x_2||_{L^2(0,t;V)}.$$

Moreover, by Lemma 2.5 of (2.11) and Theorem 3.1, we have

$$||\int_0^t S(t-s)\{Bu_1 - Bu_2\}ds||_{L^2(0,t;V)} \le C_2\sqrt{T}||Bu_1 - Bu_2||_{L^2(0,t;H)}$$

and

$$\begin{aligned} &||\sum_{0 \le t_k \le t} S(t-s)\{I_k(x_1(t_k^-)) - I_k(x_2(t_k^-))\}||_{L^2(0,t;V)} \\ &\le (3\beta)^{-1/2}2(3\beta-1)^{-1}C_{1-\beta}C_3 t^{3\beta/2} \sum_{0 \le t_k \le t} L(I_k)||x_1(t_k^-) - x_2(t_k^-)||_{L^2(0,t;V)}. \end{aligned}$$

Thus, from (3.4) it follows that

$$\begin{aligned} &||x(t; f, u_1) - x(t; f, u_2)||_{L^2(0,T;V)} \\ &\le C_2\sqrt{T}||Bu_1 - Bu_2||_{L^2(0,T;H)} + C_2\sqrt{T}L_1||x_1 - x_2||_{L^2(0,T;V)} \\ &+ (3\beta)^{-1/2}2(3\beta-1)^{-1}C_{1-\beta}C_3 t^{3\beta/2} \sum_{0 \le t_k \le t} L(I_k)||x_1(t_k^-) - x_2(t_k^-)||_{L^2(0,T;V)}. \end{aligned}$$

□

Theorem 3.1. *Under Assumptions (B),(F1), and (H) the system(4.1) is approximately controllable on* $[0, T]$.

Proof. The reachable set for the system(4.1) is given by

$$R_T = \{x(T; f, u) : u \in L^2(0, T; U)\}.$$

We will show that $D(A) \subset \overline{R_T(f)}$, i.e., for given $\varepsilon > 0$ and $\xi_T \in D(A)$, there exists $u \in L^2(0, T; U)$ such that

$$|\xi_T - x(T; f, u)| < \varepsilon, \tag{3.5}$$

where

$$x(T;,f,u) = S(T)x_0 + \int_0^T S(T-s)\{f(s,x(s,f,u)) + (Bu)(s)\}ds + \sum_{0<t_k<T} S(T-s)I_k(x(t_k^-)). \tag{3.6}$$

As $\xi_T \in D(A)$ there exists a $p \in L^2(0,T;H)$such that

$$\hat{S}p = \xi_T - S(T)x_0,$$

for instance, take $p(s) = (\xi_T - sA\xi_T) - S(s)x_0/T$. Let $u_1 \in L^2(0,T;U)$be arbitrary fixed. Since by Assumption (B) there exists $u_2 \in L^2(0,T;U)$ such that

$$|\hat{S}(p - f(\cdot, x(\cdot; f, u_1))) - \hat{S}Bu_2| < \frac{\varepsilon}{4}, \tag{3.7}$$

it follows that

$$|\xi_T - S(T)x_0 - \hat{S}f(\cdot, x(\cdot; f, u_1)) - \hat{S}Bu_2| < \frac{\varepsilon}{4}. \tag{3.8}$$

We can also choose $w_2 \in L^2(0,T;U)$ by Assumption (B) such that

$$|\hat{S}(f(\cdot, x(\cdot; f, u_2)) - f(\cdot, x(\cdot; f, u_1))) - \hat{S}Bw_2| < \frac{\varepsilon}{8} \tag{3.9}$$

$$||Bw_2||_{L^2(0,T;H)} \le q||f(\cdot, x(\cdot; f, u_2)) - f(\cdot, x(\cdot; f, u_1))||_{L^2(0,T;H)}.$$

Choose a constant r_1 satisfying

$$||x(\cdot; f, u_1)||_{C([0,T];H)} \le r_1, ||x(\cdot; f, u_2)||_{C([0,T];H)} \le r_1.$$

Therefor, in view of Lemma 3.1 and Assumption (B)

$$\begin{aligned} ||Bw_2||_{L^2(0,T;H)} &\le q||f(s, x(s; f, u_2)) - f(s, x(s; f, u_1))||_{L^2(0,T;H)} \\ &\le qL_1||x(t; f, u_1) - x(t; f, u_2)||_{L^2(0,T;V)} \\ &\le q\{1 - M_2\}^{-1}C_2L_1\sqrt{T}||Bu_1 - Bu_2||_{L^2(0,T;H)}. \end{aligned} \tag{3.10}$$

Put $u_3 = u_2 - w_2$. We determine w_3 such that

$$|\hat{S}(f(\cdot, x(\cdot; f, u_3)) - f(\cdot, x(\cdot; f, u_2))) - \hat{S}Bw_3| < \frac{\varepsilon}{8}$$

$$||Bw_3||_{L^2(0,T;H)} \le q||f(\cdot, x(\cdot; f, u_3)) - f(\cdot, x(\cdot; f, u_2))||_{L^2(0,T;H)}.$$

Let r_2 be a constant satisfying $r_2 \ge r_1$ and

$$||x(\cdot; f, u+3)||_{C([0,T];H)} \le r_2.$$

Then, in a similar way to (3.10) we have

$$\begin{aligned}||Bw_3||_{L^2(0,T;H)} &\le q||f(s,x(s;f,u_3)) - f(s,x(s;f,u_2))||_{L^2(0,T;H)}\\ &\le qL_1||x(t;f,u_3) - x(t;f,u_2)||_{L^2(0.T;V)}\\ &\le q\{1-M_2\}^{-1}C_2L_1\sqrt{T}||Bu_2 - Bu_3||_{L^2(0,T;H)}\\ &\le (q\{1-M_2\}^{-1}C_2L_1\sqrt{T})^2||Bu_1 - Bu_2||_{L^2(0,T;H)}.\end{aligned}$$

By proceeding with this process and from

$$\begin{aligned}&||B(u_n - u_{n+1})||_{L^2(0,T;H)}\\ &\quad = ||Bw_n||_{L^2(0,T;H)} \le (q\{1-M_2\}^{-1}C_2L_1\sqrt{T})^{n-1}||B(u_2 - u_1)||_{L^2(0,T;H)}.\end{aligned}$$

Here, nothing that Assumption (H) is equivalent to

$$q\{1-M_2\}^{-1}C_2L_1\sqrt{T} < 1,$$

it follows that there exists $u^* \in L^2(0,T;H)$ such that

$$\lim_{n\to\infty} Bu_n = u^* \quad \text{in} \quad L^2(0,T;H).$$

From(3.8),(3.9) it follow that

$$\begin{aligned}&|\xi_T - S(T)x_0 - \hat{S}f(\cdot,x(\cdot;f,u_2)) - \hat{S}Bu_3|\\ &\quad = |\xi_T - S(T)x_0 - \hat{S}f(\cdot,x(\cdot;f,u_1)) - \hat{S}Bu_2 + \hat{S}Bw_2\\ &\quad - [\hat{S}f(\cdot,x(\cdot;f,u_2)) - \hat{S}f(\cdot,x(\cdot;f,u_1))]|\\ &\quad < (\frac{1}{2^2} + \frac{1}{2^3})\varepsilon.\end{aligned}$$

By choosing $w_n \in L^2(0,T;U)$ by Assumption (B), such that

$$|\hat{S}(f(\cdot,x(\cdot;f,u_n)) - f(\cdot,x(\cdot;f,u_{n-1}))) - \hat{S}Bw_n| < \frac{\varepsilon}{2^{n+1}}$$

putting $u_{n+1} = u_n - w_n$ we have

$$\begin{aligned}&|\xi_T - S(T)x_0 - \hat{S}f(\cdot,x(\cdot;f,u_n)) - \hat{S}Bu_{n+1}|\\ &\quad < (\frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}})\varepsilon, \quad n = 1,2,....\end{aligned}$$

Therefor, for $\varepsilon > 0$ there exists integer N such that

$$|\hat{S}Bu_{N+1} - \hat{S}Bu_N| < \frac{\varepsilon}{2},$$

$$\begin{aligned}&|\xi_T - S(T)x_0 - \hat{S}f(\cdot,x(\cdot;f,u_N)) - \hat{S}Bu_N|\\ &\quad \le |\xi_T - S(T)x_0 - \hat{S}f(\cdot,x(\cdot;f,u_N)) - \hat{S}Bu_{N+1}| + |\hat{S}Bu_{N+1} - \hat{S}Bu_N|\\ &\quad \le (\frac{1}{2^2} + \cdots + \frac{1}{2^{N+1}})\varepsilon + \frac{\varepsilon}{2} \le \varepsilon.\end{aligned}$$

Thus, the system (3.1) is approximately controllable on $[0,T]$ as N tends to infinity. □

References

[1] H. Tanabe, *Equations of Evolution*, Pitman-London, 1979.

[2] V. Barbu, Analysis and Control of Nonlinear Infinite Dimensional Systems, Academic Press Limited, 1993.

[3] J. M. Jeong, Y. C. Kwun and J. Y. Park, *Approximate controllability for semilinear retarded functional differential equations*, J. Dynamics and Control Systems, 5(3) (1999), 329-346.

[4] Y. Kobayashi, T. Matsumoto and N. Tanaka, *Semigroups of locally Lipschitz operators associated with semilinear evolution equations*, J. Math. Anal. Appl. 330(2) (2007), 1042-1067.

[5] A. Pazy, Semigroups of Linear Operators and Applications to partial Differential Equations, Springer-Verlag Newyork, Inc. 1983.

[6] G. Webb, *Continuous nonlinear perturbations of linear accretive operator in Banach spaces*, J. Fun. Anal. 10 (1972), 191-203.

[7] J. jeong, Y. H. Kang *Controllability for trajectories of semilinear functional differential equations*, Bull. Korean Math. Soc. 55 (2018), 63-79.

[8] M. Benchohra and A. Ouahab, *Controllability results for functional semilinear differential inclusion in Frechet spaces*, Nonlinear Analysis 61(2005), 405-423.

[9] A. E. Bashirov and N. I. Mahmudov, *On concepts of controllability for deterministic and stochastic systems*, SIAM J. Control Optim. 37(1999), 1808-1821.

[10] L. Górniewicz, S. K. Ntouyas and D. O'Reran, *Controllability of semilinear differential equations and inclusions via semigroup theory in Banach spaces*, Rep. Math. Phys. 56(2005), 437-470.

[11] N. I. Mahmudov, *Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces*, SIAM J. Control Optim. 42(2006), 175-181.

[12] L. Wang, *Approximate controllability and approximate null controllability of semilinear systems*, Commun. Pure and Applied Analysis 5(2006), 953-962.

[13] R. F. Curtain and H. Zwart, An Introduction to Infinite Dimensional Linear Systems Theory, Springer-Velag, New-York, 1995.

[14] S. Nakagiri, Controllability and identifiability for linear time-delay systems in Hilbert space. Control theory of distributed parameter systems and applications, Lecture Notes in Control and Inform. Sci., 159, Springer, Berlin, 1991.

[15] R. F. Curtain and H. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory*, Springer-Velag, New-York, 1995.

[16] M. Yamamoto and J. Y. Park, *Controllability for parabolic equations with uniformly bounded nonlinear terms*, J. Optim. Theory Appl. 66(1990), 515-32.

[17] K. Naito, *Controllability of semilinear control systems dominated by the linear part*, SIAM J. Control Optim. 25 (1987), 715-722.

[18] N. Sukavanam and Nutan Kumar Tomar, *Approximate controllability of semilinear delay control system*, Nonlinear Func.Anal.Appl. 12(2007), 53-59.

[19] H. X. Zhou, *Approximate controllability for a class of semilinear abstract equations*, SIAM J. Control Optim. 21(1983), 551-565.

[20] L. Wang, *Approximate controllability for integrodifferential equations and multiple delays*, J. Optim. Theory Appl. 143(2009), 185-206.

[21] J. P. Dauer and N. I. Mahmudov, *Exact null controllability of semilinear integrodifferential systems in Hilbert spaces*, J. Math. Anal. Appl. 299(2004), 322-333.

[22] J. P. Aubin,*Un thèoréme de compacité*, C. R. Acad. Sci. 256(1963), 5042-5044.

[23] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland, 1978.

[24] G. Di Blasio, K. Kunisch and E. Sinestrari, L^2-*regularity for parabolic partial integrodifferential equations with delay in the highest-order derivatives*, J. Math. Anal. Appl. 102 (1984), 38-57.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Homoclinic solutions for a class of difference equations with asymptotically linear nonlinearity

Ali Mai*, Guowei Sun

Department of Mathematics and information technology, Yuncheng University

Shanxi, Yuncheng 044000, China

Abstract

A class of difference equations with asymptotically linear nonlinearity are considered in this paper. The existence of homoclinic solutions of the equations are obtained by using generalized saddle point theorem.

Key words: Generalized saddle point theorem; Difference equations; $(PS)_c$ sequence; Homoclinic solutions.

1 Introduction

In this paper, we consider the following difference equation

$$Lu_n - \omega u_n = f_n(u_n), \quad n \in \mathbb{Z}, \tag{1.1}$$

where

$$Lu_n = a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n$$

is a Jacobi operator ([14]), here $\{a_n\}$ and $\{b_n\}$ are real valued T-periodic sequences, and T is a positive integer.

As in the literature, a solution $u = \{u_n\}$ of (1.1) is homoclinic solution if

$$\lim_{|n|\to\infty} u_n = 0. \tag{1.2}$$

This problem appears in the following discrete nonlinear schrödinger equation

$$i\dot{\psi}_n = -\Delta\psi_n + v_n\psi_n - f_n(\psi_n), \quad n \in \mathbb{Z}, \tag{1.3}$$

*Corresponding author. E-mail address: maialiy@126.com

where

$$\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$$

is the discrete one-dimension Laplacian. And the potential $V = \{v_n\}$ is real valued T-periodic sequences, i.e., $v_{n+T} = v_n$, for all $n \in \mathbb{Z}$. Moreover, we assume that the nonlinearity $f_n(u)$ is gauge invariant, i.e.,

$$f_n(e^{i\theta}u) = e^{i\theta}f_n(u), \quad \theta \in \mathbb{R}.$$

We consider special solutions of (1.3)

$$\psi_n = u_n e^{-i\omega t},$$

where $\omega \in \mathbb{R}$ is the temporal frequency and $\{u_n\}$ is a real valued sequence such that

$$\lim_{|n|\to\infty} \psi_n = 0.$$

Such solutions are called solitons. Inserting the soliton Ansatz into (1.3), then

$$-\Delta u_n + v_n u_n - \omega u_n = f_n(u_n), \quad n \in \mathbb{Z}, \tag{1.4}$$

and

$$\lim_{|n|\to\infty} u_n = 0 \tag{1.5}$$

holds. Therefore, in order to looking for solitons of equation (1.3), we just need to get the homoclinic solutions of equation (1.4), which is a special case of (1.1) with $a_n \equiv -1$ and $b_n = 2 + v_n$.

It is well known that the operator L is a bounded and self-adjoint operator in l^2. Its spectrum $\sigma(L)$ is a union of a finite number of closed intervals and the complement $\mathbb{R}\backslash\sigma(L)$ consists of a finite number of open intervals called spectral gaps. Two of them are semi-infinite (see [14]). In particular, $T = 1$, then finite gaps do not exist. In general, finite gaps do exist. The most interesting case of equation (1.1) is when the frequency ω belongs to a finite gap. The solitons of (1.3) with the temporal frequency ω belonging to a spectral gap, in particular to a finite gap are important. Such solitons are called gap solitons. Fix any finite spectral gap and denote it by (α, β).

Discrete nonlinear schrödinger equation (DNLS) is one of the most important inherently discrete models, It appears in a great variety of applications, such as nonlinear optics, solid state, condensed matter physics and biology (see [1–3, 5, 13] and reference therein). It also has been successfully applied to the modeling of localized pulse propagation in optical fibers and wave guides, to the study of energy relaxation in solids, to the behavior of amorphous material, to the modeling of self-trapping of vibrational energy in proteins or studies related to the denaturation of the DNA double strand ([6, 7, 18]). In the past decade, the periodic DNLS equations have been considered in the physics literature ([15]). For example, results

2

on numerical simulation of gap solitons in a particular periodic DNLS equation are obtained in [4].

With the development of variational techniques, solitons of the periodic DNLS equations have become a hot topic. The existence of solitons for the periodic DNLS equations with superlinear nonlinearity (see [10, 11] and reference therein) and with saturable nonlinearity ([16, 17]) have been studied, respectively. Discrete soliton is a kind of homoclinic solutions. In this paper, we employ generalized saddle point theorem developed by Liu and Shen in [9] and obtain homoclinic solutions of equation (1.1).

The organization of this paper is as follows. In Section 2, we introduce the functional, and its critical points are solutions of the problem and remind a critical point theorem, then present the main result. The detailed proofs of the main result is given in Section 3.

2 Preliminaries and main results

Throughout this paper, we assume that

(V) $\omega \notin \sigma(L)$ and $\omega \in (\alpha, \beta)$.

(f_1) $f_n \in C(\mathbb{R}, \mathbb{R})$, $f_n(u)u \geq 0$ for all $u \in \mathbb{R}$.

(f_2) Assume that f_n is asymptotically linear at infinity, i.e.,

$$\lim_{|u| \to \infty} \frac{f_n(u)}{u} = 0. \tag{2.1}$$

(f_3) $f_n(u) = o(u)$ as $u \to 0$.

To study the homoclinic solutions, we consider the real sequence spaces

$$l^p = \left\{ u = \{u_n\}_{n \in \mathbb{Z}} : \forall\, n \in \mathbb{Z}, u_n \in \mathbb{R}, \|u\|_{l^p} = \left(\sum_{n \in \mathbb{Z}} |u_n|^p \right)^{\frac{1}{p}} < \infty \right\}. \tag{2.2}$$

Between l^p spaces the following elementary embedding holds,

$$l^q \subset l^p, \quad \|u\|_{l^p} \leq \|u\|_{l^q}, \quad 1 \leq q \leq p \leq \infty. \tag{2.3}$$

To state our results, we fix some notation. Let

$$A = L - \omega \ \text{ and } \ E = l^2(\mathbb{Z}).$$

Consider the functional J defined on E by

$$J(u) = \frac{1}{2}\ (Au, u) - \sum_{n \in \mathbb{Z}} F_n(u_n), \tag{2.4}$$

3

where $(\cdot,\cdot)$ is the inner product in E, $\|\cdot\|$ is the corresponding norm in E. $F_n(u)$ is the primitive function of $f_n(u)$, *i.e.*,

$$F_n(u) = \int_0^u f_n(s)ds.$$

Standard arguments show that the functional $J \in C^1(E,\mathbb{R})$ and equation (1.1) is easily recognized as the corresponding Euler-Lagrange equation for J. Thus, critical points of J are solutions of equation (1.1).

It is easy to get the derivative of J,

$$(J'(u), v) = (Au, v)_E - \sum_{n\in\mathbb{Z}} f_n(u_n)v_n, \quad \forall\ v \in E. \tag{2.5}$$

By (V), then we have the orthogonal decomposition $E = E^+ \oplus E^-$ corresponding to the spectral decomposition of A with respect to the positive and negative part of the spectrum, and

$$(Au, u)_E \geq (\beta-\omega)\|u\|_E^2, \qquad u \in E^+,$$

$$(Au, u)_E \leq (\alpha-\omega)\|u\|_E^2, \qquad u \in E^-.$$

For any $u, v \in E$, letting $u = u^+ + u^-$ with $u^\pm \in E^\pm$ and $v = v^+ + v^-$ with $v^\pm \in E^\pm$, we can define an equivalent inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$ on E by

$$(u, v) = (Au^+, v^+)_E - (Au^-, v^-)_E \text{ and } \|u\| = (u, u)^{\frac{1}{2}},$$

respectively. So J can be rewritten as

$$J(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \sum_{n\in\mathbb{Z}} F_n(u_n) \equiv \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - I(u). \tag{2.6}$$

Note that if ω lies in a finite spectral gap, then $\dim E^- = \infty$ and the problem (1.1) and (1.2) is strongly indefinite. Now our main result can be stated as the following:

Theorem 2.1. *Suppose that conditions (V), $(f_1)-(f_3)$ are satisfied, then equation (1.1) at least has one solution.*

Let $R > 0$. Set

$$M = \{u \in E^- : \|u\| \leq R\}.$$

Let $\{e_k\}$ be a total orthonormal sequence in E^-, we define a norm on E^- by

$$\|u\|_{E^-} = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} | < u, e_k > |.$$

Let $P_\pm : E \to E^\pm$ be the orthogonal projection of E onto $E^\pm$. We denote by τ the topology on E generated by the norm

$$\|u\|_\tau = \max\left(\|P_+u\|, \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} | < P_-u, e_k > |\right).$$

Remark 2.1. Note that if $u_n \xrightarrow{\tau} u$, then $P_+u_n \to P_+u$ and $P_-u_n \rightharpoonup P_-u$.

Definition 2.1. Let $J \in C^1(E)$, we say J is $\tau-$upper semicontinuous if $u_n \xrightarrow{\tau} u$ implies

$$J(u) \geq \overline{\lim_{n\to\infty}} J(u_n).$$

Definition 2.2. Let $J \in C^1(E)$, we say J' is weakly sequentially continuous, if $u_n \rightharpoonup u$ implies $J'(u_n) \to J'(u_n)$, as $n \to \infty$.

The purpose of this paper is to use the generalized saddle point theorem to solve some strongly indefinite problems with asymptotically linear nonlinearity. The following lemma is the generalized saddle point theorem taken from [9] and will play an important role in the proofs of our main results.

Lemma 2.1. *Assume that $J \in C^1(E, \mathbb{R})$ is $\tau-$upper semicontinuous and J' is weakly sequentially continuous. If*

$$b := \inf_{E^+} J > \sup_{\partial M} J, \quad d = \sup_{M} J < \infty,$$

then for some $c \in [b, d]$, there is a sequence $\{u_n\} \subset E$ such that

$$J(u_n) \to c \text{ and } J'(u_n) \to 0 \textit{ as } n \to \infty. \tag{2.7}$$

Such a sequence is called a Palais-Smale sequence on the level c, or a $(PS)_c$ sequence.

3 Proofs of main results

Lemma 3.1. *Assume that (V) and $(f_1) - (f_3)$ are satisfied. Then J is $\tau-$upper semicontinuous, and J' is weakly sequentially continuous.*

Proof. Let $u^{(k)} \xrightarrow{\tau} u$ and $c = \overline{\lim_{k\to\infty}} J(u^{(k)})$. Then there is a subsequence, still denoted by $\{u^{(k)}\}$ such that $J(u^{(k)}) \to c$. By Remark 2.1 we have

$$u^{(k)+} \to u^+ \quad \text{and} \quad u^{(k)-} \rightharpoonup u^-, \quad \text{as} \quad k \to \infty. \tag{3.1}$$

Passing to a subsequence if necessary, we have $u_n^{(k)} \to u_n$ for all $n \in \mathbb{Z}$, as $k \to \infty$, hence, $F_n(u_n^{(k)}) \to F_n(u_n)$. Since $F_n(u^{(k)}) \geq 0$, using the Fatou lemma we have

$$I(u) = \sum_{n\in\mathbb{Z}} \lim_{k\to\infty} F_n(u_n^{(k)}) \leq \underline{\lim_{k\to\infty}} \sum_{n\in\mathbb{Z}} F_n(u_n^{(k)}) = \underline{\lim_{k\to\infty}} I(u^{(k)}). \tag{3.2}$$

Combining (3.1) and (3.2), we have

$$\begin{aligned} -J(u) &= \frac{\|u^-\|^2}{2} - \frac{\|u^+\|^2}{2} + I(u) \\ &\leq \lim_{k\to\infty} \left(\frac{\|u^{(k)-}\|^2}{2} - \frac{\|u^{(k)+}\|^2}{2} + I(u^{(k)}) \right) \\ &= \lim_{k\to\infty} \left(-J(u^{(k)}) \right) = -c. \end{aligned}$$

So $J(u) \geq c$ and J is $\tau-$upper semicontinuous.

Finally, we show that J' is weakly sequentially continuous. Let $u^{(k)} \rightharpoonup u$ in E, we have that $u_n^{(k)} \to u_n$ for all $n \in \mathbb{Z}$, as $k \to \infty$. and there exists $M > 0$ such that $\|u^{(k)}\| \leq M$ and $\|u\| \leq M$. By (f_3), there exists constant C_0 such that $|f_n(u)| \leq C_0|u|$ for $|u| \leq M$.

For any $v \in E$ fix $0 < N \in \mathbb{N}$ such that $\sum_{|n|>N} |v_n|^2 < \frac{\varepsilon^2}{16C_0^2M^2}$. Therefore, we have

$$\begin{aligned} |I'(u^{(k)})v - I'(u)v| &\leq |\sum_{n=-N}^{N} (f_n(u_n^{(k)}) - f_n(u_n))v_n| + |\sum_{|n|>N} (f_n(u_n^{(k)}) - f_n(u_n))v_n| \\ &\leq |\sum_{n=-N}^{N} (f_n(u_n^{(k)}) - f_n(u_n))v_n| + C_0(\|u^{(k)}\| + \|u\|) \left(\sum_{|n|>N} |v_n|^2 \right)^{\frac{1}{2}} \\ &\leq |\sum_{n=-N}^{N} (f_n(u_n^{(k)}) - f_n(u_n))v_n| + \frac{\varepsilon}{2}. \end{aligned}$$

Note that $f_n(u_n^{(k)}) \to f_n(u_n)$, as $k \to \infty$, then there exists k_0 such that for $k \geq k_0$,

$$|\sum_{n=-N}^{N} (f_n(u_n^{(k)}) - f_n(u_n))v_n| < \frac{\varepsilon}{2}.$$

So $|I'(u^{(k)})v - I'(u)v| < \varepsilon$, for all $k \geq k_0$. By the definition of J', then J' is weakly sequentially continuous. $\square$

Proof of Theorem 2.1.

By (f_2) and (f_3), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f_n(u)| \leq C_\varepsilon |u|, \quad |F_n(u)| \leq C_\varepsilon |u|^2.$$

For $u \in E^+$, we have

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}} F_n(u_n) \\ &\geq \frac{1}{2}\|u\|^2 - C_\varepsilon \|u\|^2 \\ &= (\frac{1}{2} - C_\varepsilon)\|u\|^2. \end{aligned}$$

So $\inf_{E^+} J > -\infty$.

For $u \in E^-$, since $F(u) \geq 0$, we have

$$\begin{aligned} J(u) &= -\frac{1}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}} F_n(u_n) \\ &\leq -\frac{1}{2}\|u\|^2. \end{aligned}$$

For R large enough, we have

$$\inf_{E^+} J > \sup_{\partial M} J, \quad \sup_{M} J < \infty,$$

where $M = \{u \in E^- : \|u\| \leq R\}$.

By Lemma 2.1, for some $c \in \mathbb{R}$, there is a sequence $\{u^{(k)}\}$ such that

$$J(u^{(k)}) \to c \text{ and } J'(u^{(k)}) \to 0 \ as \ k \to \infty.$$

Let $\widetilde{u}^{(k)} = u^{(k)+} - u^{(k)-}$, then $\|\widetilde{u}^{(k)}\| = \|u^{(k)}\|$ and

$$\begin{aligned} \|u^{(k)}\| &= \|\widetilde{u}^{(k)}\| \geq (J'(u^{(k)}), \widetilde{u}^{(k)}) \\ &= \|u^{(k)+}\|^2 + \|u^{(k)-}\|^2 - \sum_{n\in\mathbb{Z}} f_n(u_n^{(k)})\widetilde{u}_n^{(k)} \\ &\geq \|u^{(k)}\|^2 - \sum_{n\in\mathbb{Z}} C_\varepsilon |u_n^{(k)}|(|u_n^{(k)+}| + |u_n^{(k)-}|) \\ &\geq \|u^{(k)}\|^2 - C_\varepsilon \|u^{(k)}\|\|u^{(k)+}\| - C_\varepsilon \|u^{(k)}\|\|u^{(k)-}\| \\ &= \|u^{(k)}\|^2 - C_\varepsilon \|u^{(k)}\|^2. \end{aligned}$$

It implies $\{u^{(k)}\}$ is bounded.

Next we may extract a subsequence, still denoted by $\{u^{(k)}\}$, such that $u^{(k)} \rightharpoonup u$ and $u_n^{(k)} \to u_n$ for all $n \in \mathbb{Z}$. Moreover, we have

$$(J'(u), v) = \lim_{k\to\infty} (J'(u^{(k)}), v) = 0, \ \forall v \in E,$$

so $J'(u) = 0$ and u is a homoclinic solution of (1.1). □

Acknowledgments

This work is Supported by National Natural Science Foundation of China(11526183), the Natural Science Foundation of Shanxi Province (2015021015) and Foundation of Yuncheng University(YQ-2017003, YQ-2014011).

References

[1] A. Davydov, The theory of contraction of proteins under their excitation, *J. Theor. Biol.* 38 (1973) 559-569.

[2] S. Flach, A. Gorbach, Discrete breakers-Advances in theory and applications, *Phys. Rep.* 467 (2008) 1-116.

[3] S. Flash, C. Willis, Discrete breathers, *Phys. Rep.* 295 (1998) 181-264.

[4] A. V. Gorbach, M. Johansson, Gap and out-gap breathers in a binary modulated discrete nonlinear Schrödinger model, *Eur. Phys. J. D* 29 (2004) 77-93.

[5] D. Henning, G. P. Tsironis, Wave transmission in nonlinear lattics, *Phys. Rep.* 307 (1999) 333-432.

[6] P. G. Kevrekidis, K. ϕ. Rasmussen, A. R. Bishop, The discrete nonlinear Schrödinger equation: a survey of recent results, *Int. J. Mod. Phys. B* 15 (2001) 2883-2900.

[7] Y. S. Kivshar, G. P. Agrawal, Optical Solitons: From Fibers to Photonic Crystals, Academic Press, San Diego, 2003.

[8] G. Li, A. Szulkin, An asymptotically periodic Schrödinger equation with indefinite linear part, *Commun. Contemp. Math.* 4 (2002) 763-776.

[9] S. Liu, Z. Shen, Generalized saddle point theorem and asymptotically linear problems with periodic potential, *Nonlinear Anal.* 86 (2013):52-57.

[10] A. Mai, G. Sun, Ground state solutions for second order nonlinear p-Laplacian difference equations with periodic coefficients, *J. Comput. Anal. Appl.*,2017, 22(7): 1288-1297.

[11] A. Mai, Z. Zhou, Discrete solitons for periodic discrete nonlinear Schrödinger equations, *Appl. Math. Comput.* 222(2013): 34-41.

[12] A. Pankov, N. Zakharchenko, On some discrete variational problems, *Acta. Appl. Math.* 65 (2000) 295-303.

[13] W. Su, J. Schieffer, A. Heeger, Solitons in polyacetylene, *Phys. Rev. Lett.* 42 (1979) 1698-1701.

[14] G. Teschl, Jacobi operators and completely integrable nonlinear lattices, *Mathematical Surveys and Monographs*, vol. 72, American Mathematical Society, Providence, RI, 2000.

[15] A. A. Sukhorukov, Y. S. Kivshar, Generation and stability of discrete gap solitons, *Opt. Lett.* 28 (2003) 2345-2347.

[16] H. Shi, Gap solitons in periodic discrete Schrödinger equations with nonlinearity, Acta Appl. Math. 109 (2010) 1065-1075.

[17] H. Shi, H. Zhang, Existence of gap solitons in periodic discrete nonlinear Schrödinger equations, J. Math. Anal. Appl. 361 (2010) 411-419.

[18] M. Weintein, Excitation thresholds for nonlinear localized modes on lattices, *Nonlinearity* 12 (1999) 673-691.

[19] M. Willem, Minimax Theorems, Birkhäuser, Boston, 1996.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

APPROXIMATION OF ALMOST CAUCHY'S POINTS BY CAUCHY'S POINTS

GWANG HUI KIM AND HWAN-YONG SHIN

Abstract. In this paper, we investigate Hyers–Ulam stability of Cauchy's mean value points which is a extended and generalized version of I. R. Peter and D. Popa's theorem [10] and then, as applications, we obtain Hyers-Ulam stability results of Lagrange's mean value points which refine the result of P. Găvruţă, J. Huang and Y. Li [5].

1. Introduction

The concept of Hyers–Ulam stability was raised by S. M. Ulam [11] in 1940. We are given a group G and a metric group G' with metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $d(f(x), h(x)) < \varepsilon$ for all $x \in G$? Ulam's question was partially solved by D. H. Hyers [6] in the case of approximately additive functions and when the groups in the question are Banach spaces. Due to the question of Ulam and the answer of Hyers, the stability of functional equations is called after their names. For more information of Hyers–Ulam stability, we can refer to [1, 2].

A similar problem of Ulam's question can be formulated for the mean value points : "Assume that a function f satisfies a mean value theorem with a point η. If ξ is a point near to η, does there exists a function g near to f satisfying the same mean value theorem with the point ξ?" [10].

It seems that the first result to the previous question was given by D. H. Hyers and S. M. Ulam [7] in the case of differential expressions.

Theorem 1.1. (*D. H. Hyers, S. M. Ulam, 1954,* [7]) *Let $f : \mathbb{R} \to \mathbb{R}$ be n-times differentiable in a neighborhood N of a point η. Suppose that $f^{(n)}(\eta) = 0$ and $f^{(n)}(x)$ changes sign at η. Then, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for every function $g : \mathbb{R} \to \mathbb{R}$ which is n-times differentiable in N and satisfies $|f(x) - g(x)| < \delta$ for all $x \in N$, there exists a point $\xi \in N$ such that $g^{(n)}(\xi) = 0$ and $|\xi - \eta| < \varepsilon$.*

2010 *Mathematics Subject Classification.* 39B52, 39B82, 54C65.

Key words and phrases. mean value theorem, Cauchy's mean value points, Lagrange's mean value points, Hyers–Ulam stability.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

In 2003, M. Das, T. Riedel and P. K. Sahoo [3] proved the stability problem for Flett's mean value points by using Theorem 1.1. Subsequently, some authors applied the idean from [3] to prove the Hyers–Ulam stability of various mean value points [5, 8, 9, 10]. Especially, P. Găvruţă, S.-M. Jung and Y. Li [5] proved the following stability result of Lagrange's mean value points which is a point η of a differentiable function $f : [a, b] \to \mathbb{R}$ satisfying $\frac{f(b)-f(a)}{b-a} = f'(\eta)$.

Theorem 1.2. (*P. Găvruţă, S.-M. Jung, Y. Li, 2010,* [5]) *Let a, b, η be real numbers satisfying $a < \eta < b$. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function and η is the unique Lagrange's mean value point of f in an open interval (a, b) and moreover that $f''(\eta) \neq 0$. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a differentiable function. Then, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|f(x) - g(x)| < \delta$ for all $x \in [a, b]$, then there is a Lagrange's mean value point $\xi \in (a, b)$ of g with $|\xi - \eta| < \varepsilon$.*

Hereafter, Theorem 1.2 was generalized by I. R. Peter and D. Popa [10] by proving the stability of Cauchy's mean value points which is a point η of two differentiable functions $f, g : [a, b] \to \mathbb{R}$ satisfying

$$(f(b) - f(a))g'(\eta) - (g(b) - g(a))f'(\eta) = 0.$$

Let I be an open interval which contains the interval (a, b).

Theorem 1.3. (*I. R. Peter, D. Popa, 2013,* [10]) *Assume that $f, g : I \to \mathbb{R}$ are continuously differentiable functions, η is the unique Cauchy's mean value point of the pair (f, g) in I and f, g are twice continuously differentiable in a neighborhood of η, satisfying*

$$f''(\eta)(g(b) - g(a)) - g''(\eta)(f(b) - f(a)) \neq 0.$$

Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $f_1, g_1 : (a, b) \to \mathbb{R}$ are continuously differentiable functions with the property that $|f(x) - f_1(x)| < \delta$ and $|g(x) - g_1(x)| < \delta$ for all $x \in [a, b]$ there exists a Cauchy mean value point $\xi \in (a, b)$ of (f_1, g_1) with $|\eta - \xi| < \varepsilon$.

In this paper, we prove Hyers–Ulam stability of Cauchy's mean value points which is a extended and generalized version of Theorem 1.3 and then, as applications, we obtain the stability results of Lagrange's mean value points which refine Theorem 1.2.

2. Hyers–Ulam Stability of Cauchy's mean value points

We now present a main theorem, which is a Hyers–Ulam stability of Cauchy's mean value points for real-valued differentiable functions on $[a, b]$.

Theorem 2.1. *Let $f, g, f_1, g_1 : [a, b] \to \mathbb{R}$ be countinuously differentiable functions and η be a Cauchy's mean value point of the pair (f, g) in the interval (a, b) and $N \subseteq (a, b)$ be a neighborhood of η. Suppose the following control function*

$$(f(b) - f(a))g'(x) - (g(b) - g(a))f'(x)$$

changes sign at η. Then, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|f(x) - f_1(x)| < \delta$ and $|g(x) - g_1(x)| < \delta$ for all $x \in N \cup \{a, b\}$, then there exists a point $\xi \in N$ such that ξ is a Cauchy's mean value point of (f_1, g_1) with $|\xi - \eta| < \varepsilon$.

Proof. Let $\varepsilon > 0$ be given and $N \subseteq (a, b)$ be any neighborhood of η. Consider the auxiliary function $G_{f,g}(x) : [a, b] \to \mathbb{R}$ corresponding to (f, g) defined by

$$G_{f,g}(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$$

for all $x \in [a, b]$. Evidently $G_{f,g}(x)$ is continuous on $[a, b]$ and differentiable on $[a, b]$. Further, we have

$$G'_{f,g}(x) = (f(b) - f(a))g'(x) - (g(b) - g(a))f'(x), \quad x \in [a, b].$$

Since η is the Cauchy's mean value point of (f, g), we get $G'_{f,g}(\eta) = 0$. Thus it follows from the assumption that there exists a neighborhood $(\eta - r, \eta + r) \subseteq N$ of η such that $G'_{f,g}(x)$ changes sign at η in $(\eta - r, \eta + r) \subseteq N$ for some $r > 0$ with $\eta - r > a$. Then it follows from Theorem 1.1 that there exists a $\overline{\delta} > 0$ such that for any differentiable function H on $[a, b]$ with $|H(x) - G_{f,g}(x)| < \overline{\delta}$ for x in $(\eta - r, \eta + r)$, there exists a point $\zeta \in (\eta - r, \eta + r)$ satisfying $H'(\zeta) = 0$ and $|\zeta - \eta| < \varepsilon$.

For a continuous function $f : [a, b] \to \mathbb{R}$ define

$$M_f := \max\{|f(x)| : x \in [a, b]\}$$

and analogously M_g. Define $G_{f_1,g_1}(x) : [a, b] \to \mathbb{R}$ be the corresponding auxiliary function defined as

$$G_{f_1,g_1}(x) = (f_1(b) - f_1(a))g_1(x) - (g_1(b) - g_1(a))f_1(x)$$

for all $x \in [a, b]$.

For some fixed $\lambda > 0$, let

$$\delta := \min\Big\{\frac{\overline{\delta}}{4M_f + 4M_g + 4\lambda}, \lambda\Big\}.$$

and let $f_1, g_1 : [a, b] \to \mathbb{R}$ be any differentiable functions satisfying $|f(x) - f_1(x)| < \delta$ and $|g(x) - g_1(x)| < \delta$ for all $x \in N \cup \{a, b\}$. Then one can easy to see that $G_{f_1,g_1}(x)$ is differentiable

in N. And it follows that

$$\begin{aligned} |f_1(b) - f_1(a)| &\le |f_1(b) - f(b)| + |f(b) - f(a)| + |f(a) - f_1(a)| \\ &\le 2\lambda + 2M_f. \end{aligned}$$

By the same reason we obtain that

$$|g_1(b) - g_1(a)| \le 2\lambda + 2M_g.$$

These yield that

$$\begin{aligned} |G_{f,g}(x) - G_{f_1,g_1}(x)| &= |(f(b) - f(a))g(x) - (g(b) - g(a))f(x) \\ &\quad -(f_1(b) - f_1(a))g_1(x) + (g_1(b) - g_1(a))f_1(x)| \\ &= |(f(b) - f(a))g(x) - (f_1(b) - f_1(a))g(x) \\ &\quad +(f_1(b) - f_1(a))g(x) - (f_1(b) - f_1(a))g_1(x) \\ &\quad +(g_1(b) - g_1(a))f_1(x) - (g_1(b) - g_1(a))f(x) \\ &\quad +(g_1(b) - g_1(a))f(x) - (g(b) - g(a))f(x)| \\ &\le (|f(b) - f_1(b)| + |f(a) - f_1(a)|)|g(x)| \\ &\quad +|f_1(b) - f_1(a)| \cdot |g(x) - g_1(x)| \\ &\quad +|g_1(b) - g_1(a)| \cdot |f_1(x) - f(x)| \\ &\quad +(|g_1(b) - g(b)| + |g_1(a) - g(a)|)|f(x)| \\ &\le (2M_g + |f_1(b) - f_1(a)| + |g_1(b) - g_1(a)| + 2M_f)\delta \\ &\le (4M_f + 4M_g + 4\lambda)\delta \\ &\le \bar{\delta} \end{aligned}$$

for all $x \in (\eta - r, \eta + r) \subseteq N$. Hence, there exists a point $\xi \in (\eta - r, \eta + r)$ such that $G'_{f_1,g_1}(\xi) = 0$ and $|\xi - \eta| < \varepsilon$. We note that $G'_{f_1,g_1}(\xi) = 0$ implies

$$(f_1(b) - f_1(a))g_1'(\xi) - (g_1(b) - g_1(a))f_1'(\xi) = 0.$$

Hence, the point ξ is a Cauchy's mean value point of (f_1, g_1) and the proof is complete. □

The following corollary is a refined result of Theorem 1.3.

Corollary 2.2. *Let $f, g, f_1, g_1 : [a, b] \to \mathbb{R}$ be countinuously differentiable functions and η be a Cauchy's mean value point of the pair (f, g) in the interval (a, b) and $N \subseteq (a, b)$ be a neighborhood of η. Suppose either η is unique Cauchy's mean value point of (f, g) or f, g have second derivative at η such that*

$$[f(b) - f(a)]g''(\eta) \ne [g(b) - g(a)]f''(\eta). \tag{2.1}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Then, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|f(x)-f_1(x)| < \delta$ and $|g(x)-g_1(x)| < \delta$ for all $x \in N \cup \{a, b\}$, then there exists a point $\xi \in N$ such that ξ is a Cauchy's mean value point of (f_1, g_1) with $|\xi - \eta| < \varepsilon$.

Proof. Let $G_{f,g} : [a, b] \to \mathbb{R}$ be defined as

$$G_{f,g}(x) = (f(b) - f(a))(g(x) - g(a)) - (g(b) - g(a))(f(x) - f(a))$$

for all $x \in [a, b]$. Suppose η is a unique Cauchy's mean value point of (f, g). Then we obtain that $G_{f,g}(a) = G_{f,g}(b)$ and $\eta \in (a, b)$ is a unique point such that $G'_{f,g}(\eta) = 0$. These yield that $G'_{f,g}(x)$ changes sign at η.

If f and g have second derivative and satisfy (2.1), we have $G''_{f,g}(\eta) \neq 0$. Thus associating this fact and $G'_{f,g}(\eta) = 0$, we get $G'_{f,g}(x)$ changes sign at η.

Rewriting the fact $G_{f,g}$ changes sign at η, we obtain

$$(f(b) - f(a))g'(x) - (g(b) - g(a))f'(x)$$

changes sign at η. By applying Theorem 2.1, we get the desired result. □

If we take $f_1, g_1 : [a, b] \to \mathbb{R}$ by $f_1 := h$ and $g_1 := g$ in Theorem 2.1 and Corollary 2.2, then we get the following two corollaries.

Corollary 2.3. *Let $f, g, h : [a, b] \to \mathbb{R}$ be differentiable and η be a Cauchy's mean value point of the pair (f, g) in the interval (a, b) and $N \subseteq (a, b)$ be a neighborhood of η. Suppose the following control function*

$$(f(b) - f(a))g'(x) - (g(b) - g(a))f'(x)$$

changes sign at η. Then, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|f(x) - h(x)| < \delta$ for all $x \in N \cup \{a, b\}$, then there exists a point $\xi \in N$ such that ξ is a Cauchy's mean value point of (g, h) with $|\xi - \eta| < \varepsilon$.

Corollary 2.4. *Let $f, g, h : [a, b] \to \mathbb{R}$ be countinuously differentiable functions and η be a Cauchy's mean value point of the pair (f, g) in the interval (a, b) and $N \subseteq (a, b)$ be a neighborhood of η. Suppose either η is a unique Cauchy's mean value point of (f, g) or f, g have second derivative at η such that*

$$(f(b) - f(a))g''(\eta) \neq (g(b) - g(a))f''(\eta).$$

Then, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|f(x)-h(x)| < \delta$ for all $x \in N \cup \{a, b\}$, then there exists a point $\xi \in N$ such that ξ is a Cauchy's mean value point of (g, h) with $|\xi - \eta| < \varepsilon$.

The following theorem is another type of Hyers-Ulam stability for Cauchy's mean value points.

Theorem 2.5. *Let a, b, ξ be real numbers satisfying $a < \xi < b$. Assume that $f, g : [a, b] \to \mathbb{R}$ are countinuously differentiable functions such that*

$$g'(x), \frac{f'(x)g''(x) - f''(x)g'(x)}{g'(x)^2} \neq 0$$

for all $x \in [a, b]$. If

$$\Big|\frac{f'(\xi)}{g'(\xi)} - \frac{f(b) - f(a))}{g(b) - g(a)}\Big| \leq \varepsilon \tag{2.2}$$

for some $\varepsilon > 0$, then there exists a Cauchy's mean value point η of (f, g) on (a, b) satisfying

$$|\xi - \eta| \leq \frac{\varepsilon}{min_{x\in[a,b]}\Big|\frac{f'(x)g''(x)-f''(x)g'(x)}{g'(x)^2}\Big|}.$$

Proof. Due to Cauchy's mean value theorem, there exists a point $\eta \in (a, b)$ such that

$$\frac{f'(\eta)}{g'(\eta)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Hence it follows from (2.2) that

$$\Big|\frac{f'(\xi)}{g'(\xi)} - \frac{f'(\eta)}{g'(\eta)}\Big| \leq \varepsilon.$$

If $\xi = \eta$ then the proof is clear. Otherwise, we assume that $a < \eta < \xi < b$. Since f and g have second derivative on $[a, b]$, by Lagrange's mean value theorem, there exists a point $\xi_0 \in (\eta, \xi)$ such that

$$\Big|(\xi - \eta)\Big(\frac{f'(\xi_0)g''(\xi_0) - f''(\xi_0)g'(\xi_0)}{g'(\xi_0)^2}\Big)\Big| = \Big|\frac{f'(\eta)}{g'(\eta)} - \frac{f'(\xi)}{g'(\xi)}\Big|.$$

Since f', f'', g', g'' are continuous on $[a, b]$, we obtain

$$|\xi - \eta| = \Bigg|\frac{\frac{f'(\eta)}{g'(\eta)} - \frac{f'(\xi)}{g'(\xi)}}{\frac{f'(\xi_0)g''(\xi_0)-f''(\xi_0)g'(\xi_0)}{g'(\xi_0)^2}}\Bigg| \leq \frac{\varepsilon}{min_{x\in[a,b]}\Big|\frac{f'(x)g''(x)-f''(x)g'(x)}{g'(x)^2}\Big|},$$

which complete the proof. □

3. Applications to Lagrange's mean value points

In this section, we obtain stability results of Lagrange's mean value points for the differentiable functions on $[a, b]$.

Corollary 3.1. *Let $f, g : [a, b] \to \mathbb{R}$ be countinuously differentiable functions and η be a Lagrange's mean value point of f in (a, b) and $N \subseteq (a, b)$ be a neighborhood of η. Suppose the following control function*

$$f(b) - f(a) - (b - a)f'(x)$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

changes sign at η. Then, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|f(x) - g(x)| < \delta$ for all $x \in N \cup \{a, b\}$ there exists a point $\xi \in N$ such that ξ is a Lagrange's mean value point of g with $|\xi - \eta| < \varepsilon$.

Proof. Consider the auxiliary function $G_f(x) : [a, b] \to \mathbb{R}$ corresponding to f defined by

$$G_f(x) = (f(b) - f(a))x - f(x)(b - a)$$

for all $x \in [a, b]$. Then the proof goes through the same way as that of Theorem 2.1. □

Example 3.2. Let $f : [-2\pi, 2\pi] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \cos x - 1, & \text{if} \quad x \le 0, \\ 1 - \cos x, & \text{if} \quad x > 0. \end{cases}$$

It is obvious to see that there exist three Lagrange's mean value points $-\pi, 0, \pi$ of f. Let N_i be a neighborhood of $(-1)^i\pi$ for each $i = 1, 2$. We can easily check that $f(2\pi)-f(-2\pi)-(2\pi-(-2\pi))f'(x) = -4\pi f'(x)$ changes sign at $\pm\pi$. Therefore, by Corollary 3.1, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for every differentiable function g satisfying $|f(x)-g(x)| < \delta$ for all $x \in N_i \cup \{\pm 2\pi\}$ then there exists a point $\xi_i \in N_i$ such that ξ_i is a Lagrange's mean value point of g and $|\xi_i - (-1)^i\pi| < \varepsilon$. However, $f(2\pi) - f(-2\pi) - (2\pi - (-2\pi))f'(x) = -4\pi f'(x)$ does not change sign at 0, and so we cannot apply Corollary 3.1 for the function f at the Lagrange's mean value point 0.

Let $N := (-\frac{\pi}{4}, \frac{\pi}{4})$ and $\delta > 0$ be given. And let $g : [-2\pi, 2\pi] \to \mathbb{R}$ be defined by

$$g(x) := f(x) + \frac{\delta}{1024}(x^3 - 4\pi^2 x)$$

for all $x \in [-2\pi, 2\pi]$. Then

$$|f(x) - g(x)| = \frac{\delta}{1024}|x^3 - 4\pi^2 x| < \frac{\delta}{1024}((2\pi)^3 + 4\pi^2(2\pi)) < \delta$$

for all $x \in N \cup \{\pm 2\pi\}$. But, for all $x \in N$, the following inequlity holds

$$\frac{g(2\pi) - g(-2\pi)}{4\pi} - g'(x) > 0.$$

Therefore, we can conclude that there is no Lagrange's mean value point of g in N.

The following refined result of Theorem 1.2 is obtained as a corollary of Corollary 3.1.

Corollary 3.3. *Let $f, g : [a, b] \to \mathbb{R}$ be countinuously differentiable functions and η be a Lagrange's mean value point of f in (a, b) and $N \subseteq (a, b)$ be a neighborhood of η. Suppose either η is a unique Lagrange's mean value point of f or f has second derivative at η with $f''(\eta) \ne 0$. Then, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|f(x) - g(x)| < \delta$ for all*

$x \in N \cup \{a, b\}$, then there exists a point $\xi \in N$ such that ξ is a Lagrange's mean value point of g with $|\xi - \eta| < \varepsilon$.

References

[1] J. Brzdek, W. Fechner, M. Moslehian, J. Sikorska, *Recent developments of the conditional stability of the homomorphism equation*, Banach J. Math. Anal., **9** (2015) no. 3, 278-326.

[2] K. Ciepliński, *Applications of fixed point theorems to the Hyers-Ulam stability of functional equations–a survey*, Ann. Funct. Anal., **3** (2012), no. 1, 151-164.

[3] M. Das, T. Riedel, P. K. Sahoo, *Hyers-Ulam stability of Flett's points*, Appl. Math. Lett., **16(3)** (2013), 269-271.

[4] P. Găvruţă, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184** (1994), 431-436.

[5] P. Găvruţă, S.-M. Jung, Y. Li, *Hyers–Ulam stability of mean value points*, Ann. Funct. Anal., **295** (2010), no. 2, 68-74.

[6] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci, U.S.A. **27**, (1941), 222-224.

[7] D. H. Hyers, S. M. Ulam, *On the stability of differential expressions*, Math. Mag., **28**, (1954), 59-64.

[8] H.-M. Kim, H.-Y. Shin, *Approximation of almost Sahoo–Riedel's points by Sahoo–Riedel's points*, Aequat. Mathe., **90**, (2016), 809-815.

[9] W. Lee, S. Xu, F. Ye, *Hyers-Ulam stability of Sahoo-Riedel's point*, Appl. Math. Lett., **22**, (2009), 1649-1652.

[10] I. R. Peter and D. Popa, *Stability of points in mean value theorems*, Publ. Math. Debrecen, **83/3** (2013), 375-384.

[11] S. M. Ulam, *Problems in Modern Mathematics*, Chapter 6 Wiley Interscience, New York, (1964).

Gwang Hui Kim, Department of Mathematics, Kangnam Universaty,Yongin, Gyeonggi 16979, Republic of Korea

E-mail address: ghkim@kangnam.ac.kr

Hwan-Yong Shin, Department of Mathematics, Chungnam National University,99 Daehangno, Yuseong-gu, Daejeon 34134, Republic of Korea

E-mail address: hyshin31@cnu.ac.kr

Weak Galerkin Finite Element Method for Convection-Diffusion-Reaction Problems

F. Z. Gao[a,1], **A. K. Hashim**[b,2], **S. C. Mohammed**[b,3]

[a] School of Math., Shandong Univ., Jinan, China
[b]Dept. of Math., College of Education for Pure Science
Univ. of Basrah, Basrah, Iraq

Abstract

In this paper, a weak Galerkin (WG) finite element method is proposed for solving the convection-diffusion-reaction problems. The main idea of WG finite element methods is the use of weak functions and their corresponding discrete weak derivatives in standard weak form of the model problem. We show that the continuous time WG finite element method preserves the energy conservation law as well the optimal order error estimate in L^2 norm. Numerical experiment is conducted to confirm the theoretical results.

Keywords: WG finite element method, convection-diffusion-reaction equation, energy conservation law, error estimate.

1 Introduction

The convection-diffusion-reaction processes appear in many areas of science and technology. For example, fluid dynamics, heat and mass transfer hydrology and so on. In this paper, we consider the following convection-diffusion-reaction equation:

$$\begin{aligned} u_t - \nabla\cdot(\lambda\nabla u) + b\cdot\nabla u + cu &= f, \quad (x,t)\in\Omega\times(0,T], &(1.1)\\ u(x,0) &= 0, \quad x\in\Omega, &(1.2)\\ u(x,t)|_\Gamma &= g, \quad t\in(0,T], &(1.3)\end{aligned}$$

where Ω is a bounded region in R^2, with a Lipschitz continuous boundary $\Gamma=\partial\Omega$, $u_t=\frac{\partial u}{\partial t}$, and ∇u denote the gradient of function $u=u(x,t)$. Further $\lambda>0$ is a diffusion coefficient, b is a convection coeffient and f, g are given functions.

The standard weak form of equations $(1.1)-(1.3)$ seeks $u\in L^2(0,T;H^1(\Omega))$ such that $u=g$ on $\partial\Omega\times(0,T)$ and

$$(u_t,v)+(\lambda\nabla u,\nabla v)-(bu,\nabla v)+(cu,v)=(f,v), \quad \forall v\in H_0^1(\Omega). \tag{1.4}$$

The WG finite element method refers to a general finite element technique for partial differential equation where the differential operators (e.g., gradient, divergence, curl, Laplacian) are approximated by weak forms. The method, first introduced by Wang and Ye [1] for solving a second order elliptic problems, is a newly developed finite element method. Since

[1]E-mail address: fzgao@sdu.edu.cn
[2]E-mail addresses: hkashkool@yahoo.com
[3]Corresponding author. E-mail address: cheichan.mohammed@yahoo.com

1

then, some WG finite element methods have been developed to solve other problems, such as parabolic equation [2, 3, 4], Stokes equations [5, 6], Helmholtz equation [7], Biharmonic equation [8, 9] and Navier-Stokes equations [10, 11], etc.

In general, WG finite element formulations for partial differential equation can be derived naturally by replacing usual derivatives by variational forms. The implementations of all these possible extension are based on the computation of these weak operators.

The rest of this paper is organized as follows. In section 2, we shall introduce some preliminaries and notations for Sobolev spaces. We define the weak gradient and discrete weak gradient operator and the weak finite element spaces and present semi-discrete WG finite element method for problem (1.1) − (1.3) in section 3 and section 4, respectively. In section 5, we prove the energy conservation law of the continuous time WG approximation, and in section 6 we present optimal order error estimate in L^2 norm for the WG finite element approximations. Finally, we present a numerical example to verify theory.

2 Preliminaries and notations

We use standard definitions for the Sobolev spaces $H^m(\Omega)$ and their associated inner products $(\cdot,\cdot)_{m,\Omega}$, norms $\|\cdot\|_{m,\Omega}$, and seminorms $|\cdot|_{m,\Omega}$ for $m \geq 0$ [12, 13]. For any integers $m \geq 0$ the seminorm $|\cdot|_{m,\Omega}$ is given by

$$|v|_{m,\Omega} = (\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v|^2 d\Omega)^{1/2},$$

with the usual notation

$$\alpha = (\alpha_1, \alpha_2), \quad |\alpha| = \alpha_1 + \alpha_2, \quad \partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}.$$

The Sobolev norm $\|\cdot\|_{n,\Omega}$, is given by

$$\|v\|_{n,\Omega} = (\sum_{j=0}^{n} |v|_{j,\Omega}^2)^{1/2}.$$

The space $H(div;\Omega)$ is defined as the set of vector-valued functions on Ω which, together with their divergence, are square integrable; i.e,

$$H(div;\Omega) = \{v : v \in [L^2(\Omega)]^2, \nabla\cdot v \in L^2(\Omega)\}.$$

The norm in $H(div;\Omega)$ is defined by

$$\|v\|_{H(div;\Omega)} = (\|v\|^2 + \|\nabla\cdot v\|^2)^{1/2}.$$

3 A weak Gradient operator and its approximation

In this section we introduce a weak gradient operator defined on a space of generalized functions. Let K be any polygonal domain with interior K^0 and boundary ∂K. A weak function on the region K refers to vector-valued function $v = \{v_0, v_b\}$ such that $v_0 \in L^2(K)$ and $v_b \in H^{1/2}(\partial K)$. The first component v_0 can be understood as the value of v in interior

2

of K, and the second component v_b is the value of v on the boundary of ∂K. Denote by $W(K)$ the space of weak function associated with K ; i.e.,

$$W(K) := \{v = \{v_0, v_b\} : v_0 \in L^2(K), v_b \in H^{1/2}(\partial K)\}. \tag{3.1}$$

Definition 3.1. *For any $v \in W(K)$, the weak gradient of v is defined as a linear functional $\nabla_d v$ in the dual space of $H(div, K)$ whose action on each $q \in H(div, K)$ is given by*

$$\int_K \nabla_d v \cdot q dK = -\int_K v_0 \nabla \cdot q dK + \int_{\partial K} v_b q \cdot \boldsymbol{n} ds, \tag{3.2}$$

where $\boldsymbol{n}$ is the outward normal direction to ∂K.

Next, we introduce a discrete weak gradient operator by defining ∇_d in a polynomial subspace of $H(div, K)$. To this end , for any non-negative integer $r \geq 0$ denote by $P_r(K)$ the set of polynomials on K with degree no more than r. Let $V(K, r) \subset [P_r(K)]^2$ be a subspace of the space of vector-valued polynomials of degree r. A discrete weak gradient operator, denoted by $\nabla_{d,r}$, is defined so that $\nabla_{d,r} v \in V(K, r)$ is the unique solution of the following equation

$$\int_K \nabla_{d,r} v \cdot q dK = -\int_K v_0 \nabla \cdot q dK + \int_{\partial K} v_b q \cdot \mathbf{n} ds, \quad \forall q \in V(K, r). \tag{3.3}$$

4 A weak Galerkin finite element scheme

Let T_h be triangular partition of the domain Ω with mesh size h. Assume that the partition T_h is shape regular so that the routine inverse inequality holds true (see[13]). In the general spirit of Galerkin procedure, we shall design a WG method for (1.4) by following two basic principles: first replacing $H^1(\Omega)$ by a space of discrete weak functions defined on the finite element partition T_h and the boundary of triangular elements; second replacing the classical gradient operator by a discrete weak gradient operator $\nabla_{d,r}$ for weak functions on each triangle T.

For each $T \in T_h$. Denote by $P_j(T^0)$ the set of polynomials with degree no more than j and $P_\ell(\partial T)$ the set of polynomial on ∂T with degree no more than ℓ. A discrete weak function $v = \{v_0, v_b\}$ on T refers to a weak function $v = \{v_0, v_b\}$ such that $v_0 \in P_j(T^0)$ and $v_b \in P_\ell(\partial T)$ with $j \geq 0$ and $\ell \geq 0$. Denote this space by $W(T, j, \ell)$, i.e.,

$$W(T, j, \ell) = \{v = \{v_0, v_b\} : v_0 \in P_j(T^0), v_b \in P_\ell(\partial T)\}. \tag{4.1}$$

The corresponding finite element space would be defined by patching $W(T, j, \ell)$ over all the triangles $T \in T_h$. In other words, the weak finite element space is given by

$$S_h(j, \ell) = \{v = \{v_0, v_b\} : \{v_0, v_b\}|_T \in W(T, j, \ell), \forall T \in T_h\}. \tag{4.2}$$

Denote by $S_h^0(j, \ell)$ the subspace of $S_h(j, \ell)$ with vanishing boundary values on $\partial\Omega$, i.e.,

$$S_h^0(j, \ell) = \{v = \{v_0, v_b\} \in S_h(j, \ell), v_b|_{\partial T \cap \partial\Omega} = 0, \forall T \in T_h\}. \tag{4.3}$$

To investigate the approximation properties of the discrete weak space $S_h(j, \ell)$, we define three projections in this paper. The first two are local projections defined on each triangle T:

one is $Q_h u = \{Q_0 u, Q_b u\}$, the L^2 projection of $H^1(T)$ onto $P_j(T^0) \times P_{j+1}(\partial T)$ and another is R_h, the L^2 projection of $[L^2(T)]^2$ onto $V(T, r)$. The third projection Π_h is assumed to exist and satisfy the following property: for $q \in H(div, \Omega)$ with mildly added regularity, $\Pi_h q \in H(div, \Omega)$ such that $\Pi_h q \in V(T, r)$ on each $T \in T_h$, and

$$(\nabla \cdot q, v_0)_T = (\nabla \cdot \Pi_h q, v_0)_T, \quad \forall v_0 \in P_j(T). \tag{4.4}$$

It is easy to see the following two useful identities:

$$\nabla_{d,r}(Q_h u) = R_h(\nabla u), \ \ \forall u \in H^1(T), \tag{4.5}$$

and for any $q \in H(div, \Omega)$

$$\sum_{T \in T_h} (-\nabla \cdot q, v_0)_T = \sum_{T \in T_h} (\Pi_h q, \nabla_{d,r} v)_T, \ \ \forall v = \{v_0, v_b\} \in S_h^0(j, \ell). \tag{4.6}$$

Now for any $u, v \in S_h(j, \ell)$, we introduce the following bilinear form

$$a(u, v) = (\lambda \nabla_{d,r} u, \nabla_{d,r} v) - (b u_0, \nabla_{d,r} v) + (c u_0, v_0), \tag{4.7}$$

where

$$\begin{aligned} (\lambda \nabla_{d,r} u, \nabla_{d,r} v) &= \int_\Omega \lambda \nabla_{d,r} u \cdot \nabla_{d,r} v d\Omega, \\ (b u_0, \nabla_{d,r} v) &= \int_\Omega b u_0 \cdot \nabla_{d,r} v d\Omega, \\ (c u_0, v_0) &= \int_\Omega c u_0 v_0 d\Omega. \end{aligned}$$

We pose the continuous time WG finite element method based on (3.3) and (1.4) which is to find $u_h(t) = \{u_0(\cdot, t), u_b(\cdot, t)\}$, belonging to $S_h(j, \ell))$ for $t > 0$, satisfying $u_b = Q_b g$ on $\partial\Omega$, and the following equation

$$((u_h)_t, v_0) + a(u_h, v) = (f, v_0), \qquad \forall v = \{v_0, v_b\} \in S_h^0(j, \ell), \tag{4.8}$$

where

$$a(u_h, v) = (\lambda \nabla_{d,r} u, \nabla_{d,r} v) - (b u_h, \nabla_{d,r} v) + (c u_h, v_0),$$

where, $Q_b g$ is an approximation of the boundary value in the polynomial space $P_\ell(\partial T \cap \partial\Omega)$. For simplicity, $Q_b g$ shall be taken as the standard L^2 projection for each boundary segment.

5 Energy conservation property of WG

In this section, we investigate the energy conservation property of the semi-discrete WG finite element approximation u_h. The solution u of the problem $(1.1)-(1.3)$ has the following energy preserving property on each $K \in T_h$ [2].

$$\int_{t-\Delta t}^{t+\Delta t} \int_K u_t dx dt + \int_{t-\Delta t}^{t+\Delta t} \int_{\partial K} q \cdot \mathbf{n} ds dt \ \ = \ \ \int_{t-\Delta t}^{t+\Delta t} \int_K f dx dt, \tag{5.1}$$

where $q = -\lambda \nabla u + b u$ is the flow rate of heat energy.

4

We claim that the semi-discrete WG for (1.1) $-$ (1.3) preserves the energy conservation property in (5.1). Choosing in (4.8) the test function $v = \{v_0, v_b = 0\}$ so that $v_0 = 1$ on K and $v_0 = 0$ elsewhere. We then obtain by integration over the time period $[t - \Delta t, t + \Delta t]$

$$\int_{t-\Delta t}^{t+\Delta t}\int_K u_t dxdt + \int_{t-\Delta t}^{t+\Delta t} a(u_h, v)dt \;=\; \int_{t-\Delta t}^{t+\Delta t}\int_K f dxdt, \tag{5.2}$$

where

$$a(u_h, v) \;=\; \int_K \lambda\nabla_{d,r}u_h \cdot \nabla_{d,r}v dx - \int_K bu_0 \cdot \nabla_{d,r}v dx + \int_K cu_0 dx.$$

Using the definition of operators R_h and $\nabla_{d,r}$ in (4.4), we obtain

$$\begin{aligned}\int_K \lambda\nabla_{d,r}u_h \cdot \nabla_{d,r}v dx &= \int_K R_h(\lambda\nabla_{d,r}u_h) \cdot \nabla_{d,r}v dx\\ &= -\int_K \nabla \cdot R_h(\lambda\nabla_{d,r}u_h)dx\\ &= -\int_{\partial K} R_h(\lambda\nabla_{d,r}u_h) \cdot \mathbf{n}ds, \end{aligned}\tag{5.3}$$

and

$$\begin{aligned}\int_K bu_0 \cdot \nabla_{d,r}v dx &= \int_K R_h(bu_0) \cdot \nabla_{d,r}v dx\\ &= -\int_K \nabla \cdot R_h(bu_0)dx\\ &= -\int_{\partial K} R_h(bu_0) \cdot \mathbf{n}ds. \end{aligned}\tag{5.4}$$

Now substituting (5.3) and (5.4) into (5.2) yields

$$\int_{t-\Delta t}^{t+\Delta t}\int_K u_t dxdt + \int_{t-\Delta t}^{t+\Delta t}\int_{\partial K} R_h(-\lambda\nabla_{d,r}u_h + bu_0) \cdot \mathbf{n}ds \;=\; \int_{t-\Delta t}^{t+\Delta t}\int_K f dxdt,$$

which provides a numerical flux.

$$q_h \cdot \mathbf{n} = R_h(-\lambda\nabla_{d,r}u_h + bu_0) \cdot \mathbf{n}.$$

The numerical flux $q_h \cdot \mathbf{n}$ can be verified to be continuous across the edge of each element K through a selection of the test function $v = \{v_0, v_b\}$ so that $v_0 \equiv 0$ and v_b are arbitrary.

6 Error analysis

In this section, we derive optimal order error estimate for the semi-discrete scheme (4.8) in L^2 norm. Let us begin with proving the elliptic property of WG finite element method for equation (1.1).

Lemma 6.1. *Let $S_h(j,\ell)$ be the weak finite element space defined in* (4.2) *and $a(u_h, v)$ be the bilinear form given in* (4.8)*. There exists positive constant α satisfying*

$$a(v_h, v_h) \geq \alpha(\|\nabla_{d,r} v_h\|^2 + \|v_0\|^2),$$

for all $v_h \in S_h(j,\ell)$.

Proof. Taking $u = v$ in equation (4.8) we have

$$a(v_h, v_h) \quad = \quad (\lambda \nabla_{d,r} v, \nabla_{d,r} v) - (b v_0, \nabla_{d,r} v) + (c v_0, v_0). \tag{6.1}$$

Let $A = \|b\|_{L^\infty(\Omega)}$ and $B = \|c\|_{L^\infty(\Omega)}$ be the L^∞-norm of the coefficients b and c, respectively and using Cauchy- Schwarz inequality we have.

$$\begin{aligned} |(b v_0, \nabla_{d,r} v)| &\leq \|b\|_{L^\infty(\Omega)} \|\nabla_{d,r} v\| \|v_0\|, \\ &\leq A \|\nabla_{d,r} v\| \|v_0\| \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} |(c v_0, v_0)| &\leq \|c\|_{L^\infty(\Omega)} \|v_0\|^2 \\ &\leq B \|v_0\|^2. \end{aligned} \tag{6.3}$$

Substituting (6.2) and (6.3) into (6.1) we obtain

$$a(v_h, v_h) \quad \geq \quad |\lambda| \|\nabla_{d,r} v\|^2 + A \|\nabla_{d,r} v\| \|v_0\| - B \|v_0\|^2,$$

by using Young-inequality, we have

$$\begin{aligned} a(v_h, v_h) &\geq (|\lambda| + \frac{1}{2\epsilon}) \|\nabla_{d,r} v\|^2 + (\frac{\epsilon A^2}{2} - B) \|v_0\|^2 \\ &\geq \alpha_1 \|\nabla_{d,r} v\|^2 + \alpha_2 \|v_0\|^2 \\ &\geq \alpha(\|\nabla_{d,r} v\|^2 + \|v_0\|^2), \end{aligned}$$

where $\alpha = \min\{\alpha_1, \alpha_2\}$, which completes the proof. □

Lemma 6.2. ([*2*])
For $u \in H^{1+\kappa}(\Omega)$ with $\kappa > 0$, we have

$$\|\Pi_h(\lambda \nabla u) - \lambda R_h(\nabla u)\| \quad \leq \quad C h^\kappa \|u\|_{1+\kappa}. \tag{6.4}$$

Lemma 6.3. ([*14*])
For $u \in H^{1+\kappa}(\Omega)$ with $\kappa > 0$, we have

$$\|u - \Pi_h u\| \quad \leq \quad C h^\kappa \|u\|_{1+\kappa}. \tag{6.5}$$

6.1 Continuous time WG finite element method

Our aim is to prove the following estimate in L^2 norm for the semi-discrete approximation.

Theorem 6.1. *Let $u \in H^{1+\kappa}(\Omega)$ with $\kappa > 0$ and u_h be the solutions of* (1.1) − (1.3) *and* (4.8) *respectively. Denote by $e = u_h - Q_h u$ the difference between WG approximation and the L^2 projection of the exact solution $u = u(x,t)$. Then there exists a constant C such that*

$$\|e\|^2 + \int_0^t \alpha\|e\|^2 ds \quad \leq \quad \|e(\cdot,0)\|^2 + Ch^{2\kappa}\int_0^t \|u\|^2_{1+\kappa} ds \tag{6.6}$$

Proof. Let $v = \{v_0, v_b\} \in S_h^0(j,\ell)$ be the testing function. By testing (1.1) − (1.3) against v_0, together with (4.6) we arrive at

$$\begin{aligned}
(f, v_0) &= (u_t, v_0) + \sum_{T\in T_h}(-\nabla\cdot(\lambda\nabla u), v_0)_T + \sum_{T\in T_h}(\nabla\cdot(bu), v_0) + (cu, v_0) \\
&= (u_t, v_0) + (\Pi_h(\lambda\nabla u), \nabla_{d,r} v) - (\Pi_h(bu), \nabla_{d,r} v) + (cu, v_0). \qquad (6.7)
\end{aligned}$$

Adding and subtracting the term

$$a(Q_h u, v) \equiv (\lambda\nabla_{d,r}(Q_h u), \nabla_{d,r} v) - (b(Q_0), \nabla_{d,r} v) + (c(Q_0 u), v_0),$$

on the right hand side of the equation (6.7) and using $(Q_h u_t, v_0) = (u_t, v_0)$ we obtain

$$\begin{aligned}
(f, v_0) &= (Q_h u_t, v_0) + (\Pi_h(\lambda\nabla u) - \lambda\nabla_{d,r}(Q_h u), \nabla_{d,r} v) \\
&- (\Pi_h(bu) - b(Q_0 u), \nabla_{d,r} v) + (cu - c(Q_0 u), v_0) \\
&+ (\lambda\nabla_{d,r}(Q_h u), \nabla_{d,r} v) - (b(Q_0), \nabla_{d,r}) \\
&+ (c(Q_0 u), v_0),
\end{aligned}$$

by using $R_h(\nabla u) = \nabla_{d,r}(Q_h u)$ for $u \in H^1$ and (4.8) we obtain

$$\begin{aligned}
((u_h)_t, v_0) + a(u_h, v) &= (Q_h u_t, v_0) + (\Pi_h(\lambda\nabla u) - \lambda R_h(\nabla u), \nabla_{d,r} v) \\
&- (\Pi_h(bu) - b(Q_0 u), \nabla_{d,r} v) + (cu - c(Q_0 u), v_0) \\
&+ a(Q_h u, v),
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
((u_h - Q_h)_t, v_0) + a(u_h - Q_h u, v) &= (\Pi_h(\lambda\nabla u) - \lambda R_h(\nabla u), \nabla_{d,r} v) \\
&-(\Pi_h(bu) - b(Q_0 u), \nabla_{d,r} v) + (cu - c(Q_0 u), v_0). \qquad (6.8)
\end{aligned}$$

Equation (6.8) shall be called the error equation for the WG finite element method (4.8). Substituting v in (6.8) by $e = \{u_h - Q_h u\} = \{e_0, e_b\} = \{u_0 - Q_0 u, u_b - Q_b u\}$, we have

$$\begin{aligned}
(e_t, e) + a(e, e) &= (\Pi_h(\lambda\nabla u) - \lambda R_h(\nabla u), \nabla_{d,r} e) - (\Pi_h(bu) - b(Q_0 u), \nabla_{d,r} e) \\
&+ (cu - c(Q_0 u), e).
\end{aligned}$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|e\|^2 + \beta\|\nabla_{d,r} e\|^2 + \alpha\|e\|^2 \quad = \quad \sum_{i=1}^{3} I^{(i)}, \tag{6.9}$$

where

$$\begin{aligned} I^{(1)} &= (\Pi_h(\lambda\nabla u) - \lambda R_h(\nabla u), \nabla_{d,r} e) \\ I^{(2)} &= (\Pi_h(bu) - bu_0, \nabla_{d,r} e) \\ I^{(3)} &= (cu - c(Q_0 u), e). \end{aligned}$$

To estimate $I^{(1)}$, by Cauchy-Schwarz inequality and Young inequality, we have

$$|I^{(1)}| \leq \frac{1}{2\beta}\|\Pi_h(\lambda\nabla u) - \lambda R_h(\nabla u)\|^2 + \frac{\beta}{2}\|\nabla_{d,r} e\|^2$$

by lemma (6.2), we have

$$|I^{(1)}| \leq Ch^{2\kappa}\|u\|_{1+\kappa}^2 + \frac{\beta}{2}\|\nabla_{d,r} e\|^2. \tag{6.10}$$

To estimate $I^{(2)}$, by Cauchy-Schwarz inequality and Young inequality, we have

$$|I^{(2)}| \leq \frac{1}{2\beta}\|\Pi_h(bu) - bu_0\|^2 + \frac{\beta}{2}\|\nabla_{d,r} e\|^2,$$

by lemma (6.3), we have

$$|I^{(2)}| \leq Ch^{2\kappa}\|u\|_{1+\kappa}^2 + \frac{\beta}{2}\|\nabla_{d,r} e\|^2. \tag{6.11}$$

To estimate $I^{(3)}$, again by Cauchy-Schwarz inequality, Young inequality and lemma(6.3), we have

$$\begin{aligned} |I^{(3)}| &\leq \frac{1}{2\alpha}\|cu - c(Q_0 u)\|^2 + \frac{\alpha}{2}\|e\|^2 \\ &\leq Ch^{2\kappa}\|u\|_{1+\kappa}^2 + \frac{\alpha}{2}\|e\|^2. \end{aligned} \tag{6.12}$$

Substituting (6.10), (6.11), and (6.12), into (6.9) we get

$$\frac{1}{2}\frac{d}{dt}\|e\|^2 + \beta\|\nabla_{d,r} e\|^2 + \alpha\|e\|^2 \leq Ch^{2\kappa}\|u\|_{1+\kappa}^2 + \beta\|\nabla_{d,r} e\|^2 + \frac{\alpha}{2}\|e\|^2.$$

It follows that

$$\frac{d}{dt}\|e\|^2 + \alpha\|e\|^2 \leq Ch^{2\kappa}\|u\|_{1+\kappa}^2.$$

Thus, integrating with respect to t, we obtain

$$\|e\|^2 + \int_0^t \alpha\|e\|^2 ds \leq \|e(\cdot,0)\|^2 + Ch^{2\kappa}\int_0^t \|u\|_{1+\kappa}^2 ds,$$

which completes the proof. $\square$

6.2 Optimal order of error estimation in L^2

To get an optimal order of error estimate in L^2, the idea, similar to Wheeler's projection as in [14, 15], is used where an elliptic projection E_h onto the discrete weak space $S_h(j,\ell)$ is defined as the following: Find $E_hu \in S_h(j,\ell)$ such that E_hu is the L^2 projection of the trace of u on the boundary $\partial\Omega$ and

$$\begin{aligned}(\lambda\nabla_{d,r}E_hu,\nabla_{d,r}w)+(b\cdot\nabla_{d,r}E_hu,w) &= (-\nabla\cdot(\lambda\nabla u),w)\\ &+ (-bu,\nabla w)\quad \forall w\in S_h^0(j,\ell). \qquad (6.13)\end{aligned}$$

In view of the weak formulation of the convection-diffusion-reaction problem.

$$\begin{aligned}-\nabla\cdot(\lambda\nabla u)+b\cdot\nabla u &= F,\quad in\ \Omega, &(6.14)\\ u &= g,\quad on\ \partial\Omega, &(6.15)\end{aligned}$$

this defined may be expressed by using that E_hu is the WG finite element approximation of the solution of the corresponding convection-diffusion problem with exact solution u.

Lemma 6.4. (*see*[*1*])
Assume that problem (6.14) − (6.15) *has the* $H^{1+s}(\Omega)$ *regularity* ($s\in(0,1]$). *Let* $u\in H^{1+\kappa}(\Omega)$ *be the exact solution of* (6.14) − (6.15), *and* E_hu *be a WG approximation of* u *defined in* (6.13). *Let* $Q_hu=\{Q_0u,Q_bu\}$ *be the* L^2 *projection of* u *in the corresponding finite element space. Then there exists a constant* C *such that*

$$\|Q_0u-E_hu\| \le C(h^{\kappa+1}\|F-Q_0F\|+h^{\kappa+s}\|u\|_{\kappa+1})$$

and

$$\|\nabla_{d,r}(Q_hu-E_hu\| \le Ch^{\kappa}\|u\|_{\kappa+1}.$$

Theorem 6.2. *Under the assumption of Theorem* (6.1) *and the assumption that the corresponding convection-diffusion problem has the* H^{1+s} *regularity* ($s\in(0,1]$), *there exists a constant* C *such that*

$$\begin{aligned}\|u_h(t)-Q_hu(t)\| &\le \|u_h(0)-Q_hu(0)\|+Ch^{\kappa+s}(\|\psi\|_{\kappa+1}+\int_0^t\|u_t\|_{\kappa+1}ds)\\ &+ Ch^{s+1}(\int_0^t(\|f_t-Q_0f_t\|+\|u_{tt}-Q_0u_{tt}\|)ds)\\ &+ Ch^{s+1}(\|f(0)-Q_0f(0)\|+\|u_t(0)-Q_0u_t(0)\|) \qquad (6.16)\end{aligned}$$

Proof. The error in the problem (1.1) − (1.3) is written as a sum of two terms,

$$u_h(t)-Q_hu(t) = \theta(t)+\rho(t), \qquad (6.17)$$

where

$$\theta=u_h-E_hu,\qquad \rho=E_hu-Q_hu.$$

The error bound for ρ easily by lemma (6.4) as the following [2]

$$\begin{aligned}\|\rho\| &\le C(h^{s+1}(\|f-Q_0f\|+\|u_t-Q_0u_t\|)\\ &+ h^{\kappa+s}(\|\psi\|_{\kappa+1}+\int_0^t\|u_t\|_{\kappa+1}ds)). \qquad (6.18)\end{aligned}$$

9

Now, to estimate θ, we note that by our definitions

$$\begin{aligned}
(\theta_t, w) + a(\theta, w) &= ((u_h)_t, w) + a(u_h, w) - (E_h u_t, w) - a(E_h u_h, w) \\
&= (f, w) - (E_h u_t, w) - a(E_h u_h, w) \\
&= (f, w) + (\nabla \cdot (\lambda \nabla u), w) + (b \cdot \nabla u, w) - (cu, w) - (E_h u_t, w) \\
&= (u_t, w) - (E_h u_t, w) \\
&= (Q_h u_t, w) - (E_h u_t, w) \\
&= -(\rho_t, w),
\end{aligned}$$

which is

$$(\theta_t, w) + a(\theta, w) = -(\rho_t, w), \quad \forall w \in S_h^0(j, \ell), t > 0, \tag{6.19}$$

where we have used the fact that the operator E_h commutes with time differentiation. Since $\theta \in S_h^0(j, \ell)$, we may choose $w = \theta$ in (6.19) and obtain

$$(\theta_t, \theta) + a(\theta, \theta) = -(\rho_t, \theta), \quad t > 0, \tag{6.20}$$

by using lemma (6.1) we have

$$a(\theta, \theta) \geq \alpha(\|\nabla_{d,r}\theta\|^2 + \|\theta_0\|^2) > 0.$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 = \|\theta\|\frac{d}{dt}\|\theta\| \leq \|\rho_t\|\|\theta\|,$$

and integrating with respect to t, we obtain

$$\|\theta(t)\| \leq \|\theta(0)\| + \int_0^t \|\rho_t\| ds. \tag{6.21}$$

using lemma (6.3), we have

$$\begin{aligned}
|\theta(0)\| &= \|u_h(0) - E_h u(0)\| \\
&\leq \|u_h(0) - Q_h u(0)\| + \|E_h u(0) - Q_h u(0)\| \\
&\leq \|u_h(0) - Q_h u(0)\| + C(h^{s+1}(\|f(0) - Q_0 f(0)\| \\
&+ \|u_t(0) - Q_0 u_t(0)\|) + h^{\kappa+s}\|\psi\|_{\kappa+1}),
\end{aligned} \tag{6.22}$$

and since

$$\begin{aligned}
\|\rho_t\| &= \|E_h u_t - Q_h u_t\| \\
&\leq C(h^{s+1}(\|f_t - Q_0 f_t\| + \|u_{tt} - Q_0 u_{tt}\|) \\
&+ h^{\kappa+s}\|u_t\|_{\kappa+1}).
\end{aligned} \tag{6.23}$$

Substituting (6.18) and (6.21) into (6.17), we have an optimal order of error estimate in L^2 which completes the proof. $\square$

10

7 Numerical result

In this section, we present some numerical results to illustrate the theoretical analysis in the previous section. We consider the following convection-diffusion-reaction problem.

$$u_t - \nabla \cdot (D\nabla u) + b \cdot \nabla u + cu \quad = \quad f, \qquad in\ \Omega \times J, \tag{7.1}$$

with homogeneous Dirichlet boundary condition and initial condition. The data for problem (7.1) taken as follows: let $D = 100$, Ω be a unit square, i.e., $\Omega = [0,1] \times [0,1]$, time interval be $J = (0,T) = (0,1)$, the absorption coefficient is $c = 1$ and the velocity vector has been taken as $b = (\cos(\frac{\pi}{3}), \sin(\frac{\pi}{3}))$, we can get the initial and boundary conditions and source term $f(x,t)$ according to the corresponding analysis solution of example. First, we partition the square domain $\Omega = (0,1) \times (0,1)$ in to $N \times N$ sub-square uniformly. Then we divide each square element into two triangles by the diagonal line with a negative slopeso that we complete the construction of the triangular mesh let $h = 1/N(N = 4, 8, 16, 32, 64)$ be mesh size for triangular meshes.

In the example, the analytical solution is chosen as

$$u = sin(\pi x)sin(\pi y)exp(-t).$$

Numerical error results and convergence rate are listed in Table 7.1 and convergence rate in Figure 1.

Table 7.1:numerical result

h	L^2-**error**	L^2-**order**
1/4	3.7148e-00	
1/8	9.4454e-01	1.97
1/16	2.3719e-01	1.99
1/32	5.9383e-02	2.00
1/64	1.4875e-02	2.00

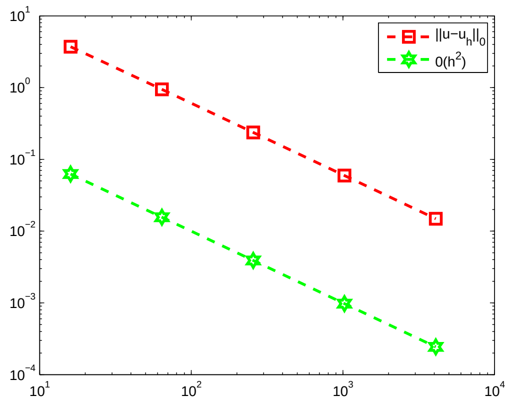


Figure 1: Convergence rate for $\kappa = 1$ and $s = 1$.

11

Acknowledgements

The first author's research is supported in part by National Natural Science Foundation of China (NSFC) no. 11871038, China Postdoctoral Science Foundation no.2014M560547, Fundamental Research Funds of Shandong University no.2017JC005.

References

[1] J. Wank, X. Ye. A weak Galerkin finite element method for the second order elliptic problem. J. Comput. Appl. Math, 241(2013) 103-115.

[2] Q.H. Li, J.P. Wang. Weak Galerkin finite element methods for parabolic equations. Numer. Meth. PDEs, 29 (2013) 2004-2024.

[3] F. Gao, L. Mu. On L^2 error estimate for weak Galerkin finite element method for parabolic problems. J. Comput. Math, 32 (2014) 195-204.

[4] F. Gao, X. Wang. A modified weak Galerkin finite element method for a class of parabolic problems. J. Comp. Appl. Math, 271 (2014) 1-19.

[5] L. Mu, J.P. Wang, X. Ye. A modified weak Galerkin finite element method for the stokes equations. J. Comp. Appl. Math, 275 (2015) 79-90.

[6] J. Wang, X. Ye. A weak Galerkin finite element method for the stokes equations. Adv. Comput. Math, 42 (2016) 155-174.

[7] L. Mu, J. Wang, X. Ye, S. Zhao. A numerical study on the weak Galerkin method for the Helmholtz equation. Commun. Comput. Phys, 15 (2014) 1461-1479.

[8] L. Mu, J.P. Wang, X. Ye. Weak Galerkin finite element method for the biharmonic equation on polytopal meshes, Numer. Meth. PDEs. 30 (2014) 1003-1029.

[9] R. Zhang, Q.L. Li. A weak Galerkin finite element scheme for the Biharmonic equations by using polynomials of reduced order. J. Sci. Comput, 64 (2015) 559-585.

[10] J. Zhang, K. Zhang, J. Li, X. Wang. A weak Galerkin finite element method for the Navier-Stokes equation. Commun. Comput. Phys, 10 (2017) 1-14.

[11] X. Lin, J. Li, Z. Chen. A weak Galerkin finite element method for the Navier-Stokes equation, J. Comp. Appl. Math, 333 (2018) 442-457.

[12] R.A. Adams. Sobolev spaces. Academic Press, New York, 1975.

[13] P.G.Ciarlet. The finite element method for elliptic problems. North-Holland, 1978.

[14] V. Thomée. Galerkin finite element method for parabolic problems. (Springer Series in Computational Mathematics), Springer-Verlag, Berlin-Heidelberg, New York, Inc., Secancns, NJ, 1984.

[15] M.F. Wheeler. A priori L^2 error estimates for Galerkin approximations to parabolic partial differential equations. SIAM Journal on Numerical Analysis, 10 (1973) 723-759.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

The Generalized Moment Problem on White Noise Spaces

A. S. Okb El Bab[1] and Hossam A. Ghany[2]
[1]Department of Mathematics, Faculty of Science, Azhar University, Naser City(11884),Cairo Egypt.
[2]Department of Mathematics, Helwan University, Sawah Street(11282), Cairo Egypt.
h.abdelghany@yahoo.com

Abstract

Our purpose in this paper, is to derive the main properties of the generalized moment functions defined on some types of white noise spaces. A new version of Wick product on some spaces of generalized functions is introduced. Applying the direct connection between the theory of construction for hypercomplex systems and white noise analysis, we setup a framework to construct a lot of spaces of generalized functions connected with different examples of hypercomplex systems.

2010 Mathematics Subject Classification: 43A62, 60H40, 30G35.
Keywords: White Noise; Wick product; moment function; generalized functions.

1 Introduction

In this paper, the main properties of the generalized moment functions defined on some types of white noise spaces are derived. A new version of Wick product with respect to non-Gaussian measures, the associated Hermite transform and the characterization theorem for the constructed spaces of generalized functions are introduced. Let Q denotes a locally compact basis on the space $\mathbb{R}^n$. The linear space of bounded continuous complex-valued functions $C_b(Q)$ is complete normed space with respect to the norm

$$\|f\|_\infty = \sup_{x \in Q} |f(x)|,$$

where f define on Q. We will denote by $C_b^\infty(Q)$ the space of infinitely differential bounded functions on Q, and by $\mathcal{S}(Q)$ the linear subspace of $C_b^\infty(Q)$ formed by the set of functions on Q such that $x^\alpha D^\beta f(x)$ is bounded on Q, where $\alpha, \beta \in \mathbb{Z}_+^n$. The space of continuous

linear functional on $\mathcal{S}(Q)$ is called tempered distribution space and is denoted by $\mathcal{S}'(Q)$. There exist many works aims to investigate white noise spaces. Some of these works devoted to deal with the construction of spaces of test, generalized functions and operators acting in these spaces using the Wiener-Itô-Segal isomorphism and various riggings of the Fock space [2, 9]. Distribution play a crucial role in the study of PDEs and quantum field theory [5, 11], where quantum field are defined as operator valued distributions. The contemporary theory of generalized functions of infinitely many variables originates from the works of Berezanskyi and Samoilenko [3] and Hida [9]. In [3], the spaces of test and generalized functions were constructed as infinite tensor products of one-dimensional spaces. In [9], the classical approach to the construction of the theory of generalized functions was, in fact, used, but all functions under consideration were functions of a point of the infinite-dimensional space on which the Gaussian measure was defined; this measure played the same role as the Lebesgue measure in the classical theory of generalized functions. This paper is organized as follows: In section 2, we give the main properties of the generalized moment functions defined on the space of rabidly decreasing functions on Q. In section 3, a new way for constructing spaces of generalized functions is given. In section 4, we derive the main relations between the construction of hypercomplex system and the Theory of white noise analysis.

2 The moment problem on $\mathcal{S}(Q)$

The elements of $\mathcal{S}(Q)$ are called rabidly decreasing functions and for each $\alpha, \beta \in \mathbb{Z}_+^n$, $\mathcal{S}(Q)$ is equipped with the family of seminorms

$$\|f\|_{\alpha,\beta} = \sup_{x\in Q} |x^\alpha D^\beta f(x)|$$

In this section, we devoted to give a full description of the integral

$$\phi(x) = \int_Q \lambda(x) d\mu(\lambda), \quad \mu \in \mathcal{M}_+(Q),$$

where $\lambda : Q \to \mathbb{C}$ belongs to the linear space of bounded continuous complex-valued functions $C_b(Q)$ and the measure μ belongs to the space of positive Radon measures $\mathcal{M}_+(Q)$. Let $s = (s_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ $(s_0 > 0)$ be an n-sequence of real numbers. We set

$$\mathcal{L}_s(x^\alpha) = s_\alpha, \quad \alpha \in \mathbb{Z}_+^n.$$

The n-sequence $s = (s_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ is called quasi-positive definite if $\mathcal{L}_s$ is quasi-positive definite (i.e., $\mathcal{L}_s(f\bar{f}) \geq 0$ for all $f \in \mathcal{S}(Q)$). The n-sequence s is called a generalized moment sequence if there exists a Radon measure μ on Q such that $x^\alpha \in L_1(\mu)$ and $s_\alpha = \int_Q x^\alpha d\mu(x)$ for all $\alpha \in \mathbb{Z}_+^n$. When such measure exists, then it is called a representing measure of the sequence s. Let $\mathcal{F} = f_1, ..., f_m$ be a finite family in $\mathcal{S}(Q)$, and

$$Q_{\mathcal{F}} = \{q \in Q; f_j(q) \geq 0, \quad i = 1, ..., m\}.$$

Clearly, we have $m_j = sup_{x\in Q} f_j(x) < \infty$, setting

$$\hat{f}_j(x) = m_j^{-1} f_j(x), \quad x \in Q \quad if \quad m_j > 0,$$

and

$$\hat{f}_j = f_j, \quad m_j = 0,$$

$j = 1, ..., m$. We define $\hat{\mathcal{F}} = \{0, 1, \hat{f}_1, ..., \hat{f}_m\}$, and we will denote by $\Delta_{\mathcal{F}}$ the set of all products of the form $f_1...f_i(1-g_1)...(1-g_i)$ for functions $f_1, ..., f_i, g_1..., g_i \in \hat{\mathcal{F}}$ and integers $i, j \geq 1$.

Theorem 2.1. Let $\Pi(\mathcal{F})$ denote the convex set of all linear mappings $\mathcal{L} : \mathcal{S}(Q) \to \mathbb{R}$ such that $\mathcal{L}(1) = 1$ and $\mathcal{L}(f) \geq 0$ for all $f \in \Delta_{\mathcal{F}}$. Then we have

$$0 \leq \mathcal{L}(f) \leq 1$$

for all $\mathcal{L} \in \Pi(\mathcal{F})$ and $f \in \Delta_{\mathcal{F}}$.

Proof. Let $f_1...f_k \in \Delta_{\mathcal{F}}$, where either $f_j \in \hat{\mathcal{F}}$ or $1 - f_j \in \hat{\mathcal{F}}$ for all $j = 1, ..., k$. We have

$$f_1...f_n = (1 - f_1) + f_1(1 - f_2) + ... + f_1...f_{k-1}(1 - f_k)$$

This implies $\mathcal{L}(1 - f) \geq 0$, whence $\mathcal{L}(f) \leq 1$.

Remark. Let $\Gamma_+(Q)$ be the positive cone generated by $\Delta_{\mathcal{F}}$. From the previous proof we notice that if $f \in \Delta_{\mathcal{F}}$, then $1 - f \in \Gamma_+(Q)$. Moreover, we notice that if $f, g \in \Delta_{\mathcal{F}}$, then $(1 - f)g \in \Gamma_+(Q)$. In particular, if $\mathcal{L} : \mathcal{S}(Q) \to \mathbb{R}$ is positive on $\Delta_{\mathcal{F}}$, then $\mathcal{L}((1 - f)g) \geq 0$ for all $f, g \in \Delta_{\mathcal{F}}$. Finally, we notice that $\mathcal{L}(1) = 0$ implies $\mathcal{L} = 0$.

Lemma 2.2. Let $\mathcal{L}$ be an extreme point of the convex set $\Pi(\mathcal{F})$. Then $\mathcal{L}$ is multiplicative on $\mathcal{S}(Q)$.

Proof. Suppose $f \in \Delta_{\mathcal{F}}$ be fixed. Sufficiently, we need to prove that

$$\mathcal{L}(fg) = \mathcal{L}(f)\mathcal{L}(g) \quad for \quad all \quad g \in \Delta_{\mathcal{F}}$$

Let $d = \mathcal{L}(f)$. We have the following possibilities:

1. If $0 < d < 1$, we consider the linear functionals $\mathcal{L}_1(h) = d^{-1}\mathcal{L}(fh)$ and $\mathcal{L}_2(h) = (1 - d)^{-1}\mathcal{L}((1 - f)h), \quad h \in \mathcal{S}(Q)$. Clearly, $\mathcal{L}_1, \mathcal{L}_2 \in \Pi(\mathcal{F})$. since $\mathcal{L} = d\mathcal{L}_1 + (1 - d)\mathcal{L}_2$ and $\mathcal{L}$ is an extreme point of $\Pi(\mathcal{F})$, this implies $\mathcal{L} = \mathcal{L}_1$, whence $\mathcal{L}(fg) = \mathcal{L}(f)\mathcal{L}(g)$.

2. If $d = 0$, then the functional $\mathcal{L}_0(h) = \mathcal{L}(fh)$ is positive on $\Delta_{\mathcal{F}}$ and $\mathcal{L}_0(1) = 0$, applying the above remark implies $\mathcal{L}_0 = 0$, whence $\mathcal{L}(fg) = 0 = \mathcal{L}(f)\mathcal{L}(g)$.

3. If $d = 1$, we use the above discussion to the functional $\mathcal{L}_1(g) = \mathcal{L}((1 - f)g)$, and obtain $\mathcal{L}(fg) = \mathcal{L}(g) = \mathcal{L}(f)\mathcal{L}(g)$.

Theorem 2.3. For every linear functional $\mathcal{L} \in \Pi(\mathcal{F})$ there exists a uniquely probability measure μ on Q such that

$$\mathcal{L}(f) = \int_Q f d\mu$$

for all $f \in \mathcal{S}(Q)$.

Proof. Let $\mathcal{L}_0 \in \Pi(\mathcal{F})$ be an extreme point. Then $\mathcal{L}_0$ is multiplicative on $\mathcal{S}(Q)$, by the above lemma. Thus, for the sequence $\gamma = (\gamma_1, ..., \gamma_n) \in \mathbb{R}^n$ defined by $\mathcal{L}_0(t_j) = \gamma_j$, we have $\mathcal{L}_0(f) = f(\gamma)$ for all $f \in \mathcal{S}(Q)$. But we have $0 \leq \mathcal{L}_0(f) \leq 1, \quad f \in \Delta_{\mathcal{F}}$, by Theorem 2.1, we obtain that

$$|\mathcal{L}_0(f)| = |f(\gamma)| \leq ||f||_Q = sup_{t\in Q}|f(t)|, \quad f \in \mathcal{S}(Q).$$

If $f \in \Pi(\mathcal{F})$ is of the form $\mathcal{L} = \sum_{j\in I} c_j L_j$, where $c_j \geq 0, \quad \sum_{j\in I} c_j = 1$, $\mathcal{L}_j$ an extreme point of $\Pi(\mathcal{F})$, then

$$|\mathcal{L}(f)| \leq \sum_{j\in I} c_j|\mathcal{L}_j(f)| \leq \sum_{j\in I} c_j||f||_Q = ||f||_Q, \quad f \in \mathcal{S}(Q).$$

Let $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}(\gamma_0 > 0$ be a generalized moment sequence. Then the linear form $\mathcal{L} = \gamma_0^{-1}$ is an element of $\Pi(\mathcal{F})$, and by using the result obtained from the above Theorem we have:

Corollary 2.4. Let Q is compact and $\mathcal{F} = f_0 = 1, f_1, ..., f_m$ be a finite family which generates the space $\mathcal{S}(Q)$. An n-sequence of real numbers $s = (s_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ $(s_0 > 0)$ is a generalized moment sequence if and only if the linear form $\mathcal{L}_s$ is nonnegative on the set $\Delta_{\mathcal{F}}$.

3 The spaces of generalized functions

This section is devoted to give the main relations between the construction of hypercomplex system and the Theory of white noise analysis. We will consider the following rigging of a Hilbert space H_0 with positive and negative spaces H_+ and H_-:

$$H_- \supseteq H_0 \supseteq H_+. \tag{3.1}$$

Let $\mathbf{I}_0^+ : H_- \longrightarrow H_+$ be the canonical isometry transferring the negative space H_- onto the positive space H_+. A biorthogonal basis $(p_n, q_n)_{n=0}^\infty$ in the space H_0 can be understood as sequences $(p_n)_{n=0}^\infty \subset H_+$ and $(q_n = \mathbf{I}_0^- p_n)_{n=0}^\infty \subset H_-$, where the first sequence is an orthogonal basis in the positive space H_+ and the second is an orthogonal basis in the negative space H_-. Hence, these systems of sequences p_n and q_n are biorthogonal:

$$(p_n, q_n)_{H_0} = \delta_{n,m} h_n, \quad h_n = \|p_n\|_{H_+}^2 = \|q_n\|_{H_-}^2, \quad n, m \in \mathbb{Z}_+, \tag{3.2}$$

for all $\varphi \in H_+$,

$$\varphi = \sum_{n=0}^{\infty} \varphi_n p_n, \quad \varphi_n = (\varphi, q_n)_{H_0} h_n^{-1}, \quad \sum_{n=0}^{\infty} |\varphi_n|^2 h_n = \|\varphi\|^2_{H_+} < \infty, \tag{3.3}$$

for all $\xi \in H_-$,

$$\xi = \sum_{n=0}^{\infty} \xi_n q_n, \quad \xi_n = (\xi, p_n)_{H_0} h_n^{-1}, \quad \sum_{n=0}^{\infty} |\xi_n|^2 h_n = \|\xi\|^2_{H_-} < \infty, \tag{3.4}$$

$$(\xi, \varphi)_{H_0} = \sum_{n=0}^{\infty} \xi_n \overline{\varphi_n} h_n. \tag{3.5}$$

Let $(p_n)_{n=0}^{\infty}$ be an arbitrary total sequence of vectors p_n of a Hilbert space H_0. It is easy to prove that such sequence $(h_n)_{n=0}^{\infty}$ of positive numbers h_n exists for which the set of test functions

$$H_+ = \left\{ \varphi = \sum_{n=0}^{\infty} \varphi_n \, p_n \mid \varphi_n \in \mathbb{C} : \|\varphi\|^2_{H_+} = \sum_{n=0}^{\infty} |\varphi_n|^2 \, h_n < \infty \right\}, \tag{3.6}$$

with the corresponding scalar product is the positive space with respect to H_0. Note that, it is necessary to assume in addition the fulfilment of the following necessary and sufficient condition on $(p_n)_{n=0}^{\infty}$: an arbitrary sequence $(\varphi^{(i)})_{i=0}^{\infty}$ of vectors $\varphi^{(i)} \in H_+$ with finite sequences of coordinates $\varphi_n^{(i)}$ which is fundamental in H_+ and converges to 0 in H_0 must converge to 0 in H_+. This condition will always be fulfilled in our case. Similarly, for the negative space H_-, by replacing p_n by q_n, we have the set of generalized functions as follows

$$H_- = \left\{ \xi = \sum_{n=0}^{\infty} \xi_n \, q_n \mid \xi_n \in \mathbb{C} : \|\xi\|^2_{H_-} = \sum_{n=0}^{\infty} |\xi_n|^2 \, h_n < \infty \right\}. \tag{3.7}$$

As pointed out from [1-3], there exists a quasinuclear rigging such that, the zero space H_0 is a hypercomplex system $L_2(Q, dm(p))(p \in Q)$ and we assume that

$$\mathbf{I}_\chi^+ : H_+ \longrightarrow H_1^\chi, \qquad \mathbf{I}_\chi^- : H_- \longrightarrow H_{-1}^\chi.$$

such that

$$\langle \mathbf{I}_\chi^- \xi, \mathbf{I}_\chi^+ \varphi \rangle_{L_2(Q,dm(p))} = \langle \xi, \varphi \rangle_{H_0}, \quad \xi \in H_-, \varphi \in H_+.$$

So, we have a biunitary map $\{\mathbf{I}_\chi^-, \mathbf{I}_\chi^+\}$. This mapping transfers the rigging of the space H_0 to a rigging of the hypercomplex space $L_2(Q, dm(p))$:

$$\begin{array}{ccccc} H_- & \supseteq & H_0 & \supseteq & H_+. \\ \big\downarrow \mathbf{I}_\chi^- & & & & \big\downarrow \mathbf{I}_\chi^+ \\ H_{-1}^\chi & \supseteq & L_2(Q, dm(p)) & \supseteq & H_1^\chi, \end{array} \tag{3.8}$$

Hence, we consider the space H_1^χ is a positive space of the form

$$H_1^\chi = \left\{ \varphi = \sum_{n=0}^{\infty} \varphi_n \, \chi_n \mid \, : \|\varphi\|^2_{H_1^\chi} = \sum_{n=0}^{\infty} |\varphi_n|^2 \, (n!)^2 K^n < \infty \right\}, \tag{3.9}$$

where $p_n = \chi_n, h_n = (n!)^2 K^n, n \in \mathbb{Z}_+$, ($K > 1$ is a fixed sufficiently large number), and consists of continuous functions on Q. Similarly, for the space H_1^χ, we have

$$H_{-1}^\chi = \left\{ \xi = \sum_{n=0}^{\infty} \xi_n \, \chi_n \mid \, : \|\xi\|^2_{H_{-1}^\chi} = \sum_{n=0}^{\infty} |\xi_n|^2 \, (n!)^2 K^n < \infty \right\}, \tag{3.10}$$

The system $(\chi_n, q_n^\chi)_{n=0}^\infty$, where $q_n^\chi = \mathbf{I}_1^- \chi_n \in H_{-1}^\chi$, is a biorthogonal basis of the space $L_2(Q, dm(p))$. It is essential to introduce the rigging of the hypercomplex space $L_2(Q, dm(p))$ by means of projective and inductive limits of Hilbert spaces which are constructed by rules of type (3.6), (3.8) and (3.9). For every $q \in \mathbb{N}$, we define the Hilbert space of type (3.6):

$$H_q^\chi = \left\{ \varphi = \sum_{n=0}^{\infty} \varphi_n \chi_n \in H_0 : \|\varphi\|^2_{H_q^\chi} = \sum_{n=0}^{\infty} |\varphi_n|^2 (n!)^2 K^{qn} < \infty \right\}. \tag{3.11}$$

Then, we have the rigging:

$$(\Psi^\chi)' \supseteq H_{-q}^\chi \supseteq L_2(Q, dm(p)) \supseteq H_q^\chi \supseteq \Psi^\chi, \tag{3.12}$$

$$\Psi^\chi = \operatorname{pr} \lim_{q \in \mathbb{N}} H_q^\chi = \bigcap_{q \in \mathbb{N}} H_q^\chi, \qquad (\Psi^\chi)' = \operatorname{ind} \lim_{q \in \mathbb{N}} H_{-q}^\chi = \bigcup_{q \in \mathbb{N}} H_{-q}^\chi,$$

$$H_{-q}^\chi = \left\{ \xi = \sum_{n=0}^{\infty} \xi_n q_n^\chi : \|\xi\|^2_{H_{-q}^\chi} = \sum_{n=0}^{\infty} |\xi_n|^2 (n!)^2 K^{-qn} < \infty \right\}, \tag{3.13}$$

with the action

$$(\xi, \varphi)_{L_2(Q, dm(p))} = \sum_{n=0}^{\infty} \xi_n \overline{\varphi_n} (n!)^2 K^{qn}, \quad \varphi \in H_q^\chi, \quad \xi \in H_{-q}^\chi.$$

To illustrate the above result, we give the following example

Example 3.1. In the classical case when $H_0 := L_2(\mathbb{R}, dx)$ with respect to the Lebesgue measure dx and ordinary convolution. Then, the generalized character $\chi(x, \lambda) = e^{\lambda x}$ $(\lambda \in \mathbb{C})$ and $\chi_n(x) = x^n$ $(x \in \mathbb{R}, \ n \in \mathbb{Z}_+)$. Therefore, the space (3.11) consists of entire functions $\varphi(x)$ and $\varphi_n(x)$ are the Taylor coefficients of $\varphi(x)$. Formula (3.2) gives their representation as the Fourier coefficients using the scalar product $(\xi, \varphi)_{H_0}$, $(\xi \in H_{-1}^\chi, \ \varphi \in H_1^\chi)$.

Remark. Obviously, such a generalization gives the possibility of constructing a lot of spaces of generalized functions connected with different examples of hypercomplex systems.

4 The generalized Wick product

In this section, we devoted to introduce a new version of Wick product with respect to non-Gaussian measures, the associated Hermite transform and the characterization theorem for the constructed spaces of generalized functions. Wick is the first one introduced the product between two functions in white noise space, so this product carry his name [13]. He was used as a tool to renormalize certain infinite quantities in quantum field theory. Later on, the Wick product was considered, in a stochastic ordinary and partial differential equations (see, e.g., [6,8,10]). Under the assumption that $\|\chi\|_{H_0}^2 \leq C^n$ for some $C > 0$, we define a new Wick product, called χ-Wick product on the space H_{-q}^{χ}. Then, we give the definition of the χ-Hermite transform and apply it to establish a characterization theorem for the space H_{-q}^{χ}.

Definition 4.1. Let $\xi = \sum_{m=0}^{\infty} \xi_m q_m^{\chi}$, $\eta = \sum_{n=0}^{\infty} \eta_n q_n^{\chi} \in H_{-q}^{\chi}$ with $\xi_m, \eta_n \in \mathbb{C}$. The χ-Wick product of ξ, η, denoted by $\xi \diamond_{\chi} \eta$, is defined by the formula

$$\xi \diamond_{\chi} \eta = \sum_{m,n=0}^{\infty} \xi_m \eta_n q_{m+n}^{\chi}. \tag{4.1}$$

It is important to show that the spaces $H_{-q}^{\chi}, H_q^{\chi}$ are closed under χ-Wick product.

Lemma 4.2. If $\xi, \eta \in H_{-q}^{\chi}$ and $\varphi, \psi \in H_q^{\chi}$, we have

(i) $\xi \diamond_{\chi} \eta \in H_{-q}^{\chi}$,

(ii) $\varphi \diamond_{\chi} \psi \in H_q^{\chi}$.

Proof. If $\xi = \sum\limits_{m=0}^{\infty} \xi_m q_m^{\chi}$, $\eta = \sum\limits_{n=0}^{\infty} \eta_n q_n^{\chi} \in H_{-q}^{\chi}$, then for some $q_1 \in \mathbb{N}$ we have

$$\sum_{m=0}^{\infty} |\xi_m|^2 K^{-q_1 m} < \infty \text{ and } \sum_{n=0}^{\infty} |\eta_n|^2 K^{-q_1 n} < \infty. \tag{4.2}$$

We note that

$$\xi \diamond_{\chi} \eta = \sum_{m,n=0}^{\infty} \xi_m \eta_n q_{m+n}^{\chi} = \sum_{l=0}^{\infty} \left(\sum_{m+n=l}^{\infty} \xi_m \eta_n \right) q_l^{\chi} = \sum_{l=0}^{\infty} \zeta_l q_l^{\chi}, \tag{4.3}$$

where $\zeta_l = \sum\limits_{m+n=l}^{\infty} \xi_m \eta_n$. With $q = q_1 + p$ we have

$$\begin{aligned}
\sum_{l=0}^{\infty} |\zeta_l|^2 K^{-ql} &= \sum_{l=0}^{\infty} \left| \sum_{m+n=l}^{\infty} \xi_m \eta_n \right|^2 K^{-q_1 l} K^{-pl} \\
&\leq \sum_{l=0}^{\infty} \left(\sum_{m+n=l}^{\infty} |\xi_m|^2 K^{-q_1 m} \right) \left(\sum_{m+n=l}^{\infty} |\eta_n|^2 K^{-q_1 n} \right) K^{-pl} \\
&\leq \left(\sum_{l=0}^{\infty} K^{-pl} \right) \left(\sum_{m=0}^{\infty} |\xi_m|^2 K^{-q_1 m} \right) \left(\sum_{n=0}^{\infty} |\eta_n|^2 K^{-q_1 n} \right) \\
&< \infty,
\end{aligned} \tag{4.4}$$

which proves $(\boldsymbol{i})$. The proof of $(\boldsymbol{ii})$ is similar. ■

The following important algebraic properties of the χ-Wick product follow directly from Definition 4.1.

Lemma 4.3. For each $\xi, \eta, \zeta \in H_{-q}^{\chi}$, we get

$(\boldsymbol{i})$ $\xi \diamond_{\chi} \eta = \eta \diamond_{\chi} \xi$ (Commutative law),

$(\boldsymbol{ii})$ $\xi \diamond_{\chi} (\eta \diamond_{\chi} \zeta) = (\xi \diamond_{\chi} \eta) \diamond_{\chi} \zeta$ (Associative law),

$(\boldsymbol{ii})$ $\xi \diamond_{\chi} (\eta + \zeta) = \xi \diamond_{\chi} \eta + \xi \diamond_{\chi} \zeta$ (Distributive law).

Remark. According to Lemmas 4.2. and 4.3., we can conclude that the spaces H_{-q}^{χ} and H_{q}^{χ} form topological algebras with respect to the χ-Wick product.

From the above arguement, the χ-Wick product satisfies all the ordinary algebraic rules for multiplication. But, there are some problems when limit operations are involved. To treat these situations it is convenient to apply a transformation, called the χ-Hermite transform, which converts χ-Wick products into ordinary (complex) products and convergence in H_{-q}^{χ} into bounded, pointwise convergence in a certain neighborhood of 0 in $\mathbb{C}$.

Definition 4.4. Let $\xi = \sum_{n=0}^{\infty} \xi_n q_n^{\chi} \in H_{-q}^{\chi}$ with $\xi_n \in \mathbb{C}$. Then, the χ-Hermite transform of ξ, denoted by $\mathcal{H}_{\chi}\xi$, is defined by

$$\mathcal{H}_{\chi}\xi(z) = \sum_{n=0}^{\infty} \xi_n z^n \in \mathbb{C} \quad \text{(when convergent)}. \tag{4.5}$$

In the following, we define for $0 < M, q < \infty$ the neighborhoods of zero in $\mathbb{C}$ which denoted it by $\mathbb{O}_{q,M}(0)$:

$$\mathbb{O}_{q,M}(0) = \left\{ z \in \mathbb{C} : \sum_{n=0}^{\infty} |z^n|^2 K^{qn} < M^2 \right\}. \tag{4.6}$$

It is easy to see that

$$q \leq p,\ N \leq M \Rightarrow \mathbb{O}_{q,N}(0) \subseteq \mathbb{O}_{q,M}(0). \tag{4.7}$$

Note that, if $\xi = \sum_{n=0}^{\infty} \xi_n q_n^{\chi} \in H_{-q}^{\chi}, z \in \mathbb{O}_{q,M}(0)$ for some $0 < M, q < \infty$, we have the estimate

$$\begin{aligned} \sum_{n=0}^{\infty} |\xi_n||z^n| &= \sum_{n=0}^{\infty} |\xi_n||z^n| K^{-\frac{qn}{2}} K^{\frac{qn}{2}} \\ &\leq \left(\sum_{n=0}^{\infty} |\xi_n|^2 K^{-qn} \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} |z^n|^2 K^{qn} \right)^{\frac{1}{2}} \\ &< M \left(\sum_{n=0}^{\infty} |\xi_n|^2 K^{-qn} \right)^{\frac{1}{2}} \\ &< \infty. \end{aligned} \tag{4.8}$$

The conclusion above can be stated as follows:

Proposition 4.5. If $\xi \in H^{\chi}_{-q}$, then $\mathcal{H}_{\chi}\xi$ converges for all $z \in \mathbb{O}_q(M)$ for all $q, M < \infty$.

Proposition 4.6. If $\xi, \eta \in H^{\chi}_{-q}$, then

$$\mathcal{H}_{\chi}(\xi \diamond_{\chi} \eta)(z) = \mathcal{H}_{\chi}\xi(z).\mathcal{H}_{\chi}\eta(z). \tag{4.9}$$

for all z such that $\mathcal{H}_{\chi}\xi$ and $\mathcal{H}_{\chi}\eta$ exist.

Proof. The proof is an immediate consequence of Definitions 4.1. and 4.4.

Let $\xi = \sum_{n=0}^{\infty} \xi_n q^{\chi}_n \in H^{\chi}_{-q}$, with $\xi_n \in \mathbb{R}$. Then, the number $\xi_0 = \mathcal{H}_{\chi}\xi(0) \in \mathbb{R}$ is called the generalized expectation of ξ and is denoted by $\mathbb{E}(\xi)$. Suppose that $V \ni z \mapsto f(z) \in \mathbb{C}$ is an analytic function, where V is a neighborhood of $\mathbb{E}(\xi)$. Assume that the Taylor series of f around $\mathbb{E}(\xi)$ has coefficients in $\mathbb{R}$. Then, the χ-Wick version $f^{\diamond_{\chi}}$ of f is defined by

$$H^{\chi}_{-q} \ni \xi \mapsto f^{\diamond_{\chi}}(\xi) = \mathcal{H}^{-1}\big(f \circ \mathcal{H}_{\chi}(\xi)\big) \in H^{\chi}_{-q}. \tag{4.10}$$

Example 4.7. If the function $f : \mathbb{C} \to \mathbb{C}$ is entire, then $f^{\diamond_{\chi}}$ is defined for all $\xi \in H^{\chi}_{-q}$. For example, the χ-Wick exponential is defined by

$$\exp^{\diamond_{\chi}}(\xi) = \sum_{j=0}^{\infty} \frac{1}{j!}\xi^{\diamond_{\chi} n}. \tag{4.11}$$

5 Concluding Remarks

The space of continuous linear functional on $\mathcal{S}(Q)$ are called tempered distributions, and is denoted by $\mathcal{S}'(Q)$. Let $L \in \mathcal{S}'(Q)$ and $\alpha \in \mathbb{Z}^d_+$. The weak derivative $D^{\alpha}L$ (or the derivative of the sense of distributions) is given by

$$(D^{\alpha}L)(f) = (-1)^{|\alpha|}L(D^{\alpha}f) \tag{5.1}$$

for $f \in (Q)$. This corresponds to $D^{\alpha}L\{g\} = L\{D^{\alpha}g\}$. Note that distribution always has a weak derivative. A function f is completely monotonic if for each $\alpha \in \mathbb{Z}^n_+$, $(-1)^{|\alpha|}D^{\alpha}f(x) \geq 0$ on $\mathbb{R}^n_+$; see [4, 7, 12] for many properties of completely monotonic functions. Bernstien's theorem asserts that f is completely monotonic if and only if $f(x) = \int_{\mathbb{R}^n} e^{-x.t}d\mu(t)$ where μ is a positive measure supported on a subset of $\mathbb{R}^n_+$. If assume that $Q = \mathbb{R}^n$. So, $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Let x^{α} be denote the product $x_1^{\alpha_1} ... x_n^{\alpha_n}$, $\mathbb{Z}^n_+$ denote the set of n-tuples $(\alpha_1, ..., \alpha_n)$ where each α_i is a non-negative integer, $|\alpha| = \sum_{i=1}^{n} \alpha_i$ and D^{α} denote the partial differential operator $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$. Then, we obtain the special case $\mathcal{S}(Q) = \mathcal{S}(\mathbb{R}^n)$ is the space of rabidly decreasing function on $\mathbb{R}^n$ (so-called Schwartz space) and its dual $\mathcal{S}'(Q) = \mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distribution on $\mathbb{R}^n$.

References

[1] Yu. M. Berezansky and A. A. Kalyuzhnyi, *Harmonic Analysis in Hypercomplex Systems*, Naukova Dumka,Kyiv, Ukrania, 1992. (in Russian; English transl.: Kluwer: Dordrecht-Boston-London, 1996).

[2] Yu. M. Berezansky and Yu. G. Kondratiev, *Spectral methods in infinite dimensional analysis*, vol. 1, 2, Kluwer, Dordrecht 1995.

[3] Yu. M. Berezanskyi and Yu. S. Samoilenko, Nuclear spaces of functions of infinitely many variables, Ukr. Mat. Zh., 25, No. 6, 723-737 (1973).

[4] C. Berg, J. P. R. Christensen, P. Ressel, *Harmonic Analysis on Semigroups: Theory of Positive Definite and Related Functions*, Springer-Verlag: Berlin, Heidelberg, New York, 1984.

[5] I. M. Gelfand and G. E. Shilov, Generalized Function, Academic Press,Inc., 1964.

[6] H. A. Ghany, Exact solutions for stochastic generalized Hirota-Satsuma coupled KdV equations, *Chinese Journal of Physics*, vol. **49**, no. 4, (2011) 926-940.

[7] H. A. Ghany, Harmonic analysis in the product of commutative hypercomplex systems. *Journal of Computational Analysis and Applications,* **25** (5), (2018) 875-888.

[8] H. A. Ghany and M. A. Qurashi, Travelling Solitary Wave Solutions for Stochastic Kadomtsev-Petviashvili Equation. *Journal of Computational Analysis and Applications,* **21** (1), (2015) 121-131.

[9] T. Hida, Analysis of Brownian Functionals, Carleton Math. Lect. Notes, No. 13 (1975).

[10] H. Holden, B. Øsendal, J. Ubøe and T. Zhang, *Stochastic partial differential equations,* Springer Science+Business Media, LLC, (2010).

[11] J. Lighthill, Introduction to Fourier Analysis and Generalized Function,cambridge university press,1958.

[12] A. S. Okb El Bab and H. A. Ghany, Harmonic analysis on hypergroups systems, *AIP Conf. Proc.*, **312** (2010) 1309.

[13] G. C. Wick, The evaluation of the collinear matrix, *Phys. Rev* **80** (1950), 268-272.

QUADRATIC TYPE FUNCTIONAL INCLUSIONS ON SQUARE-SYMMETRIC GROUPOIDS AND HYERS-ULAM STABILITY

GWANG HUI KIM AND HWAN-YONG SHIN

Abstract. We consider that a set-valued map $F : X \to \mathcal{P}_0(Y)$ satisfying the functional inclusion $F(x * y)\Diamond F(x * y^{-1}) \subseteq \sigma_\Diamond(F(x)\Diamond F(y))$(or $\sigma_\Diamond(F(x)\Diamond F(y)) \subseteq \sigma_\Diamond(F(x * y)\Diamond F(x * y^{-1})))$ admits a unique selection $f : X \to Y$ satisfying the functional equation $f(x*y)\diamond f(x*y^{-1}) = \sigma_\diamond(f(x) \diamond f(y))$ in appropriate conditions, where $(X, *)$, $(Y, \diamond)$ are square-symmetric groupoids and $\Diamond$ is the extension of $\diamond$ to the collection $\mathcal{P}_0(Y)$ of all nonempty subsets of Y.

1. Introduction

Let $(X, *), (Y, \diamond)$ be groupoids with binary operations. If the binary oepration $*$ satisfies the following inequality

$$(x * y) * (x * y) = (x * x) * (y * y), \quad x, y \in X$$

then the operation $*$ is called square-symmetric. Note that the square symmetric $*$ implies that $\sigma_*(x) := x*x$ is an endomorphism. A binary operation $*$ such that σ_* is an automorphism of $(X, *)$ is called divisible and the corresponding groupoid is said to be a divisible groupoid. The triple $(Y, \diamond, d)$ is called a metric groupoid if $(Y, \diamond)$ is a groupoid, (Y, d) is a metric space and $\diamond$ is a continuous operation with respect to the topology of (Y, d). For a nonempty set Y we denote by $\mathcal{P}_0(Y)$ the collection of all nonempty subsets of Y. The diameter of a set $A \in \mathcal{P}_0(Y)$ is defined by

$$\delta(A) := \sup\{d(x, y)|x, y \in A\}.$$

The Lipschitz modulus of a function $f : X \to Y$ is the smallest real extended number L with the property

$$d(f(x), f(y)) \leq Ld(x, y), \quad x, y \in Y.$$

The Lipschitz modulus of a function f is denoted by $Lipf$. A selection of a set-valued mapping $F : X \to \mathcal{P}_0$ is a single-valued map $f : X \to Y$ with the property $f(x) \in F(x)$ for all $x \in X$.

2010 *Mathematics Subject Classification.* 39B72, 54C60.

Key words and phrases. Hyers–Ulam stability; square-symmetric groupoid; functional inclusion.

1

In a linear normed space $(Y, \|\cdot\|)$ we define the following families of sets

$$\begin{aligned} c(Y) &:= \{A : A \in \mathcal{P}_0(Y), A \text{ is convex set}\} \\ ccl(Y) &:= \{A : A \in \mathcal{P}_0(Y), A \text{ is closed and convex set}\} \\ cc(Y) &:= \{A : A \in \mathcal{P}_0(Y), A \text{ is compact and convex set}\}. \end{aligned}$$

The theory of stability of functional equations had been formulated by Ulam [14]. In 1941, Hyers [3] had answered affirmatively the question of Ulam for Banach spaces and it represents the starting point of the Hyers–Ulam stability of functional equations. Let us recall the Hyers' result.

Theorem 1.1. [3] *Let X be a linear normed space, Y a Banach space and $\varepsilon > 0$. If a function $f : X \to Y$ satisfies the following inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon, \quad x, y \in X \tag{1.1}$$

then there exists a unique additive function $g : X \to Y$ such that

$$\|f(x) - g(x)\| \le \varepsilon, \quad x \in X. \tag{1.2}$$

Smajdor [13] and Gajda and Ger [2] observed an interesting connection between the stability of the Cauchy functional equation and set-valued functions satisfying $F(x + y) \subseteq F(x) + F(y)$. If $f : X \to Y$ satisfies (1.1), then the set-valued mapping $F : X \to \mathcal{P}_0$ defined by

$$F(x) = f(x) + \overline{B}(0, \varepsilon), \quad x \in X,$$

where $\overline{B}(0, \varepsilon)$ is the closed ball in Y centered at 0 and radius $\varepsilon > 0$, implies that $F(x + y) \subseteq F(x) + F(y)$ for $x, y \in X$, and the function g from relation (1.2) is an additive selection of F. Naturally Gajda and Ger [2] considered under what conditions a set-valued mapping with $F(x + y) \subseteq F(x) + F(y)$ admits an additive selection and they obtained the following theorem.

Theorem 1.2. [2] *Let $(S, +)$ be a commutative semigroup with zero element, X a Banach space over $\mathbb{R}$ and $F : S \to ccl(X)$ a set-valued mapping with convex and closed values such that $F(x + y) \subseteq F(x) + F(y)$ for $x, y \in S$ and $\sup_{x \in S} \delta(F(x)) < \infty$. Then F admits a unique additive selection.*

For the last two decades, many mathematicians have developed Theorem 1.2 [6, 9, 10, 11] and investigated various properties of functional inclusion and its connectedness of Hyers–Ulam stability of functional equations [4, 5, 7, 8, 12].

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

The aim of this paper is to study some properties for set-valued mappings satisfying the following quadratic type functional inclusions

$$\sigma_{\lozenge}(F(x)\lozenge F(y)) \subseteq F(x*y)\lozenge F(x*y^{-1})$$
$$F(x*y)\lozenge F(x*y^{-1}) \subseteq \sigma_{\lozenge}(F(x)\lozenge F(y))$$

and obtain Hyers-Ulam stability of functional equation.

2. Main results

Let $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Throughout this section, suppose that the operation $\lozenge$ satisfies the following condition : for all $\varepsilon > 0$ there exists $\eta > 0$ such that if $\delta(A), \delta(B) < \eta, A, B \in \mathcal{P}_0(Y)$, then

$$\delta(A\lozenge B) < \varepsilon$$

and we assume that X and Y have unique identity id_X and id_Y respectively.

If the operation $\diamond$ satisfies that

$$(x_1 \diamond y_1) \diamond (x_2 \diamond y_2) = (x_1 \diamond x_2) \diamond (y_1 \diamond y_2)$$

for all $x_1, x_2, y_1, y_2 \in Y$, then we say $\diamond$ is bisymmetric operation.

Lemma 2.1. [9] *If $(Y, \diamond)$ is a groupoid with a bisymmetric operation, then $\sigma_\diamond$ is increasing endomorphism of $(\mathcal{P}_0(Y), \lozenge, \subseteq)$.*

Now, we present the main theorem of this paper.

Theorem 2.2. *Let $(X, *)$ be a square-symmetric divisible groupoid, $(Y, \diamond, d)$ a complete metric bisymmetric divisible groupoid and $F : X \to \mathcal{P}_0(Y)$ with $F(id_X) = \{id_Y\}$ a set-valued mapping such that*

$$\sigma_{\lozenge}(F(x)\lozenge F(y)) \subseteq F(x*y)\lozenge F(x*y^{-1}) \tag{2.1}$$

for all $x, y \in X$. Assume that

$$\lim_{m\to\infty} \delta(F\circ\sigma_*^{-m}(x))Lip(\sigma_\diamond^{2m}) = 0, \quad \textit{and} \tag{2.2}$$
$$\sigma_{\lozenge}^{2n}\circ F\circ\sigma_*^{-n}(x) \in cl(Y)$$

for all $x \in X$ and $n \in \mathbb{N}_0$. Then there exists a unique selection $f : X \to Y$ of F such that

$$\sigma_\diamond(f(x)\diamond f(y)) = f(x*y)\diamond f(x*y^{-1}) \tag{2.3}$$

for all $x, y \in X$.

Proof. First we prove that there exists a selection of F satisfying (2.3). Consider the set valued mapping $F_n : X \to \mathcal{P}_0(Y)$ corresponding to F defined by

$$F_0 := F, \quad F_n := \sigma_{\Diamond}^{2n} \circ F \circ \sigma_*^{-n}. \tag{2.4}$$

for each $n \in \mathbb{N}$. Letting x, y by $\sigma_*^{-n-1}(x)$ in (2.14) respectively, we get

$$\sigma_{\Diamond}^2 \circ F(\sigma_*^{-n-1}(x)) \subseteq F(\sigma_*^{-n}(x)) \tag{2.5}$$

for all $x \in X$. By composing $\sigma_{\diamond}^{2n}$ to the both sides of (2.5) and using Lemma 2.1, we obtain

$$\sigma_{\Diamond}^{2n+2} \circ F \circ \sigma_*^{-n-1}(x) \subseteq \sigma_{\Diamond}^{2n} \circ F \circ \sigma_*^{-n}(x)$$

for all $x \in X$ and $n \in \mathbb{N}_0$. This means that $\{F_n(x)\}_{n=0}^{\infty}$ is a decreasing sequence of closed subsets of the Banach space Y. Let $s, t \in F_m(x)$ for some fixed $m \in \mathbb{N}$. Denoting $\sigma_{\diamond}^{-2m}(s) = u$, $\sigma_{\diamond}^{-2m}(t) = v$, we have

$$\begin{aligned} d(s,t) &= d(\sigma_{\diamond}^{2m}(u), \sigma_{\diamond}^{2m}(v)) \le Lip(\sigma_{\diamond}^{2m}) \cdot d(u,v) \\ &\le Lip(\sigma_{\diamond}^{2m})\delta(F \circ \sigma_*^{-m}(x)) \end{aligned}$$

and this implies that

$$\delta(F_m(x)) \le Lip(\sigma_{\diamond}^{2m}) \cdot \delta(F \circ \sigma_*^{-m}(x)) \tag{2.6}$$

for all $x \in X$. Taking the limit $m \to \infty$ of (2.6), we find that

$$\lim_{m \to \infty} \delta(F_m(x))$$

for all $x \in X$. It is follows from the Cantor intersection theorem in the complete metric spaces that

$$\bigcap_{n=0}^{\infty} F_n(x) \tag{2.7}$$

is singleton $f(x)$. Since the function $f : X \to Y$ satisfies $f(x) \in F_0(x) = F(x)$ for all $x \in X$, f is a selection of F.

Putting x, y for $\sigma_*^{-n}(x)$ and $\sigma_*^{-n}(y)$, respectively in (2.14) and applying $\sigma_{\Diamond}^{2n}$ to the both sides of (2.14), we arrive at

$$\sigma_{\Diamond}(F_n(x) \Diamond F_n(y)) \subseteq F_n(x * y) \Diamond F_n(x * y^{-1}) \tag{2.8}$$

for all $x, y \in X$ and $n \in \mathbb{N}_0$. Since $\{f(x)\} = \bigcap_{n=0}^{\infty} F_n(x), x \in X$, we have $\sigma_{\diamond}(f(x) \diamond f(y)) \in \sigma_{\Diamond}(F_n(x) \Diamond F_n(y))$, for all $x, y \in X$, $n \in \mathbb{N}_0$. Therefore, in view of (2.8), we get

$$d(\sigma_{\diamond}(f(x) \diamond f(y)), f(x * y) \diamond f(x * y^{-1})) \le \delta(F_n(x * y) \diamond F_n(x * y^{-1})) \tag{2.9}$$

for all $x, y \in X$ and $n \in \mathbb{N}_0$. Taking the limit $n \to \infty$ of (2.9), it is reduced to the equation

$$\sigma_{\diamond}(f(x) \diamond f(y)) = f(x * y) \diamond f(x * y^{-1}), \text{ for all } x, y \in X. \tag{2.10}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

To show the uniqueness of f, assume that $g : X \to Y$ is a selection of F such that

$$\sigma_\diamond(g(x) \diamond g(y)) = g(x * y) \diamond g(x * y^{-1}), \text{ for all } x, y \in X. \tag{2.11}$$

From (2.10) and (2.11), it follows that

$$f(x) = \sigma_\diamond^{2n} \circ f \circ \sigma_*^{-n}(x),$$
$$g(x) = \sigma_\diamond^{2n} \circ g \circ \sigma_*^{-n}(x)$$

for all $x \in X, n \in \mathbb{N}$. Hence, for $x \in X$ and $n \in \mathbb{N}$, we see that

$$\begin{aligned} d(f(x), g(x)) &= d(\sigma_\diamond^{2n} \circ f \circ \sigma_*^{-n}(x), \sigma_\diamond^{2n} \circ g \circ \sigma_*^{-n}(x)) \\ &= Liq(\sigma_\diamond^{2n}) d(f \circ \sigma_*^{-n}(x), g \circ \sigma_*^{-n}(x)) \\ &\leq Liq(\sigma_\diamond^{2n}) \delta(F \circ \sigma_\circ^{-n}(x)). \end{aligned}$$

Taking $n \to \infty$, we arrive at the desired conclusion. □

Next, we are going to establish another theorem about the inclusion (2.14).

Theorem 2.3. *Let $(X, *)$ be a square-symmetric divisible groupoid, $(Y, \diamond, d)$ a metric bisymmetric divisible groupoid and A a divisible subgroupoid of $(\mathcal{P}_0(Y), \Diamond)$. Suppose that $F : X \to A$ with $F(id_X) = \{id_Y\}$ is a set-valued mapping subject to the condition* (2.14) *and satisfying*

$$\lim_{n\to\infty} \delta(F \circ \sigma_*^n(x)) Lip(\sigma_\diamond^{-2n}) = 0, \quad x \in X. \tag{2.12}$$

Then F is single-valued mapping and

$$\sigma_\Diamond(F(x) \Diamond F(y)) = F(x * y) \Diamond F(x * y^{-1}), \quad \textit{for all } x, y \in X. \tag{2.13}$$

Proof. Consider the function $G_n : X \to A$ corresponding to F defined by

$$G_0 := F, \quad G_n := \sigma_\Diamond^{-2n} \circ F \circ \sigma_*^n$$

for each $n \in \mathbb{N}$. Replacing x, y by $\sigma_*^n(x)$ in (2.12) respectively, and then composing on both sides by $\sigma_\Diamond^{-2n-2}$, we have

$$\sigma_\Diamond^{-2n} \circ F \circ \sigma_*^n(x) \subseteq \sigma_\Diamond^{-2n-2} \circ F \circ \sigma_*^{n+1}(x)$$

for all $x, y \in X$. This means that $\{G_n(x)\}_{n=0}^\infty$ is an increasing sequence of $(A, \Diamond)$. By the similar argument in the proof of Theorem 2.2, we see that

$$\lim_{n\to\infty} \delta(G_n(x)) \leq \lim_{n\to\infty} \delta(F \circ \sigma_*^n(x)) Lip(\sigma_\diamond^{-2n}) = 0, \quad \text{for all } x \in X.$$

It implies that $\delta(G_n(x)) = 0$ for every $n \in \mathbb{N}_0$ and $G_n(x)$ is single-valued for all $n \in \mathbb{N}_0$. Therefore, in view of (2.14), $G_0 = F$ satisfies (2.13) and the proof is completed. □

Corollary 2.4. *Let $(X,*)$ be a square-symmetric divisible groupoid, $(Z,\|\cdot\|)$ a Banace space over $\mathbb{R}$, $p,q\in\mathbb{R}$, $p+q\neq 0$, $p+q\neq 1$, and $F: X\to c(Z)$ with $F(id_X)=\{0_Z\}$ a set-valued mapping such that*

$$p(p+q)F(x)+q(p+q)F(y))\subseteq pF(x*y)+qF(x*y^{-1}) \tag{2.14}$$

for all $x,y\in X$. Assume that there exists $M>0$ such that

$$\delta(F(x))\le M, \quad and$$
$$F\circ\sigma_*^{-n}(x)\in cl(Y)$$

for all $x,y\in X$ and $n\in\mathbb{Z}$. Then there exists a unique selection $f: X\to Z$ of F such that

$$p(p+q)f(x)+q(p+q)f(y)=pf(x*y)+qf(x*y^{-1}),\quad x,y\in X. \tag{2.15}$$

Proof. Consider the operation $\diamond: Z\times Z\to Z$ is defined by

$$x\diamond y=px+qy,\quad x,y\in Z,$$

where $p,q\in\mathbb{R}$ are given real numbers. Then the triple $(Z,\diamond,\|\cdot\|)$ is a metric groupoid with a bisymmetric operation. For all $U,V\in\mathcal{P}_0(Z)$, the operation $\Diamond$ is naturally defined by

$$U\Diamond V=pU+qV$$

and we have $\sigma_\Diamond(U)=(p+q)U$ and in general, $\sigma_\Diamond^n(U)=(p+q)^nU$, for all $n\in\mathbb{N}$. And we get

$$Lip(\sigma_\diamond^{2n})=|p+q|^{2n},\quad n\in\mathbb{Z}.$$

If $|p+q|<1$, we have

$$\sigma_\Diamond^{2n}\circ F\circ_*^{-n}(x)=(p+q)^{2n}F\circ\sigma_*^{-n}(x)\in cl(Z)$$

and

$$\delta(F\circ\sigma_*^{-n})\le M|p+q|^{2n},\quad x\in X, n\in\mathbb{N}_0,$$

thus, by Theorem 2.2, there exists a unique selection of F satisfying (2.15).

If $|p+q|>1$, we obtain

$$\delta(F\circ\sigma_*^{n})Lip(\sigma_\diamond^{-2n})\le\frac{M}{|p+q|^{2n}},\quad x\in X, n\in\mathbb{N}_0.$$

By using Theorem 2.3, F is single-valued mapping satisfying (2.15). We arrive at the desired conclusion. $\square$

Corollary 2.5. *Let $(X,*)$ be a square-symmetric divisible groupoid, $(Z,\|\cdot\|)$ a Banach space over $\mathbb{R}$, $p,q,\varepsilon>0, p+q<1$, and $z\in Z$. Assume that $f:X\to Z$ is a function satisfying*

$$\|pf(x*y)+qf(x*y^{-1})-p(p+q)f(x)-q(p+q)f(y)-z\|\le\varepsilon,\quad x,y\in X.$$

Then there exists a unique function $g:X\to Z$ satisfying

$$pg(x*y)+qg(x*y^{-1})=p(p+q)g(x)+q(p+q)g(y)+z,\quad x,y\in X \tag{2.16}$$

and

$$\|f(x)-g(x)\|\le\frac{\varepsilon}{(1-p-q)(p+q)},\quad x\in X. \tag{2.17}$$

Proof. Consider the auxiliary set-valued mapping $G_f : X\to ccl(Z)$ corresponding to f defined by

$$G_f(x)=f(x)+\frac{1}{(1-p-q)(p+q)}(\overline{B}(0,\varepsilon)-z),\quad \text{if } x\in X-\{id_X\}$$

and $G_f(id_X)=\{0_Z\}$. Then we obtain

$$\begin{aligned}
p(p+q)G_f(x)+q(p+q)G_f(y) &= p(p+q)f(x)+\frac{p(p+q)}{(1-p-q)(p+q)}(\overline{B}(0,\varepsilon)-z)\\
&\quad +q(p+q)f(y)+\frac{q(p+q)}{(1-p-q)(p+q)}(\overline{B}(0,\varepsilon)-z)\\
&\subseteq pf(x*y)+qf(x*y^{-1})+(\overline{B}(0,\varepsilon)-z)\\
&\quad +\frac{(p+q)^2}{(1-p-q)(p+q)}(\overline{B}(0,\varepsilon)-z)\\
&= pf(x*y)+\frac{p}{(1-p-q)(p+q)}(\overline{B}(0,\varepsilon)-z)\\
&\quad +qf(x*y^{-1})+\frac{q}{(1-p-q)(p+q)}(\overline{B}(0,\varepsilon)-z)\\
&= pG_f(x*y)+qG_f(x*y^{-1})
\end{aligned}$$

for all $x,y\in X$. By the definition of $\delta(G_f(x))$, we have

$$\delta(G_f(x))\le\frac{2\varepsilon}{(1-p-q)(p+q)}$$

for all $x\in X$. Since all conditions of Corollary 2.4 are equipped, G_f has a unique selection $h:X\to Z$ such that

$$ph(x*y)+qh(x*y^{-1})=p(p+q)h(x)+q(p+q)h(y),\quad x,y\in X.$$

Defining the function $g:X\to Z$ as

$$g(x)=h(x)+\frac{z}{(1-p-q)(p+q)}$$

for all $x \in X$, we see that the function g satisfies (2.16) and (2.17). $\square$

Next, we will introduce some theorems and corollaries which are obtained by the similar proofs of Theorem 2.2, 2.3 and Corollary 2.4, 2.5.

Theorem 2.6. *Let $(X, *)$ be a square-symmetric divisible groupoid, $(Y, \diamond, d)$ a complete metric bisymmetric divisible groupoid and $F : X \to \mathcal{P}_0(Y)$ with $F(id_X) = \{id_Y\}$ a set-valued mapping such that*

$$F(x * y)\Diamond F(x * y^{-1}) \subseteq \sigma_{\Diamond}(F(x)\Diamond F(y)) \tag{2.18}$$

for all $x, y \in X$. Assume that

$$\begin{aligned} &\lim_{m\to\infty} \delta(F \circ \sigma_*^m(x))Lip(\sigma_{\Diamond}^{-2m}) = 0, \quad \text{and} \\ &\sigma_{\Diamond}^{2n} \circ F \circ \sigma_*^{-n}(x) \in cl(Y) \end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}_0$. Then there exists a unique selection $f : X \to Y$ of F such that

$$\sigma_{\diamond}(f(x) \diamond f(y)) = f(x * y) \diamond f(x * y^{-1})$$

for all $x, y \in X$.

Theorem 2.7. *Let $(X, *)$ be a square-symmetric divisible groupoid, $(Y, \diamond, d)$ a metric bisymmetric divisible groupoid and A a divisible subgroupoid of $(\mathcal{P}_0(Y), \Diamond)$. Suppose that $F : X \to A$ with $F(id_X) = \{id_Y\}$ is a set-valued mapping subject to the condition (2.18) and satisfying*

$$\lim_{n\to\infty} \delta(F \circ \sigma_*^{2n}(x))Lip(\sigma_{\Diamond}^{2n}) = 0, \quad x \in X.$$

Then F is single valued and

$$\sigma_{\Diamond}(F(x)\Diamond F(y)) = F(x * y)\Diamond F(x * y^{-1}), \quad \text{for all } x, y \in X.$$

Corollary 2.8. *Let $(X, *)$ be a square-symmetric divisible groupoid, $(Z, \|\cdot\|)$ a Banace space over $\mathbb{R}$, $p, q \in \mathbb{R}$, $p + q \neq 0$, $p + q \neq 1$, and $F : X \to c(Z)$ with $F(id_X) = \{0_Z\}$ a set-valued mapping subject to the condition (2.18). Assume that there exists $M > 0$ such that*

$$\begin{aligned} &\delta(F(x)) \leq M, \quad \text{and} \\ &F \circ \sigma_*^{-n}(x) \in cl(Z) \end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}_0$. Then there exists a unique selection $f : X \to Z$ of F such that

$$p(p + q)f(x) + q(p + q)f(y) = pf(x * y) + qf(x * y^{-1}), \quad x, y \in X.$$

Corollary 2.9. *Let $(X, *)$ be a square-symmetric divisible groupoid, $(Z, \|\cdot\|)$ a Banach space over $\mathbb{R}$, $p, q, \varepsilon > 0, p+q > 1$, and $z \in Z$. Assume that $f : X \to Z$ is a function satisfying*

$$\|pf(x*y) + qf(x*y^{-1}) - p(p+q)f(x) - q(p+q)f(y) - z\| \le \varepsilon, \quad x, y \in X.$$

Then there exists a unique function $g : X \to Z$ satisfying

$$pg(x*y) + qg(x*y^{-1}) = p(p+q)g(x) + q(p+q)g(y) + z, \quad x, y \in X$$

and

$$\|f(x) - g(x)\| \le \frac{\varepsilon}{(p+q-1)(p+q)}, \quad x \in X.$$

Remark. If $(X, +, \cdot)$ is a vector space and $*$ is defined by $x * y = x + y$ and $z = 0$, $p = q = 1$, $y^{-1} = -y$ in Corollary 2.9, then it is a same result of Czerwik [1].

Acknowledgment

The second author of this work was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant number: 2015R1D1A1A01058083).

References

[1] S. Czerwik, *On the stability of the quadratic mapping in normed space*, Bull. Abh. Math. Sem. Univ. Hamburg, **62** (1992), 59-64.

[2] Z. Gajda, R. Ger. *Subadditive multifunctions and Hyers–Ulam stability*, Numer. Math. **80**, (1987), 281-291.

[3] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci., **27**, (1941), 222–224.

[4] G.H. Kim, *On the stability of homogeneous functional equations with degree t and n-variables*, Mathematical Inequalities and Applications, **6**, (2003), 675-688.

[5] G.H. Kim, *On the stability of functional equations on a square-symmetric groupoid*, Nonlinear Analysis: Theory, Methods and Applications, **63**, (2005), 2559-2568.

[6] K. Nikodem, D. Popa, *On selections of general linear inclusions*, Publ. Math. Debrecen **75**, (2009), 239-249.

[7] M. Piszczek, *On selections of set-valued inclusions in a single variable with applications to several variables*, Results. Math., **64**, (2013), 1-12.

[8] M. Piszczek, *The properties of functional inclusions and Hyers–Ulam stability*, Aequat. Math., **85**, (2013), 111-118.

[9] D. Popa, *Functional inclusions on square-symmetric grupoids and Hyers–Ulam stability*, Math. Inequal. Appl., **3**, (2004), 419–428.

[10] D. Popa, *A property of a functional inclusion connected with Hyers–Ulam stability*, J. Math. Inequal., **4**, (2009), 591-598.

[11] J. Brzdek, D. Popa, B. Xu, *Selections of set-valued maps satisfying a linear inclusion in a single variable*, Nonlinear Analysis, **74**, (2011), 324-330.

[12] J. Sikorska, *Set-valued Orthogonal Additivity*,Set-Valued Var. Anal. **23**, (2015), 547-557.
[13] W. Smajdor, *Superadditive set-valued functions*, Glas. Math. **21**, (1986), 343-348.
[14] S. M. Ulam, Problems in Modern Mathematics, Chapter 6, Wiley Interscience, New York, (1964).

Gwang Hui Kim, Department of Mathematics, Kangnam Universaty,Yongin, Gyeonggi 16979, Republic of Korea
E-mail address: ghkim@kangnam.ac.kr

Hwan-Yong Shin, Department of Mathematics, Chungnam National University,99 Daehangno, Yuseong-gu, Daejeon 34134, Republic of Korea
E-mail address: hyshin31@cnu.ac.kr

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Explicit identities involving r-Bell polynomials

Cheon Seoung Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Korea

Abstract : In this paper, we study differential equations arising from the generating functions of the r-Bell polynomials. We give explicit identities for the r-Bell polynomials.

Key words : Differential equations, Bell polynomials, r-Bell polynomials.

2000 Mathematics Subject Classification : 05A19, 11B68, 11S40, 11S80, 11B83, 34A30, 65L99.

1 Introduction

The moments of the Poisson distribution are well-known to be connected to the combinatorics of the Bell and Stirling numbers(see [1, 4, 5]). As is well known, the Bell numbers B_n are given by the generating function

$$e^{(e^t-1)} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{1.1}$$

The Bell polynomials $B_n(\lambda)$ are given by the generating function

$$e^{\lambda(e^t-1)} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}. \tag{1.2}$$

The Bell polynomials $B_n(\lambda)$ satisfy the relation $B_n(\lambda) = E_\lambda[Z^n], n \in \mathbb{N}$, where Z is a Poisson random variable with parameter $\lambda > 0$.

The r-Bell polynomials $G_n(x,r)$ are defined by the exponential generating function:

$$\sum_{n=0}^{\infty} G_n(x,r) \frac{t^n}{n!} = e^{rt+x(e^t-1)}, \text{ (see [4])}, \tag{1.3}$$

where, r may be real or complex numbers. Note that $B_n(x) = G_n(x,0)$. The first few examples of r-Bell polynomials $G_n(x,r)$ are

$$\begin{aligned}
&G_0(x,r) = 1,\\
&G_1(x,r) = r + x,\\
&G_2(x,r) = r^2 + x + 2rx + x^2,\\
&G_3(x,r) = r^3 + x + 3rx + 3r^2x + 3x^2 + 3rx^2 + x^3,\\
&G_4(x,r) = r^4 + x + 4rx + 6r^2x + 4r^3x + 7x^2 + 12rx^2\\
&\qquad\qquad + 6r^2x^2 + 6x^3 + 4rx^3 + x^4,\\
&G_5(x,r) = r^5 + x + 5rx + 10r^2x + 10r^3x + 5r^4x + 15x^2 + 35rx^2\\
&\qquad\qquad + 30r^2x^2 + 10r^3x^2 + 25x^3 + 30rx^3 + 10r^2x^3 + 10x^4 + 5rx^4 + x^5.
\end{aligned}$$

From (1.2) and (1.3), we see that

$$\begin{aligned}\sum_{n=0}^{\infty} G_n(x,r)\frac{t^n}{n!} &= e^{(e^t-1)x}e^{rt} \\ &= \left(\sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty} r^m\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}B_k(x)r^{n-k}\right)\frac{t^n}{n!}.\end{aligned} \tag{1.4}$$

Comparing the coefficients on both sides of (1.4), we obtain

$$G_n(x,r) = \sum_{k=0}^{n}\binom{n}{k}B_k(x)r^{n-k} \quad (n \geq 0).$$

Similarly we also have

$$G_n(x+y,r) = \sum_{k=0}^{n}\binom{n}{k}G_k(x,r)B_{n-k}(y).$$

Recently, many mathematicians have have studied the differential equations arising from the generating function of special polynomials(see [2, 3, 6, 7, 8, 9]). In this paper, we study differential equations arising from the generating function of r-Bell polynomials. We give explicit identities for the r-Bell polynomials.

2 Explicit identities involving r-Bell polynomials

Differential equations arising from the generating functions of special polynomials are studied by many authors in order to give explicit identities for special polynomials(see [7, 8, 13]). In this section, we study differential equations arising from the generating functions of r-Bell polynomials.

Let

$$F = F(t,x,r) = \sum_{n=0}^{\infty} G_n(x,r)\frac{t^n}{n!} = e^{rt+(e^t-1)x}, \quad x, r \in \mathbb{C}. \tag{2.1}$$

Then, by (2.1), we have

$$\begin{aligned}F^{(1)} = \frac{d}{dt}F(t,x,r) = \frac{d}{dt}\left(e^{rt+(e^t-1)x}\right) \\ &= e^{rt+(e^t-1)x}(r+xe^t) \\ &= re^{rt+(e^t-1)x} + xe^{(r+1)t+(e^t-1)x} \\ &= rF(t,x,r) + xF(t,x,r+1),\end{aligned} \tag{2.2}$$

$$\begin{aligned}F^{(2)} &= \frac{d}{dt}F^{(1)} = rF^{(1)}(t,x,r) + xF^{(1)}(t,x,r+1) \\ &= r^2F(t,x,r) + x(2r+1)F(t,x,r+1) + x^2F(t,x,r+2),\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}F^{(3)} &= \frac{d}{dt}F^{(2)} \\ &= r^2F^{(1)}(t,x,r) + x(2r+1)F^{(1)}(t,x,r+1) + x^2F^{(1)}(t,x,r+2) \\ &= r^3F(t,x,r) + x\left(r^2+(2r+1)(r+1)\right)F(t,x,r+1) \\ &\quad + x^2(3r+3)F(t,x,r+2) + x^3F(t,x,r+3).\end{aligned}$$

Continuing this process, we can guess that

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x,r) \\ = \sum_{i=0}^{N} a_i(N,x,r)F(t,x,r+i), (N=0,1,2,\ldots). \tag{2.4}$$

Taking the derivative with respect to t in (2.4), we get

$$\begin{aligned} F^{(N+1)} &= \frac{dF^{(N)}}{dt} = \sum_{i=0}^{N} a_i(N,x,r)F^{(1)}(t,x,r+i) \\ &= \sum_{i=0}^{N} a_i(N,x,r)\left\{(r+i)F(t,x,r+i) + xF(t,x,r+i+1)\right\} \\ &= \sum_{i=0}^{N} a_i(N,x,r)(r+i)F(t,x,r+i) \\ &\qquad + x\sum_{i=0}^{N} a_i(N,x,r)F(t,x,r+(i+1)) \\ &= \sum_{i=0}^{N}(r+i)a_i(N,x,r)F(t,x,r+i) \\ &\qquad + x\sum_{i=1}^{N+1} a_{i-1}(N,x,r)F(t,x,r+i). \end{aligned} \tag{2.5}$$

On the other hand, by replacing N by $N+1$ in (2.4), we get

$$F^{(N+1)} = \sum_{i=0}^{N+1} a_i(N+1,x,r)F(t,x,r+i). \tag{2.6}$$

Comparing the coefficients on both sides of (2.5) and (2.6), we obtain

$$a_0(N+1,x,r) = ra_0(N,x,r), \quad a_{N+1}(N+1,x,r) = xa_N(N,x,r), \tag{2.7}$$

and

$$a_i(N+1,x,r) = (r+i)a_{i-1}(N,x,r) + xa_{i-1}(N,x,r), (1 \le i \le N). \tag{2.8}$$

In addition, by (2.4), we get

$$F(t,x,r) = F^{(0)}(t,x,r) = a_0(0,x,r)F(t,x,r). \tag{2.9}$$

By (2.9), we get

$$a_0(0,x,r) = 1. \tag{2.10}$$

It is not difficult to show that

$$\begin{aligned} &rF(t,x,r) + xF(t,x,r+1) \\ &= F^{(1)}(t,x,r) \\ &= \sum_{i=0}^{1} a_i(1,x,r)F(t,x,r+1) \\ &= a_0(1,x,r)F(t,x,r) + a_1(1,x,r)F(t,x,r+1). \end{aligned} \tag{2.11}$$

Thus, by (2.11), we also get

$$a_0(1,x,r) = r, \quad a_1(1,x,r) = x. \tag{2.12}$$

From (2.7), we note that

$$a_0(N+1,x,r) = ra_0(N,x,r) = \cdots = r^N a_0(1,x,r) = r^{N+1}, \tag{2.13}$$

and

$$a_{N+1}(N+1,x,r) = xa_N(N,x,r) = \cdots = x^N a_1(1,x,r) = x^{N+1}. \tag{2.14}$$

For $i = 1,2,3$ in (2.8), we have

$$a_1(N+1,x,r) = x\sum_{k=0}^{N}(r+1)^k a_0(N-k,x,r), \tag{2.15}$$

$$a_2(N+1,x,r) = x\sum_{k=0}^{N-1}(r+2)^k a_1(N-k,x,r), \tag{2.16}$$

and

$$a_3(N+1,x,r) = x\sum_{k=0}^{N-2}(r+3)^k a_2(N-k,x,r). \tag{2.17}$$

Continuing this process, we can deduce that, for $1 \le i \le N$,

$$a_i(N+1,x,r) = x\sum_{k=0}^{N-i+1}(r+i)^k a_{i-1}(N-k,x,r). \tag{2.18}$$

Here, we note that the matrix $a_i(j,x,r)_{0\le i,j\le N+1}$ is given by

$$\begin{pmatrix} 1 & r & r^2 & r^3 & \cdots & r^{N+1} \\ 0 & x & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & x^2 & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & x^3 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x^{N+1} \end{pmatrix}$$

Now, we give explicit expressions for $a_i(N+1,x,r)$. By (2.15), (2.16), and (2.17), we get

$$a_1(N+1,x,r) = x\sum_{k_1=0}^{N}(r+1)^{k_1} a_0(N-k_1,x,r) = \sum_{k_1=0}^{N}(r+1)^{k_1} r^{N-k_1},$$

$$\begin{aligned} a_2(N+1,x,r) &= x\sum_{k_2=0}^{N-1}(r+2)^{k_2} a_1(N-k_2,x,r) \\ &= x^2\sum_{k_2=0}^{N-1}\sum_{k_1=0}^{N-1-k_2}(r+1)^{k_1}(r+2)^{k_2} r^{N-k_2-k_1-1}, \end{aligned}$$

and

$$\begin{aligned} &a_3(N+1,x,r) \\ &= x\sum_{k_3=0}^{N-2}(r+3)^{k_3} a_2(N-k_3,x,r) \\ &= x^3\sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-2-k_3}\sum_{k_1=0}^{N-2-k_3-k_2}(r+3)^{k_3}(r+2)^{k_2}(r+1)^{k_1} r^{N-k_3-k_2-k_1-2}. \end{aligned}$$

Continuing this process, we have

$$\begin{aligned} &a_i(N+1,x,r) \\ &= x^i \sum_{k_i=0}^{N-i+1} \sum_{k_{i-1}=0}^{N-i+1-k_i} \cdots \sum_{k_1=0}^{N-i+1-k_i-\cdots-k_2} \left(\prod_{l=1}^{i}(r+l)^{k_l}\right) r^{N-i+1-\sum_{l=1}^{i} k_l}. \end{aligned} \tag{2.19}$$

Therefore, by (2.19), we obtain the following theorem.

Theorem 2.1 *For $N = 0, 1, 2, \dots$, the differential equation*

$$F^{(N)} = \sum_{i=0}^{N} a_i(N,x,r)e^{it}F(t,x,r)$$

has a solution

$$F = F(t,x,r) = e^{rt+(e^t-1)x},$$

where

$$\begin{aligned} &a_0(N,x,r) = r^N, \\ &a_N(N,x,r) = x^N, \\ &a_i(N,x,r) = x^i \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_i-\cdots-k_2} \left(\prod_{l=1}^{i}(r+l)^{k_l}\right) r^{N-i-\sum_{l=1}^{i} k_l}, \\ &\qquad (1 \le i \le N). \end{aligned}$$

From (2.1), we note that

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x,r) = \sum_{k=0}^{\infty} G_{k+N}(x,r)\frac{t^k}{k!}. \tag{2.20}$$

From Theorem 1 and (2.20), we can derive the following equation:

$$\begin{aligned} \sum_{k=0}^{\infty} G_{k+N}(x,r)\frac{t^k}{k!} &= F^{(N)} = \left(\sum_{i=0}^{N} a_i(N,x,r)e^{it}\right) F(t,x,r) \\ &= \sum_{i=0}^{N} a_i(N,x,r)\left(\sum_{l=0}^{\infty} i^l \frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} G_m(x,r)\frac{t^m}{m!}\right) \\ &= \sum_{i=0}^{N} a_i(N,x,r)\left(\sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{k}{m} i^{k-m} G_m(x,r)\frac{t^k}{k!}\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{N}\sum_{m=0}^{k}\binom{k}{m} i^{k-m} a_i(N,x,r)G_m(x,r)\right)\frac{t^k}{k!}. \end{aligned} \tag{2.21}$$

By comparing the coefficients on both sides of (2.21), we obtain the following theorem.

Theorem 2.2 *For $k, N = 0, 1, 2, \dots$, we have*

$$G_{k+N}(x,r) = \sum_{i=0}^{N}\sum_{m=0}^{k}\binom{k}{m} i^{k-m} a_i(N,x,r)G_m(x,r), \tag{2.22}$$

where where

$$\begin{aligned} &a_0(N,x,r) = r^N, \quad a_N(N,x,r) = x^N, \\ &a_i(N,x,r) = x^i \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_i-\cdots-k_2} \left(\prod_{l=1}^{i}(r+l)^{k_l}\right) r^{N-i-\sum_{l=1}^{i} k_l}, \\ &\qquad (1 \le i \le N). \end{aligned}$$

Let us take $k = 0$ in (2.22). Then, we have the following corollary.

Corollary 2.3 *For $N = 0, 1, 2, \ldots,$ we have*

$$G_N(x, r) = \sum_{i=0}^{N} a_i(N, x, r).$$

For $N = 0, 1, 2, \ldots,$ the functional equation $F^{(N)} = \sum_{i=0}^{N} a_i(N, x, r)e^{it}F(t, x, r)$ has a solution $F = F(t, x, r) = e^{rt+(e^t-1)x}$. Here is a plot of the surface for this solution. In Figure 1(left), we

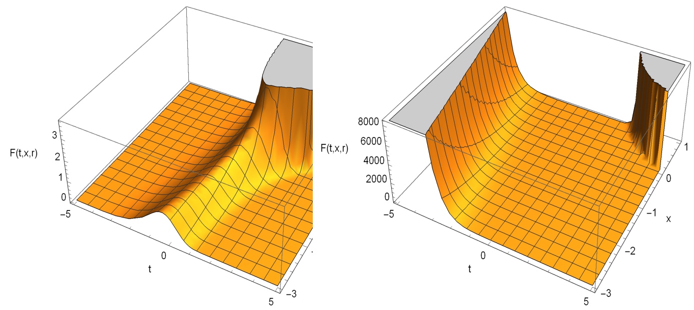


Figure 1: The surface for the solution $F(t, x, r)$

choose $-3 \leq x \leq 1, -5 \leq t \leq 5$, and $r = -2$. In Figure 1(right), we choose $-3 \leq x \leq 3, -5 \leq t \leq 5$, and $r = 2$.

Acknowledgement: This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

REFERENCES

1. Roberto B. Corcino and Cristina B. Corcino, *On generalized Bell polynomials,* Discrete Dynamics in Nature and Society, **2011** (2011), Article ID 623456, 21 pages.

2. T. Kim, D.S. Kim, C.S. Ryoo, H.I. Kwon, *Differential equations associated with Mahler and Sheffer-Mahler polynomials*, submitted for publication.

3. T. Kim, D.S. Kim, *Identities involving degenerate Euler numbers and polynomials arising from non-linear differential equations*, J. Nonlinear Sci. Appl., **9**(2016), 2086-2098.

4. I. Mező, *The r-Bell numbers*, J. Integer Seq., **13** (2010), Article 10.9.8.

5. N. Privault, *Genrealized Bell polynomials and the combinatorics of Poisson central moments*, The Electronic Journal of Combinatorics, **18**(2011), #54

6. C.S. Ryoo, *Differential equations associated with tangent numbers,* J. Appl. Math. & Informatics, **34** (2016), 487-494.

7. C. S. Ryoo, *Differential equations associated with generalized Bell polynomials and their zeros*, Open Mathematics, **14** (2016), 807-815.

8. C. S. Ryoo, *A numerical investigation on the structure of the zeros of the degenerate Euler-tangent mixed-type polynomials*, J. Nonlinear Sci. Appl., **10** (2017), 4474-4484

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

A CLASS INVOLVING DERIVATIVES OF RATIO OF THE ANALYTIC FUNCTIONS

JI HYANG PARK, VIRENDRA KUMAR, AND NAK EUN CHO

Abstract. The class of functions defined using linear combination of the derivatives of ratio of the normalized analytic function with the identity function is considered in this manuscript. Further, the sharp bounds on the Hankel determinants and estimates on the higher order Schwarzian derivatives for the first three consecutive derivatives are investigated.

1. Introduction

Let $\mathcal{A}$ be the family of functions f in the open unit disk $\mathbb{D}$ and satisfying the normalization conditions $f(0) = 0 = f'(0) - 1$. Let the collection $\mathcal{S} \subset \mathcal{A}$ contains univalent functions in $\mathbb{D}$. An analytic function f is subordinate to another analytic function g if there is an analytic function w with $|w(z)| \leq |z|$ and $w(0) = 0$ such that $f(z) = g(w(z))$ and we write $f \prec g$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. The classes $\mathcal{S}^*$ and $\mathcal{K}$ of starlike and convex functions, respectively, are defined by $\operatorname{Re}(zf'(z)/f(z)) > 0$ and $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$. There are several sufficient conditions for functions to be univalent. Among them the simplest one is to verify $\operatorname{Re} f'(z) > 0$ in $z \in \mathbb{D}$. However, there are several other sufficient conditions for univalency were investigated in the recent years. Obradovič [17] proved that if $f \in \mathcal{A}$ satisfy $|f''(z)| < 1/2$, then f is convex in $\mathbb{D}$. Later, this condition was generalized by Frasin [7]. For $0 < \gamma \leq 1$, Tunseki [24] investigated the conditions on the expressions $f'(z) - (1-\gamma)f(z)/z$ and $zf''(z) - \gamma f'(z)$ for the sufficient conditions of starlikeness and convexity. Frasin [8] obtained some sufficient conditions on $f'''(z)$ for starlikeness and convexity. In particular, he proved that when the function $f \in \mathcal{A}$ with $f''(0) = 0$ satisfies $|f'''(z)| < 1$, then f is starlike in $\mathbb{D}$ and if $|f'''(z)| < 1/2$, then f is convex in $\mathbb{D}$, see [8, Cororllary 2, Cororllary 3, p. 65].

Motivated by this, in 2010, Uyanik *et al.* [25] introduced and investigated a new subclass of $\mathcal{A}$ defined using the linear combination of the derivatives of ratio of the normalized analytic function with the identity function. For $\beta_1, \beta_2 \in \mathbb{C}, \lambda > 0$ and $f \in \mathcal{A}$ he defined $\mathcal{V}(\beta_1, \beta_2, \lambda)$ as follows:

$$\left|\beta_1 z \left(\frac{f(z)}{z}\right)' + \beta_2 z^2 \left(\frac{f(z)}{z}\right)''\right| \leq \lambda.$$

2010 *Mathematics Subject Classification.* 30C45, 30C50, 30C80.

Key words and phrases. Univalent function, Coefficient bound, Hankel determinant.

1

He obtained sufficient condition for normalized analytic functions to be in the class $\mathcal{V}(\beta_1, \beta_2, \lambda)$. He proved that the nth coefficient of functions in this class is bounded by $\lambda/((n-1)(|\beta_1| + (n-2)|\beta_2|))$.

It is well-known that the function in the class $\mathcal{S}$ satisfy $|a_n| \leq 2$ $(n = 2, 3, \cdots)$. Moreover, if $\sum_{n=2}^{\infty} n|a_n| \leq 1$, then $f \in \mathcal{S}^*$ and if $\sum_{n=2}^{\infty} n^2|a_n| \leq 1$, then $f \in \mathcal{K}$. There is another important quantity related to coefficients, called the Hankel determinant, which enable us to determine the necessary condition on coefficient functional for functions belonging to a given class of functions. For given natural numbers n, q, the Hankel determinant $H_{q,n}(f)$ of a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_1 = 1$ is defined by means of the following determinant

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

It is easy to see that the functional $H_{2,1}(f) = a_3 - a_2^2$ is the well-known Fekete-Szegö functional. However, the second Hankel determinant is given by $H_{2,2}(f) := a_2a_4 - a_3^2$. Further, the third Hankel determinant is $H_{3,\,1}(f) := a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$. The Hankel determinant $H_{q,n}(f)$ for the class $\mathcal{S}$ was investigated by Pommerenke [19] and Hayman [10]. For more details, see [4,5,11,13,19,21] and the references cited therein.

The Schwarzian derivative of a locally univalent function f, defined by

$$\mathbf{S}(f)(z) := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

The Schwarzian derivative is an important quantity in Univalent Function Theory. Further properties were investigated by Nehari [16]. He obtained the necessary and sufficient conditions for $f \in \mathcal{S}$. The higher order Schwarzian derivative [9, 23]), is defined by $\sigma_3(f) = \mathbf{S}(f)$ and for any integer $n \geq 4$, it is given by

$$\sigma_{n+1}(f) = (\sigma_n(f))' - (n-1)\sigma_n(f)\frac{f''}{f'}.$$

Droff and Szynal [6] studied the higher order Schwarzian derivative for convex functions. Now $\sigma_n(f)(0) =: \mathbf{S}_n$ and $\mathbf{S}_3 = \sigma_3(f)(0) = 6(a_3 - a_2^2)$, $\mathbf{S}_4 = \sigma_4(f)(0) = 24(a_4 - 3a_2a_3 + 2a_2^3)$ and $\mathbf{S}_5 = \sigma_5(f)(0) = 24(5a_5 - 20a_2a_4 - 9a_3^2 + 48a_3a_2^2 - 24a_2^4)$. The sharp bound on $|\mathbf{S}_i|$ $(i = 2, 3, 4)$, for $f \in \mathcal{K}$, investigated by Droff and Szynal. The generalization of their work, recently, carried out in [3] by Cho *et al.*

We shall investigate, the estimates on the Hankel determinants and the higher order Schwarzian derivatives by associating the functions of the class under consideration with the Carathéodory functions. Now we recall those results which shall be needed for investigation of our results. Let $\mathcal{P}$ denote the class of Carathéodory [1, 2] functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (z \in \mathbb{D}). \tag{1.1}$$

Let $\mathcal{B}$ be the class of analytic functions $w(z) = \sum_{n=1}^{\infty} c_n z^n$ $(z \in \mathbb{D})$ and satisfying the condition $|w(z)| < 1$ for $z \in \mathbb{D}$. The function $w \in \mathcal{B}$ and $p \in \mathcal{P}$ are related as $p(z) = (1 + w(z))/(1 - w(z))$. Consider a functional $\Psi(w) = |c_3 + \alpha c_1 c_2 + \beta c_1^3|$ for $w \in \mathcal{B}$ and α, $\beta \in \mathbb{R}$.

Lemma 1.1. [20, Lemma 2, p. 128] *If $w \in \mathcal{B}$, then for any real numbers α and β the following sharp estimate $\Psi(w) \le \Phi(\alpha, \beta)$ holds, where*

$$\Phi(\alpha,\beta) = \begin{cases} 1, & \text{if } (\alpha,\beta) \in \Omega_1 \cup \Omega_2, \\ |\beta|, & \text{if } (\alpha,\beta) \in \Omega_3 \cup \Omega_4 \cup \Omega_5 \\ \frac{2}{3}(|\alpha|+1)\left(\frac{|\alpha|+1}{3(|\alpha|+\beta+1)}\right)^{1/2}, & \text{if } (\alpha,\beta) \in \Omega_6 \cup \Omega_7. \end{cases}$$

Here the sets Ω_i's are defined by
$\Omega_1 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : |\alpha| \le 1/2, \ \ -1 \le \beta \le 1\right\}$,
$\Omega_2 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : \frac{1}{2} \le |\alpha| \le 2, \ \ \frac{4}{27}(|\alpha|+1)^3 - (|\alpha|+1) \le \beta \le 1\right\}$,
$\Omega_3 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : |\alpha| \le 2, \ \ \beta \ge 1\right\}$,
$\Omega_4 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : 2 \le |\alpha| \le 4, \ \ \beta \ge \frac{1}{12}(\alpha^2+8)\right\}$, *and*
$\Omega_5 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : |\alpha| \ge 4, \ \beta \ge \frac{2}{3}(|\alpha|-1)\right\}$.
$\Omega_6 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : \frac{1}{2} \le |\alpha| \le 2, \ -\frac{2}{3}(|\alpha|+1) \le \beta \le \frac{4}{27}(|\alpha|+1)^3 - (|\alpha|+1)\right\}$,
$\Omega_7 := \left\{(\alpha,\beta) \in \mathbb{R}^2 : |\alpha| \ge 2, \ -\frac{2}{3}(|\alpha|+1) \le \beta \le \frac{2|\alpha|(|\alpha|+1)}{\alpha^2+2|\alpha|+4}\right\}$.

Lemma 1.2. [14,15, Libera and Zlotkiewicz] *If $p \in \mathcal{P}$ has the form given by* (1.1) *with $p_1 \ge 0$, then*

$$2p_2 = p_1^2 + x(4 - p_1^2) \tag{1.2}$$

and

$$4p_3 = p_1^3 + 2p_1(4-p_1^2)x - p_1(4-p_1^2)x^2 + 2(4-p_1^2)(1-|x|^2)y \tag{1.3}$$

for some x and y such that $|x| \le 1$ and $|y| \le 1$.

Lemma 1.3. [22, Ravichandran and Verma] *Let $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{a}$ satisfy the inequalities $0 < \hat{\alpha} < 1$, $0 < \hat{a} < 1$ and*

$$8a(1-a)[(\hat{\alpha}\hat{\beta} - 2\hat{\gamma})^2 + (\hat{\alpha}(\hat{a}+\hat{\alpha}) - \hat{\beta})^2] + \hat{\alpha}(1-\hat{\alpha})(\hat{\beta} - 2\hat{a}\hat{\alpha})^2 \le 4\hat{a}\hat{\alpha}^2(1-\hat{\alpha})^2(1-\hat{a}).$$

If $p \in \mathcal{P}$ has the form given by (1.1), *then*

$$|\hat{\gamma}p_1^4 + \hat{a}p_2^2 + 2\hat{\alpha}p_1p_3 - (3/2)\hat{\beta}p_1^2p_2 - p_4| \le 2.$$

Lemma 1.4. [18, Ohno and Sugawa] *For any real numbers a, b and c, let the quantity $Y(a,b,c)$ be given by*

$$Y(a,b,c) = \max_{z \in \overline{\mathbb{D}}} \left\{|a + bz + cz^2| + 1 - |z|^2\right\},$$

where $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$. If $ac \ge 0$, then

$$Y(a,b,c) = \begin{cases} |a| + |b| + |c|, & \text{if } |b| \ge 2(1-|c|), \\ 1 + |a| + \frac{b^2}{4(1-|c|)}, & \text{if } |b| < 2(1-|c|). \end{cases}$$

Further, if $ac < 0$, *then*

$$Y(a,b,c) = \begin{cases} 1 - |a| + \frac{b^2}{4(1-|c|)}, & \text{if } -4ac(c^{-2}-1) \le b^2 \text{ and } |b| < 2(1-|c|), \\ 1 + |a| + \frac{b^2}{4(1+|c|)}, & \text{if } b^2 < \min\{4(1+|c|)^2, -4ac(c^{-2}-1)\}, \\ R(a,b,c), & \text{otherwise,} \end{cases}$$

where

$$R(a,b,c) = \begin{cases} |a| + |b| - |c|, & \text{if } |c|(|b|+4|a|) \le |ab|, \\ -|a| + |b| + |c|, & \text{if } |ab| \le |c|(|b|-4|a|), \\ (|c|+|a|)\sqrt{1 - \frac{b^2}{4ac}}, & \text{otherwise.} \end{cases}$$

2. Coefficient bounds

The following theorem gives the sharp upper bound for Fekete-Szegö functional and Hankel determinant for functions in the class $\mathcal{V}(\beta_1, \beta_2, \lambda)$.

Theorem 2.1. *Let* $0 < \beta_1 < 1, 0 < \beta_2 < 1$ *and* $f \in \mathcal{V}(\beta_1, \beta_2, \lambda)$. *Then the following sharp inequalities hold:*

(1) $|a_3 - \mu a_2^2| \le \frac{\lambda}{2\beta_1+\beta_2} \max\left\{1; \frac{2\lambda(\beta_1+\beta_2)|\mu|}{{\beta_2}^2}\right\}, \ \mu \in \mathbb{C}$.

(2) $|a_2 a_4 - {a_3}^2| \le \frac{\lambda^2(\beta_1+\beta_2)^2}{3\beta_1(\beta_1+2\beta_2)({\beta_1}^2+2\beta_1\beta_2+4{\beta_2}^2)}$.

(3) $|a_2 a_3 - a_4| \le \begin{cases} \frac{\lambda}{3(\beta_1+2\beta_2)}, & 0 < \lambda \le \frac{(3\sqrt{2}-2){\beta_1}^2+(3\sqrt{2}-2)\beta_1\beta_2}{3\beta_1+6\beta_2}; \\ \frac{2\beta_1(\beta_1+\beta_2)+3\lambda(\beta_1+2\beta_2)}{9\sqrt{3}\beta_1(\beta_1+\beta_2)(\beta_1+2\beta_2)}\lambda, & \lambda > \frac{(3\sqrt{2}-2){\beta_1}^2+(3\sqrt{2}-2)\beta_1\beta_2}{3\beta_1+6\beta_2}. \end{cases}$

Proof. Since $f \in \mathcal{V}(\beta_1, \beta_2, \lambda)$, it follows that there exists a Schwarz function $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots \in \mathcal{B}$ such that

$$\beta_1 z \left(\frac{f(z)}{z}\right)' + \beta_2 z^2 \left(\frac{f(z)}{z}\right)'' = \lambda w(z)). \tag{2.1}$$

In the view of interconnection $w(z) = (p(z)-1)/(p(z)+1) \in \mathbf{B}$ if and only if $p \in \mathcal{P}$ between the Schwarz function w and the Carathéodory function $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$, from (2.1), we get

$$a_2 = \frac{\lambda p_1}{2\beta_1}, \quad a_3 = \frac{\lambda(2p_2 - {p_1}^2)}{8(\beta_1+\beta_2)}, \tag{2.2}$$

and

$$a_4 = \frac{\lambda(4p_3 - 4p_1p_2 + {p_1}^3)}{24(\beta_1+2\beta_2)}, \quad a_5 = \frac{\lambda(8p_4 - 8p_1p_3 - 4{p_2}^2 + 6{p_1}^2p_2 - {p_1}^4)}{64(\beta_1+3\beta_2)}. \tag{2.3}$$

(1) From (2.2), Using the result [see [12]], for any complex number μ,

$$\left|p_2 - \mu p_1^2\right| \le 2\max\{1; |2\mu - 1|\},$$

we have

$$\begin{aligned} |a_3 - \mu {a_2}^2| &= \frac{\lambda}{4(\beta_1+\beta_2)}\left[p_2 - \frac{{\beta_1}^2 + 2\mu\lambda(\beta_1+\beta_2)}{2{\beta_1}^2}p_1^2\right] \\ &= \frac{\lambda}{2(\beta_1+\beta_2)}\max\left\{1, \frac{2\lambda(\beta_1+\beta_2)}{{\beta_1}^2}|\mu|\right\}. \end{aligned} \tag{2.4}$$

The equality holds in case of the function f defined by (2.1) with choice of the function $w(z) = z$.

(2) Using (2.2) and (2.3), we have

$$\begin{aligned} a_2a_4 - {a_3}^2 = \frac{\lambda^2}{192\beta_1(\beta_1+\beta_2)^2(\beta_1+2\beta_2)} &\left[({\beta_1}^2 + 2\beta_1\beta_2 + 4{\beta_2}^2){p_1}^4\right. \\ &-4({\beta_1}^2 + 2\beta_1\beta_2 + 4{\beta_2}^2){p_1}^2p_2 - 12\beta_1(\beta_1+2\beta_2){p_2}^2 \\ &\left.+16(\beta_1+\beta_2)^2p_1p_3\right]. \end{aligned} \tag{2.5}$$

Putting equivalent expressions for p_2 and p_3 in terms of p_1 from (1.2) and (1.3) in (2.5), we have

$$\begin{aligned} a_2a_4 - {a_3}^2 = \frac{\lambda^2}{192\beta_1(\beta_1+\beta_2)^2(\beta_1+2\beta_2)} &\left[\{-3\beta_1(\beta_1+2\beta_2)(4-{p_1}^2) + 4(\beta_1+\beta_2)^2{p_1}^2\}\right. \\ &\left.\times(4-{p_1}^2)x^2 + 8p_1(4-{p_1}^2)(\beta_1+\beta_2)^2(1-|x|^2)y\right]. \end{aligned} \tag{2.6}$$

Because $p \in \mathcal{P}$, and the class $\mathcal{P}$ is invariant under rotation, without loss of any generality, we can set $p_1 = |p_1| =: s \in [0, 2]$. Further, since $|x| \le 1$ and $|y| \le 1$ for some $x, y \in \mathbb{C}$, using this facts and the triangle inequality in (2.6) we can write

$$|a_2a_4 - {a_3}^2| \quad \le \quad T\left[\left|-\frac{3\beta_1(\beta_1+2\beta_2)(4-s^2) + 4(\beta_1+\beta_2)^2s^2}{8(\beta_1+\beta_2)^2}x^2\right| + s(1-|x|^2)\right], \tag{2.7}$$

where

$$T := \frac{\lambda^2(4-s^2)}{24\beta_1(\beta_1+2\beta_2)}.$$

We note that for $s = p_1 = 0$, and $s = p_1 = 2$ from (2.7), we have $|a_2a_4 - {a_3}^2| \le \lambda^2/4(\beta_1+\beta_2)^2$ and $|a_2a_4 - {a_3}^2| = 0$, respectively.

Now we assume that $s \in (0, 2)$. Then, form (2.7), we obtain

$$|a_2a_4 - {a_3}^2| \le \frac{\lambda^2}{24\beta_1(\beta_1+2\beta_2)}s(4-s^2)F(a,b,c), \tag{2.8}$$

where

$$F(a,b,c) := |a + bx + cx^2| + 1 - |x|^2,$$

with

$$a := 0, \quad b := 0 \text{ and } \ c := -\frac{3\beta_1(\beta_1+2\beta_2)(4-s^2) + 4(\beta_1+\beta_2)^2s^2}{8(\beta_1+\beta_2)^2s}.$$

Here it is easily seen that $ac = 0$. Here we have two cases now:

(i) When $0 < \beta_1 < \beta_2(\sqrt{3}-1)$ and $s^* \leq s < 2$, we obtain $|b| \geq 2(1-|c|)$. Therefore, by Lemma 1.4, we have

$$\begin{aligned} |a_2a_4 - {a_3}^2| &\leq \frac{\lambda^2(\beta_1+\beta_2)^2 s(4-s^2)}{24\beta_1(\beta_1+\beta_2)^2(\beta_1+2\beta_2)} F(a,b,c) \\ &= \frac{\lambda^2(\beta_1+\beta_2)^2 s(4-s^2)}{24\beta_1(\beta_1+\beta_2)^2(\beta_1+2\beta_2)} \left(\frac{3\beta_1(\beta_1+2\beta_2)(4-s^2)+4(\beta_1+\beta_2)^2 s^2}{8(\beta_1+\beta_2)^2 s} \right) \\ &= \frac{\lambda^2}{192\beta_1(\beta_1+\beta_2)^2(\beta_1+2\beta_2)} g(s), \end{aligned}$$

where $g : [s^*, 2) \to \mathbb{R}$ is defined by

$$g(s) := 3\beta_1(\beta_1+2\beta_2)(4-s^2) + 4(\beta_1+\beta_2)^2(4-s^2)s^2.$$

Cleanly, g has maximum at

$$s = s_1 := \frac{2\sqrt{-{\beta_1}^2 - 2\beta_1\beta_2 + {\beta_2}^2}}{\sqrt{{\beta_1}^2 + 2\beta_1\beta_2 + 4{\beta_2}^2}},$$

we have

$$\begin{aligned} |a_2a_4 - {a_3}^2| &\leq \frac{\lambda^2}{192\beta_1(\beta_1+\beta_2)^2(\beta_1+2\beta_2)} g(s_1) \\ &= \frac{\lambda^2(\beta_1+\beta_2)^2}{3\beta_1(\beta_1+2\beta_2)({\beta_1}^2+2\beta_1\beta_2+4{\beta_2}^2)}. \end{aligned}$$

(ii) When $0 < \beta_1 \leq \beta_2(\sqrt{3}-1)$ and $0 < s < s^*$, $\beta_1 > \beta_2(\sqrt{3}-1) > 0$ and $0 < s < 2$, we obtain $|b| < 2(1-|c|)$. Therefore, by Lemma 1.4, we have

$$\begin{aligned} |a_2a_4 - {a_3}^2| &\leq \frac{\lambda^2(\beta_1+\beta_2)^2 s(4-s^2)}{24\beta_1(\beta_1+\beta_2)^2(\beta_1+2\beta_2)} F(a,b,c) \\ &= \frac{\lambda^2(4-s^2)s}{24\beta_1(\beta_1+2\beta_2)} \\ &= \frac{\lambda^2}{24\beta_1(\beta_1+2\beta_2)} h(s), \end{aligned}$$

where the function $h : (0,2) \to \mathbb{R}$ is defined by

$$h(s) := (4-s^2)s.$$

Further computation reveals that h has its maximum at $s = s_2 := 2/\sqrt{3}$, and thus we have

$$|a_2a_4 - {a_3}^2| \leq \frac{\lambda^2}{24\beta_1(\beta_1+2\beta_2)} h(s_2) = \frac{2\sqrt{3}\lambda^2}{27\beta_1(\beta_1+2\beta_2)}.$$

Therefore, from (i) and (ii), we conclude that

$$|a_2a_4 - {a_3}^2| \leq \frac{\lambda^2(\beta_1+\beta_2)^2}{3\beta_1(\beta_1+2\beta_2)({\beta_1}^2+2\beta_1\beta_2+4{\beta_2}^2)}.$$

The equality holds in case of the function defined in (2.1) with

$$w(z) = \frac{z(u_0 - 2z^2)}{2 - u_0 z},$$

where $u_0 = 2\sqrt{-{\beta_1}^2 - 2\beta_1\beta_2 + {\beta_2}^2}/\sqrt{{\beta_1}^2 + 2\beta_1\beta_2 + 4{\beta_2}^2}$.

(3) To find the estimates on the functional $|a_2a_3 - a_4|$, we shall express the coefficients (a_i) in terms of Schwarz's coefficients (c_i). From (2.1), we have

$$a_2 = \frac{c_1\lambda}{\beta_1}, \quad a_3 = \frac{c_2\lambda}{2(\beta_1+\beta_2)}, \quad a_4 = \frac{c_3\lambda}{3(\beta_1+2\beta_2)}, \; a_5 = \frac{c_4\lambda}{4(\beta_1+3\beta_2)}. \tag{2.9}$$

Using (2.10), we get

$$\begin{aligned} |a_2a_3 - a_4| &= \left| -\frac{\lambda^2 c_1c_2}{2\beta_1(\beta_1+\beta_2)} + \frac{\lambda c_3}{3(\beta_1+2\beta_2)} \right| \\ &= \frac{\lambda}{3(\beta_1+2\beta_2)} \left| -\frac{3\lambda(\beta_1+2\beta_2)}{2\beta_1(\beta_1+\beta_2)} c_1c_2 + c_3 \right| \\ &= \frac{\lambda}{3(\beta_1+2\beta_2)} \Phi(\mu,\nu), \end{aligned}$$

where $\Phi(\mu,\nu) := |c_3 + \mu c_1c_2 + \nu {c_1}^3|$ with

$$\mu := -\frac{3\lambda(\beta_1+2\beta_2)}{2\beta_1(\beta_1+\beta_2)}, \text{ and } \nu := 0.$$

Assume that Ω_i's are as defined in lemma 1.1 with μ and ν as given above. We now complete the proof in the following cases.

- (i) Suppose that $0 < \lambda \le \beta_1(\beta_1+\beta_2)/3(\beta_1+2\beta_2)$, then we see that $-1/2 \le \mu \le 1/2$ and $-1 \le \nu \le 1$. So, we conclude that $(\mu,\nu) \in \Omega_1$.
- (ii) Let $\beta_1(\beta_1+\beta_2)/3(\beta_1+2\beta_2) \le \lambda \le \beta_1(3\sqrt{3}-2)(\beta_1+\beta_2)/3(\beta_1+2\beta_2)$. Then we can easily verify that $-2 \le \mu \le -1/2$ and $(4/27)(|\mu|+1)^3 - (|\mu|+1) \le \nu \le 1$ holds and we get $(\mu,\nu) \in \Omega_2$.
- (iii) Let $\beta_1(3\sqrt{3}-2)(\beta_1+\beta_2)/3(\beta_1+2\beta_2) \le \lambda \le 4\beta_1(\beta_1+\beta_2)/3(\beta_1+2\beta_2)$. Now we see that $-2 \le \mu \le -1/2$ and $-2(|\mu|+1)/3 \le \lambda \le (4/27)(|\mu|+1)^3 - (|\mu|+1)$ hold for all such positive values of λ and so $(\mu,\nu) \in \Omega_6$.
- (iv) Let $\lambda \ge 4\beta_1(\beta_1+\beta_2)/3(\beta_1+2\beta_2)$. Then we see that the conditions $\mu \le -2$ and $-(2/3)(|\mu|+1) \le \nu \le 2|\mu|(|\mu|+1)/(\mu^2+|\mu|+4)$ hold. Therefore, $(\mu,\nu) \in \Omega_7$.

Now by using Lemma 1.1, the cases (i) and (ii), we conclude that if

$$0 < \lambda \le \beta_1(3\sqrt{3}-2)(\beta_1+\beta_2)/3(\beta_1+2\beta_2),$$

then $\Phi(\mu,\nu) \le 1$. Further, the (iii) and (iv) hold, then

$$\Phi(\mu,\nu) \le (2\beta_1(\beta_1+\beta_2) + 3\lambda(\beta_1+2\beta_2))/3\sqrt{3}\beta_1(\beta_1+\beta_2)$$

for $\lambda \ge \beta_1(3\sqrt{3}-2)(\beta_1+\beta_2)/3(\beta_1+2\beta_2)$. The result is sharp in case of the function f defined by (2.1) with choice of the Schwarz function $w(z) = z^3$ and $w(z) = z(t_1 +$

$z)/(1+t_1z)$, respectively, where

$$t_1 = \left(\frac{|\mu|-1}{3(|\mu|-1-\nu)}\right)^{\frac{1}{2}}.$$

This ends the proof. ∎

Remark 2.2. In the case, when β_1 and β_2 are real numbers, from the result [25, Corollary 2.1, p.383], we conclude that

$$|a_n| \leq \frac{\lambda}{(n-1)(|\beta_1|+(n-2)|\beta_2|)}.$$

Using the above results, we deduce the following estimates on the third Hankel determinant:

Corollary 2.3. *Let* $0<\beta_1<1, 0<\beta_2<1$ *and* $f \in \mathcal{V}(\beta_1,\beta_2;\lambda)$. *Then the following holds:*

$$|H_{3,1}(f)| \leq \begin{cases} \tau_1\lambda^2, & 0<\lambda\leq\lambda_1; \\ \tau_2\lambda, & \lambda_1<\lambda\leq\lambda_2; \\ \tau_3, & \lambda\geq\lambda_2, \end{cases}$$

where

$$\tau_1 := \frac{\begin{array}{r}24\lambda(\beta_1+\beta_2)^2(\beta_1+2\beta_2)(\beta_1+3\beta_2)+\beta_1({\beta_1}^2+2\beta_1\beta_2+4{\beta_2}^2)\\ \times(5\beta_1+6\beta_2)(5\beta_1+14\beta_2)\end{array}}{2(\alpha g_3+\beta h_3)B_1^2},$$

$$\tau_2 := \frac{\begin{array}{r}6\lambda^2(\beta_1+\beta_2)(\beta_1+2\beta_2)(\beta_1+3\beta_2)+4\lambda\beta_1(\beta_1+3\beta_2)({\beta_1}^2+2\beta_1\beta_2+4{\beta_2}^2)\\ +9\beta_1(\beta_1+2\beta_2)^2({\beta_1}^2+2\beta_1\beta_2+4{\beta_2}^2)\end{array}}{\beta_1(\beta_1+2\beta_2)^2(\beta_1+3\beta_2)({\beta_1}^2+2\beta_1\beta_2+4{\beta_2}^2)}$$

and

$$\tau_3 := \frac{\mu^2\left(1296\mu^4+3456\mu^3+2304\mu^2+1740\mu+1015\right)}{5184(12\mu+7)}.$$

Theorem 2.4. *Let* $f \in \mathcal{V}(\beta_1,\beta_2;\lambda)$, *then the following sharp inequalities hold:*

(1) *If* $0<\beta_1<1$ *and* $0<\beta_2<1$, *then*

$$|\mathbf{S}_3| \leq \begin{cases} \frac{3\lambda}{\beta_1+\beta_2}, & 0<\lambda\leq\frac{{\beta_1}^2}{2(\beta_1+\beta_2)}; \\ \frac{6\lambda^2}{{\beta_1}^2}, & \lambda>\frac{{\beta_1}^2}{2(\beta_1+\beta_2)}. \end{cases}$$

(2) *(a) If either of the set of conditions* $0<\lambda\leq\lambda^*$ *or* ${\lambda_1}^*\leq\lambda\leq\lambda^{**}$ *and*

$$\{9(\beta_1+2\beta_2)\lambda+2\beta_1(\beta_1+\beta_2)\}[\{9(\beta_1+2\beta_2)\lambda+2\beta_1(\beta_1+\beta_2)\}^2-27{\beta_1}^2(\beta_1+\beta_2)^2] \\ \leq 324(\beta_1+2\beta_2)\lambda^2(\beta_1+\beta_2)^3$$

holds, then

$$|\mathbf{S}_4| \leq \frac{24\lambda}{3(\beta_1+2\beta_2)}.$$

(b) If either of the set of conditions ${\lambda_2}^* \leq \lambda \leq 4{\lambda_1}^*$ *and* $\frac{9}{4}\sqrt{\frac{\beta_1(\beta_1+2\beta)}{6}} \leq \beta_1+\beta_2$ *or* $\lambda \geq 4{\lambda_1}^*$ *holds, then*

$$|\mathbf{S}_4| \leq \frac{48\lambda^4}{{\beta_1}^3}.$$

(3) *If* $0<\beta_1<1, 0<\beta_2<1$ *and* $0<\lambda<3\beta_1(\beta_1+2\beta_2)/8(\beta_1+3\beta_3)$, *then*

$$|\mathbf{S}_5| \leq \frac{720\lambda}{\beta_1+3\beta_2}.$$

Proof. Let $f \in \mathcal{V}(\beta_1, \beta_2; \lambda)$. Then, to find the estimates on the higher order Schwarzian derivatives, we shall express the coefficients (a_i) in terms of Schwarz's coefficients (c_i). From (2.1), we have

$$a_2 = \frac{c_1\lambda}{\beta_1}, \quad a_3 = \frac{c_2\lambda}{2(\beta_1+\beta_2)}, \quad a_4 = \frac{c_3\lambda}{3(\beta_1+2\beta_2)}, \; a_5 = \frac{c_4\lambda}{4(\beta_1+3\beta_2)}. \tag{2.10}$$

Using first part of Theorem 2.1, we have

$$\begin{aligned} |\mathbf{S}_3| &= 6|a_3-{a_2}^2)| \\ &\leq \frac{3\lambda}{\beta_1+\beta_2}\max\left\{1, \frac{2(\beta_1+\beta_2)\lambda}{{\beta_1}^2}\right\}. \end{aligned}$$

The function for the equality holds by (2.1) with the choice $w(z)=z$.

Now we consider the estimate on $|S_4|$. From (2.10), we obtain

$$\begin{aligned} \mathbf{S}_4 &= 24(a_4-3a_2a_3+2{a_3}^2) \\ &= \frac{24\lambda}{3(\beta_1+2\beta_2)}\left[\frac{6(\beta_1+2\beta_2)\lambda^2}{{\beta_1}^3}{c_1}^3 - \frac{9(\beta_1+2\beta_2)\lambda}{2\beta_1(\beta_1+\beta_2)}c_1c_2+c_3\right] \\ &= \frac{24\lambda}{3(\beta_1+2\beta_2)}\Upsilon(\mu,\nu) \end{aligned}$$

where $\Upsilon(\mu,\nu) := c_3+\mu c_1c_2+\nu {c_1}^3$ with

$$\mu := -\frac{9(\beta_1+2\beta_2)\lambda}{2\beta_1(\beta_1+\beta_2)}, \;\; \text{and} \;\; \nu := \frac{6(\beta_1+2\beta_2)\lambda^2}{{\beta_1}^3}.$$

Assume that Ω_i's are as defined in Lemma 1.1 with μ and ν as given above. We now complete the proof with the following cases.

(i) Suppose that $0<\lambda\leq{\lambda_1}^*$. In this case, we see that $-1/2\leq\mu\leq1/2$ holds. Moreover, $-1\leq\nu\leq1$ holds if and only if $0<\lambda\leq{\lambda_2}^*$, where ${\lambda_1}^* := \beta_1(\beta_1+\beta_2)/9(\beta_1+2\beta_2)$ and ${\lambda_2}^* := \beta_1\sqrt{\beta_1/6(\beta_1+2\beta_2)}$. Thus, for all $0<\lambda\leq\min\{{\lambda_1}^*, {\lambda_2}^*\}$, we conclude that $(\mu,\nu)\in\Omega_1$.

(ii) Next suppose that ${\lambda_1}^*<\lambda\leq4{\lambda_1}^*$, then we see that the condition $-2\leq\mu\leq-1/2$ holds. Further, $(4/27)(\mu+1)^3-(\mu+1)\leq\nu\leq1$ holds if and only if $0<\lambda\leq{\lambda_2}^*$

and

$$\{9(\beta_1+2\beta_2)\lambda+2\beta_1(\beta_1+\beta_2)\}[\{9(\beta_1+2\beta_2)\lambda+2\beta_1(\beta_1+\beta_2)\}^2 - 27{\beta_1}^2(\beta_1+\beta_2)^2] \leq 324(\beta_1+2\beta_2)\lambda^2(\beta_1+\beta_2)^3. \quad (2.11)$$

So, if ${\lambda_1}^* \leq \lambda \leq \lambda^{**} := \min\{4{\lambda_1}^*, {\lambda_2}^*\}$ and (2.11) hold, then $(\mu,\nu)\in\Omega_2$.

(iii) Let ${\lambda_2}^* \leq \lambda \leq {\lambda_3}^* := 4(\beta_1+\beta_2)/9(\beta_1+2\beta_2)$ and $\beta_1+\beta_2 \geq 9\sqrt{\beta_1(\beta_1+2\beta_2)}/4\sqrt{6}$. Then, we can easily verify that $|\mu|\leq 2$ and $\nu\geq 1$. Therefore, $(\mu,\nu)\in\Omega_5$.

(iv) Let $4{\lambda_1}^* \leq \lambda \leq 8{\lambda_1}^*$. Now we see that $-4\leq\mu\leq -2$ and $\nu \geq (\mu^2+8)/12$ hold for all such positive values of λ and hence $(\mu,\nu)\in\Omega_6$.

(v) Let $\lambda \geq 8{\lambda_1}^*$. Then we see that $\mu\leq -4$ and $\nu\geq 2(|\mu|-1)$, so $(\mu,\nu)\in\Omega_7$.

Now by using Lemma 1.1 and the cases (i) and (ii), we conclude that if $0<\lambda\leq \min\{{\lambda_1}^*, {\lambda_2}^*\}$ or ${\lambda_1}^* \leq \lambda \leq \lambda^{**}$ and (2.11) hold, then $\Upsilon(\mu,\nu)\leq 1$. Further, from the cases $(iii)-(v)$ and Lemma 1.1, we conclude that $\Upsilon(\mu,\nu)\leq\nu$, for ${\lambda_2}^*\leq\lambda\leq{\lambda_3}^*$ and $\beta_1+\beta_2 \geq 9\sqrt{\beta_1(\beta_1+2\beta_2)}/4\sqrt{6}$ or $\lambda\geq 4{\lambda_1}^*$. The result is sharp in case of the function f defined by (2.1) with choice of the Schwarz function $w(z)=z^3$ and $w(z)=z$, respectively. This completes the proof.

Now we find the estimate on $|\mathbf{S}_5|$. Using 2.2 and 2.3, we get

$$\begin{aligned}\mathbf{S}_5 &= 24(5a_5-20a_2a_4-9a_3^2+48a_2^2a_3-24a_2^4)\\ &= \frac{-720\lambda}{\beta_1+3\beta_2}[\hat{\gamma}p_1^4+\hat{a}p_2^2+2\hat{\alpha}p_1p_3-(3/2)\hat{\beta}p_1^2p_2-p_4)]\\ &= \frac{-720\lambda}{\beta_1+3\beta_2}\Psi(\hat{\gamma},\hat{a},\hat{\alpha},\hat{\beta}), \end{aligned}\quad (2.12)$$

where $\Psi(\hat{\gamma},\hat{a},\hat{\alpha},\hat{\beta}) := \hat{\gamma}p_1^4+\hat{a}p_2^2+2\hat{\alpha}p_1p_3-(3/2)\hat{\beta}p_1^2p_2-p_4$ with the parameters $\hat{\gamma},\hat{a},\hat{\alpha}$ and $\hat{\beta}$ are given by

$$\hat{\gamma} := \frac{\beta_1+3\beta_2}{15}\left(\frac{15}{8(\beta_1+3\beta_2)}+\frac{27\lambda}{8(\beta_1+\beta_2)^2}+\frac{10\lambda}{\beta_1(\beta_1+2\beta_2)}+\frac{36\lambda^2}{{\beta_1}^2(\beta_1+\beta_2)}+\frac{36\lambda^3}{{\beta_1}^4}\right),$$

$$\hat{a} := \frac{\beta_1+3\beta_2}{6}\left(\frac{8\lambda}{\beta_1(\beta_1+2\beta_2)}+\frac{3}{\beta_1+3\beta_2}\right),$$

$$\hat{\alpha} := \frac{\beta_1+3\beta_2}{5}\left(\frac{5}{2(\beta_1+3\beta_2)}+\frac{9\lambda}{2(\beta_1+\beta_2)^2}\right)$$

and

$$\hat{\beta} := \frac{2(\beta_1+3\beta_2)}{45}\left(\frac{45}{4(\beta_1+3\beta_2)}+\frac{27\lambda}{2(\beta_1+\beta_2)^2}+\frac{20\lambda}{\beta_1(\beta_1+2\beta_2)}+\frac{72\lambda^2}{{\beta_1}^2(\beta_1+\beta_2)}\right).$$

We assume that $0<\beta_1<1$ and $0<\beta_2<1$ and $0<\lambda<3\beta_1(\beta_1+2\beta_2)/8(\beta_1+3\beta_3)$. Under these conditions, it is a simple matter to verify that $0<\hat{\alpha}<1$ and $0<\hat{a}<1$. Moreover, with these restrictions all conditions of Lemma 1.3 are fulfilled and thus, we get $|\Psi(\hat{\gamma},\hat{a},\hat{\alpha},\hat{\beta})|\leq 2$. Thus, the result follows from (2.12). ∎

Acknowledgement

The third author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2016R1D1A1A09916450).

References

[1] C. Carathéodory, Über den variabilitätsbereich der fourier'schen konstanten von positiven harmonischen funktionen, Rend. Circ. Mat. Palermo **32** (1911), 193–217.

[2] C. Carathéodory, Über den variabilitätsbereich der coeffizienten von potenzreihen, die gegebene werte nicht annehmen, Math. Ann. **64** (1907), no. 1, 95–115.

[3] N. E. Cho, V. Kumar and V. Ravichandran, Sharp bounds on the higher order Schwarzian derivatives for Janowski classes, submitted.

[4] N. E. Cho, B. Kowalczyk, O. S. Kwon, A. Lecko and Y. J. Sim The bounds of some determinants for starlike functions of order alpha, Bull. Malays. Math. Sci. Soc. (2017), 13 pp.

[5] N. E. Cho, B. Kowalczyk, O. S. Kwon, A. Lecko and Y. J. Sim, Some coefficient inequalities related to the Hankel determinant for strongly starlike functions of order alpha, J. Math. Ineq. **11**, 2 (2017), 429–439.

[6] M. Dorff and J. Szynal, Higher order Schwarzian derivatives for convex univalent functions, Tr. Petrozavodsk. Gos. Univ. Ser. Mat. **15** (2009), 7–11.

[7] B. A. Frasin, New sufficient conditions for univalence, Gen. Math. **17** (2009), no. 3, 91–98.

[8] B. A. Frasin, New sufficient conditions for analytic and univalent functions, Acta Univ. Apulensis Math. Inform. No. 17 (2009), 61–67.

[9] R. Harmelin, Aharonov invariants and univalent functions, Israel J. Math. **43** (1982), no. 3, 244–254.

[10] W. K. Hayman, On the second Hankel determinant of mean univalent functions, Proc. London Math. Soc. **18** (1968), no. 3, 77–94.

[11] A. Janteng, S. A. Halim and M. Darus, Hankel determinant for starlike and convex functions, Int. J. Math. Anal. (Ruse) **1** (2007), no. 13-16, 619–625.

[12] F. R. Keogh and E. P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc. **20** (1969), 8–12.

[13] S. K. Lee, V. Ravichandran and S. Supramaniam, Bounds for the second Hankel determinant of certain univalent functions, J. Inequal. Appl. Art. ID. 281 (2013), 17 pp.

[14] R. J. Libera and E. J. Zlotkiewicz, Early coefficients of the inverse of a regular convex function, Proc. Amer. Math. Soc. **85** (1982), no. 2, 225–230.

[15] R. J. Libera and E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivatives in $\mathcal{P}$, Proc. Amer. Math. Soc. **87** (1983), no. 2, 251–257.

[16] Z. Nehari, The Schwarzian derivative and schlicht functions, Bull. Amer. Math. Soc. **55** (1949), 545–551.

[17] M. Obradovič, Simple suients conditions for starlikeness, Matematicki Vesnik. 49 (1997), 241–244.

[18] R. Ohno and T. Sugawa, Coefficient estimates of analytic endomorphisms of the unit disk fixing a point with applications to concave functions, Kyoto J. Math. (advance publication, 9 June 2017), doi: 10.1215/21562261-2017-0015.

[19] C. Pommerenke, On the coefficients and Hankel determinants of univalent functions, J. London Math. Soc. **41** (1966), 111–122.

[20] D. V. Prokhorov and J. Szynal, Inverse coefficients for (α, β)-convex functions, Ann. Univ. Mariae Curie-Skłodowska Sect. A **35** (1981), 125–143.

[21] C. Pommerenke, On the Hankel determinants of univalent functions, Mathematika **14** (1967), 108–112.

[22] V. Ravichandran and S. Verma, Bound for the fifth coefficient of certain starlike functions, C. R. Math. Acad. Sci. Paris **353**, 6 (2015), 505–510.
[23] E. Schippers, Distortion theorems for higher order Schwarzian derivatives of univalent functions, Proc. Amer. Math. Soc. **128** (2000), no. 11, 3241–3249.
[24] N. Tuneski, Some simple sufficient conditions for starlikeness and convexity, Appl. Math. Lett. **22** (2009), no. 5, 693–697.
[25] N. Uyanik, S. Owa, E. Kadioğlu, Some properties of functions associated with close-to-convex and starlike of order α, Appl. Math. Comput. **216** (2010), no. 2, 381–387.

(Ji Hyang Park) Department of Applied Mathematics, Pukyong National University, Busan 48513, Korea

E-mail address: `jihyang1022@naver.com`

(Virendra Kumar) Department of Applied Mathematics, Pukyong National University, Busan 48513, Korea

E-mail address: `vktmaths@yahoo.in`

(Nak Eun Cho) Corresponding Author, Department of Applied Mathematics, Pukyong National University, Busan 48513, Korea

E-mail address: `necho@pknu.ac.kr`

EXPLICIT FORMULAE OF CAUCHY POLYNOMIALS WITH A q PARAMETER IN TERMS OF r-WHITNEY NUMBERS

F. A. SHIHA

Abstract. The Cauchy polynomials with a q parameter were recently defined, and several arithmetical properties were studied. In this paper, we establish explicit formulae for computing the Cauchy polynomials with a q parameter in terms of r-Whitney numbers of the first kind. We also obtain several properties and combinatorial identities.

AMS (2010) Subject Classification: 05A15, 05A19, 11B73, 11B75.
Key Words. Cauchy numbers and polynomials, r-Whitney numbers, Stirling numbers.

1. Introduction

The Cauchy polynomials of the first kind $c_n(z)$ are defined by

$$c_n(z) = \int_0^1 (x-z)_n \, dx, \tag{1.1}$$

and the Cauchy polynomials of the second kind $\hat{c}_n(z)$ are defined by

$$\hat{c}_n(z) = \int_0^1 (-x+z)_n \, dx, \tag{1.2}$$

where $(y)_n = \prod_{i=0}^{n-1}(y-i)$ is the falling factorial with $(y)_0 = 1$. The exponential generating function of these polynomials are

$$\sum_{n=0}^{\infty} c_n(z) \frac{t^n}{n!} = \frac{t}{(1+t)^z \, \ln(1+t)}. \tag{1.3}$$

$$\sum_{n=0}^{\infty} \hat{c}_n(z) \frac{t^n}{n!} = \frac{t(1+t)^z}{(1+t) \, \ln(1+t)}. \tag{1.4}$$

(see [7, 4]). When $z = 0$, $c_n(0) = c_n$ and $\hat{c}_n(0) = \hat{c}_n$ are the Cauchy numbers of the first and second kind (see [2, 9, 12, 8]).

Recently [5] obtained a representation of the integer values of Cauchy polynomials in terms of r-Stirling numbers of the first kind $s_r(n,k)$ [3]. For all integers $n, r \geq 0$,

$$c_n(r) = \sum_{k=0}^{n} s_r(n+r, k+r) \frac{1}{k+1}, \tag{1.5}$$

$$\hat{c}_n(-r) = \sum_{k=0}^{n} (-1)^k \, s_r(n+r, k+r) \frac{1}{k+1}. \tag{1.6}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Given variables y and m and a positive integer k, define the generalized rising and falling factorials of order k with increment m by

$$[y|m]_k = \prod_{j=0}^{k-1}(y+jm), \qquad [y|m]_0 = 1$$

$$(y|m)_k = \prod_{j=0}^{k-1}(y-jm), \qquad (y|m)_0 = 1.$$

Komatsu [6] introduced the Poly-Cauchy polynomials and numbers with a q parameter, and the Cauchy polynomials and numbers with a q parameter as special cases.

Let q be a real number with $q \neq 0$, Komatsu [6] defined the Cauchy polynomials with a q parameter of the first kind $c_n^q(z)$ by

$$c_n^q(z) = \int_0^1 (x-z|q)_n \, dx \tag{1.7}$$

and the Cauchy polynomials with a q parameter of the second kind $\hat{c}_n^q(z)$ by

$$\hat{c}_n^q(z) = \int_0^1 (-x+z|q)_n \, dx. \tag{1.8}$$

The exponential generating functions are

$$\sum_{n=0}^{\infty} c_n^q(z)\frac{t^n}{n!} = (1+qt)^{\frac{-z}{q}} \sum_{k=0}^{\infty}\left(\frac{\ln(1+qt)}{q}\right)^k \frac{1}{k!}\,\frac{1}{k+1}, \tag{1.9}$$

$$\sum_{n=0}^{\infty} \hat{c}_n^q(z)\frac{t^n}{n!} = (1+qt)^{\frac{z}{q}} \sum_{k=0}^{\infty}\left(-\frac{\ln(1+qt)}{q}\right)^k \frac{1}{k!}\,\frac{1}{k+1}. \tag{1.10}$$

If $z=0$, then $c_n^q(0) = c_n^q$ and $\hat{c}_n^q(0) = \hat{c}_n^q$ are the Cauchy numbers with q parameter of the first and second kind, respectively. If $q=1$, then $c_n^1(z) = c_n(z)$ and $\hat{c}_n^1(z) = \hat{c}_n(z)$.

The r-Whitney numbers of the first and second kind were introduced by Mezö [10]. For non-negative integers n and k with $0 \le k \le n$, let $w(n,k) = w_{q,r}(n,k)$ denote the r-Whitney numbers of the first kind, which are defined by

$$q^n (x)_n = \sum_{k=0}^{n} w(n,k)\,(qx+r)^k. \tag{1.11}$$

Let $W(n,k) = W_{q,r}(n,k)$ denote the r-Whitney numbers of the second kind, which are defined by

$$(qx+r)^n = \sum_{k=0}^{n} q^k\, W(n,k)\,(x)_k. \tag{1.12}$$

Usually r is taken to be a non-negative integer and q a positive integer, but both may also be regarded as real numbers [11]. The exponential generating function of $w(n,k)$ is given by [10]

$$\sum_{n\ge k} w(n,k)\frac{t^n}{n!} = (1+qt)^{\frac{-r}{q}}\left(\frac{\ln(1+qt)}{q}\right)^k \frac{1}{k!}, \tag{1.13}$$

2. Basic results

Replace x by $\frac{x-r}{q}$ in (1.11), then the r-Whitney numbers of the first kind $w(n,k)$ are given by

$$(x-r|q)_n = \prod_{j=0}^{n-1}(x-r-jq) = \sum_{k=0}^{n} w(n,k)\, x^k, \qquad q \neq 0, \tag{2.1}$$

Using (1.7), we get the following theorem.

Theorem 1. *The Cauchy polynomials with q parameter of the first kind $c_n^q(r)$, $q \neq 0$ can be written explicitly as*

$$c_n^q(r) = \sum_{k=0}^{n} w(n,k)\, \frac{1}{k+1}. \tag{2.2}$$

The first few polynomials are
$c_0^q(r) = 1$,
$c_1^q(r) = -r + \frac{1}{2}$,
$c_2^q(r) = r^2 + (q-1)r - \frac{1}{2}q + \frac{1}{3}$,
$c_3^q(r) = -r^3 - \frac{3}{2}(2q-1)r^2 + (-2q^2+3q-1)r + q^2 - q + \frac{1}{4}$,
$c_4^q(r) = r^4 + (6q-2)r^3 + (11q^2-9q+2)r^2 + (6q^3-11q^2+6q-1)r - 3q^3 + \frac{11}{3}q^2 - \frac{3}{2}q + \frac{1}{5}$.

Remark 1. *If $r = 0$, then $c_n^q(0) = c_n^q$ are the Cauchy numbers with q parameter of the first kind* [6]

$$c_n^q = \int_0^1 (x|q)_n\, dx = \sum_{k=0}^{n} q^{n-k}\, s(n,k)\, \frac{1}{k+1},$$

where $s(n,k)$ are the Stirling numbers of the first kind.

If $q = 1$, we have $c_n^1(r) = c_n(r)$ and $w_{1,r}(n,k)$ are reduced to $s_r(n+r, k+r)$, and hence we obtain the explicit formula (1.5).

From (1.13), we can easily derive the exponential generating function of $c_n^q(r)$ as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} c_n^q(r)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} w(n,k)\frac{1}{k+1}\,\frac{t^n}{n!} \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} w(n,k)\frac{t^n}{n!}\,\frac{1}{k+1} \\
&= (1+qt)^{\frac{-r}{q}} \sum_{k=0}^{\infty}\left(\frac{\ln(1+qt)}{q}\right)^{k} \frac{1}{k!}\,\frac{1}{k+1} \\
&= (1+qt)^{\frac{-r}{q}} \sum_{k=0}^{\infty}\left(\frac{\ln(1+qt)}{q}\right)^{k+1} \frac{1}{(k+1)!}\,\frac{q}{\ln(1+qt)} \\
&= \frac{q(1+qt)^{\frac{-r}{q}}}{\ln(1+qt)} \sum_{k=1}^{\infty}\left(\frac{\ln(1+qt)}{q}\right)^{k} \frac{1}{k!} \\
&= \frac{q(1+qt)^{\frac{-r}{q}}}{\ln(1+qt)} \left((1+qt)^{\frac{1}{q}} - 1\right).
\end{aligned}$$

When $r=0$, we get the exponential generating function of c_n^q

$$\sum_{n=0}^{\infty} c_n^q \frac{t^n}{n!} = \frac{q}{\ln(1+qt)}\left((1+qt)^{\frac{1}{q}}-1\right)$$

According to (2.1),

$$(-x-r|q)_n = \prod_{j=0}^{n-1}(-x-r-jq) = \sum_{k=0}^{n} w(n,k)\,(-1)^k\,x^k, \qquad q\neq 0. \tag{2.3}$$

Using (1.7), we get the following theorem.

Theorem 2. *The Cauchy polynomials with q parameter of the second kind $\hat{c}_n^q(r)$, $q\neq 0$ can be written explicitly as*

$$\hat{c}_n^q(-r) = \sum_{k=0}^{n}(-1)^k\,w(n,k)\,\frac{1}{k+1}. \tag{2.4}$$

The first few polynomials are
$\hat{c}_0^q(r)=1$,
$\hat{c}_1^q(r)=r-\frac{1}{2}$,
$\hat{c}_2^q(r)=r^2-(q+1)r+\frac{1}{2}q+\frac{1}{3}$,
$\hat{c}_3^q(r)=r^3-\frac{3}{2}(2q+1)r^2+(2q^2+3q+1)r-q^2-q-\frac{1}{4}$,
$\hat{c}_4^q(r)=r^4-(6q+2)r^3+(11q^2+9q+2)r^2-(6q^3+11q^2+6q+1)r+3q^3+\frac{11}{3}q^2+\frac{3}{2}q+\frac{1}{5}$.

Remark 2. *If $r=0$, then $\hat{c}_n^q(0)=\hat{c}_n^q$ are the Cauchy numbers with q parameter of the second kind* [6]

$$\hat{c}_n^q = \int_0^1(-x|q)_n\,dx = \sum_{k=0}^{n} q^{n-k}\,s(n,k)\,\frac{(-1)^k}{k+1},$$

Similarly, we can obtain the exponential generating function of $\hat{c}_n^q(r)$:

$$\begin{aligned}\sum_{n=0}^{\infty}\hat{c}_n^q(r)\frac{t^n}{n!} &= (1+qt)^{\frac{r}{q}}\sum_{k=0}^{\infty}\left(-\frac{\ln(1+qt)}{q}\right)^k\frac{1}{k!}\,\frac{1}{k+1}\\ &= \frac{q(1+qt)^{\frac{r}{q}}}{\ln(1+qt)}\left(1-(1+qt)^{\frac{-1}{q}}\right).\end{aligned} \tag{2.5}$$

And

$$\sum_{n=0}^{\infty}\hat{c}_n^q\frac{t^n}{n!} = \frac{q}{\ln(1+qt)}\left(1-(1+qt)^{\frac{-1}{q}}\right). \tag{2.6}$$

Replace x by $\frac{x-r}{q}$ in (1.12), then the r-Whitney numbers of the second kind $W(n,k)$ are given by

$$x^n = \sum_{k=0}^{n} W(n,k)(x-r|q)_k = \sum_{k=0}^{n} W(n,k)\prod_{j=0}^{k-1}(x-r-jq), \qquad q\neq 0. \tag{2.7}$$

Thus, the relation between $c_n^q(r)$, $\hat{c}_n^q(r)$ and $W(n,k)$ can be obtained as follows:

$$\sum_{k=0}^{n} W(n,k)\,c_k^q(r) = \int_0^1\sum_{k=0}^{n} W(n,k)(x-r|q)_k\,dx = \int_0^1 x^n\,dx = \frac{1}{n+1} \tag{2.8}$$

(2.9)
$$\sum_{k=0}^{n} W(n,k)\,\hat{c}_k^q(-r) = \int_0^1 \sum_{k=0}^{n} W(n,k)(-x-r|q)_k\,dx = \int_0^1 (-1)^n\,x^n\,dx = \frac{(-1)^n}{n+1}$$

Cheon et al. [1] gave the following representation of $w(n,k)$ in terms of $s(n,k)$

$$w(n,k) = \sum_{i=k}^{n} \binom{n}{i} (-1)^{n-i}\,q^{i-k}\,[r|q]_{n-i}\,s(i,k).$$

Hence,

Corollary 1. *The Cauchy polynomials $c_n^q(r)$ can be computed by using $s(n,k)$ as follows:*

$$\begin{aligned} c_n^q(r) &= \sum_{k=0}^{n}\sum_{i=k}^{n} \binom{n}{i} (-1)^{n-i}\,q^{i-k}\,[r|q]_{n-i}\,s(i,k)\,\frac{1}{k+1} \\ &= \sum_{i=0}^{n}\sum_{k=0}^{i} \binom{n}{i} (-1)^{n-i}\,q^{i-k}\,[r|q]_{n-i}\,s(i,k)\,\frac{1}{k+1}. \end{aligned} \tag{2.10}$$

When $q=1$, we obtain the identity

$$c_n(r) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i}\,[r|1]_{n-i} c_i. \tag{2.11}$$

The r-Whitney numbers $w_{q,r}(n,k)$ satisfy the following identity [1].

$$w_{q,r+s}(n,k) = \sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} [r|q]_{n-j}\,w_{q,s}(j,k), \tag{2.12}$$

hence, we obtain the following theorem.

Theorem 3. *For $n \geq 0$, we have*

$$c_n^q(r+s) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} [r|q]_{n-j}\,c_j^q(s). \tag{2.13}$$

Proof.

$$\begin{aligned} c_n^q(r+s) &= \sum_{k=0}^{n} w_{q,r+s}(n,k)\,\frac{1}{k+1} \\ &= \sum_{k=0}^{n}\sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} [r|q]_{n-j}\,w_{q,s}(j,k)\,\frac{1}{k+1} \\ &= \sum_{j=0}^{n}\sum_{k=0}^{j} (-1)^{n-j} \binom{n}{j} [r|q]_{n-j}\,w_{q,s}(j,k)\,\frac{1}{k+1} \\ &= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} [r|q]_{n-j}\,c_j^q(s). \end{aligned}$$

□

Remark 3. *For $s=0$, we get*

$$c_n^q(r)=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}[r|q]_{n-j}\,c_j^q. \tag{2.14}$$

For $q=1$, we get

$$c_n(r+s)=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}[r|1]_{n-j}\,c_j(s). \tag{2.15}$$

Acknowledgement 1. *The author thank Prof. István Mezö for reading carefully the paper and giving helpful suggestions.*

References

[1] G. S. Cheon and J. H. Jung, r-Whitney numbers of Dowling lattices, *Discrete Math.* **312**, 2337–2348 (2012).

[2] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.

[3] A. Z. Broder, The r-Stirling numbers, *Discrete Math.* **49**, 241-259 (1984).

[4] K. Kamano and T. Komatsu, Poly-Cauchy polynomials, *Mosc. J. Comb. Number Theory* **3**, 61–87 (2013).

[5] T. Komatsu and I. Mezö, Several explicit formulae of Cauchy polynomials in terms of r-Stirling numbers, *Acta Math. Hungar.* **148** 2, 522–529 (2016).

[6] T. Komatsu, Poly-Cauchy numbers with a q parameter, *Ramanujan J.* **31**, 353–371, (2013).

[7] T. Komatsu, Poly-Cauchy numbers, *Kyushu J. Math.* **67**, 143–153 (2013).

[8] T. Komatsu, Sums of products of Cauchy numbers, including Poly-Cauchy numbers, *J. Discrete Math.* **2013**, Article ID 373927, 10 pages (2013).

[9] D. Merlini, R. Sprugnoli and M. C. Verri, The Cauchy numbers, *Discrete Math.* **306**, 1906-1920 (2006).

[10] I. Mezö, A new formula for the Bernoulli polynomials, *Results Math.* **58**, 329–335 (2010).

[11] M. Shattuck, Identities for Generalized Whitney and Stirling Numbers, *J. Integer Seq.* **20**, Article 17.10.4. (2017)

[12] F-Z. Zhao, Sums of products of Cauchy numbers, *Discrete Math.* **309**, 3830–3842 (2009).

Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, EGYPT.

E-mail address: fshiha@yahoo.com, fshiha@mans.edu.eg

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Global dynamics of Chikungunya virus with two routes of infection

A. M. Elaiw[a], S. E. Almalki[a,b] and A. Hobiny[a]

[a]Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia.

[b]Jeddah College of Technology, Technical and Vocational Training Corporation

Emails: a_m_elaiw@yahoo.com (A. Elaiw), samialmalki0@gmail.com (S. E. Almalki)

Abstract

In this paper, we address the stability analysis of within-host Chikungunya virus (CHIKV) infection models with antibodies. We incorporate two modes of infections, attaching a CHIKV to a host monocyte, and contacting an infected monocyte with an uninfected monocyte. The global stability analysis of the equilibria are established using Lyapunov method. The existence and global stability of the steady states are determined by the basic reproduction number $\mathcal{R}_0$. We have proven that the CHIKV-free equilibrium E_0 is globally asymptotically stable when $\mathcal{R}_0 \leq 1$, and the infected equilibrium E_1 is globally asymptotically stable when $\mathcal{R}_0 > 1$. The theoretical results are confirmed by numerical simulations.

1 Introduction

During last decades, many researchers have developed and analyzed several mathematical models human pathogens (see e.g. [1]-[16]). Chikungunya virus (CHIKV) is an alphavirus causes chikungunya fever. CHIKV is a mosquito-transmitted and is transmitted by the Aedes albopictus and Aedes agypti mosquito. Most of authors develop the mathematical models to describe the disease transmission mosquito and human populations. Recently, Wang and Liu [16] have proved a mathematical model for the within-host CHIKV dynamics as:

$$\dot{s} = \beta - \delta s - \eta s y, \tag{1}$$

$$\dot{y} = \eta s y - \epsilon y, \tag{2}$$

$$\dot{p} = \pi y - c p - r x p, \tag{3}$$

$$\dot{x} = \lambda + \rho x p - m x, \tag{4}$$

Here, s, y, p and x are the concentrations of uninfected monocytes, infected monocytes, CHIKV pathogen and antibodies, respectively. β and δ represent the birth rate and death rate constants of the uninfected monocytes, respectively. The monocytes become infected at rate ηsy, where η is the infection rate constant. Constants ϵ, c and m represent, respectively, the death rate of the infected monocytes, CHIKV and antibodies. Constant π is the generation rate of the CHIKV from actively infected monocytes. Antibodies attack the CHIKV at rate rxp. Once antigen is encountered, the antibodies expand at a constant rate λ and proliferate at rate ρxp. In a very recent work, Elaiw et al. [17], [18] have studied the global stability analysis of a class of CHIKV dynamics models. The models presented in [16]-[18] assume that the uninfected monocyte becomes infected by contacting with CHIKV(CHIKV-to-monocyte transmission). Kristin and Mork [19] reported that the CHIKV can also spread by infected-to-monocyte transmission. Viral danamics models with both cellular and viral infections have been studied in several works [20]-[24]. However, the dynamics of CHIKV with two routes of infection did not studied before.

Our aim is to propose and analyse a CHIKV dynamics model with two routes of infection. We calculate the basic reproduction number $\mathcal{R}_0$, and construct Lyapunov functions to prove the global stability of the equilbria.

2 CHIKV dynamics model

We investigate the following CHIKV dynamics model with CHIKV-to-monocyte and infected-to-monocyte with two routes of infection:

$$\dot{s} = \beta - \delta s - \eta_1 sp - \eta_2 sy, \tag{5}$$

$$\dot{y} = \eta_1 sp + \eta_2 sy - \epsilon y, \tag{6}$$

$$\dot{p} = \pi y - cp - rxp, \tag{7}$$

$$\dot{x} = \lambda + \rho xp - mx. \tag{8}$$

Here, the uninfected monocytes become infected at rate $(\eta_1 y + \eta_2 p)s$, where η_1 and η_2 are the CHIKV-monocyte and infected-monocyte incidence constants, respectively.

2.1 Nonnegativity and boundedness

Lemma 1 *There exist $M_1, M_2, M_3 > 0$, such that the following compact set is positively invariant for system (5)-(8)*

$$\Gamma_1 = \{(s, y, p, x) \in \mathbb{R}^4_{\geq 0} : 0 \leq s, y \leq M_1, 0 \leq p \leq M_2, 0 \leq x \leq M_3\}$$

Proof. We have

$$\dot{s}\mid_{s=0}=\beta>0, \qquad \dot{y}\mid_{y=0}=\eta_1 sp\geq 0, \quad \text{for all} \quad s,p\geq 0,$$

$$\dot{p}\mid_{p=0}=\pi y\geq 0, \qquad \text{for all} \quad y\geq 0, \qquad \dot{x}\mid_{x=0}=\lambda>0.$$

Thus $\mathbb{R}^4_{\geq 0}$ positively invariant with respect to system (5)-(8). Let us define

$$F_1(t)=s(t)+y(t),$$
$$F_2(t)=p(t)+\frac{r}{\rho}x(t).$$

Then from Eqs. (5)-(8) we get

$$\begin{aligned}\dot{F}_1(t)&=\beta-\delta s(t)-\epsilon y(t)\\&\leq\beta-\sigma_1(s(t)+y(t))\\&=\beta-\sigma_1F_1(t),\end{aligned}$$

where, $\sigma_1=min\{\delta,\epsilon\}$. Hence $F_1(t)\leq M_1$, if $F_1(0)\leq M_1$, where $M_1=\frac{\beta}{\sigma_1}$. It follows that $0\leq s(t),y(t)\leq M_1$ if $0\leq s(0)+y(0)\leq M_1$. Moreover, we have

$$\begin{aligned}\dot{F}_2(t)&=\pi y(t)-cp(t)+\frac{r}{\rho}\lambda-\frac{mr}{\rho}x(t)\\&\leq\pi M_1+\frac{r}{\rho}\lambda-\sigma_2\left(p(t)+\frac{r}{\rho}x(t)\right)\\&=\pi M_1+\frac{r}{\rho}\lambda-\sigma_2F_2(t),\end{aligned}$$

where, $\sigma_2=min\{c,m\}$. Hence $F_2(t)\leq M_2$, if $F_2(0)\leq M_2$, where $M_2=\frac{\pi M_1+\frac{r}{\rho}\lambda}{\sigma_2}$. Since $p(t)$ and $x(t)$ are all nonnegative, then $0\leq p(t)\leq M_2$ and $x(t)\leq M_3$ if $0\leq p(0)+\frac{r}{\rho}x(0)\leq M_2$, where $M_3=\frac{\rho M_2}{r}$. ∎

2.2 Equilbria

We define the basic reproduction number

$$\mathcal{R}_0=\frac{(\eta_1\pi m+\eta_2 cm+\eta_2 r\lambda)\beta}{\epsilon\delta(cm+r\lambda)}.$$

Lemma 2 *(i) if $\mathcal{R}_0\leq 1$, then there exists only one equilbrium $E_0\in\Gamma_1$ (ii) if $\mathcal{R}_0>1$, then there exist two equilbria $E_0\in\Gamma_1$ and $E_1\in\overset{\circ}{\Gamma}_1$, where $\overset{\circ}{\Gamma}_1$ is the interior of Γ_1.*

Proof. Any equilbrium satisfying

$$\beta-\delta s-\eta_1 sp-\eta_2 sy=0, \tag{9}$$

$$\eta_1 sp+\eta_2 sy-\epsilon y=0, \tag{10}$$

$$\pi y-cp-rxp=0, \tag{11}$$

$$\lambda+\rho xp-mx=0. \tag{12}$$

By solving Eqs. (9)-(12) we we get two equilbria a CHIKV-free equilbrium $E_0 = (s_0, 0, 0, x_0)$, where $s_0 = \frac{\beta}{\delta}$ and $x_0 = \frac{\lambda}{m}$. Moreover we have

$$C_1 p^3 + C_2 p^2 + C_3 p + C_4 = 0,$$

where

$$\begin{aligned}
C_1 &= -c\pi\epsilon\eta_1\rho^2 - c^2\epsilon\eta_2\rho^2,\\
C_2 &= 2cm\pi\epsilon\eta_1\rho + 2c^2m\epsilon\eta_2\rho + \pi r\epsilon\eta_1\lambda\rho + 2cr\epsilon\eta_2\lambda\rho - c\pi\delta\epsilon\rho^2 + \pi^2\beta\eta_1\rho^2 + c\pi\beta\eta_2\rho^2\\
C_3 &= -cm^2\pi\epsilon\eta_1 - c^2m^2\epsilon\eta_2 - m\pi r\epsilon\eta_1\lambda - 2cmr\epsilon\eta_2\lambda - r^2\epsilon\eta_2\lambda^2 + 2cm\pi\delta\epsilon\rho - 2m\pi^2\beta\eta_1\rho\\
&\quad - 2cm\pi\beta\eta_2\rho + \pi r\delta\epsilon\lambda\rho - \pi r\beta\eta_2\lambda\rho,\\
C_4 &= -cm^2\pi\delta\epsilon + m^2\pi^2\beta\eta_1 + cm^2\pi\beta\eta_2 - m\pi r\delta\epsilon\lambda + m\pi r\beta\eta_2\lambda.
\end{aligned}$$

Let define a function $X(p)$ as:

$$X(p) = C_1 p^3 + C_2 p^2 + C_3 p + C_4 = 0.$$

Then we obtain

$$\begin{aligned}
X(0) &= C_4,\\
X\left(\frac{m}{\rho}\right) &= -\frac{mr^2\epsilon\eta_2\lambda^2}{\rho} < 0.
\end{aligned}$$

The constant C_4 can be written as

$$C_4 = m\pi\delta\epsilon(cm + r\lambda)\left(\frac{(\eta_1\pi m + \eta_2 cm + \eta_2 r\lambda)\beta}{\epsilon\delta(cm + r\lambda)} - 1\right)$$

Then $C_4 > 0$ if the following condition is satisfied

$$\frac{(\eta_1\pi m + \eta_2 cm + \eta_2 r\lambda)\beta}{\epsilon\delta(cm + r\lambda)} > 1, \tag{13}$$

then there exists $p_1 \in (0, \frac{m}{\rho})$ such that $X(p_1) = 0$. Therefore, if condition (13) is satisfied, then

$$\begin{aligned}
s_1 &= \frac{\epsilon c(m - \rho p_1) + \epsilon r\lambda}{\eta_1\pi(m - \rho p_1) + \eta_2 c(m - \rho p_1) + \eta_2 r\lambda} > 0,\\
y_1 &= \frac{p_1(c(m - \rho p_1) + r\lambda)}{\pi(m - \rho p_1)} > 0, \qquad x_1 = \frac{\lambda}{m - \rho p_1} > 0.
\end{aligned}$$

Then an infected equilbrium $E_1 = (s_1, y_1, p_1, x_1)$ exists when $\mathcal{R}_0 > 1$.

Now we show that $E_0 \in \Gamma_1$ and $E_1 \in \overset{\circ}{\Gamma}_1$. Clearly, $E_0 \in \Gamma_1$. From the equilbrium conditions of E_1 we have

$$\beta = \delta s_1 + \eta_1 s_1 p_1 + \eta_2 s_1 y_1 \Rightarrow \delta s_1 + \epsilon y_1 = \beta \Rightarrow 0 < s_1 < \frac{\beta}{\delta} \le M_1, 0 < y_1 < \frac{\beta}{\epsilon} \le M_1.$$

Moreover, from Eqs. (11) and (12) we have

$$\begin{aligned}
cp_1 &= \pi y_1 + \frac{r}{\rho}\lambda - \frac{mr}{\rho}x_1 \Rightarrow cp_1 + \frac{mr}{\rho}x_1 = \pi y_1 + \frac{r}{\rho}\lambda < \pi M_1 + \frac{r}{\rho}\lambda \\
p_1 &< \frac{\pi M_1 + \frac{r}{\rho}\lambda}{c} \le M_2, x_1 < \frac{\rho}{r}\frac{\pi M_1 + \frac{r}{\rho}\lambda}{m} \le \frac{\rho M_2}{r} = M_3.
\end{aligned}$$

It follows that, $E_1 \in \overset{\circ}{\Gamma}$. ■

3 Global properties

Define a function $G(z) = z - 1 - \ln z$.

Theorem 1 *If $\mathcal{R}_0 \le 1$, then E_0 is globally asymptotically stable in Γ_1.*

Proof. Letting $\mathcal{R}_0 \le 1$ and constructing a Lyapunov function $U_0(s, y, p, x)$ as:

$$U_0(s, y, p, x) = s_0 G\left(\frac{s}{s_0}\right) + y + \frac{\eta_1 s_0}{c + rx_0}p + \frac{r\eta_1 s_0}{\rho(c + rx_0)}x_0 G\left(\frac{x}{x_0}\right).$$

Calculating $\frac{dU_0}{dt}$ along system (5)-(8) we obtain

$$\begin{aligned}
\frac{dU_0}{dt} &= \left(1 - \frac{s_0}{s}\right)\left(\beta - \delta s - \eta_1 sp - \eta_2 sy\right) + \eta_1 sp + \eta_2 sy - \epsilon y \\
&+ \frac{\eta_1 s_0}{c + rx_0}\left(\pi y - cp - rxp\right) + \frac{r\eta_1 s_0}{\rho(c + rx_0)}\left(1 - \frac{x_0}{x}\right)\left(\lambda + \rho xp - mx\right) \\
&= \left(1 - \frac{s_0}{s}\right)\left(\beta - \delta s\right) + \eta_2 s_0 y - \epsilon y + \frac{\eta_1 s_0}{c + rx_0}\pi y + \frac{r\eta_1 s_0}{\rho(c + rx_0)}\left(1 - \frac{x_0}{x}\right)\left(\lambda - mx\right)
\end{aligned}$$

$$\begin{aligned}
\frac{dU_0}{dt} &= -\delta\frac{(s - s_0)^2}{s} + \epsilon\left(\frac{\eta_2 s_0}{\epsilon} + \frac{\eta_1 s_0 \pi}{\epsilon(c + rx_0)} - 1\right)y - \frac{r\eta_1 s_0 m}{\rho(c + rx_0)}\frac{(x - x_0)^2}{x} \\
&= -\delta\frac{(s - s_0)^2}{s} - \frac{r\eta_1 s_0 m}{\rho(c + rx_0)}\frac{(x - x_0)^2}{x} + \epsilon(\mathcal{R}_0 - 1)y. \qquad (14)
\end{aligned}$$

Since $\mathcal{R}_0 \le 1$, then for all $s, y, p, x > 0$ we have $\frac{dU_0}{dt} \le 0$. Let $W_0 = \{(s, y, p, x) : \frac{dU_0}{dt} = 0\}$. It can be easily shown that $\frac{dU_0}{dt} = 0$ at E_0. Appling LaSalle's invariance principle, we get E_0 is globally asymptotically stable when $\mathcal{R}_0 \le 1$. ■

Theorem 2 *If $\mathcal{R}_0 > 1$, then E_1 is globally asymptotically stable in $\overset{\circ}{\Gamma}_1$.*

Proof. Define

$$U_1(s, y, p, x) = s_1 G\left(\frac{s}{s_1}\right) + y_1 G\left(\frac{y}{y_1}\right) + \frac{\eta_1 s_1 p_1}{\pi y_1}p_1 G\left(\frac{p}{p_1}\right) + \frac{r}{\rho}\frac{\eta_1 s_1 p_1}{\pi y_1}x_1 G\left(\frac{x}{x_1}\right).$$

Calculating $\frac{dU_1}{dt}$ along the trajectories of (5)-(8) we obtain

$$\begin{aligned}\frac{dU_1}{dt} &= \left(1-\frac{s_1}{s}\right)\left(\beta-\delta s-\eta_1 sp-\eta_2 sy\right)+\left(1-\frac{y_1}{y}\right)\left(\eta_1 sp+\eta_2 sy-\epsilon y\right)\\ &+\frac{\eta_1 s_1 p_1}{\pi y_1}\left(1-\frac{p_1}{p}\right)\left(\pi y-cp-rxp\right)+\frac{r\eta_1 s_1 p_1}{\rho\pi y_1}\left(1-\frac{x_1}{x}\right)\left(\lambda+\rho xp-mx\right)\\ &=\left(1-\frac{s_1}{s}\right)\left(\beta-\delta s\right)+\eta_1 s_1 p+\eta_2 s_1 y-\eta_1 sp\frac{y_1}{y}-\eta_2 sy_1-\epsilon y+\epsilon y_1+\frac{\eta_1 s_1 p_1}{y_1}y-\eta_1 s_1 p_1\frac{p_1 y}{py_1}\\ &-\frac{\eta_1 s_1 p_1}{\pi y_1}cp+\frac{\eta_1 s_1 p_1}{\pi y_1}cp_1+\frac{\eta_1 s_1 p_1}{\pi y_1}rxp_1-\frac{r\eta_1 s_1 p_1}{\pi y_1}x_1 p+\frac{r\eta_1 s_1 p_1}{\rho\pi y_1}\left(1-\frac{x_1}{x}\right)\left(\lambda-mx\right).\end{aligned}$$

Applying the equilbrium conditions for E_1

$$\beta=\eta_1 s_1 p_1+\eta_2 s_1 y_1+\delta s_1,\quad \epsilon y_1=\eta_1 s_1 p_1+\eta_2 s_1 y_1,\quad cp_1=\pi y_1-rx_1 p_1,\quad \lambda=mx_1-\rho x_1 p_1.$$

we get

$$\begin{aligned}\frac{dU_1}{dt} &= -\delta\frac{(s-s_1)^2}{s}+\left(1-\frac{s_1}{s}\right)\left(\eta_1 s_1 p_1+\eta_2 s_1 y_1\right)\\ &-\eta_1 s_1 p_1\frac{spy_1}{s_1 p_1 y}-\eta_2 s_1 y_1\frac{s}{s_1}+\eta_1 s_1 p_1+\eta_2 s_1 y_1-\eta_1 s_1 p_1\frac{p_1 y}{py_1}+\eta_1 s_1 p_1\\ &-2\frac{\eta_1 s_1 p_1}{\pi y_1}rx_1 p_1+\frac{\eta_1 s_1 p_1}{\pi y_1}rxp_1+\frac{\eta_1 s_1 p_1}{\pi y_1}rx_1 p_1\frac{x_1}{x}-\frac{r\eta_1 s_1 p_1 m}{\rho\pi y_1}\frac{(x-x_1)^2}{x}.\end{aligned}\tag{15}$$

Eq. (15) can be simplified as:

$$\begin{aligned}\frac{dU_1}{dt} &= -\delta\frac{(s-s_1)^2}{s}+\eta_2 s_1 y_1\left[2-\frac{s_1}{s}-\frac{s}{s_1}\right]+\eta_1 s_1 p_1\left[3-\frac{s_1}{s}-\frac{spy_1}{s_1 p_1 y}-\frac{p_1 y}{py_1}\right]\\ &-\frac{\eta_1 s_1 p_1}{\pi y_1}rx_1 p_1\left[2-\frac{x}{x_1}-\frac{x_1}{x}\right]-\frac{r\eta_1 s_1 p_1 m}{\rho\pi y_1}\frac{(x-x_1)^2}{x}\\ &=-\delta\frac{(s-s_1)^2}{s}-\frac{\eta_2 y_1(s-s_1)^2}{s}+\eta_1 s_1 p_1\left[3-\frac{s_1}{s}-\frac{spy_1}{s_1 p_1 y}-\frac{p_1 y}{py_1}\right]\\ &+\frac{\eta_1 s_1 p_1}{\pi y_1}rp_1\frac{(x-x_1)^2}{x}-\frac{r\eta_1 s_1 p_1 m}{\rho\pi y_1}\frac{(x-x_1)^2}{x}\\ &=-(\delta+\eta_2 y_1)\frac{(s-s_1)^2}{s}-\frac{\eta_1 s_1 p_1}{\pi y_1}\frac{r\lambda}{\rho x_1}\frac{(x-x_1)^2}{x}+\eta_1 s_1 p_1\left[3-\frac{s_1}{s}-\frac{spy_1}{s_1 p_1 y}-\frac{p_1 y}{py_1}\right].\end{aligned}$$

We use the following arithmetic mean-geometric mean inequality rule. If $a_i\geq 0$, $i=1,2,...,n$, then

$$\frac{1}{n}\sum_{i=1}^{n}a_i\geq\sqrt[n]{\prod_{i=1}^{n}a_i},\tag{16}$$

where equality holding if and only if $a_1=a_2=...=a_n$. It follows that

$$\frac{1}{3}\left(\frac{s_1}{s}+\frac{spy_1}{s_1 p_1 y}+\frac{p_1 y}{py_1}\right)\geq 1.$$

Therefore, $\frac{dU_1}{dt}\leq 0$ for all $s,y,p,x>0$ and $\frac{dU_1}{dt}=0$ if and only if $s=s_1, y=y_1, p=p_1$ and $x=x_1$. It follows that the global stability of E_1 is induced from LaSalle's invariance principle. ■

Table 1: The value of the parameters of model (5)-(8).

Parameter	Value	Parameter	Value
β	2	δ	0.1
η_1, η_2	varied	ϵ	0.5
π	4	c	0.1
r	0.5	λ	1.4
m	1	ρ	0.2

4 Numerical Simulations

We will use the values of the parameters given in Table 1. Moreover, we similate the system with three different initial values as:

IV1: $s(0) = 14.0, y(0) = 1.0, p(0) = 1.0$, and $x(0) = 1.0$,

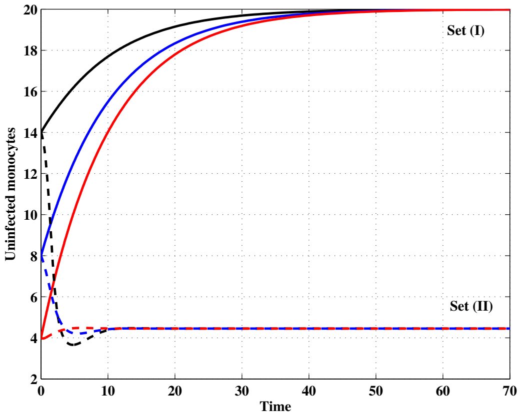
IV2: $s(0) = 8.0, y(0) = 2.0, p(0) = 3.0$, and $x(0) = 4.0$,

IV3: $s(0) = 4.0, y(0) = 3.5, p(0) = 6.0$, and $x(0) = 7.0$.

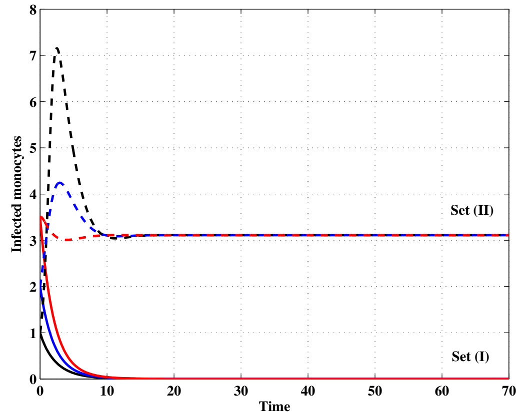
Then we consider two sets of the values of η_1 and η_2 as follows:

Set (I): We choose $\eta_1 = \eta_2 = 0.001$. The value of $\mathcal{R}_0$ is computed as $\mathcal{R}_0 = 0.2400 < 1$. Figure 1 shows that, the concentrations of the uninfected monocytes and B cells return to their values $s_0 = \frac{\beta}{\delta} = 20$ and $x_0 = \frac{\lambda}{m} = 1.4$, respectively. On the other hand, the concentrations of infected monocytes and CHIKV particles are declining and reaching zero for the initial values IV1-IV3. This shows that, E_0 is GAS which agrees with the result of Theorem 1.

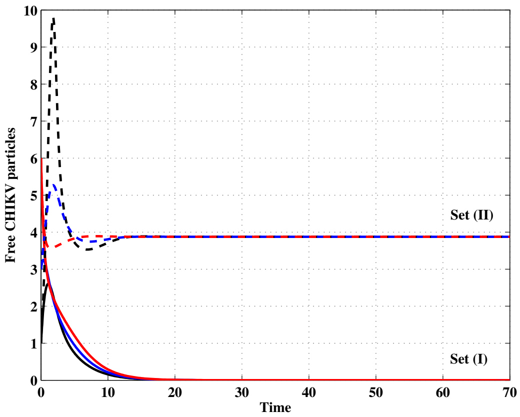
Set (II): We take $\eta_1 = \eta_2 = 0.05$. Then, we calculate $\mathcal{R}_0 = 12.0 > 1$. We comput the equilbria as $E_0(20.0, 0, 0, 1.4)$ and $E_1 = (4.45, 3.10, 3.87, 6.22)$. Figure 1 shows that when $\mathcal{R}_0 > 1$, the states of the system tend to E_1 for all the three initial values IV1-IV3. This confirms that the validity of Theorem 2.



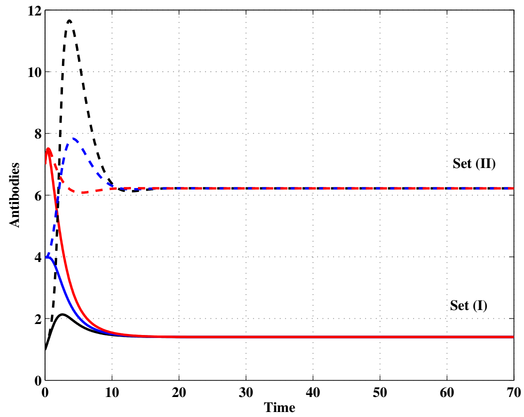
(a) Uninfected monocytes.



(b) Infected monocytes.



(c) Free CHIKV particles.



(d) Antibodies.

Figure 1: The simulation of trajectories of system (5)-(8).

References

[1] M. A. Nowak and C. R. M. Bangham, *Population dynamics of immune responses to persistent viruses*, Science, **272** (1996) 74-79.

[2] X. Li and S. Fu, *Global stability of a virus dynamics model with intracellular delay and CTL immune response*, Mathematical Methods in the Applied Sciences, **38** (2015), 420-430.

[3] A. M. Elaiw, N. H. AlShamrani and K. Hattaf, *Dynamical behaviors of a general humoral immunity viral infection model with distributed invasion and production*, International Journal of Biomathematics, 10(3), (2017) Article ID 1750035.

[4] A. M. Elaiw,, A. A. Raezah, and K. Hattaf, *Stability of HIV-1 infection with saturated virus-target and infected-target incidences and CTL immune response*, International Journal of Biomathematics, Vol. 10, No. 5 (2017), 1750070.

[5] A. M. Elaiw and S.A. Azoz, *Global properties of a class of HIV infection models with Beddington-DeAngelis functional response,* Mathematical Methods in the Applied Sciences, **36** (2013), 383-394.

[6] A. M. Elaiw, *Global properties of a class of HIV models*, Nonlinear Analysis: Real World Applications, **11** (2010), 2253-2263.

[7] A. M. Elaiw, and N. A. Almuallem, *Global dynamics of delay-distributed HIV infection models with differential drug efficacy in cocirculating target cells*, Mathematical Methods in the Applied Sciences, **39** (2016), 4-31.

[8] G. Huang, Y. Takeuchi and W. Ma, *Lyapunov functionals for delay differential equations model of viral infections*, SIAM J. Appl. Math., **70**(7) (2010), 2693-2708.

[9] K. Wang, A. Fan, and A. Torres, *Global properties of an improved hepatitis B virus model*, Nonlinear Analysis: Real World Applications, **11** (2010), 3131-3138.

[10] A. U. Neumann, N. P. Lam, H. Dahari, D. R. Gretch, T. E. Wiley, T. J, Layden, and A. S. Perelson, *Hepatitis C viral dynamics in vivo and the antiviral efficacy of interferon-alpha therapy*, Science, **282** (1998), 103-107.

[11] L. Wang, M. Y. Li, and D. Kirschner, *Mathematical analysis of the global dynamics of a model for HTLV-I infection and ATL progression*, Mathematical Biosciences, **179** (2002) 207-217.

[12] H. Shu, L. Wang and J. Watmough, *Global stability of a nonlinear viral infection model with infinitely distributed intracellular delays and CTL imune responses*, SIAM Journal of Applied Mathematics, **73**(3) (2013), 1280-1302.

[13] A. M. Elaiw and N. H. AlShamrani, *Global stability of humoral immunity virus dynamics models with nonlinear infection rate and removal*, Nonlinear Analysis: Real World Applications, **26**, (2015), 161-190.

[14] A. M. Elaiw and N. H. AlShamrani, *Stability of a general delay-distributed virus dynamics model with multi-staged infected progression and immune response*, Mathematical Methods in the Applied Sciences, **40(3)** (2017), 699-719.

[15] M. Li, and H. Shu, *Global dynamics of a mathematical model for HTLV-I infection of CD4+ T-cells*, Applied Mathematical Modelling, **35**(7) (2011) 3587-3595

[16] Y. Wang, X. Liu, *Stability and Hopf bifurcation of a within-host chikungunya virus infection model with two delays,* Mathematics and Computers in Simulation, **138** (2017), 31-48.

[17] A. M. Elaiw, T. O. Alade and S. M. Alsulami, *Stability of a within-host Chikungunya virus dynamics model with latency*, Journal of Computational Analysis and Applications, **26**(5) 2019, 777-790.

[18] A. M. Elaiw, T. O. Alade and S. M. Alsulami, *Analysis of latent CHIKV dynamics model with time delays*, Journal of Computational Analysis and Applications, **27**(1) 2019, 19-36.

[19] Kristin M. Long and Mark T. Heise, *Protective and Pathogenic Responses to Chikungunya Virus Infection*, Curr Trop Med Rep. **2**(1) (2015), 13-21.

[20] J. Wang, J. Lang, X. Zou, *Analysis of an age structured HIV infection model with virus-to-cell infection and cell-to-cell transmission*, Nonlinear Analysis: Real World Applications, **34** (2017), 75-96.

[21] F. Li and J. Wang, *Analysis of an HIV infection model with logistic target cell growth and cell-to-cell transmission*, Chaos, Solitons and Fractals, **81** (2015), 136-145.

[22] X. Lai and X. Zou , *Modeling cell-to-cell spread of HIV-1 with logistic target cell growth*, Journal of Mathematical Analysis and Applications, **426** (2015), 563–584.

[23] X. Lai, X. Zou, *Modelling HIV-1 virus dynamics with both virus-to-cell infection and cell-to-cell transmission*, SIAM Journal of Applied Mathematics, **74** (2014), 898–917.

[24] Y. Yang, L. Zou and S. Ruanc, *Global dynamics of a delayed within-host viral infection model with both virus-to-cell and cell-to-cell transmissions*, Mathematical Biosciences, **270** (2015), 183-191.

Weighted norm inequalities of θ-type Calderón-Zygmund operators and commutators on λ-central Morrey space

Yanqi Yang and Shuangping Tao*

(*College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, Gansu, China*, Email:taosp@nwnu.edu.cn)

Abstract: In this paper, the weighted boundedness for θ-type Calderón-Zygmund operators T_θ is established on the λ-central Morrey space. Futhermore, the weighted norm inequalities for commutators of $[b, T_\theta]$ generated by T_θ and BMO functions on the weighted λ-central Morrey space is also given.

Keywords: θ-type Calderón-Zygmund operator; weighted λ-central Morrey space; commutator
2010 MR Subject Classification: 42B20, 42B25, 42B35.

1 Introduction and notation

The theory of Calderón-Zygmund operators has played very important roles in modern harmonic analysis with lots of extensive applications in the others fields of mathematics, which has been extensively studied (see [7-10, 16-17], for instance). In 1985, Yabuta introduced certain θ-type Calderón-Zygmund operators to facilitate his study of certain classes of pseudodifferential operators (see [36]). Following the terminology of Yabuta, we recall the so-called θ-type Calderón-Zygmund operators. Let θ be a non-negative and non-decreasing function on $\mathbb{R}^+ = (0, \infty)$ satisfying

$$\int_0^1 \frac{\theta(t)}{t} \mathrm{d}t < \infty. \tag{1.1}$$

A measurable function $K(\cdot, \cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is said to be a θ-type Calderón-Zygmund kernel if it satisfies

$$|K(x, y)| \le C|x - y|^{-n} \tag{1.2}$$

and

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \le C\theta\left(\frac{|x - x'|}{|x - y|}\right)|x - y|^{-n}, \quad \text{as} \quad |x - y| \ge 2|x - x'|. \tag{1.3}$$

Definition 1.1[36] Let T_θ be a linear operator from $\mathcal{S}(\mathbb{R}^n)$ into its dual $\mathcal{S}'(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class. One can say that T_θ is a θ-type Calderón-Zygmund operator if it satisfies the following conditions:

(1) T_θ can be extended to be a bounded linear operator on $L^2(\mathbb{R}^n)$;

(2) there is a θ-type Calderón-Zygmund kernel $K(x, y)$ such that

$$T_\theta f(x) := \int_{\mathbb{R}^n} K(x, y) f(y) \mathrm{d}y, \quad \text{as} \quad f \in C_c^\infty(\mathbb{R}^n) \quad \text{and} \quad x \notin \mathrm{supp} f. \tag{1.4}$$

*Corresponding author and Email:taosp@nwnu.edu.cn(by S. Tao); 18709498755@126.com (by Y. Yang)

It is easy to see that the classical Calderón-Zygmund operator with standard kernel is a special case of θ-type operator T_θ as $\theta(t) = t^\delta$ with $0 < \delta \leq 1$. Given a locally integrable function b, the commutator generated by T_θ and b is defined by

$$[b, T_\theta]f(x) = b(x)T_\theta f(x) - T_\theta(b \cdot f)(x) = \int_{\mathbb{R}^n} [b(x) - b(y)]K(x, y)f(y)\mathrm{d}y. \tag{1.5}$$

Such type of operators is extensively applied in PDE with non-smooth area. Many authors concentrates on the boundedness of this operators on various function spaces, we refer the reader to see [19-20, 27, 29-30, 33-35] for its developments and applications. In [27], Quek-Yang established the boundedness of T_θ on spaces such as weighted Lebesgue spaces and weak Lebesgue spaces, weighted Hardy spaces and weak Hardy spaces. Ri-Zhang obtained the bounedness of T_θ on Hardy spaces with non-doubling measures and non-homogeneous metric measure spaces in [29-30]. Wang proved the boundedness of T_θ and $[b, T_\theta]$ on the generalized weighted Morrey spaces in [33]. Inspired by the results mentioned previously, a natural and interesting problem is to consider whether the θ-type Calderón-Zygmund operators T_θ and their commutators $[b, T_\theta]$ are bounded on λ-central Morrey space or not. The purpose of this paper is to give an surely answer.

On the other hand, the well-known Morrey spaces which introduced originally by Morrey [23] in relation to the study of partial differential equations, were widely investigated during last decades, including the study of classical operators of harmonic analysis in various generalizations of these spaces. Morrey-type spaces appeared to be quite useful in the study of the local behavior of the solutions of partial differential equations, a priori estimates and other topics. They are also widely used in applications to regularity properties of solutions to PDE including the study of Navier-Stokes equations (see [32] and references therein). The ideas of Morrey (see [23]) were further developed by Campanato in 1964 (see [11]). In 1975, Adam proved the boundedness of Riesz potential on the classical Morrey space in [1]. Later, in 1987, the boundedness of singular integrals and Hardy-Littlewood maximal functions on Morrey spaces was obtained By Chiarenza and Frasca in [13]. A more systematic study of these (and even more general) spaces, we refer the readers to see [2-3, 6, 26, 28, 31].

In [5], Beurling introduced a pair of dual Banach spaces, A^q and $B^{q'}$ with $1/q + 1/q' = 1$. After that, Feichtinger found the folling way to describe B^q as

$$\|f\|_{B^q} = \sup_{k\geq 0}(2^{-kn/q}\|f\chi_k\|_{L^q}) < \infty, \tag{1.6}$$

where χ_0 is the characteristic function of the unit ball defined by $\{x \in \mathbb{R}^n : |x| \leq 1\}$ and χ_k is the characteristic function of the annulus, that is $\{x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^k\}$ with $k \in \mathbb{Z}^+$. By duality, the beurling algebra A^q can be written as

$$\|f\|_{A^q} = \sum_{k=0}^{\infty} 2^{kn/q'}\|f\chi_k\|_{L^q} < \infty. \tag{1.7}$$

Later, a new Hardy space HA^q related to the Beurling algebra A^q was introduced by Chen and Lau (see [12]). Denotes $B(0, R)$ be a cube centered at the origin with the side-length $R > 0$. Let

$f_{B(0,R)} = \frac{1}{|B(0,R)|}\int_{B(0,R)} f(x)\mathrm{d}x$ be the integral average of f on B. Then using duality, the dual space of HA^q can be described by $\mathrm{CBMO^q}$ with the following norm,

$$\|f\|_{\mathrm{CBMO}^q} = \sum_{R\geq 1}\left(\frac{1}{|B(0,R)|}\int_{B(0,R)} |f(x)-f_{B(0,R)}|^q \mathrm{d}x\right)^{1/q} < \infty. \tag{1.8}$$

Later, Lu and Yang (see [21-22]) introduced the homogeneous new Hardy type space $\mathrm{H\dot{A}}_q$ and they proved that the dual space of $\mathrm{H\dot{A}}_q$ can be written by

$$\|f\|_{\mathrm{C\dot{B}MO}^q} = \sum_{R\geq 0}\left(\frac{1}{|B(0,R)|}\int_{B(0,R)} |f(x)-f_{B(0,R)}|^q \mathrm{d}x\right)^{1/q} < \infty. \tag{1.9}$$

Obviously, the space of $\mathrm{C\dot{B}MO}^q$ is the homogeneous central bounded mean oscillation space depending on q and it can be regarded as an extention of the classical BMO since the famous John-Nirenberg inequality no longer hold in such space.

Alverez, Lakey and Guzmán-Partida introduced the λ-central bounded mean oscillation space and the λ-central Morrey space in 2000 (see [4]), respectively.

Definition 1.2[4] Let $\lambda < 1/n$ and $1 < q < \infty$. Then we say that a function $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to the λ-central bounded mean oscillation space $\mathrm{C\dot{B}MO}^{q,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{C\dot{B}MO}^{q,\lambda}} = \sum_{R>0}\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)} |f(x)-f_{B(0,R)}|^q \mathrm{d}x\right)^{1/q} < \infty. \tag{1.10}$$

Definition 1.3[4] Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. Then the λ-central Morrey space $\dot{B}^{q,\lambda}(\mathbb{R}^n)$ is defined of all functions $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ by the following norm

$$\|f\|_{\dot{B}^{q,\lambda}} = \sum_{R>0}\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)} |f(x)|^q \mathrm{d}x\right)^{1/q} < \infty. \tag{1.11}$$

It is very important to study the weighted norm inequalities for some integral operators on classical L^p spaces, one may see [14, 24-25] et al. for more details. In 2009, Komori-Furuya and Shirai (see [18]) defined the weighted Morrey space and showed the boundedness of some classical integral operators and their commutators on the weighted Morrey spaces. In this paper, we will prove the weighted boundedness of θ-type Calderón-Zygmund operator T_θ on the weighted λ-central Morrey space. Before giving the main results, we introduce the following definitions.

Definition 1.4[37] Let $\lambda \in \mathbb{R}$ and $1 < q < \infty$. Then the weighted λ-central Morrey space $\dot{\mathrm{B}}^{q,\lambda}_{\omega_1,\omega_2}(\mathbb{R}^n)$ is defined by

$$\left\{ f \in \dot{\mathrm{B}}^{q,\lambda}_{\omega_1,\omega_2}(\mathbb{R}^n) : \|f\|_{\dot{\mathrm{B}}^{q,\lambda}_{\omega_1,\omega_2}} = \sum_{R>0}\left(\frac{1}{\omega_1(B(0,R))^{1+\lambda q}}\int_{B(0,R)} |f(x)|^q \omega_2(x)\mathrm{d}x\right)^{1/q} < \infty \right\}, \tag{1.12}$$

where ω_1 and ω_2 are non-negative and local integrable functions. Moreover, if $\omega_1 = \omega_2 = \omega$, we denote $\dot{\mathrm{B}}^{q,\lambda}_{\omega_1,\omega_2}(\mathbb{R}^n) = \dot{\mathrm{B}}^{q,\lambda}_{\omega}(\mathbb{R}^n)$.

Definition 1.5[37] Let $\lambda < 1/n$ and $1 < q < \infty$. Then we say that a function $f \in L^q_{\text{loc}}(\mathbb{R}^n)$ belongs to the weighted λ-central bounded mean oscillation space $\text{C}\dot{\text{B}}\text{MO}^{q,\lambda}_\omega(\mathbb{R}^n)$ if

$$\|f\|_{\text{C}\dot{\text{B}}\text{MO}^{q,\lambda}_\omega} = \sum_{R>0} \left(\frac{1}{\omega(B(0,R))^{1+\lambda q}} \int_{B(0,R)} |f(x) - f_{B,\omega}|^q \omega(x) \mathrm{d}x \right)^{1/q} < \infty, \tag{1.13}$$

where the definition of $f_{B,\omega}$ is $f_{B,\omega} = \frac{1}{\omega(B)} \int_B f(x)\omega(x)\mathrm{d}x$.

Definition 1.6[25] We say a non-negative function $\omega(x)$ belongs to the Muckenhoupt class A_p with $1 < p < \infty$ if there exist a constant $C > 1$ such that

$$\left(\frac{1}{|Q|} \int_Q \omega(x)\mathrm{d}x \right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{1-p'}\mathrm{d}x \right)^{p-1} < \infty,$$

where $1/p + 1/p' = 1$ and $[\omega]_{A_p}$ denotes the infimum of C. Moreover, we define $A_\infty = \bigcup_{1<p<\infty} A_p$.

Obviously, by the classical Hölder inequality, there is $A_p \subset A_q \subset A_\infty$ for $1 < p < q < \infty$.

Our results can be stated as follows.

Theorem 1.1 Let T_θ be defined by (1.4) with θ satisfies (1,1). Suppose that $1 < p < \infty$, $\lambda < 0$ and $\omega(x) \in A_p$, then there exists a constant $C > 0$ independent of f, such that, for any $f \in \dot{\text{B}}^{p,\lambda}_\omega$,

$$\|T_\theta(f)\|_{\dot{\text{B}}^{p,\lambda}_\omega} \leq C\|f\|_{\dot{\text{B}}^{p,\lambda}_\omega}.$$

Theorem 1.2 Let $[b, T_\theta]$ be defined by (1.5) with θ satisfies $\int_0^1 \frac{\theta(t)}{t} |\log t| dt < \infty$. Suppose that $1 < p < \infty$, $1/p = 1/p_1 + 1/p_2$, $b \in \text{C}\dot{\text{B}}\text{MO}^{p_1,\lambda_1}_\omega$, $\omega(x) \in A_p$ and $\lambda = \lambda_1 + \lambda_2$ with $\lambda_i < 0 (i = 1, 2)$, then there exists a constant $C > 0$ independent of f, such that, for any $f \in \dot{\text{B}}^{p_2,\lambda_2}_\omega$,

$$\|[b, T_\theta]f\|_{\dot{\text{B}}^{p,\lambda}_\omega} \leq C\|b\|_{\text{C}\dot{\text{B}}\text{MO}^{p_1,\lambda_1}_\omega} \|f\|_{\dot{\text{B}}^{p_2,\lambda_2}_\omega}.$$

Let us give some necessary notations. Throughout the paper C will denote a positive constant whose value may change at each appearance. In the following, unless otherwise stated, for any real number $p > 1$, we denote p' by $1/p + 1/p' = 1$. Moreover, we say that a weight ω satisfies the doubling condition if there exists a constant D, such that for any cube $Q \in \mathbb{R}^n$, we have $\omega(2Q) \leq D\omega(Q)$. For simplicity, we denote $\omega \in \Delta_2$ if ω satisfies the doubling condition.

2 Preliminary Lemmas

Lemmas 2.1[15] If $\omega \in A_p$ for some $1 \leq p < \infty$, then $\omega \in \Delta_2$. More precisely, for all $\alpha > 1$, we have

$$\omega(\alpha Q) \leq \alpha^{np}[\omega]_{A_p}\omega(Q).$$

Lemmas 2.2[27] Let $1 < p < \infty$ and $\omega \in A_p$. Then, the θ-type Calderón-Zygmund operator T_θ is bounded on L^p_ω.

Lemmas 2.3[18] If $\omega \in \Delta_2$, then there exists a constant $D > 1$ such that for any cube B,

$$\omega(2B) \geq D\omega(B).$$

Lemmas 2.4[37] If $\omega \in A_p$ for some $1 \leq p < \infty$, then for any $k \in \mathbb{Z}^+$, $s < 0$ and any cube $B \in \mathbb{R}^n$,

$$\omega(2^k B)^s \leq D_1^{ks} \omega(B)^s,$$

where D_1 is a positive constant which belongs to the interval $(1, 2)$.

3 Proof of Theorems

Proof of Theorem 1.1. For a fixed cube $B = B(0, R)$, we may decompose $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$. Then we obain

$$\begin{aligned}\frac{1}{\omega(B)^{1+\lambda p}} \int_B |T(f)(x)|^p \omega(x) \mathrm{d}x \leq \frac{1}{\omega(B)^{1+\lambda p}} \int_B |T(f_1)(x)|^p \omega(x) \mathrm{d}x \\ \frac{1}{\omega(B)^{1+\lambda p}} \int_B |T(f_2)(x)|^p \omega(x) \mathrm{d}x =: C(I_1 + I_2).\end{aligned}$$

From Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}I_1 &= \frac{1}{\omega(B)^{1+\lambda p}} \int_B |T(f_1)(x)|^p \omega(x) \mathrm{d}x \\ &\leq \frac{1}{\omega(B)^{1+\lambda p}} \int_{2B} |f(x)|^p \omega(x) \mathrm{d}x \\ &\leq C \|f\|^p_{\dot{B}^{p,\omega}_\omega} \frac{\omega(2B)^{1+\lambda p}}{\omega(B)^{1+\lambda p}}.\end{aligned}$$

As $1 + \lambda p \geq 0$, by using Lemma 2.1, then there exists a constant $C > 0$ independent of f such that

$$I_1 \leq C \|f\|^p_{\dot{B}^{p,\omega}_\omega}. \tag{3.1}$$

On the other hand, by using Lemma 2.4, we can also get (3.1) with an similar argument in the case of $1 + \lambda p < 0$.

Next let's estimate I_2. Noting that $x \in B$ and $y \in (2B)^c$, then there exists a constant $C > 0$ such that $|y| < C|x - y|$. Thus, we have

$$|T_\theta(f_2)| \leq \int_{\mathbb{R}^n} |K(x, y)| |f(y)| \mathrm{d}y \leq C \int_{|y|>2r} 1/|y|^n |f(y)| \mathrm{d}y$$

Furthermore, by using Definition 1.6 and the Hölder's inequality, we can get

$$\begin{aligned}&\int_{|y|>2r} 1/|y|^n |f(y)| \mathrm{d}y = \sum_{j=1}^{\infty} \int_{2^j r<|y|<2^{j+1}r} 1/|y|^n |f(y)| \mathrm{d}y \\ &\leq \sum_{j=1}^{\infty} \frac{1}{|2^j B|} \left(\int_{2^{j+1}B} |f(y)|^p \omega(y) \mathrm{d}y \right)^{\frac{1}{p}} \left(\int_{2^{j+1}B} \omega(y)^{\frac{-p'}{p}} \mathrm{d}y \right)^{\frac{1}{p'}} \\ &\leq \sum_{j=1}^{\infty} \frac{1}{2^j B} \left(\frac{1}{\omega(2^{j+1}B)^{1+\lambda p}} \int_{2^{j+1}B} |f(y)|^p \omega(y) \mathrm{d}y \right)^{\frac{1}{p}} \omega(2^{j+1}B)^{\frac{1+\lambda p}{p}} \left[\left(\int_{|2^{j+1}B|} \omega(y)^{1-p'} \mathrm{d}y \right)^{p-1} \right]^{\frac{1}{p}} \\ &\leq C \|f\|_{\dot{B}^{\lambda,q}_\omega} \sum_{j=1}^{\infty} (\omega(2^{j+1}B))^\lambda.\end{aligned}$$

Then, by the fact $\lambda<0$ and Lemma 2.4, we get

$$I_2=\frac{1}{\omega(B)^{1+\lambda p}}\int_B\int|T_\theta(f_2)(x)|^p\omega(x)\mathrm{d}x\le C\frac{\omega(2^{j+1}B)^{\lambda p}}{\omega(B)^{\lambda p}}\|f\|^p_{\dot{\mathrm{B}}^{\lambda,q}_\omega}\le C\|f\|^p_{\dot{\mathrm{B}}^{\lambda,q}_\omega} \tag{3.2}$$

Combing with the estimates of I_1 and I_2, we finish the proof of **Theorem 1.1**. □

Proof of Theorem 1.2. For a fixed cube $B=B(0,R)$, we decompose $f=f_1+f_2$ as in the proof of Theorem 1.1. Then we have

$$\begin{aligned}\frac{1}{\omega(B)^{1+\lambda p}}\int_B|[b,T_\theta]f(x)|^p\omega(x)\mathrm{d}x\le&\frac{1}{\omega(B)^{1+\lambda p}}\int_B|[b,T_\theta]f_1(x)|^p\omega(x)\mathrm{d}x\\&+\frac{1}{\omega(B)^{1+\lambda p}}\int_B|[b,T_\theta]f_2(x)|^p\omega(x)\mathrm{d}x=:I+II.\end{aligned}$$

To estimate I, we may split as

$$\begin{aligned}I\le&\frac{1}{\omega(B)^{1+\lambda p}}\int_B|b(x)-b_{B,\omega}|^p|T_\theta(f_1)(x)|^p\omega(x)\mathrm{d}x\\&+\frac{1}{\omega(B)^{1+\lambda p}}\int_B|T_\theta(f_1(b-b_{B,\omega}))(x)|^p\omega(x)\mathrm{d}x\\=:&\,I_1+I_2.\end{aligned}$$

First, we give the estimate of I_1. Noting that $p<p_2$, by the Hölder's inequality and Lemma 2.2, one has

$$\begin{aligned}I_1=&\frac{1}{\omega(B)^{1+\lambda p}}\int_B|b(x)-b_{B,\omega}|^p\omega(x)^{\frac{p}{p_1}}|T_\theta(f_1)(x)|^p\omega(x)^{1-\frac{p}{p_1}}\mathrm{d}x\\\le&\frac{1}{\omega(B)^{1+\lambda p}}\left(\int_B|b(x)-b_{B,\omega}|^{p_1}\omega(x)\right)^{\frac{p}{p_1}}\left(\int_B|T_\theta(f_1)(x)|^{p_2}\omega(x)\mathrm{d}x\right)^{1-\frac{p}{p_1}}\\\le&C\frac{1}{\omega(B)^{1+\lambda p}}\left(\frac{1}{\omega(B)^{1+\lambda_1p_1}}\int_B|b(x)-b_{B,\omega}|^{p_1}\omega(x)\right)^{\frac{p}{p_1}}\omega(B)^{\frac{p}{p_1}+\lambda_1p}\left(\int_{2B}|f_1(x)|^{p_2}\omega(x)\mathrm{d}x\right)^{\frac{p}{p_2}}\\\le&C\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}}\|f\|^p_{\dot{B}^{p_2,\lambda_2}_\omega}\left(\frac{\omega(2B)^{\frac{1}{p_2}+\lambda_2}}{\omega(B)^{\frac{1}{p_2}+\lambda_2}}\right)^p.\end{aligned}$$

If $\frac{1}{p_2}+\lambda_2\ge0$, we can use Lemma 2.1 to get $\frac{\omega(2B)^{\frac{1}{p_2}+\lambda_2}}{\omega(B)^{\frac{1}{p_2}+\lambda_2}}\le C$. Moreover, we can also use Lemma 2.4 to get the same estimate for the case of $\frac{1}{p_2}+\lambda_2<0$. Thus, we have

$$I_1\le C\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_\omega}\|f\|^p_{\dot{\mathrm{B}}^{p_2,\lambda_2}_\omega} \tag{3.3}$$

For I_2, by the Hölder's inequality, we can obtain

$$\begin{aligned}I_2\le&\frac{1}{\omega(B)^{1+\lambda p}}\int_{2B}|f(x)(b(x)-b_{B,\omega})|^p\omega(x)\mathrm{d}x\\\le&\frac{1}{\omega(B)^{1+\lambda p}}\left(\int_{2B}(|b(x)-b_{B,\omega}|^p\omega(x)^{\frac{p}{p_1}})^{\frac{p_1}{p}}\mathrm{d}x\right)^{\frac{p}{p_1}}\left((\int_{2B}|f(x)|^p\omega(x)^{1-\frac{p}{p_1}})^{\frac{p_1}{p_1-p}}\mathrm{d}x\right)^{1-\frac{p}{p_1}}\\\le&\frac{1}{\omega(B)^{1+\lambda p}}\left(\frac{1}{\omega(2B)^{1+\lambda_1p_1}}\int_{2B}|b(x)-b_{B,\omega}|^{p_1}\omega(x)\mathrm{d}x\right)^{\frac{p}{p_1}}\omega(2B)^{\frac{p}{p_1}+\lambda_1p}\\&\times\left(\frac{1}{\omega(2B)^{1+\lambda_2p_2}}\int_{2B}|f(x)|^{p_2}\omega(x)\right)^{\frac{p}{p_2}}\omega(2B)^{\frac{p}{p_2}+\lambda_2p}\\\le&C\|f\|^p_{\dot{B}^{p_2,\lambda_2}_\omega}\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_\omega}\frac{\omega(2B)^{1+\lambda p}}{\omega(B)^{1+\lambda p}}.\end{aligned}$$

If $1+\lambda p>0$, we can use Lemma 2.1 to get $\frac{\omega(2B)^{1+\lambda p}}{\omega(B)^{1+\lambda p}}\leq C$. Moreover, in the case of $1+\lambda p<0$, we can also get the above estimate by using Lemma 2.4 with a similar argument.

Combining the estimates of I_1 and I_2, we have

$$I\leq C\|f\|^p_{\dot{B}^{p_2,\lambda_2}_{\omega}}\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_{\omega}}. \tag{3.4}$$

Now we are going to give the estimate of II. First, we may give the following estimates.

$$\begin{aligned}
|[b,T_\theta]f_2(x)|^p &\leq C\left(\int_{\mathbb{R}^n}\frac{|b(x)-b(y)|}{|x-y|^n}|f_2(y)|\right)^p\\
&\leq C\left(\int_{|y|>2r}\frac{|f(x)|}{|x_0-y|^n}(|b(x)-b_{B,\omega}|+|b_{B,\omega}-b(y)|)\mathrm{d}y\right)^p\\
&\leq C\left(\int_{|y|>2r}\frac{|f(x)|}{|x_0-y|^n}\mathrm{d}y\right)^p|b(x)-b_{B,\omega}|^p\\
&\quad+C\left(\int_{|y|>2r}\frac{|f(x)|}{|x_0-y|^n}\mathrm{d}y|b(y)-b_{B,\omega}|\mathrm{d}y\right)^p.
\end{aligned}$$

Thus, we can decompose II as

$$\begin{aligned}
II &= \frac{1}{\omega(B)^{1+\lambda p}}\int_B|[b,T_\theta]f_2(x)|^p\omega(x)\mathrm{d}x\\
&\leq \frac{1}{\omega(B)^{1+\lambda p}}\int_B\left(\int_{|y|>2r}\frac{|f(x)|}{|x_0-y|^n}\mathrm{d}y\right)^p|b(x)-b_{B,\omega}|^p\omega(x)\mathrm{d}x\\
&+\frac{1}{\omega(B)^{1+\lambda p}}\int_B\left(\int_{|y|>2r}\frac{|f(x)|}{|x_0-y|^n}\mathrm{d}y|b(y)-b_{B,\omega}|\mathrm{d}y\right)^p\omega(x)\mathrm{d}x\\
&= II_1+II_2.
\end{aligned}$$

For II_1, by the same estimate as in the proof of Theorem 1.1, we can obtain that

$$\int_{|y|>2r}1/|y|^n|f(y)|\mathrm{d}y\leq C\|f\|_{\dot{B}^{\lambda_2,q_2}_{\omega_2}}\sum_{j=1}^{\infty}\omega(2^{j+1}B)^{\lambda_2},$$

which implies

$$\begin{aligned}
II_1 &\leq \|f\|^p_{\dot{B}^{\lambda_2,q_2}_{\omega_2}}\sum_{j=1}^{\infty}\frac{\omega(2^{j+1}B)^{\lambda_2 p}}{\omega(B)^{1+\lambda p}}\int_B|b(x)-b_{B,\omega}|^p\omega(x)\mathrm{d}x\\
&\leq \|f\|^p_{\dot{B}^{\lambda_2,q_2}_{\omega_2}}\sum_{j=1}^{\infty}\frac{\omega(2^{j+1}B)^{\lambda_2 p}}{\omega(B)^{1+\lambda p}}\left(\int_B|b(x)-b_{B,\omega}|^{p_1}\omega(x)\mathrm{d}x\right)^{p/p_1}\times\left(\int_B\omega(x)^{\frac{p_1-p}{p}}\frac{p_1}{p_1-p}\mathrm{d}x\right)^{1-p/p_1}\\
&\leq C\|f\|^p_{\dot{B}^{\lambda_2,q_2}_{\omega_2}}\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_{\omega}}\omega(B)^{p/p_1+\lambda_1 p+1-p/p_1}\sum_{j=1}^{\infty}\frac{\omega(2^{j+1}B)^{\lambda_2 p}}{\omega(B)^{1+\lambda p}}\\
&= C\|f\|^p_{\dot{B}^{\lambda_2,q_2}_{\omega_2}}\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_{\omega}}\sum_{j=1}^{\infty}\frac{\omega(2^{j+1}B)^{\lambda_2 p}}{\omega(B)^{\lambda_2 p}}\\
&\leq C\|f\|^p_{\dot{B}^{\lambda_2,q_2}_{\omega_2}}\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_{\omega}},
\end{aligned}$$

where in the last inequality we use the fact $\lambda_2<0$ and Lemma 2.4.

7

Next, we turn to estimate II_2. Noticing that $1/p = 1/p_1 + 1/p_2$, by using the Hölder's inequality, then we have

$$\begin{aligned}
\int_{|y|>2r} \frac{|f(y)|}{|y|^n}|b(y)-b_{B,\omega}|\mathrm{d}y &= \sum_{j=1}^{\infty}\int_{2^j r<|y|<2^{j+1}r} \frac{|f(y)|}{|y|^n}|b(y)-b_{B,\omega}|\mathrm{d}y \\
&\le \sum_{j=1}^{\infty}\frac{1}{|2^jB|}\int_{2^{j+1}B\setminus 2^jB} |f(y)||b(y)-b_{B,\omega}|\mathrm{d}y \\
&\le \sum_{j=1}^{\infty}\frac{1}{|2^jB|}\left(\int_{2^{j+1}B} |f(y)||b(y)-b_{B,\omega}|\omega(y)^{\frac{1}{p}}\omega(y)^{\frac{1}{-p}}\right)\mathrm{d}y \\
&\le \sum_{j=1}^{\infty}\frac{1}{|2^jB|}\left(\int_{2^{j+1}B} |f(y)|^{p_2}\omega(y)\mathrm{d}y\right)^{\frac{1}{p_2}}\left(\int_{2^{j+1}B} |b(y)-b_{B,\omega}|^{p_1}\omega(y)\mathrm{d}y\right)^{\frac{1}{p_1}} \\
&\quad\times\left(\int_{2^{j+1}B}\omega(y)^{-\frac{p'}{p}}\mathrm{d}y\right)^{\frac{1}{p'}}.
\end{aligned}$$

By the fact that $\omega(x)\in A_p$, we get

$$\begin{aligned}
\left(\int_{2^{j+1}B}\omega(y)^{-\frac{p'}{p}}\mathrm{d}y\right)^{\frac{1}{p'}} &= \left[\left(\int_{2^{j+1}B}\omega(y)^{1-p'}\mathrm{d}y\right)^{p-1}\right]^{\frac{1}{p}} \\
&\le C\left(\frac{|2^{j+1}B|}{\omega(2^{j+1}B)}\right)^{\frac{1}{p}}|2^{j+1}B|^{\frac{p-1}{p}} \\
&= C\frac{|2^{j+1}B|}{\omega(2^{j+1}B)^{1/p_1+1/p_2}}.
\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}
\int_{|y|>2r} \frac{|f(y)|}{|y|^n}|b(y)-b_{B,\omega}|\mathrm{d}y &\le C\sum_{j=1}^{\infty}\frac{1}{|2^jB|}\|f\|_{\dot{B}^{\lambda_2,q_2}_{\omega}}\omega(2^{j+1}B)^{1/p_2+\lambda_2} \\
&\quad\times\left(\int_{2^{j+1}B}|b(y)-b_{B,\omega}|^{p_1}\omega(y)\mathrm{d}y\right)^{\frac{1}{p_1}}\times\frac{|2^{j+1}B|}{\omega(2^{j+1}B)^{1/p_1+1/p_2}} \\
&\le C\|f\|_{\dot{B}^{\lambda_2,q_2}_{\omega}}\sum_{j=1}^{\infty}\omega(2^{j+1}B)^{\lambda_2-1/p_1}\left(\int_{2^{j+1}B}|b(y)-b_{B,\omega}|^{p_1}\omega(y)\mathrm{d}y\right)^{\frac{1}{p_1}} \\
&\le C\|f\|_{\dot{B}^{\lambda_2,q_2}_{\omega}}\sum_{j=1}^{\infty}\omega(2^{j+1}B)^{\lambda_2-1/p_1}\left(\int_{2^{j+1}B}|b(y)-b_{2^{j+1}B,\omega}|^{p_1}\omega(y)\mathrm{d}y\right)^{\frac{1}{p_1}} \\
&\quad+C\|f\|_{\dot{B}^{\lambda_2,q_2}_{\omega}}\sum_{j=1}^{\infty}\omega(2^{j+1}B)^{\lambda_2-1/p_1}\left(\int_{2^{j+1}B}|b_{B,\omega}-b_{2^{j+1}B,\omega}|^{p_1}\omega(y)\mathrm{d}y\right)^{\frac{1}{p_1}} \\
&=: C(II_{21}+II_{22}).
\end{aligned}$$

For II_{21}, by the definition of $\mathrm{C\dot{B}MO}^{p,\lambda}_{\omega}(\mathbb{R}^n)$, the fact $\lambda<0$ and Lemma 2.4, we have

$$\begin{aligned}\frac{1}{\omega(B)^{1+p\lambda}}\int_B II_{21}^p\cdot\omega(x)\mathrm{d}x&\le\frac{\|f\|^p_{\dot{B}^{\lambda_2,q_2}_{\omega}}\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_{\omega}}}{\omega(B)^{1+\lambda p}}\int_B\sum_{j=1}^{\infty}\omega(2^{j+1}B)^{\lambda p}\mathrm{d}x\\&\le C\|f\|^p_{\dot{B}^{\lambda_2,q_2}_{\omega}}\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_{\omega}}\sum_{j=1}^{\infty}\frac{\omega(2^{j+1}B)^{\lambda p}}{\omega(B)^{\lambda p}}\\&\le C\|f\|^p_{\dot{B}^{\lambda_2,q_2}_{\omega}}\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_{\omega}}\end{aligned}$$

Next, we will give the estimate of II_2. First, we have the following inequality

$$|b_{B,\omega}-b_{2^{j+1}B,\omega}|\le\sum_{k=0}^{j}|b_{2^{k+1}B,\omega}-b_{2^kB,\omega}|.$$

Then for any $0<k\le j$, we obtain

$$\begin{aligned}|b_{2^{k+1}B,\omega}-b_{2^kB,\omega}|&\le\frac{1}{\omega(2^kB)}\int_{2^kB}|b_{2^{k+1}B,\omega}-b(y)|\omega(y)\mathrm{d}y\\&\le\frac{1}{\omega(2^kB)}\left(\int_{2^kB}|b_{2^{k+1}B,\omega}-b(y)|^{p_1}\omega(y)\mathrm{d}y\right)^{1/p_1}\left(\int_{2^kB}(\omega(y)^{1-\frac{1}{p_1}})^{\frac{p_1}{p_1-1}}\mathrm{d}y\right)^{1-1/p_1}\\&\le C\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_{\omega}}\omega(2^kB)^{\lambda_1}.\end{aligned}$$

Using the Lemma 2.4 and the fact $\lambda_1<0$, we get

$$\sum_{k=0}^{j}|b_{2^{k+1}B,\omega}-b_{2^kB,\omega}|\le C\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_{\omega}}\omega(B)^{\lambda_1}D_1^{(j+1)\lambda_1},$$

where D_1 is a positive constant and belongs to the interval $(1,2)$.

Thus, using Lemma 2.4 again, we obtain

$$\begin{aligned}\frac{1}{\omega(B)^{1+p\lambda}}\int II_{22}^p\cdot\omega(x)\mathrm{d}x&\le C\|f\|^p_{\dot{B}^{\lambda_2,q_2}_{\omega}}\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_{\omega}}\frac{\left(\sum_{j=1}^{\infty}\omega(2^{j+1}B)^{\lambda_2-\frac{1}{p_1}+\frac{1}{p_1}}D_1^{(j+1)\lambda}\omega(B)^{\lambda_1}\right)^p}{\omega(B)^{p\lambda}}\\&\le C\|f\|^p_{\dot{B}^{\lambda_2,q_2}_{\omega}}\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_{\omega}}\sum_{j=1}^{\infty}D_1^{(j+1)\lambda_2p}D_1^{(j+1)\lambda_1p}\\&\le C\|f\|^p_{\dot{B}^{\lambda_2,q_2}_{\omega}}\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_{\omega}}\sum_{j=1}^{\infty}D_1^{(j+1)\lambda p}\\&\le C\|f\|^p_{\dot{B}^{\lambda_2,q_2}_{\omega}}\|b\|^p_{\mathrm{C\dot{B}MO}^{p_1,\lambda_1}_{\omega}}.\end{aligned}$$

Combining the estimates of $I,\ II,\ II_1,\ II_2,\ II_{21}$ and II_{22}, we finishe the proof of **Theorem 1.2**. $\square$

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

This work is supported by National Natural Science Foundation of China (Grant No. 11561062).

References

[1] D. R. Adams, A note on Riesz potentials, Duke Math. J., 42(1975), 765-788.

[2] D. R. Adams, Morrey spaces, Lecture Notes Appl. Numer. Harmon. Anal., Birkaäuser, 2015.

[3] D. R. Adams, J.Xiao, Morrey spaces in harmonic analysis, Ark. Mat., 50(2012),201-230.

[4] J. Alvarez, J. Lakey, M. Guzmán-Partida, spaces of bounded λ-central mean oscilation, Morrey spaces, and λ-central Carleson measures, Collect Math, 51(2000), 1-47.

[5] A. Beurling, Construction and analysis of some convolution algebras, Ann. Inst. Fourier (Grenoble), 14(1964), 1-32.

[6] Y. Brudnyi, Spaces defined by local approximations, Tr. Mosk. Mat. Obs., 24(1971), 69-132 (inRus-sian); Engl. transl. Trans. Moscow Math. Soc., (24) 1971, 73-139.

[7] A. Calderón, A. Zygmund, On the existence of certain singular integrals, Acta Math., 88(1952), 85-139.

[8] A. Calderón, A. Zygmund, A note on the interpolation of sublinear operators., Amer. J. Math., 78(1956), 282-288.

[9] A. Calderón, A. Zygmund, On singular integrals., Amer. J. Math., 78(1956), 289-309.

[10] A. Calderón, A. Zygmund, A note on singular integrals, Studia Math., 65(1979), 77-87.

[11] S. Campanato, Proprietàdi una famiglia di spazi funzioni, Ann. Sc. Norm. Super. Pisa., 18(1964), 137-160.

[12] Y. Chen, K. Lau, Some new classes of of Hardy spaces, J. Funct. Anal., 84(1989), 255-278.

[13] F. Chiarenza, M. Frasca, Morrey spaces and Hardy-Littlewood maximal function, Rend Mat Appl., 7(1987), 273-279.

[14] R. Coifman, C. Fefferman, Weighted norm inequalities for maximal and singular integrals, Studia Math, 51(1974), 241-250.

[15] L. Grafakos, Classical and modern Frouier analysis, GAT 249, Springer Science+Business Media, LLC, 2008.

[16] S. He, J. Zhou, Vector-valued maximal multilinear Calderon-Zygmund operator with nonsmooth kernel on weighted Morrey space, Journal of Pseudo-differential Operators and Applications, 8(2017), 213-239.

[17] G. Hu, Weighted vector-valued estimates for a non-standard Calderon-Zygmund operator, Nonlinear Analysis-theory Methods and Applications, 165(2017), 143-162.

[18] Y. Komori-Furuya, S. Shirai, Weighted Morrey spaces and a singular integral operator, Math Nachr, 282(2009), 219-231.

[19] J. Lan, Weak type endpoint estimates for multilinear θ-type Calderón-Zygmund operators, Acta Mathematicae Applicatae Sinica, English Series, 21(2005), 615-622.

[20] G. Lu, S. Tao, Bilinear θ-Type Calderón-Zygmund Operators on Non-homogeneous Generalized Morrey Spaces, to appear in J. of Computational Analysis and Applications, 26(4)(2019), 650-670.

[21] S. Lu, D. Yang, The Littlewood-Paley function and φ-transform characterization of a new Hardy space HK_2 associated with Herz space, Studia Math, 101(1992), 285-298.

[22] S. Lu, D. Yang, The central BMO space and Littlewood operators, Approx. Theory Appl., 11(1995), 72-94.

[23] C. B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc., 43(1938), 126-166.

[24] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc. 156(1972), 207-226.

[25] B. Muckenhoupt, R. Wheeden, Weighted norm inequalities for fractional integral, Trans. Amer. Math. Soc., 192(1974), 261-274.

[26] J. Peetre, On the theory of $L_{p,\lambda}$ spaces, J. Funct. Anal., 4(1969), 71-87.

[27] T. Quek, D. Yang, On weighted weak hardy spaces over $\mathbb{R}^n$, Acta Mathe. Sinica (English Series), 16(2000), 141-160.

[28] H. Rafeiro, N. Samko, Morrey-Campanato spaces: anoverview, in: Karlovich Y, Rodino L, Silbermann B, Spitkovsky I M(Eds.), Operator Theory, Pseudo-Differential Equations and Methematical Physics, Advances and Applications, 228(2013), 293-323.

[29] C. Ri, Z. Zhang, Boundedness of θ-type Calderón-Zygmund operators on Hardy spaces with non-doubling measures, Journal of inequalities and Applications, 323(2015), 1-10.

[30] C. Ri, Z. Zhang, Boundedness of θ-type Calderón-Zygmund operators on non-homogeneous metric measure spaces, Front. Math. China, 11(2016), 141-153.

[31] Y. Sawano, H. Gunawan, V. Guliyev, H.Tanaka, Morrey spaces and related function spaces, Journal of Function spaces, Volume 2014, Article ID 867192.

[32] H. Triebel, Local function spaces, Heat and Navier – Stokes Equations, EMS Tracts Math. 20, 2013.

[33] H. Wang, Boundedness of θ-type Calderón-zygmund operators and commutators in the genralized weighted Morrey spaces, Journal of Function spaces, Volume 2016, Article ID 1309348, 18 pages.

[34] R. Xie, L. Shu, On multilinear commutators of θ-type Calderón-zygmund operators, Analysis in Theory and Applications, 24(2008), 260-270.

[35] R. Xie, L. Shu, L^p boundedness of for the maximal multilinear singular integral operator of θ-type Calderón-zygmund kernel, J. Sys. Sci. and Math. Scis., 29(2009), 519-526.

[36] K. Yabuta, Generalizations of Calderón-Zygmund operators, Studia Mathematica, 82(1985), 17-31.

[37] X. Yu, H. Zhang, G. Zhao, Weighted boundedness of some integrals operators on weighted λ-central Morrey space, Appl. Math. J. Chinese Univ., 31(2016), 331-342.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Stability of latent CHIKV infection model with CHIKV-to-monocyte and infected-to-monocyte transmissions

A. M. Elaiw[a], S. E. Almalki[a,b] and A. Hobiny[a]

[a]Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia.
[b]Jeddah College of Technology, Technical and Vocational Training Corporation
Emails: a_m_elaiw@yahoo.com (A. Elaiw),
samialmalki0@gmail.com (S. E. Almalki)

Abstract

We investigate the global stability of within-host Chikungunya virus (CHIKV) infection model with CHIKV-to-monocyte and infected-to-monocyte transmissions. We take into account the antibody immune response. The model incorporates both latently infected monocytes which do not generate the CHIKV, and actively infected monocytes. The global stability analysis of the equilibria are established using Lyapunov method. The theoretical results are confirmed by numerical simulations. The effect of latently infection has been discussed.

1 Introduction

Chikungunya virus (CHIKV) is an alphavirus causes chikungunya fever. CHIKV is a mosquito-transmitted and is transmitted by the Aedes albopictus and Aedes agypti mosquito. Mathematical models have been constructed to describe the CHIKV transmission in mosquito and human populations [1]-[7]). Modeling and analysis of within-host CHIKV dynamics have first studied in [8]. The model presented in [8] has negelected the latently infected monocytes. Therefore, Elaiw et al. [9] have modified the model by considering five compartments, uninfected monocytes (s), latently infected monocytes (w), actively infected monocytes (y), CHIKV pathogen (p) and antibodies (x). The model is given as:

$$\dot{s} = \beta - \delta s - \frac{\eta s p}{1+\theta p}, \tag{1}$$

$$\dot{w} = (1-n)\frac{\eta s p}{1+\theta p} - (b+d)w, \tag{2}$$

$$\dot{y} = n\frac{\eta s p}{1+\theta p} + dw - \epsilon y, \tag{3}$$

$$\dot{p} = \pi y - cp - rxp, \tag{4}$$

$$\dot{x} = \lambda + \rho xp - mx, \tag{5}$$

where, β and δ represent the generation and death rate constants of the uninfected monocytes, respectively. The uninfected monocytes become infected at rate ηsp, where η is the infection rate constant. θ is the saturation constant. Constants b, ϵ, c and m represent, respectively, the death rate of the latently infected monocytes, actively infected monocytes, CHIKV and antibodies. We assume that a fraction $(1-n)$ of the CHIKV-contacted monocytes becomes latently infected monocytes and the remaining n becomes actively infected monocytes, where $0 < n < 1$. The latently infected monocytes are transmitted to actively infected monocytes at rate bw. Constant π is the generation rate of the CHIKV from actively infected monocytes. Antibodies attack the CHIKV at rate rxp. Once antigen is encountered, the antibodies expand at a constant rate λ and proliferate at rate ρxp. Latently infected cells have been considered in viral infection models in several papers (see e.g. [11]-[15]).

Model (1)-(5) assumes that the uninfected monocyte becomes infected by contacting with CHIKV(CHIKV-to-monocyte transmission). Kristin and Mork [16] reported that the CHIKV can also spread by infected-to-monocyte transmission. Cellular and viral infections have been considered in several viral infection models [17]-[20]. However, the dynamics of CHIKV with CHIKV-to-monocyte and infected-to-monocyte transmissions did not studied before.

The aim of the present paper is to construct and analyze a CHIKV dynamics model with both CHIKV-to-monocyte and infected-to-monocyte transmissions. The model incorporates two types of infected monocytes, latently infected monocytes which do not generate the CHIKV, and actively infected monocytes. We use Lyapunov method to prove the global stability of the proposed model .

2 Presentation of the model and mathematical problem

We propose a CHIKV model as:

$$\dot{s} = \beta - \delta s - \eta_1 sp - \eta_2 sy, \tag{6}$$

$$\dot{w} = (1-n)(\eta_1 sp + \eta_2 sy) - (b+d)w, \tag{7}$$

$$\dot{y} = n(\eta_1 sp + \eta_2 sy) + dw - \epsilon y, \tag{8}$$

$$\dot{p} = \pi y - cp - rxp, \tag{9}$$

$$\dot{x} = \lambda + \rho xp - mx, \tag{10}$$

Here, the uninfected monocytes become infected at rate $(\eta_1 y + \eta_2 p)s$, where η_1 and η_2 are constants.

2.1 Basic properties

Lemma 1 *There exist $M_1, M_2, M_3 > 0$, such that the following compact set is positively invariant for system (6)-(10)*

$$\Gamma_2 = \{(s,w,y,p,x) \in \mathbb{R}^5_{\geq 0} : 0 \leq s,w,y \leq M_1^L, 0 \leq p \leq M_2^L, 0 \leq x \leq M_3^L\}$$

Proof. We have

$$\begin{aligned}
&\dot{s}\mid_{s=0} = \beta > 0,\\
&\dot{w}\mid_{w=0} = (1-n)(\eta_1 sp + \eta_2 sy) \geq 0, \quad \text{for all} \;\; s,p \geq 0,\\
&\dot{y}\mid_{y=0} = n(\eta_1 sp) + dw \geq 0, \quad \text{for all} \;\; s,p,w \geq 0,\\
&\dot{p}\mid_{p=0} = \pi y \geq 0, \quad \text{for all} \;\; y \geq 0,\\
&\dot{x}\mid_{x=0} = \lambda > 0.
\end{aligned}$$

Then, $\mathbb{R}^5_{\geq 0}$ is positively invariant for system (6)-(10). We let

$$\begin{aligned}
H_1(t) &= s(t) + w(t) + y(t),\\
H_2(t) &= p(t) + \frac{r}{\rho}x(t),
\end{aligned}$$

then

$$\begin{aligned}
\dot{H}_1(t) &= \beta - \delta s(t) - bw(t) - \epsilon y(t)\\
&\leq \beta - \sigma_1^L(s(t) + w(t) + y(t))\\
&= \beta - \sigma_1^L H_1(t),
\end{aligned}$$

where, $\sigma_1^L = min\{\delta, b, \epsilon\}$. Hence $H_1(t) \leq M_1^L$, if $H_1(0) \leq M_1^L$, where $M_1^L = \frac{\beta}{\sigma_1^L}$. Hence , $0 \leq s(t), w(t), y(t) \leq M_1^L$ if $0 \leq s(0) + w(0) + y(0) \leq M_1^L$. Moreover, we have

$$\begin{aligned}\dot{H}_2(t) &= \pi y(t) - cp(t) + \frac{r}{\rho}\lambda - \frac{mr}{\rho}x(t) \\ &\leq \pi M_1^L + \frac{r}{\rho}\lambda - \sigma_2\left(p(t) + \frac{r}{\rho}x(t)\right) \\ &= \pi M_1^L + \frac{r}{\rho}\lambda - \sigma_2 H_2(t),\end{aligned}$$

where, σ_2 is defined before. Hence $H_2(t) \leq M_2^L$, if $H_2(0) \leq M_2^L$, where $M_2^L = \frac{\pi M_1^L + \frac{r}{\rho}\lambda}{\sigma_2}$. Thus, $0 \leq p(t) \leq M_2^L$ and $x(t) \leq M_3^L$ if $0 \leq p(0) + \frac{r}{\rho}x(0) \leq M_2^L$, where $M_3^L = \frac{\rho M_2^L}{r}$. ■

2.2 Equilibria

We define the basic reproduction number as:

$$\mathcal{R}_0^L = \frac{\beta(d + bn)(\eta_1\pi m + \eta_2 cm + \eta_2 r\lambda)}{\epsilon\delta(cm + r\lambda)(b + d)}.$$

Lemma 2 *(i) if $\mathcal{R}_0^L \leq 1$, then there exists only one equilibrium E_0, (ii) if $\mathcal{R}_0^L > 1$, then there exist two equilibria E_0 and E_1.*

Proof. The equilibria of system (6)-(10) satisfying

$$\beta - \delta s - \eta_1 sp - \eta_2 sy = 0 \tag{11}$$

$$(1 - n)(\eta_1 sp + \eta_2 sy) - (b + d)w = 0, \tag{12}$$

$$n(\eta_1 sp + \eta_2 sy) + dw - \epsilon y = 0, \tag{13}$$

$$\pi y - cp - rxp = 0, \tag{14}$$

$$\lambda + \rho xp - mx = 0. \tag{15}$$

Solving Eqs. (11)-(15) there exists a CHIKV-free equilibrium $E_0 = (s_0, 0, 0, 0, x_0)$, where $s_0 = \frac{\beta}{\delta}$ and $x_0 = \frac{\lambda}{m}$. From Eqs. (11)-(15) we have

$$s = \frac{\pi\beta}{\pi(\delta + p\eta_1) + p(c + rx)\eta_2}, \tag{16}$$

$$w = \frac{(1 - n)p\beta(\pi\eta_1 + (c + rx)\eta_2)}{(b + d)(\pi(\delta + p\eta_1) + p(c + rx)\eta_2)}, \tag{17}$$

$$y = \frac{p(c + rx)}{\pi}, \tag{18}$$

$$x = \frac{\lambda}{m - \rho p}. \tag{19}$$

Substituting from Eqs. (16)-(19) into (13) we get

$$D_1 p^3 + D_2 p^2 + D_3 p + D_4 = 0,$$

4

where

$$\begin{aligned}
D_1 &= -c(b+d)\epsilon(\pi\eta_1 + c\eta_2)\rho^2,\\
D_2 &= 2bcm\pi\epsilon\eta_1\rho + 2cdm\pi\epsilon\eta_1\rho + 2bc^2m\epsilon\eta_2\rho + 2c^2dm\epsilon\eta_2\rho + b\pi r\epsilon\eta_1\lambda\rho + d\pi r\epsilon\eta_1\lambda\rho + 2bcr\epsilon\eta_2\lambda\rho\\
&\quad + 2cdr\epsilon\eta_2\lambda\rho - bc\pi\delta\epsilon\rho^2 - cd\pi\delta\epsilon\rho^2 + d\pi^2\beta\eta_1\rho^2 + bn\pi^2\beta\eta_1\rho^2 + cd\pi\beta\eta_2\rho^2 + bcn\pi\beta\eta_2\rho^2,\\
D_3 &= -bcm^2\pi\epsilon\eta_1 - cdm^2\pi\epsilon\eta_1 - bc^2m^2\epsilon\eta_2 - c^2dm^2\epsilon\eta_2 - bm\pi r\epsilon\eta_1\lambda - dm\pi r\epsilon\eta_1\lambda - 2bcmr\epsilon\eta_2\lambda\\
&\quad - 2cdmr\epsilon\eta_2\lambda - br^2\epsilon\eta_2\lambda^2 - dr^2\epsilon\eta_2\lambda^2 + 2bcm\pi\delta\epsilon\rho + 2cdm\pi\delta\epsilon\rho - 2dm\pi^2\beta\eta_1\rho - 2bmn\pi^2\beta\eta_1\rho\\
&\quad - 2cdm\pi\beta\eta_2\rho - 2bcmn\pi\beta\eta_2\rho + b\pi r\delta\epsilon\lambda\rho + d\pi r\delta\epsilon\lambda\rho - d\pi r\beta\eta_2\lambda\rho - bn\pi r\beta\eta_2\lambda\rho,\\
D_4 &= -bcm^2\pi\delta\epsilon - cdm^2\pi\delta\epsilon + dm^2\pi^2\beta\eta_1 + bm^2n\pi^2\beta\eta_1 + cdm^2\pi\beta\eta_2 + bcm^2n\pi\beta\eta_2 - bm\pi r\delta\epsilon\lambda\\
&\quad - dm\pi r\delta\epsilon\lambda + dm\pi r\beta\eta_2\lambda + bmn\pi r\beta\eta_2\lambda.
\end{aligned}$$

Let

$$X_2(p) = D_1p^3 + D_2p^2 + D_3p + D_4 = 0.$$

Then

$$\begin{aligned}
X_2(0) &= D_4,\\
X_2\left(\frac{m}{\rho}\right) &= -\frac{(b+d)mr^2\epsilon\eta_2\lambda^2}{\rho} < 0.
\end{aligned}$$

D_4 can be written as:

$$D_4 = m\pi(b\delta\epsilon + d\delta\epsilon)(cm + r\lambda)\left(\mathcal{R}_0^L - 1\right).$$

Then $D_4 > 0$ if $\mathcal{R}_0^L > 1$. Then there exists $p_1 \in (0, \frac{m}{\rho})$ such that $X_2(p_1) = 0$.If $\mathcal{R}_0^L > 1$, then system (6)-(10) has infected equilibrium $E_1 = (s_1, y_1, p_1, x_1)$, where

$$\begin{aligned}
s_1 &= \frac{\pi\beta}{\pi(\delta + p_1\eta_1) + p_1(c + rx_1)\eta_2} > 0, \qquad & w_1 &= \frac{(1-n)p_1\beta(\pi\eta_1 + (c + rx_1)\eta_2)}{(b+d)(\pi(\delta + p_1\eta_1) + p_1(c + rx_1)\eta_2)} > 0,\\
y_1 &= \frac{p_1(c(m - \rho p_1) + r\lambda)}{\pi(m - \rho p_1)} > 0, & x_1 &= \frac{\lambda}{m - \rho p_1} > 0.
\end{aligned}$$

■

3 Global properties

Define a function $G(z) = z - 1 - \ln z$.

Theorem 1 *If $\mathcal{R}_0^L \leq 1$, then E_0 is globally asymptotically stable in Γ_2.*

Proof. Let

$$V_0 = s_0 G\left(\frac{s}{s_0}\right) + \frac{d}{bn+d}w + \frac{b+d}{bn+d}y + \frac{\eta_1 s_0}{c + rx_0}p + \frac{r\eta_1 s_0}{\rho(c + rx_0)}x_0 G\left(\frac{x}{x_0}\right).$$

Calculating $\frac{dV_0}{dt}$ along system (6)-(10) we obtain

$$\begin{aligned}
\frac{dV_0}{dt} &= \left(1-\frac{s_0}{s}\right)\left(\beta-\delta s-\eta_1 sp-\eta_2 sy\right)+\frac{d}{bn+d}\left[(1-n)(\eta_1 sp+\eta_2 sy)-(b+d)w\right]\\
&+\frac{b+d}{bn+d}\left[n(\eta_1 sp+\eta_2 sy)+dw-\epsilon y\right]+\frac{\eta_1 s_0}{c+rx_0}\left(\pi y-cp-rxp\right)\\
&+\frac{r\eta_1 s_0}{\rho(c+rx_0)}\left(1-\frac{x_0}{x}\right)\left(\lambda+\rho xp-mx\right)\\
&=\left(1-\frac{s_0}{s}\right)\left(\beta-\delta s\right)+\eta_2 s_0 y-\frac{b+d}{bn+d}\epsilon y+\frac{\eta_1 s_0}{c+rx_0}\pi y+\frac{r\eta_1 s_0}{\rho(c+rx_0)}\left(1-\frac{x_0}{x}\right)\left(\lambda-mx\right)
\end{aligned}$$

$$\begin{aligned}
\frac{dV_0}{dt} &= -\delta\frac{(s-s_0)^2}{s}+\frac{\epsilon(b+d)}{bn+d}\left(\frac{\eta_2 s_0(bn+d)}{\epsilon(b+d)}+\frac{\eta_1 s_0\pi(bn+d)}{\epsilon(b+d)(c+rx_0)}-1\right)y-\frac{r\eta_1 s_0 m}{\rho(c+rx_0)}\frac{(x-x_0)^2}{x}\\
&=-\delta\frac{(s-s_0)^2}{s}-\frac{r\eta_1 s_0 m}{\rho(c+rx_0)}\frac{(x-x_0)^2}{x}+\frac{\epsilon(b+d)}{bn+d}(\mathcal{R}_0-1)y. \qquad (20)
\end{aligned}$$

Since $\mathcal{R}_0\leq 1$, then $\frac{dV_0}{dt}\leq 0$ for all $s,w,y,p,x>0$. Let $D_0=\{(s,w,y,p,x):\frac{dV_0}{dt}=0\}$. One can show that $D_0=\{E_0\}$. LaSalle's invariance principle implies that E_0 is globally asymptotically stable when $\mathcal{R}_0\leq 1$. ■

Theorem 2 *If $\mathcal{R}_0>1$, then E_1 is globally asymptotically stable in $\overset{\circ}{\Gamma}_2$.*

Proof. Let

$$\begin{aligned}
V_1(s,w,y,p,x) &= s_1 G\left(\frac{s}{s_1}\right)+\frac{d}{bn+d}w_1 G\left(\frac{w}{w_1}\right)+\frac{b+d}{bn+\alpha}y_1 G\left(\frac{y}{y_1}\right)\\
&+\frac{\eta_1 s_1 p_1}{\pi y_1}p_1 G\left(\frac{p}{p_1}\right)+\frac{r\eta_1 s_1 p_1}{\rho\pi y_1}x_1 G\left(\frac{x}{x_1}\right).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{dV_1}{dt} &= \left(1-\frac{s_1}{s}\right)\left(\beta-\delta s-\eta_1 sp-\eta_2 sy\right)+\frac{d}{bn+d}\left(1-\frac{w_1}{w}\right)\left((1-n)(\eta_1 sp+\eta_2 sy)-(b+d)w\right)\\
&+\frac{b+d}{bn+d}\left(1-\frac{y_1}{y}\right)\left(n(\eta_1 sp+\eta_2 sy)+dw-\epsilon y\right)+\frac{\eta_1 s_1 p_1}{\pi y_1}\left(1-\frac{p_1}{p}\right)\left(\pi y-cp-rxp\right)\\
&+\frac{r\eta_1 s_1 p_1}{\rho\pi y_1}\left(1-\frac{x_1}{x}\right)\left(\lambda+\rho xp-mx\right)\\
&=\left(1-\frac{s_1}{s}\right)\left(\beta-\delta s\right)+\eta_1 s_1 p+\eta_2 s_1 y-\frac{\eta_1 d(1-n)}{bn+d}\frac{spw_1}{w}-\frac{\eta_2 d(1-n)}{bn+d}\frac{syw_1}{w}+\frac{d(b+d)}{bn+d}w_1\\
&-\frac{n\eta_1(b+d)}{bn+d}\frac{spy_1}{y}-\frac{n\eta_2(b+d)}{bn+d}\frac{syy_1}{y}-\frac{d(b+d)}{bn+d}\frac{wy_1}{y}-\frac{b+d}{bn+d}\epsilon y+\frac{b+d}{bn+d}\epsilon y_1+\eta_1 s_1 p_1\frac{y}{y_1}-\eta_1 s_1 p_1\frac{yp_1}{y_1 p}\\
&-\frac{\eta_1 s_1 p_1}{\pi y_1}cp+\frac{\eta_1 s_1 p_1}{\pi y_1}cp_1+\frac{\eta_1 s_1 p_1}{\pi y_1}rxp_1-\frac{r\eta_1 s_1 p_1}{\pi y_1}x_1 p+\frac{r\eta_1 s_1 p_1}{\rho\pi y_1}\left(1-\frac{x_1}{x}\right)\left(\lambda-mx\right).
\end{aligned}$$

Applying the conditions for E_1

$$\begin{aligned}
&\beta=\delta s_1+\eta_1 s_1 p_1+\eta_2 s_1 y_1,\quad (b+d)w_1=(1-n)(\eta_1 s_1 p_1+\eta_2 s_1 y_1),\\
&\frac{b+d}{bn+d}\epsilon y_1=\eta_1 s_1 p_1+\eta_2 s_1 y_1,\quad cp_1=\pi y_1-rx_1 p_1,\quad \lambda=mx_1-\rho x_1 p_1
\end{aligned}$$

we get

$$\begin{aligned}
\frac{dV_1}{dt} &= -\delta\frac{(s-s_1)^2}{s} + \frac{d(1-n)}{bn+d}\left(1-\frac{s_1}{s}\right)\left(\eta_1 s_1 p_1 + \eta_2 s_1 y_1\right) \\
&+ \frac{n(b+d)}{bn+d}\left(1-\frac{s_1}{s}\right)\left(\eta_1 s_1 p_1 + \eta_2 s_1 y_1\right) + 3\frac{d(1-n)}{bn+d}\eta_1 s_1 p_1 \\
&- \frac{d(1-n)}{bn+d}\eta_1 s_1 p_1 \frac{spw_1}{s_1 p_1 w} - \frac{d(1-n)}{bn+d}\eta_1 s_1 p_1 \frac{wy_1}{w_1 y} - \frac{d(1-n)}{bn+d}\eta_1 s_1 p_1 \frac{yp_1}{y_1 p} \\
&+ 2\frac{d(1-n)}{bn+d}\eta_2 s_1 y_1 - \frac{d(1-n)}{bn+d}\eta_2 s_1 y_1 \frac{syw_1}{s_1 y_1 w} - \frac{d(1-n)}{bn+d}\eta_2 s_1 y_1 \frac{wy_1}{w_1 y} \\
&+ 2\frac{n(b+d)}{bn+d}\eta_1 s_1 p_1 - \frac{n(b+d)}{bn+d}\eta_1 s_1 p_1 \frac{spy_1}{s_1 p_1 y} - \frac{n(b+d)}{bn+d}\eta_1 s_1 p_1 \frac{yp_1}{y_1 p} \\
&+ \frac{n(b+d)}{bn+d}\eta_2 s_1 y_1 - \frac{n(b+d)}{bn+d}\eta_2 s_1 y_1 \frac{s}{s_1} - 2\frac{\eta_1 s_1 p_1}{\pi y_1} r x_1 p_1 \\
&+ \frac{\eta_1 s_1 p_1}{\pi y_1} r x p_1 + \frac{\eta_1 s_1 p_1}{\pi y_1} r x_1 p_1 \frac{x_1}{x} - \frac{mr\eta_1 s_1 p_1}{\rho \pi y_1}\frac{(x-x_1)^2}{x}.
\end{aligned} \tag{21}$$

Eq. (21) can be simplified as:

$$\begin{aligned}
\frac{dV_1}{dt} &= -\delta\frac{(s-s_1)^2}{s} - \frac{n(b+d)}{bn+d}\eta_2 y_1 \frac{(s-s_1)^2}{s} - \frac{\eta_1 s_1 p_1 r\lambda}{\pi y_1 \rho x_1}\frac{(x-x_1)^2}{x} \\
&+ \frac{d(1-n)}{bn+d}\eta_1 s_1 p_1 \left[4 - \frac{s_1}{s} - \frac{spw_1}{s_1 p_1 w} - \frac{wy_1}{w_1 y} - \frac{yp_1}{y_1 p}\right] \\
&+ \frac{n(b+d)}{bn+d}\eta_1 s_1 p_1 \left[3 - \frac{s_1}{s} - \frac{spy_1}{s_1 p_1 y} - \frac{yp_1}{y_1 p}\right] \\
&+ \frac{d(1-n)}{bn+d}\eta_2 s_1 y_1 \left[3 - \frac{s_1}{s} - \frac{syw_1}{s_1 y_1 w} - \frac{y_1 w}{y w_1}\right].
\end{aligned} \tag{22}$$

Using the arithmetic mean-geometric mean inequality we find that the last three terms of Eq. (22) are less that or equal zero. Thus, $\frac{dV_1}{dt} \le 0$ for all $s, w, y, p, x > 0$ and $\frac{dU_1}{dt} = 0$ at E_1. The global stability of E_1 is induced from LaSalle's invariance principle. ■

4 Numerical Simulations

We perform the numerical simulation of model (6)-(10) using Matlab.

4.1 Effect of the parameters η_1 and η_2

We simulate the system with three different initial values as:

IV1: $s(0) = 18.0, w(0) = 0.2, y(0) = 0.2, p(0) = 1.0$, and $x(0) = 1$,
IV2: $s(0) = 16.0, w(0) = 0.6, y(0) = 1.0, p(0) = 2.0$, and $x(0) = 2.5$,
IV3: $s(0) = 12.0, w(0) = 1.0, y(0) = 1.5, p(0) = 2.5$, and $x(0) = 3.0$.

We fix the value of $n = 0.7$ and the other parameters are given in Table 1. Then we consider two sets of the values of η_1 and η_2 as follows:

Table 1: The parameters's values.

Parameter	Value	Parameter	Value
β	2	δ	0.1
η_1, η_2	varied	ϵ	0.5
π	4	c	0.1
r	0.4	λ	1.4
m	1	ρ	0.2
n	varied	d	0.1
b	0.3		

Set (I): We choose $\eta_1 = \eta_2 = 0.001$. We compute $\mathcal{R}_0 = 0.2189 < 1$. From Figure 1 we can see that, the concentrations of the uninfected monocytes and B cells return to their values $s_0 = \frac{\beta}{\delta} = 20$ and $x_0 = \frac{\lambda}{m} = 1.4$, respectively. On the other hand, the concentrations of latently infected monocytes, actively infected monocytes and CHIKV particles are declining and reaching zero for all the three initial values IV1-IV3. This demonstrates that, there exists one equilibrium E_0 which is globally asymptotically stable. This result agrees the result of Theorem 1.

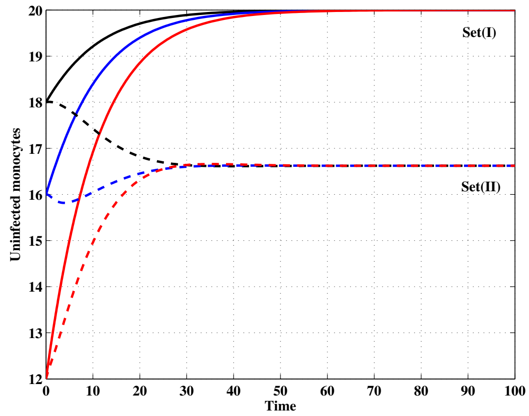
Set (II): We take $\eta_1 = \eta_2 = 0.008$. Then, we calculate $\mathcal{R}_0 = 1.7510 > 1$, $E_0(20.0, 0, 0, 0, 1.4)$ and $E_1 = (16.62, 0.253, 0.523, 2.016, 2.346)$. From Figure 1 we see that when $\mathcal{R}_0 > 1$, the solutions of the system starting at IV1-IV3 will tend to E_1. This agrees the results of Theorem 2.

4.2 Effect of the parameter n

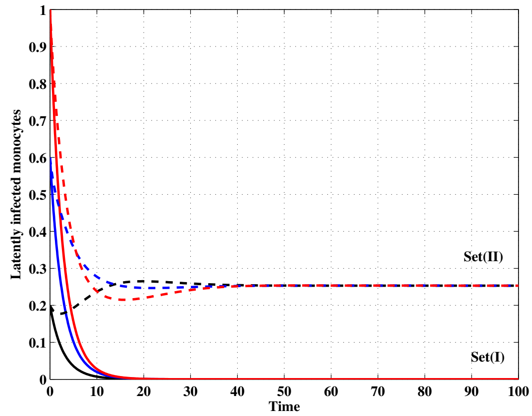
In this case, we use the values of the parameters given in Table 1 and we choose $\eta_1 = \eta_2 = 0.008$ and n is selected. We consider

IV4: $s(0) = 17, w(0) = 0.1, y(0) = 0.4, p(0) = 1.0$, and $x(0) = 2.0$.

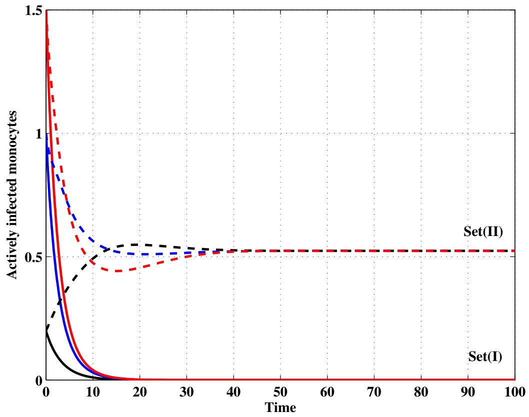
In Table 2, we calclaute the value $\mathcal{R}_0$ and the equilibria for different values of n. From the table we observe the value $\mathcal{R}_0$ is increased as n increased which means the solution of system will converge to E_0 if the values of n are small and they will converge to E_1 if values of n are large. Figure 2 supports the results of Theorem 2



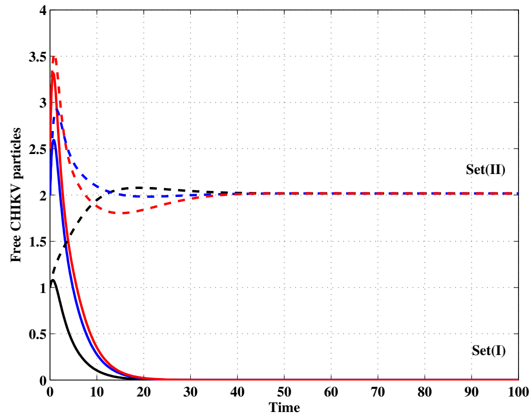
(a) Uninfected monocytes.



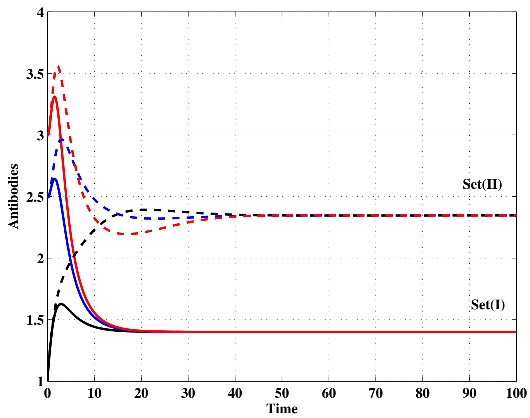
(b) Latently infected monocytes.



(c) Actively infected monocytes.



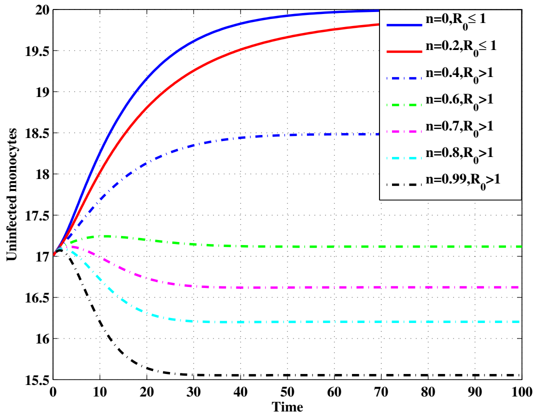
(d) Free CHIKV particles.



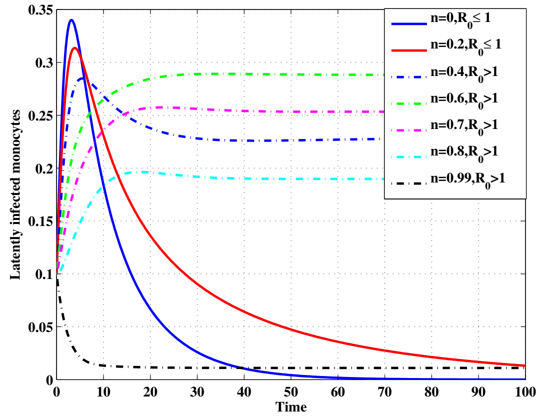
(e) Antibodies.

Figure 1: Numerical solutions of system (6)-(10) with selected values of η_1 and η_2.

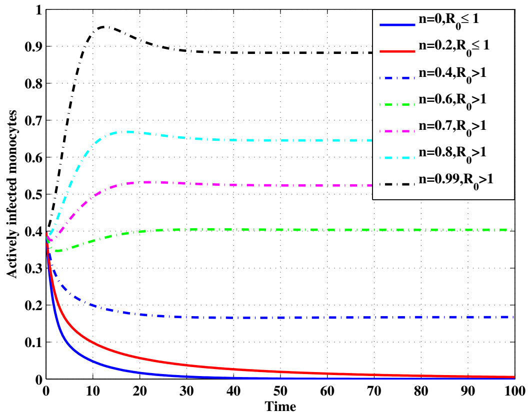
9



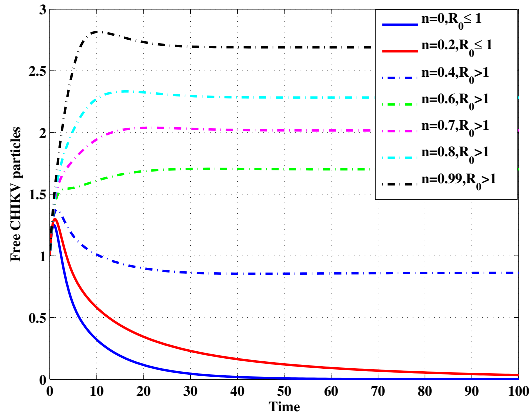
(a) Uninfected monocytes.



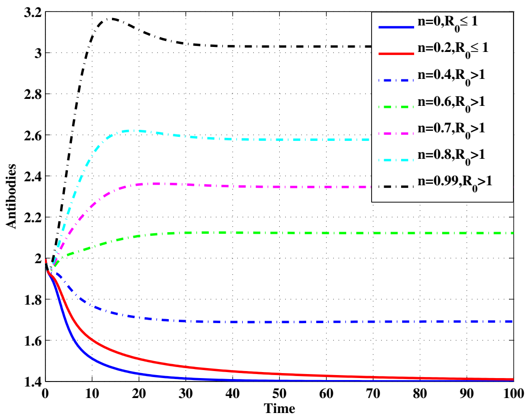
(b) Latently infected monocytes.



(c) Actively infected monocytes.



(d) Free CHIKV particles.



(e) Antibodies.

Figure 2: Numerical solutions of system (6)-(10) with selected values of n.

Table 2: The values of equilibria and $\mathcal{R}_0$ for system (6)-(10) with different values of n.

n	Steady states	$\mathcal{R}_0$
0.000001	$E_0 = (20, 0, 0, 0, 1.4)$	0.5648
0.2	$E_0 = (20, 0, 0, 0, 1.4)$	0.9038
0.256795	$E_0 = (20, 0, 0, 0, 1.4)$	1
0.4	$E_1 = (18.5, 0.2283, 0.1674, 0.8621, 1.6917)$	1.2427
0.6	$E_1 = (17.1178, 0.2882, 0.4035, 1.7011, 2.122)$	1.5816
0.7	$E_1 = (16.6222, 0.2533, 0.5236, 2.0166, 2.3463)$	1.7510
0.8	$E_1 = (16.2037, 0.1898, 0.6454, 2.2832, 2.5766)$	1.9205
0.99	$E_1 = (15.5546, 0.0111, 0.8824, 2.69, 3.0303)$	2.2424

References

[1] Y. Dumont, F. Chiroleu, *Vector control for the chikungunya disease*, Mathematical Biosciences and Engineering, **7** (2010), 313-345.

[2] Y. Dumont, J. M. Tchuenche, *Mathematical studies on the sterile insect technique for the chikungunya disease and aedes albopictus*, Journal of Mathematical Biology **65**(5) (2012), 809-854.

[3] Y. Dumont, F. Chiroleu, C. Domerg, *On a temporal model for the chikungunya disease: modeling, theory and numerics*, Mathematical Biosciences, **213**, (2008), 80-91.

[4] D. Moulay, M. Aziz-Alaoui, M.Cadivel, *The chikungunya disease: modeling, vector and transmission global dynamics,* Mathematical Biosciences, **229** (2011) 50-63.

[5] D. Moulay, M. Aziz-Alaoui, H. D. Kwon, *Optimal control of chikungunya disease: larvae reduction, treatment and prevention,* Mathematical Biosciences and Engineering, **9** (2012), 369-392.

[6] C. A. Manore, K. S. Hickmann, S. Xu, H. J. Wearing, J. M. Hyman, *Comparing dengue and chikungunya emergence and endemic transmission in A. aegypti and A. albopictus*, Journal of Theoretical Biology **356** (2014), 174-191.

[7] X. Liu, and P. Stechlinski, Application of control strategies to a seasonal model of chikungunya disease, Applied Mathematical Modelling, **39** (2015), 3194-3220.

[8] Y. Wang, X. Liu, *Stability and Hopf bifurcation of a within-host chikungunya virus infection model with two delays,* Mathematics and Computers in Simulation, **138** (2017), 31-48.

[9] A. M. Elaiw, T. O. Alade and S. M. Alsulami, *Stability of a within-host Chikungunya virus dynamics model with latency*, Journal of Computational Analysis and Applications, **26**(5) 2019, 777-790.

[10] A. M. Elaiw, T. O. Alade and S. M. Alsulami, *Analysis of latent CHIKV dynamics model with time delays*, Journal of Computational Analysis and Applications, **27**(1) 2019, 19-36.

[11] D.S. Callaway, and A.S. Perelson, *HIV-1 infection and low steady state viral loads*, Bull. Math. Biol., **64** (2002), 29-64.

[12] B. Buonomo, and C. Varglobally asymptotically stable-De-Le, *Global stability for an HIV-1 infection model including an eclipse stage of infected cells*, Journal of Mathematical Analysis and Applications, **385** (2012), 709-720.

[13] A. M. Elaiw and S.A. Azoz, *Global properties of a class of HIV infection models with Beddington-DeAngelis functional response,* Mathematical Methods in the Applied Sciences, **36** (2013), 383-394.

[14] A. M. Elaiw and N. H. AlShamrani, *Global stability of humoral immunity virus dynamics models with nonlinear infection rate and removal*, Nonlinear Analysis: Real World Applications, **26**, (2015), 161-190.

[15] A. M. Elaiw, *Global properties of a class of HIV models*, Nonlinear Analysis: Real World Applications, **11** (2010), 2253-2263.

[16] Kristin M. Long and Mark T. Heise, *Protective and Pathogenic Responses to Chikungunya Virus Infection*, Curr Trop Med Rep. **2**(1) (2015), 13-21.

[17] J. Wang, J. Lang, X. Zou, *Analysis of an age structured HIV infection model with virus-to-cell infection and cell-to-cell transmission*, Nonlinear Analysis: Real World Applications, **34** (2017), 75-96.

[18] X. Lai and X. Zou , *Modeling cell-to-cell spread of HIV-1 with logistic target cell growth*, Journal of Mathematical Analysis and Applications, **426** (2015), 563–584.

[19] X. Lai, X. Zou, *Modelling HIV-1 virus dynamics with both virus-to-cell infection and cell-to-cell transmission*, SIAM Journal of Applied Mathematics, **74** (2014), 898–917.

[20] Y. Yang, L. Zou and S. Ruanc, *Global dynamics of a delayed within-host viral infection model with both virus-to-cell and cell-to-cell transmissions*, Mathematical Biosciences, **270** (2015), 183-191.

OPTIMAL BOUNDS FOR TOADER MEAN IN TERMS OF GEOMETRIC AND CONTRAHARMONIC MEANS*

WEI-MAO QIAN[1,2], WEN ZHANG[3], AND YU-MING CHU[4,**]

Abstract. In this paper, we present the best possible parameters α_1, α_2, α_3 and β_1, β_2, β_3 such that the double inequalities

$$C^{\alpha_1}(a,b)G^{1-\alpha_1}(a,b) < T(a,b) < C^{\beta_1}(a,b)G^{1-\beta_1}(a,b),$$
$$\alpha_2 C(a,b) + (1-\alpha_2)G(a,b) < T(a,b) < \beta_2 C(a,b) + (1-\beta_2)G(a,b),$$
$$\frac{\alpha_3}{G(a,b)} + \frac{1-\alpha_3}{C(a,b)} < \frac{1}{T(a,b)} < \frac{\beta_3}{G(a,b)} + \frac{1-\beta_3}{C(a,b)}$$

hold for all $a,b>0$ with $a\neq b$, where $G(a,b)=\sqrt{ab}$, $C(a,b)=\left(a^2+b^2\right)/(a+b)$ and $T(a,b)=2\int_0^{\pi/2}\sqrt{a^2\cos^2(t)+b^2\sin^2(t)}dt/\pi$ are the geometric, contraharmonic and Toader means of a and b, respectively.

1. Introduction

The Toader mean $T(a,b)$ [1-5] of two positive real numbers a and b is defined by

$$T(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2(t)+b^2\sin^2(t)}dt \tag{1.1}$$
$$=\begin{cases}\frac{2a}{\pi}\mathcal{E}\left(\sqrt{1-(b/a)^2}\right), & a>b,\\ \frac{2b}{\pi}\mathcal{E}\left(\sqrt{1-(a/b)^2}\right), & a<b,\\ a, \ a=b,\end{cases}$$

where $\mathcal{E}(r)=\int_0^{\pi/2}\left(1-r^2\sin^2(t)\right)^{1/2}dt$ $(r\in[0,1])$ is the complete elliptic integral of the second kind [6-30]. The Toader mean $T(a,b)$ is well known in mathematical literature for many years, it satisfies

$$T(a,b)=R_E\left(a^2,b^2\right),$$

where

$$R_E(a,b)=\frac{1}{\pi}\int_0^{\infty}\frac{[a(t+b)+b(t+a)]t}{(t+a)^{3/2}(t+b)^{3/2}}dt$$

2010 *Mathematics Subject Classification.* Primary: 26E60; Secondary: 33E05.

Key words and phrases. Toader mean, geometric mean, contraharmonic mean, complete elliptic integral.

*The research was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11301127, 11701176, 11626101, 11601485), the Science and Technology Research Program of Zhejiang Educational Committee (Grant No. Y201635325) and the Natural Science Foundation of Huzhou City (Grant No. 2018YZ07).

**Corresponding author: Yu-Ming Chu, Email: chuyuming2005@126.com.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

stands for the symmetric complete elliptic integral of the second kind (See [31-33]), therefore it cannot be expressed in terms of the elementary transcendental functions.

Recently, the Toader mean $T(a,b)$ has been the subject of intensive research. In particular, many remarkable inequalities for the Toader mean can be found in the literature [34-41].

Let $G(a,b)=\sqrt{ab}$ [42-48], $A(a,b)=(a+b)/2$ [49-57], $C(a,b)=(a^2+b^2)/(a+b)$ [58-61], and $M_p(a,b)=[(a^p+b^p)/2]^{1/p}$ [62-73] and $M_0(a,b)=\sqrt{ab}$ be respectively the geometric, arithmetic, contraharmonic and pth power means of a and b. Then it is well known that power mean $M_p(a,b)$ is strictly increasing with respect to $p\in\mathbb{R}$ for all fixed $a,b>0$ with $a\neq b$, and the inequalities

$$G(a,b)=M_0(a,b)<A(a,b)=M_1(a,b)<C(a,b)=M_2(a,b) \tag{1.2}$$

hold for all $a,b>0$ with $a\neq b$.

Vuorinen [74] conjectured that

$$T(a,b)>M_{3/2}(a,b) \tag{1.3}$$

for all $a,b>0$ with $a\neq b$. This conjecture was proved by Qiu and Shen [75], and Barnard et al. [76].

Alzer and Qiu [77] proved that the inequality

$$T(a,b)<T_\lambda(a,b) \tag{1.4}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\lambda\geq \log 2/(\log\pi-\log 2)=1.5349\cdots$.

From (1.2)-(1.4) we clearly see that

$$G(a,b)<T(a,b)<C(a,b) \tag{1.5}$$

for all $a,b>0$ with $a\neq b$.

Motivated by (1.5), it is natural to ask what are the best possible parameters α_1, α_2, α_3 and β_1, β_2, β_3 such that the double inequalities

$$C^{\alpha_1}(a,b)G^{1-\alpha_1}(a,b)<T(a,b)<C^{\beta_1}(a,b)G^{1-\beta_1}(a,b),$$

$$\alpha_2 C(a,b)+(1-\alpha_2)G(a,b)<T(a,b)<\beta_2 C(a,b)+(1-\beta_2)G(a,b),$$

$$\frac{\alpha_3}{G(a,b)}+\frac{1-\alpha_3}{C(a,b)}<\frac{1}{T(a,b)}<\frac{\beta_3}{G(a,b)}+\frac{1-\beta_3}{C(a,b)}$$

hold for all $a,b>0$ with $a\neq b$? The main purpose of this paper is to answer this question.

2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

Let $r\in[0,1]$, $\mathcal{K}(r)=\int_0^{\pi/2}(1-r^2\sin^2(t))^{-1/2}dt$ and $\mathcal{E}(r)=\int_0^{\pi/2}\left(1-r^2\sin^2(t)\right)^{1/2}dt$ be respectively the complete elliptic integrals of the first and second kinds. Then it is well known that $\mathcal{K}(r)$ is strictly increasing and $\mathcal{E}(r)$ is strictly decreasing on $[0,1]$,

$$\mathcal{K}(0)=\mathcal{E}(0)=\pi/2,\ \mathcal{K}(1)=\infty,\ \mathcal{E}(1)=1, \tag{2.1}$$

and $\mathcal{K}(r)$ and $\mathcal{E}(r)$ satisfy the formulas (See[17, Appendix E, pp. 474-475])

$$\frac{d\mathcal{K}(r)}{dr}=\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r(1-r^2)},\ \frac{d\mathcal{E}(r)}{dr}=\frac{\mathcal{E}(r)-\mathcal{K}(r)}{r}, \tag{2.2}$$

$$\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right)=\frac{2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{1+r}. \tag{2.3}$$

Lemma 2.1. *(See [78]) Let $-\infty<a<b<\infty$, $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are the functions $[f(x)-f(a)]/[g(x)-g(a)]$ and $[f(x)-f(b)]/[g(x)-g(b)]$. If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2.2. *(See [79]) The function $r\to(1-r^2)^\lambda\mathcal{K}(r)$ is strictly decreasing from $[0,1]$ onto $[0,\pi/2]$ if $\lambda\geq 1$.*

Lemma 2.3. *Let $f_1(r)=[\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)]/r^2$. Then $f_1(r)$ is strictly increasing from $(0,1]$ onto $(\pi/4,1]$.*

Proof. Let $g_1(r)=\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)$ and $g_2(r)=2r$. Then from (2.1) and (2.2) together with Lemma 2.2 we clearly see that

$$f_1(r)=\frac{g_1(r)}{g_2(r)},\ g_1(0)=g_2(0)=0,\ f_1(1)=1, \tag{2.4}$$

$$\frac{g_1'(r)}{g_2'(r)}=\frac{1}{2}\mathcal{K}(r),\ \lim_{r\to 0}f_1(r)=\lim_{r\to 0}\frac{g_1'(r)}{g_2'(r)}=\frac{\pi}{4}. \tag{2.5}$$

Therefore, Lemma 2.3 follows from (2.4) and (2.5) together with Lemma 2.1 and the monotonicity of $\mathcal{K}(r)$ on $[0,1]$. □

Lemma 2.4. *Let $f_2(r)=[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)]/(1+r^2)$. Then $f_2(r)$ is strictly decreasing from $[0,1]$ onto $[1,\pi/2]$.*

Proof. It follows from (2.1) and Lemma 2.2 that

$$f_2(0)=\frac{\pi}{2},\ f_2(1)=1. \tag{2.6}$$

Differentiating $f_2(r)$ gives

$$f_2'(r)=\frac{r}{(1+r^2)^2}\left[(1-r^2)f_1(r)-2\mathcal{E}(r)\right], \tag{2.7}$$

where $f_1(r)$ is given by Lemma 2.3.

From (2.7) and Lemma 2.3 together with the monotonicity of $\mathcal{E}(r)$ on $[0,1]$ we get

$$f_2'(r)<\frac{r}{(1+r^2)^2}\left[(1-r^2)-2\right]=-\frac{r}{1+r^2}<0 \tag{2.8}$$

for $r\in(0,1)$.

Therefore, Lemma 2.4 follows easily from (2.6) and (2.8). □

Lemma 2.5. *Let $p\in\mathbb{R}$, $f_1(r)$ and $f_2(r)$ be respectively defined by Lemmas 2.3 and 2.4, and*

$$f(r)=\frac{f_1(r)}{f_2(r)}+\frac{2(1-p)}{1-r^2}-(1+p). \tag{2.9}$$

Then the following statements are true:
(1) $f(r)>0$ for all $r\in(0,1)$ if $p=1/2$;
(2) $f(r)<0$ for all $r\in(0,1)$ if $p=1$;

Proof. Let $f_3(r)=f_1(r)/f_2(r)$, then Lemmas 2.3 and 2.4 lead to

$$f_3(0^+)=\frac{1}{2},\ f_3(1^-)=1, \tag{2.10}$$

and $f_3(r)$ is strictly increasing on $(0,1)$.

For part (1), if $p=1/2$, then (2.9) becomes

$$f(r)=f_3(r)+\frac{1}{1-r^2}-\frac{3}{2}. \tag{2.11}$$

From (2.10) and (2.11) together with the monotonicity of $f_3(r)$ we clearly see that

$$f(r)>f_3(0^+)+1-\frac{3}{2}=0$$

for all $r\in(0,1)$.

For part (2), if $p=1$, then (2.9) becomes

$$f(r)=f_3(r)-2. \tag{2.12}$$

Therefore, $f(r)<1-2=-1<0$ for all $r\in(0,1)$ follows from (2.10) and (2.12) together with the monotonicity of $f_3(r)$. □

Lemma 2.6. *Let $q\in\mathbb{R}$, $f_1(r)$ be defined by Lemma 2.3, and*

$$g(r)=\frac{2}{\pi}f_1(r)+\frac{1-q}{\sqrt{1-r^2}}-2q. \tag{2.13}$$

Then the following statements are true:
(1) $g(r)>0$ for all $r\in(0,1)$ if $q=1/2$;
(2) there exists $r_0\in(0,1)$ such that $g(r)<0$ for $r\in(0,r_0)$ and $g(r)>0$ for $r\in(r_0,1)$ if $q=2/\pi$.

Proof. For part (1), if $q=1/2$, then (2.13) becomes

$$g(r)=\frac{2}{\pi}f_1(r)+\frac{1}{2\sqrt{1-r^2}}-1. \tag{2.14}$$

It follows from Lemma 2.3 and (2.14) that

$$g(r)>\frac{2}{\pi}\times\frac{\pi}{4}+\frac{1}{2}-1=0$$

for all $r\in(0,1)$.

For part (2), if $q=2/\pi$, then Lemma 2.3 and (2.13) lead to

$$g(0^+)=-\frac{3(4-\pi)}{2\pi}<0,\ g(1^-)=\infty, \tag{2.15}$$

and $g(r)$ is strictly increasing on $(0,1)$.

Therefore, part (2) follows from (2.15) and the monotonicity of $g(r)$. □

Lemma 2.7. *Let*

$$h(r)=\frac{2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{\pi}-\frac{(2-r^2)(1+r^2)}{4+r^2}$$

Then $h(r)>0$ for all $r\in(0,1)$.

Proof. Simple computations lead to

$$h(0)=0, \tag{2.16}$$

$$\begin{aligned} h'(r) &= \frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{\pi r}+\frac{2r(r^4+8r^2-2)}{(4+r^2)^2} \\ &= \frac{r}{(4+r^2)^2}\left[\frac{(4+r^2)^2}{\pi}f_1(r)+2(r^4+8r^2-2)\right], \end{aligned} \tag{2.17}$$

where $f_1(r)$ is defined by Lemma 2.3.

It follows from Lemma 2.3 and (2.17) that

$$h'(r)>\frac{r}{(4+r^2)^2}\left[\frac{(4+r^2)^2}{\pi}\times\frac{\pi}{4}+2(r^4+8r^2-2)\right]=\frac{9r^3(8+r^2)}{4(4+r^2)^2}>0 \tag{2.18}$$

for $r\in(0,1)$.

Therefore, Lemma 2.7 follows easily from (2.16) and (2.18). □

3. Main Results

Theorem 3.1. *The double inequality*

$$C^{\alpha_1}(a,b)G^{1-\alpha_1}(a,b)<T(a,b)<C^{\beta_1}(a,b)G^{1-\beta_1}(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leq 1/2$ and $\beta_1\geq 1$.

Proof. Since $C(a,b)$, $T(a,b)$ and $G(a,b)$ are symmetric and homogeneous of degree 1. Without loss of generality, we assume that $a>b>0$. Let $r=(a-b)/(a+b)\in(0,1)$ and $p\in\mathbb{R}$. Then from (1.1) and (2.3) we get

$$T(a,b)=\frac{2}{\pi}A(a,b)\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right], \tag{3.1}$$

$$G(a,b)=A(a,b)\sqrt{1-r^2},\ C(a,b)=A(a,b)(1+r^2). \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\frac{\log[T(a,b)]-\log[G(a,b)]}{\log[C(a,b)]-\log[G(a,b)]}=\frac{\log\left[\frac{2}{\pi}\left(2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right)\right]-\frac{1}{2}\log(1-r^2)}{\log(1+r^2)-\frac{1}{2}\log(1-r^2)}, \tag{3.3}$$

$$\begin{aligned} &\log[T(a,b)]-\{p\log[C(a,b)+(1-p)\log[G(a,b)]\} \\ &=\log\left[\frac{2}{\pi}\left(2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right)\right]-\left\{p\log(1+r^2)+\frac{1}{2}(1-p)\log(1-r^2)\right\}. \end{aligned} \tag{3.4}$$

Let

$$F(r)=\log\left[\frac{2}{\pi}\left(2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right)\right]-\left\{p\log(1+r^2)+\frac{1}{2}(1-p)\log(1-r^2)\right\}. \tag{3.5}$$

Then simple computations lead to

$$F(0)=0, \tag{3.6}$$

$$F'(r)=\frac{r}{1+r^2}f(r), \tag{3.7}$$

where $f(r)$ is defined by (2.9).

We divide the proof into two cases.

Case 1 $p=1/2$. Then Lemma 2.5(1) and (3.7) lead to the conclusion that $F(r)$ is strictly increasing on $(0,1)$. Therefore,

$$T(a,b) > C^{1/2}(a,b)G^{1/2}(a,b) \tag{3.8}$$

follows from (3.4)-(3.6) and the monotonicity of $F(r)$ on $(0,1)$.

Case 2 $p=1$. Then Lemma 2.5(2) and (3.7) lead to the conclusion that $F(r)$ is strictly decreasing on $(0,1)$. Therefore,

$$T(a,b) < C(a,b) \tag{3.9}$$

follows from (3.4)-(3.6) and the monotonicity of $F(r)$ on $(0,1)$.

Note taht

$$\lim_{r\to 0^+} \frac{\log\left[\frac{2}{\pi}\left(2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right)\right]-\frac{1}{2}\log(1-r^2)}{\log(1+r^2)-\frac{1}{2}\log(1-r^2)}=\frac{1}{2}, \tag{3.10}$$

$$\lim_{r\to 1^-} \frac{\log\left[\frac{2}{\pi}\left(2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right)\right]-\frac{1}{2}\log(1-r^2)}{\log(1+r^2)-\frac{1}{2}\log(1-r^2)}=1. \tag{3.11}$$

Therefore, Theorem 3.2 follows from (3.8) and (3.9) together with the following statements.

• If $p>1/2$, then (3.3) and (3.10) imply that there exists $\delta_1\in(0,1)$ such that

$$T(a,b) < C^{p}(a,b)G^{1-p}(a,b)$$

for all $(a-b)/(a+b)\in(0,\delta_1)$.

•• If $p<1$, then (3.3) and (3.11) imply that there exists $\delta_2\in(0,1)$ such that

$$T(a,b) > C^{p}(a,b)G^{1-p}(a,b)$$

for all $(a-b)/(a+b)\in(1-\delta_2,1)$. □

Theorem 3.2. *The double inequality*

$$\alpha_2 C(a,b)+(1-\alpha_2)G(a,b) < T(a,b) < \beta_2 C(a,b)+(1-\beta_2)G(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_2\leq 1/2$ and $\beta_2\geq 2/\pi$.

Proof. Without loss of generality, we assume that $a>b>0$. Let $r=(a-b)/(a+b)\in(0,1)$ and $q\in\mathbb{R}$. Then from (3.1) and (3.2) we have

$$\frac{T(a,b)-G(a,b)}{C(a,b)-G(a,b)}=\frac{\frac{2}{\pi}\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]-\sqrt{1-r^2}}{(1+r^2)-\sqrt{1-r^2}}, \tag{3.12}$$

$$\begin{aligned} &T(a,b)-[qC(a,b)+(1-q)G(a,b)] \\ &= A(a,b)\left\{\frac{2}{\pi}\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]-[q(1+r^2)+(1-q)\sqrt{1-r^2}]\right\}. \end{aligned} \tag{3.13}$$

Let

$$G(r)=\frac{2}{\pi}\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]-[q(1+r^2)+(1-q)\sqrt{1-r^2}]. \tag{3.14}$$

Then simple computations lead to

$$G(0^+)=0, \tag{3.15}$$

$$G(1^-)=\frac{4}{\pi}-2q, \tag{3.16}$$

$$G'(r)=rg(r), \tag{3.17}$$

where $g(r)$ is defined by (2.13).

We divide the proof into two cases.

Case 1 $q = 1/2$. Then Lemma 2.6(1) and (3.17) lead to the conclusion that $G(r)$ is strictly increasing on $(0, 1)$. Therefore,

$$T(a,b) > \frac{1}{2}C(a,b) + \frac{1}{2}G(a,b) \tag{3.18}$$

follows from (3.13)-(3.15) and the monotonicity of $G(r)$.

Case 2 $q = 2/\pi$. Then (3.16) becomes

$$G(1^-) = 0. \tag{3.19}$$

It follows from Lemma 2.6(2) and (3.17) that there exists $r_0 \in (0, 1)$ such that $G(r)$ is strictly decreasing on $(0, r_0)$ and strictly increasing on $(r_0, 1)$. Therefore,

$$T(a,b) < \frac{2}{\pi}C(a,b) + \left(1 - \frac{2}{\pi}\right)G(a,b) \tag{3.20}$$

follows from (3.13)-(3.15) and (3.19) together with the piecewise monotonicity of $G(r)$.

Note that

$$\lim_{r\to 0^+} \frac{\frac{2}{\pi}\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right] - \sqrt{1-r^2}}{(1+r^2) - \sqrt{1-r^2}} = \frac{1}{2}, \tag{3.21}$$

$$\lim_{r\to 1^-} \frac{\frac{2}{\pi}\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right] - \sqrt{1-r^2}}{(1+r^2) - \sqrt{1-r^2}} = \frac{2}{\pi}. \tag{3.22}$$

Therefore, Theorem 3.2 follows easily from (3.12), (3.18) and (3.20)-(3.22). $\square$

Theorem 3.3. *The double inequality*

$$\frac{\alpha_3}{G(a,b)} + \frac{1-\alpha_3}{C(a,b)} < \frac{1}{T(a,b)} < \frac{\beta_3}{G(a,b)} + \frac{1-\beta_3}{C(a,b)}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \le 0$ and $\beta_3 \ge 1/2$.

Proof. Without loss of generality, we assume that $a > b > 0$. Let $r = (a-b)/(a+b) \in (0, 1)$, then from (3.1) and (3.2) we have

$$\begin{aligned} &\frac{1}{T(a,b)} - \frac{1}{2}\left[\frac{1}{G(a,b)} + \frac{1}{C(a,b)}\right] \\ &= \frac{1}{2}\left[\frac{\pi}{2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)} - \frac{1}{\sqrt{1-r^2}} - \frac{1}{1+r^2}\right]. \end{aligned} \tag{3.23}$$

Let

$$H(r) = \frac{\pi}{2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)} - \frac{1}{\sqrt{1-r^2}} - \frac{1}{1+r^2}. \tag{3.24}$$

Then Lemma 2.7 and $\sqrt{1-r^2} < 1 - r^2/2$ lead to

$$\begin{aligned} H(r) &< \frac{\pi}{2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)} - \frac{1}{1-\frac{r^2}{2}} - \frac{1}{1+r^2}. \\ &= \frac{\pi}{2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)} - \frac{4+r^2}{(2-r^2)(1+r^2)} < 0. \end{aligned} \tag{3.25}$$

Therefore,

$$\frac{1}{C(a,b)} < \frac{1}{T(a,b)} < \frac{1}{2}\left[\frac{1}{G(a,b)} + \frac{1}{C(a,b)}\right] \tag{3.26}$$

follows from Theorem 3.1 and (3.23)-(3.25).

Let $\lambda \in \mathbb{R}$ and $r \in (0,1)$. Then making use of (1.1) and Taylor expansion we get

$$\frac{1}{T(1+r,1-r)} = \frac{\pi}{2}\frac{1}{2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)} = 1 - \frac{1}{4}r^2 + o(r^2),$$

$$\frac{\lambda}{G(1+r,1-r)} + \frac{1-\lambda}{C(1+r,1-r)} = \frac{\lambda}{\sqrt{1-r^2}} + \frac{1-\lambda}{1+r^2} = 1 + \frac{3p-2}{2}r^2 + o(r^2),$$

$$\begin{aligned}\frac{1}{T(1+r,1-r)} &- \left[\frac{\lambda}{G(1+r,1-r)} + \frac{1-\lambda}{C(1+r,1-r)}\right] \\ &= -\frac{3}{2}\left(\lambda - \frac{1}{2}\right)r^2 + o(r^2).\end{aligned} \tag{3.27}$$

Note that

$$\begin{aligned}\lim_{r\to 1^-}&\left\{\frac{1}{T(1+r,1-r)} - \left[\frac{\lambda}{G(1+r,1-r)} + \frac{1-\lambda}{C(1+r,1-r)}\right]\right\} \\ &= \frac{\pi}{4} - \lim_{r\to 1^-}\left[\frac{\lambda}{\sqrt{1-r^2}} + \frac{1-\lambda}{1+r^2}\right] = -\infty\end{aligned} \tag{3.28}$$

if $\lambda > 0$.

Therefore, Theorem 3.3 follows from (3.26) and the following statements.

• If $\lambda < 1/2$, then (3.27) implies that there exists $\delta_3 \in (0,1)$ such that

$$\frac{1}{T(1+r,1-r)} > \frac{\lambda}{G(1+r,1-r)} + \frac{1-\lambda}{C(1+r,1-r)}$$

for $r \in (0, \delta_3)$.

•• If $\lambda > 0$, then there exists $\delta_4 \in (0,1)$ such that

$$\frac{1}{T(1+r,1-r)} < \frac{\lambda}{G(1+r,1-r)} + \frac{1-\lambda}{C(1+r,1-r)}$$

for $r \in (1-\delta_4, 1)$. □

Let $r \in (0,1)$, $a = 1$, $b = \sqrt{1-r^2}$. Then Theorems 3.1-3.3 lead to Corollary 3.4.

Corollary 3.4. *The double inequalities*

$$\frac{\pi}{2}\frac{\sqrt{2-r^2}\sqrt[8]{1-r^2}}{\sqrt{1+\sqrt{1-r^2}}} < \mathcal{E}(r) < \frac{\pi}{2}\frac{2-r^2}{\sqrt{1+\sqrt{1-r^2}}},$$

$$\frac{\pi}{4}\frac{2-r^2+\sqrt[4]{1-r^2}(1+\sqrt{1-r^2})}{1+\sqrt{1-r^2}} < \mathcal{E}(r) < \frac{2(2-r^2)+(\pi-2)\sqrt[4]{1-r^2}(1+\sqrt{1-r^2})}{2(1+\sqrt{1-r^2})},$$

$$\frac{\pi(2-r^2)\sqrt[4]{1-r^2}}{(2-r^2)(1+\sqrt{1-r^2})\sqrt[4]{1-r^2}} < \mathcal{E}(r) < \frac{\pi}{2}\frac{2-r^2}{\sqrt{1+\sqrt{1-r^2}}}$$

holds for all $r \in (0,1)$.

References

[1] Gh. Toader, Some mean values related to the arithmetic-geometric mean, J. Math. Anal. Appl., 1998, **218**(2), 358–368.

[2] Y.-M. Chu, M.-K. Wang, S.-L. Qiu and Y.-F. Qiu, Sharp generalized Seiffert mean bounds for Toader mean, Abstr. Appl. Anal., 2011, **2011**, Article ID 605259, 8 pages.

[3] Y.-M. Chu and M.-K. Wang, Inequalities between arithmetic-geometric, Gini, and Toader means, Abstr. Appl. Anal., 2012, **2012**, Article ID 830585, 11 pages.

[4] W.-M. Qian, Z.-H. Zhang and Y.-M. Chu, Sharp bounds for Toader-Qi mean in terms of harmonic and geometric means, J. Math. Inequal., 2017, **11**(1), 121–127.

[5] W.-M. Qian and Y.-M. Chu, Sharp bounds for a special quasi-arithmetic mean in terms of arithmetic and geometric means with two parameters, J. Inequal. Appl., 2017, **2017**, Article 274, 10 pages.

[6] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, Hölder mean inequalities for the complete elliptic integrals, Integral Transforms Spec. Funct., 2012, **23**(7), 521–527.

[7] Y.-M. Chu, M.-K. Wang, Y.-P. Jiang and S.-L. Qiu, Concavity of the complete elliptic integrals of the second kind with respect to Hölder means, J. Math. Anal. Appl., 2012, **395**(2), 637–642.

[8] Y.-M. Chu, M.-K. Wang and Y.-F. Qiu, On Alzer and Qiu's conjecture for complete elliptic integral and inverse hyperbolic tangent function, Abstr. Appl. Anal., 2011, **2011**, Article ID 697547, 7 pages.

[9] Y.-M. Chu, M.-K. Wang, S.-L. Qiu and Y.-P. Jiang, Bounds for complete elliptic integrals of the second kind with applications, Comput. Math. Appl., 2012, **63**(7), 1177–1184.

[10] Y.-M. Chu and T.-H. Zhao, Convexity and concavity of the complete elliptic integrals with respect to Lehmer mean, J. Inequal. Appl., 2015, **2015**, Article 396, 6 pages.

[11] T.-R. Huang, S.-Y. Tan, X.-Y. Ma and Y.-M. Chu, Monotonicity properties and bounds for the complete p-elliptic integrals, J. Inequal. Appl., 2018, **2018**, Article 239, 11 pages.

[12] M.-K. Wang and Y.-M. Chu, Asymptotical bounds for complete elliptic integrals of the second kind, J. Math. Anal. Appl., 2013, **402**(1), 119–126.

[13] M.-K. Wang and Y.-M. Chu, Refinements of transformation inequalities for zero-balanced hypergeometric function, Acta Math. Sci., 2017, **37B**(3), 607–622.

[14] M.-K. Wang and Y.-M. Chu, Landen inequalities for a class of hypergeometric functions with applications, Math. Inequal. Appl., 2018, **21**(2), 521–537.

[15] M.-K. Wang, Y.-M. Chu and Y.-P. Jiang, Ramanujan's cubic transformation inequalities for zero-balanced hypergeometric functions, Rocky Mountain J. Math., 2016, **46**(2), 679–691.

[16] M.-K. Wang, Y.-M. Chu and S.-L. Qiu, Some monotonicity properties of generalized elliptic integrals with applications, Math. Inequal. Appl., 2013, **16**(3), 671–677.

[17] M.-K. Wang, Y.-M. Chu, S.-L. Qiu and Y.-P. Jiang, Convexity of the complete elliptic integrals of the first kind with respect to Hölder means, J. Math. Anal. Appl., 2012, **388**(2), 1141–1146.

[18] M.-K. Wang, Y.-M. Chu and Y.-Q. Song, Asymptotical formulas for Gaussian and generalized hypergeometric functions, Appl. Math. Comput., 2016, **276**, 44–60.

[19] M.-K. Wang, Y.-M. Li and Y.-M. Chu, Inequalities and infinite product formula for Ramanujan generalized modular equation function, Ramanujan J., 2018, **46**(1), 189–200.

[20] M.-K. Wang, S.-L. Qiu and Y.-M. Chu, Infinite series formula for Hübner upper bound function with applications to Hersch-Pfluger distortion function, Math. Inequal. Appl., 2018, **21**(3), 629–648.

[21] M.-K. Wang, S.-L. Qiu, Y.-M. Chu and Y.-P. Jiang, Generalized Hersch-Pfluger distortion function and complete elliptic integrals, J. Math. Anal. Appl., 2012, **385**(1), 221–229.

[22] Zh.-H. Yang and Y.-M. Chu, A monotonicity property involving the generalized elliptic integral of the first kind, Math. Inequal. Appl., 2017, **20**(3), 729–735.

[23] Zh.-H. Yang, Y.-M. Chu and M.-K. Wang, Monotonicity criterion for the quotient of power series with applications, J. Math. Anal. Appl., 2015, **428**(1), 587–604.

[24] Zh.-H. Yang, Y.-M. Chu and W. Zhang, Monotonicity of the ratio for the complete elliptic integral and Stolarsky mean, J. Inequal. Appl., 2016, **2016**, Article 176, 10 pages.

[25] Zh.-H. Yang, Y.-M. Chu and W. Zhang, Accurate approximations for the complete elliptic integral of the second kind, J. Math. Anal. Appl., 2016, **438**(2), 875–888.

[26] Zh.-H. Yang, W.-M. Qian and Y.-M. Chu, Monotonicity properties and bounds involving the complete elliptic integrals of the first kind, Math. Inequal. Appl., 2018, **21**(4), 1185–1199.
[27] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind, J. Math. Anal. Appl., 2018, **462**(2), 1714–1726.
[28] H. Wang, W.-M. Qian and Y.-M. Chu, Optimal bounds for Gaussian arithmetic-geometric mean with applications to complete elliptic integral, J. Funct. Spaces, 2016, **2016**, Article ID 3698463, 6 pages.
[29] Zh.-H. Yang, Y.-M. Chu and X.-H. Zhang, Sharp Stolarsky mean bounds for the complete elliptic integral of the second kind, J. Nonlinear Sci. Appl., 2017, **10**(3), 929–936.
[30] T.-H. Zhao, M.-K. Wang, W. Zhang and Y.-M. Chu, Quadratic transformation inequalities for Gaussian hypergeometric function, J. Inequal. Appl., 2018, **2018**, Article 251, 15 pages.
[31] E. Neuman, Bounds for symmetric elliptic integrals, J. Approx. Theory, 2003, **122**(2), 249–259.
[32] H. Kazi and E. Neuman, Inequalities and bounds for elliptic integrals, J. Approx. Theory, 2007, **146**(2), 212–226.
[33] H. Kazi and E. Neuman, Inequalities and bounds for elliptic integrals II, In: Special Functions and Orthogonal Polynomials, 127–138, Contemp. Math. 471, Amer. Math. Soc., Providence, RI, 2008.
[34] Y.-M. Chu, M.-K. Wang and S.-L. Qiu, Optimal combinations bounds of root-square and arithmetic means for Toader mean, Proc. Indian Acad. Sci. Math. Sci., 2012, **122**(1), 41–51.
[35] Y.-M. Chu and M.-K. Wang, Optimal Lehmer mean bouns for the Toader mean, Results Math., 2012, **61**(3-4), 223–229.
[36] J.-F. Li, W.-M. Qian and Y.-M. Chu, Sharp bounds for Toader mean in terms of arithmetic, quadratic, and Neuman means, J. Inequal. Appl., 2015, Article 277, 9 pages.
[37] Y.-Q. Song, W.-D. Jiang, Y.-M. Chu and D.-D. Yan, Optimal bounds for Toader mean in terms of arithmetic and contraharmonic means, J. Math. Inequal., 2013, **7**(4), 751–757.
[38] W.-H. Li and M.-M. Zheng, Some inequalities for bounding Toader mean, J. Funct. Spaces Appl., 2013, Article ID 394194, 5 pages.
[39] Y.-M. Chu, M.-K. Wang and X.-Y. Ma, Sharp bounds for Toader mean in terms of contraharmonic mean with applications, J. Math. Inequal., 2013, **7**(2), 161–166.
[40] Y. Hua and F. Qi, The best bounds for Toader mean in terms of the centroidal and arithmetic means, Filomat, 2014, **28**(4), 775–780.
[41] Y. Hua and F. Qi, A double inequality for bounding Toader mean by the centroidal mean, Proc. Indian Acad. Sci. Math. Sci., 2014, **124**(4), 527–531.
[42] W.-M. Gong, Y.-Q. Song, Y.-M. Chu and M.-K. Wang, A sharp double inequality between Seiffert, arithmetic, and geoemtric means, Abstr. Appl. Anal., 2012, **2012**, Article ID 684834, 7 pages.
[43] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, Best possible inequalities among harmonic, geometric, logarithmic and Seiffert means, Math. Inequal. Appl., 2012, **15**(2), 415–422.
[44] M.-K. Wang, Z.-K. Wang and Y.-M. Chu, An optimal double inequality between geometric and identric means, Appl. Math. Lett., 2012, **25**(3), 471–475.
[45] Y.-M. Chu, C. Zong and G.-D. Wang, Optimal convex combination bounds of Seiffert and geometric means for arithmetic mean, J. Math. Inequal., 2011, **5**(2), 429–434.
[46] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, An optimal double inequality between Seiffert and geometric means, J. Appl. Math., 2011, **2011**, Article ID 261237, 6 pages.
[47] Y.-M. Chu and M.-K. Wang, Optimal inequalities between harmonic, geometric, logarithmic, and arithmetic-geometric means, J. Appl. Math., 2011, **2011**, Article ID 618929, 9 pages
[48] B.-Y. Long and Y.-M. Chu, Optimal inequalities for generalized logarithmic, arithmetic, and geometric means, J. Inequal. Appl., 2010, **2010**, Article ID 806825, 10 pages.
[49] A. Iqbal, M. Adil Khan, S. Ullah, Y.-M. Chu and A. Kashuri, Hermite-Hadamard type inequalities pertaining conformable fractional integrals and their applications, AIP Advances, 2018, **8**, Article ID 075101, 18 pages, DOI: 10.1063/1.5031954.
[50] Y.-M. Chu, M. Adil Khan, T. Ali and S. S. Dragomir, Inequalities for α-fractional differentiable functions, J. Inequal. Appl., 2017, **2017**, Article 93, 12 pages.
[51] M. Adil Khan, S. Begum, Y. Khurshid and Y.-M. Chu, Ostrowski type inequalities involving conformable fractional integrals, J. Inequal. Appl., 2018, **2018**, Article 70, 14 pages.

[52] M. Adil Khan, Y. Khurshid, T.-S. Du and Y.-M. Chu, Generalized of Hermite-Hadamard type inequalities via conformable fractional integrals, J. Funct. Spaces, 2018, **2018**, Article ID 5357463, 12 pages.
[53] M. Adil Khan, Y.-M. Chu, T. U. Khan and J. Khan, Some new inequalities of Hermite-Hadamard type for s-convex functions with applications, Open Math., 2017, **15**, 1414–1430.
[54] M. Adil Khan, Z. M. Al-sahwi and Y.-M. Chu, New estimations for Shannon and Zipf-Mandelbrot entropies, Entropy, 2018, **20**, Article 608, 10 pages, DOI: 10.3390/e2008608.
[55] M. Adil Khan, Y.-M. Chu, A. Kashuri and R. Liko, Hermite-Hadamard type fractional integral inequalities for $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions, J. Comput. Anal. Appl., 2019, **26**(8), 1487–1503.
[56] Y.-M. Chu, Y.-F. Qiu, M.-K. Wang and G.-D. Wang, The optimal convex combination bounds of arithmetic and harmonic means for the Seiffert mean, J. Inequal. Appl., 2010, **2010**, Article ID 436457, 7 pages.
[57] X.-M. Zhang and Y.-M. Chu, Convexity of the integral arithmetic mean of a convex function, Rocky Mountain J. Math., 2010, **40**(3), 1061–1068.
[58] H.-Z. Xu, Y.-M. Chu and W.-M. Qian, Sharp bounds for Sándor-Yang means in terms of arithmeitc and contra-Harmonic means, J. Inequal. Appl., 2018, **2018**, Article 127, 13 pages.
[59] Z.-Y. He, Y.-M. Chu and M.-K. Wang, Optimal bounds for Neuman means in terms of harmonic and contraharmonic means, J. Appl. Math., 2013, **2013**, Article ID 807623, 4 pages.
[60] Y.-M. Chu and S.-W. Hou, Sharp bounds for Seiffert mean in terms of contraharmonic mean, Abstr. Appl. Anal., 2012, **2012**, Article ID 425175, 6 pages.
[61] Z.-Y. He, W.-M. Qian, Y.-L. Jiang, Y.-Q. Song and Y.-M. Chu, Bounds for the combinations of Neuman-Sándor, arithmetic, and Second Seiffert means in terms of contraharmonic mean, Abstr. Appl. Anal., 2013, **2013**, Article ID 903982, 5 pages.
[62] W.-F. Xia, W. Janous and Y.-M. Chu, The optimal convex combination bounds of arithmetic and harmonic means in terms of power mean, J. Math. Inequal., 2012, **6**(2), 241–248.
[63] Y.-M. Chu and B.-Y. Long, Bounds of the Neuman-Sándor mean using power and identric means, Abstr. Appl. Anal., 2013, **2013**, Article ID 832591, 6 pages.
[64] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, Sharp power mean bounds for the combination of Seiffert and geometric means, Abstr. Appl. Anal., 2010, **2010**, Article ID 108920, 12 pages.
[65] Y.-M. Chu, S.-L. Qiu, and M.-K. Wang, Sharp inequalities involving the power mean and complete ellipitc integral of the first kind, Rocky Mountain J. Math., 2013, **43**(5), 1489–1496.
[66] Y.-M. Chu, M.-K. Wang and Y.-F. Qiu, An optimal double inequality between power-type Heron and Seiffert means, J. Inequal. Appl., 2010, **2010**, Article ID 146945, 11 pages.
[67] Y.-M. Chu, S.-S. Wang and C. Zong, Optimal lower power mean bound for the convex combiantion of harmonic and logarithmic means, Abstr. Appl. Anal., 2011, **2011**, Article ID 520648, 9 pages.
[68] Y.-M. Chu and W.-F. Xia, Two optimal double inequalities between power mean and logarithmic mean, Comput. Math. Appl., 2010, **60**(1), 83–89.
[69] Y.-M. Chu and T.-H. Zhao, Concavity of the error function with respect to Hölder means, Math. Inequal. Appl., 2016, **19**(2), 589–595.
[70] Y.-M. Li, B.-Y. Long and Y.-M. Chu, Sharp bounds by the power mean for the generalized Heronian mean, J. Inequal. Appl., 2012, **2012**, Article 129, 12 pages.
[71] B.-Y. Long and Y.-M. Chu, Optimal power mean bounds for the weighted geoemtric mean of classical means, J. Inequal. Appl., 2010, **2010**, Article ID 905679, 6 pages.
[72] M.-K. Wang, Y.-M. Chu, Y.-F. Qiu and S.-L. Qiu, An optimal power mean inequality for the complete elliptic integrals, Appl. Math. Lett., 2011, **24**(6), 887–890.
[73] G.-D. Wang, X.-H. Zhang and Y.-M. Chu, A power mean inequality involving the complete elliptic integrals, Rocky Mountain J. Math., 2014, **44**(5), 1661–1667.
[74] M. Vuorinen, Hypergeometric functions in geometric function theory, In: Special functions and differential equations (Madras, 1977), 119-126, Allied Publ., New Delhi, 1998.
[75] S.-L. Qiu and J.-M. Shen, On two problems concerning means, J. Hangzhou Inst. Electron. Eng., 1997, **17**(3), 1–7 (in Chinese).
[76] R. W. Barnard, K. Pearce and K. C. Richards, An inequality invoving the generalized hypergeometric function and the arc length of an ellipse, SIAM J. Math. Anal., 2000, **31**(3), 693–699.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

[77] H. Alzer and S.-L. Qiu, Monotonicity theorems and inequalities for the complete elliptic integrals, J. Comput. Appl. Math., 2004, **172**(2), 289–312.
[78] G. D. Anderson, S.-L. Qiu, M. K. Vamanamurthy and M. Vuorinen, Generalized elliptic integrals and modular equations, Pacific J. Math., 2000, **192**(1), 1–37.
[79] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Conformal Invariants, Inequalities, and Quasiconformal Maps, John Wiley & Songs, New York, 1997.

Wei-Mao Qian, [1]College of Science, Hunan City University, Yiyang 413000, Hunan, China; [2]School of Continuing Education, Huzhou Vocational and Technological College, Huzhou 313000, Zhejiang, China
E-mail address: qwm661977@126.com

Wen Zhang, [3]Friedman Brain Institute, Icahn School of Medicine at Mount Sinai, New York, NY 10029, USA
E-mail address: zhang.wen81@gmail.com

Yu-Ming Chu (Corresponding author), [4]Department of Mathematics, Huzhou University, Huzhou 313000, Zhejiang, China
E-mail address: chuyuming2005@126.com

OPTIMAL BOUNDS FOR TOADER-QI MEAN WITH APPLICATIONS*

WEN-MAO QIAN[1,2], WEN ZHANG[3], AND YU-MING CHU[4,**]

ABSTRACT. In the article, we find the best possible parameters α_1, α_2, α_3, β_1, β_2 and β_3 such that the double inequalities

$$A^{\alpha_1}(a,b)H^{1-\alpha_1}(a,b) < TQ(a,b) < \beta_1 A(a,b) + (1-\beta_1)H(a,b),$$

$$\frac{[\alpha_2 A(a,b) + (1-\alpha_2)H(a,b)]A(a,b)}{L(a,b)} < TQ(a,b) < \frac{[\beta_2 A(a,b) + (1-\beta_2)H(a,b)]A(a,b)}{L(a,b)},$$

$$\sqrt{[\alpha_3 L(a,b) + (1-\alpha_3)H(a,b)]A(a,b)} < TQ(a,b) < \sqrt{[\beta_3 L(a,b) + (1-\beta_3)H(a,b)]A(a,b)}$$

hold for all $a,b>0$ with $a\neq b$, where $A(a,b)=(a+b)/2$, $H(a,b)=2ab/(a+b)$, $L(a,b)=(b-a)/(\log b-\log a)$ and $TQ(a,b)=2\int_0^{\pi/2} a^{\cos^2\theta}b^{\sin^2\theta}d\theta/\pi$ are the arithmetic, harmonic, logarithmic and Toader-Qi means of a and b, respectively. As applications, we present new bounds for the modified Bessel function of the first kind $I_0(t)=\sum_{n=0}^{\infty} t^{2n}/[2^{2n}(n!)^2]$.

1. Introduction

Let $a,b>0$ with $a\neq b$. Then the arithmetic mean $A(a,b)$ [1-7], harmonic mean $H(a,b)$ [8-16], logarithmic mean $L(a,b)$ [17-22] and Toader-Qi mean $TQ(a,b)$ [23, 24] are defined by

$$A(a,b)=\frac{a+b}{2},\quad H(a,b)=\frac{2ab}{a+b}, \tag{1.1}$$

$$L(a,b)=\frac{b-a}{\log b-\log a},\quad TQ(a,b)=\frac{2}{\pi}\int_0^{\pi/2} a^{\cos^2\theta}b^{\sin^2\theta}d\theta, \tag{1.2}$$

respectively. Recently, the arithmetic mean $A(a,b)$, harmonic mean $H(a,b)$, logarithmic mean $L(a,b)$ have attracted the attention of many researchers, and many remarkable inequalities for these means and related special functions can be found in the literature [25-59].

Very recently, Qi et al. [24] proved that the identity

$$TQ(a,b)=\sqrt{ab}\,I_0\left(\log\sqrt{b/a}\right) \tag{1.3}$$

2010 *Mathematics Subject Classification.* Primary: 26E60; Secondary: 33C10.

Key words and phrases. Toader-Qi mean, modified Bessel function, arithmetic mean, harmonic mean, logarithmic mean.

*The research was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11301127, 11701176, 11626101, 11601485), the Science and Technology Research Program of Zhejiang Educational Committee (Grant No. Y201635325) and the Natural Science Foundation of Huzhou City (Grant No. 2018YZ07).

**Corresponding author: Yu-Ming Chu, Email: chuyuming2005@126.com.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

and the inequalities

$$L(a,b) < TQ(a,b) < \frac{A(a,b)+G(a,b)}{2} < \frac{2A(a,b)+G(a,b)}{3} < I(a,b)$$

hold for all $a,b>0$ with $a\neq b$, where

$$I_\nu(t) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\nu+1)} \left(\frac{t}{2}\right)^{2n+\nu} \tag{1.4}$$

is the modified Bessel function of the first kind [60], $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ is the classical gamma function [61-69], and $G(a,b) = \sqrt{ab}$ [70-72] and $I(a,b) = (b^b/a^a)^{1/(b-a)}/e$ [73-75] are respectively the geometric and inentric means of a and b.

Yang and Chu [76, 77], and Yang, Chu and Song [78] proved that the inequalities

$$\lambda_1\sqrt{L(a,b)A(a,b)} < TQ(a,b) < \mu_1\sqrt{L(a,b)A(a,b)},$$

$$L^{\lambda_2}(a,b)A^{1-\lambda_2}(a,b) < TQ(a,b) < \mu_2 L(a,b) + (1-\mu_2)A(a,b),$$

$$TQ(a,b) > L_p(a,b)$$

$$\lambda_3\sqrt{L(a,b)I(a,b)} < TQ(a,b) < \mu_3\sqrt{L(a,b)I(a,b)},$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\lambda_1 \leq \sqrt{2/\pi}$, $\mu_1 \geq 1$, $\lambda_2 \geq 3/4$, $\mu_2 \leq 3/4$, $p \leq 3/2$, $\lambda_3 \leq \sqrt{e/\pi}$ and $\mu_3 \geq 1$, where $L_p(a,b) = [(b^p-a^p)/(p(\log b - \log a))]^{1/p}$ is the p-order logarithmic mean of a and b.

In [79], the authors proved that $p_1 = 0$, $q_1 = 1/4$, $p_2 = 0$ and $q_2 = 1/2-\sqrt{2}/4$ are the best possible parameters on the interval $[0,1/2]$ such that the double inequalities

$$H[p_1a+(1-p_1)b, p_1b+(1-p_1)a] < TQ(a,b) < H[q_1a+(1-q_1)b, q_1b+(1-q_1)a],$$

$$G[p_2a+(1-p_2)b, p_2b+(1-p_2)a] < TQ(a,b) < G[q_2a+(1-q_2)b, q_2b+(1-q_2)a]$$

hold for all $a,b>0$ with $a\neq b$.

The main purpose of the article is to present the best possible parameters α_1, α_2, α_3, β_1, β_2 and β_3 such that the double inequalities

$$A^{\alpha_1}(a,b)H^{1-\alpha_1}(a,b) < TQ(a,b) < \beta_1 A(a,b) + (1-\beta_1)H(a,b),$$

$$\frac{[\alpha_2 A(a,b)+(1-\alpha_2)H(a,b)]A(a,b)}{L(a,b)} < TQ(a,b) < \frac{[\beta_2 A(a,b)+(1-\beta_2)H(a,b)]A(a,b)}{L(a,b)},$$

$$\sqrt{[\alpha_3 L(a,b)+(1-\alpha_3)H(a,b)]A(a,b)} < TQ(a,b) < \sqrt{[\beta_3 L(a,b)+(1-\beta_3)H(a,b)]A(a,b)}$$

hold for all $a,b>0$ with $a\neq b$, and find the new bounds for the modified Bessel function $I_0(t)$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

Lemma 2.1. *(See [80, Theorem 2.18]) The identity*

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!}{2^{2n}n!}\sqrt{\pi}$$

holds for all $n\in\mathbb{N}$.

Lemma 2.2. *(See [81]) Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two real sequences with $b_n>0$ and $\lim_{n\to\infty}a_n/b_n=s$. Then the power series $\sum_{n=0}^{\infty}a_nt^n$ is convergent for all $t\in\mathbb{R}$ and*

$$\lim_{t\to\infty}\frac{\sum_{n=0}^{\infty}a_nt^n}{\sum_{n=0}^{\infty}b_nt^n}=s$$

if the power series $\sum_{n=0}^{\infty}b_nt^n$ is convergent for all $t\in\mathbb{R}$.

Lemma 2.3. *(See [82, Lemma 2.2]) The double inequality*

$$\frac{1}{(x+a)^{1-a}}<\frac{\Gamma(x+a)}{\Gamma(x+1)}<\frac{1}{x^{1-a}}$$

holds for all $x>0$ and $a\in(0,1)$.

Lemma 2.4. *(See [80, Theorem 1.25]) Let $a,b\in\mathbb{R}$ with $a<b$, $f,g:[a,b]\mapsto\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)},\qquad\frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotonic, then the monotonicity in the conclusion is also strict.

Lemma 2.5. *(See [83], [84, Lemma 2.1]) Let $A(t)=\sum_{k=0}^{\infty}a_kt^k$ and $B(t)=\sum_{k=0}^{\infty}b_kt^k$ be two real power series converging on $(-r,r)$ ($r>0$) with $b_k>0$ for all k. If the non-constant sequence $\{a_k/b_k\}_{k=0}^{\infty}$ is increasing (decreasing) for all k, then the function $t\mapsto A(t)/B(t)$ is strictly increasing (decreasing) on $(0,r)$.*

Lemma 2.6. *(See [85, (3.5)]) The identity*

$$I_\lambda(t)I_\mu(t)=\sum_{n=0}^{\infty}\frac{\Gamma(2n+\lambda+\mu+1)}{n!\Gamma(n+\lambda+\mu+1)\Gamma(n+\lambda+1)\Gamma(n+\mu+1)}\left(\frac{t}{2}\right)^{2n+\lambda+\mu}$$

holds for all $\lambda,\mu>-1$ and $t\in\mathbb{R}$.

Lemma 2.7. *The identities*

$$\cosh(t)I_0(t)=\sum_{n=0}^{\infty}\frac{(4n)!}{2^{2n}[(2n)!]^3}t^{2n},$$

$$\sinh(t)I_0(t)=\sum_{n=0}^{\infty}\frac{(4n+2)!}{2^{2n+1}[(2n+1)!]^3}t^{2n+1},$$

$$\cosh(t)I_1(t)=\sum_{n=0}^{\infty}\frac{(4n+1)!}{2^{2n+1}(n+1)(2n+1)[(2n)!]^3}t^{2n+1}$$

hold for all $t\in\mathbb{R}$, where $\sinh(t)=(e^t-e^{-t})/2$ and $\cosh(t)=(e^t+e^{-t})/2$ are the hyperbolic sine and cosine functions, respectively.

Proof. It follows from (1.4), and Lemmas 2.1 and 2.6 that

$$I_{-1/2}(t)=\sqrt{\frac{2}{\pi t}}\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}=\sqrt{\frac{2}{\pi t}}\cosh(t),$$

$$I_{1/2}(t)=\sqrt{\frac{2}{\pi t}}\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}=\sqrt{\frac{2}{\pi t}}\sinh(t),$$

$$\cosh(t)I_0(t)=\sqrt{\frac{\pi t}{2}}I_{-1/2}(t)I_0(t)$$

$$=\sqrt{\frac{\pi t}{2}}\sum_{n=0}^{\infty}\frac{\Gamma\left(2n+\frac{1}{2}\right)}{\left[n!\Gamma\left(n+\frac{1}{2}\right)\right]^2}\left(\frac{t}{2}\right)^{2n-1/2}=\sum_{n=0}^{\infty}\frac{(4n)!}{2^{2n}[(2n)!]^3}t^{2n},$$

$$\sinh(t)I_0(t)=\sqrt{\frac{\pi t}{2}}I_{1/2}(t)I_0(t)$$

$$=\sqrt{\frac{\pi t}{2}}\sum_{n=0}^{\infty}\frac{\Gamma\left(2n+\frac{3}{2}\right)}{\left[n!\Gamma\left(n+\frac{3}{2}\right)\right]^2}\left(\frac{t}{2}\right)^{2n+1/2}=\sum_{n=0}^{\infty}\frac{(4n+2)!}{2^{2n+1}[(2n+1)!]^3}t^{2n+1},$$

$$\cosh(t)I_1(t)=\sqrt{\frac{\pi t}{2}}I_{-1/2}(t)I_1(t)$$

$$=\sqrt{\frac{\pi t}{2}}\sum_{n=0}^{\infty}\frac{\Gamma\left(2n+\frac{3}{2}\right)}{(n+1)\left(n+\frac{3}{2}\right)\left[n!\Gamma\left(n+\frac{1}{2}\right)\right]^2}\left(\frac{t}{2}\right)^{2n+1/2}$$

$$=\sum_{n=0}^{\infty}\frac{(4n+1)!}{2^{2n+1}(n+1)(2n+1)[(2n)!]^3}t^{2n+1}.$$

□

Lemma 2.8. *The function $f(t)=\log[I_0(t)]/[\log\cosh(t)]$ is strictly increasing from $(0,\infty)$ onto $(1/2,1)$.*

Proof. Let $f_1(t)=\log[I_0(t)]$, $f_2(t)=\log\cosh(t)$, and a_n and b_n be defined by

$$a_n=\frac{(4n+1)!}{2^{2n+1}(n+1)(2n+1)[(2n)!]^3},\quad b_n=\frac{(4n+2)!}{2^{2n+1}[(2n+1)!]^3}. \tag{2.1}$$

Then from (1.4), Lemma 2.7 and (2.1) we clearly see that

$$f(t)=\frac{f_1(t)}{f_2(t)}=\frac{f_1(t)-f_1(0^+)}{f_2(t)-f_2(0^+)}, \tag{2.2}$$

$$\frac{a_n}{b_n}=1-\frac{1}{2(n+1)}, \tag{2.3}$$

$$\frac{a_0}{b_0}=\frac{1}{2},\quad \lim_{n\to\infty}\frac{a_n}{b_n}=1-\lim_{n\to\infty}\frac{1}{2(n+1)}=1, \tag{2.4}$$

$$\frac{f_1'(t)}{f_2'(t)}=\frac{\cosh(t)I_1(t)}{\sinh(t)I_0(t)}=\frac{\sum_{n=0}^{\infty}a_nt^{2n}}{\sum_{n=0}^{\infty}b_nt^{2n}}. \tag{2.5}$$

From Lemma 2.5, (2.3) and (2.5) we know that the function $f_1'(t)/f_2'(t)$ is strictly increasing on $(0,\infty)$. Then Lemma 2.4 and (2.2) lead to the conclusion that $f(t)$ is strictly increasing on $(0,\infty)$.

Therefore, Lemma 2.8 follows from Lemma 2.2 and (2.4) together with the monotonicity of $f(t)$. □

Lemma 2.9. *The function* $g(t)=[\cosh(t)I_0(t)-1]/[\cosh(2t)-1]$ *is strictly decreasing from* $(0,\infty)$ *onto* $(0,3/8)$.

Proof. Let $n\in\mathbb{N}$, and c_n and d_n be defined by

$$c_n=\frac{(4n+4)!}{2^{2n+2}[(2n+2)!]^3},\qquad d_n=\frac{2^{2n+2}}{(2n+2)!}. \tag{2.6}$$

Then Lemmas 2.1 and 2.7 together with (2.6) lead to

$$\frac{c_0}{d_0}=\frac{3}{8}, \tag{2.7}$$

$$\frac{c_n}{d_n}=\frac{(4n+4)!}{2^{4n+4}\Gamma(2n+3)(2n+2)!}=\frac{\Gamma\left(2n+\frac{5}{2}\right)}{\sqrt{\pi}\Gamma(2n+3)}, \tag{2.8}$$

$$g(t)=\frac{\sum_{n=0}^{\infty}\frac{(4n)!}{2^{2n}[(2n)!]^3}t^{2n}-1}{\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n)!}t^{2n}-1}=\frac{\sum_{n=0}^{\infty}c_nt^{2n}}{\sum_{n=0}^{\infty}d_nt^{2n}}, \tag{2.9}$$

$$\frac{c_{n+1}}{d_{n+1}}-\frac{c_n}{d_n}=-\frac{(n+2)(2n+3)(8n+13)(4n+4)!}{2^{4n+5}[(2n+4)!]^2}<0 \tag{2.10}$$

for all $n\in\mathbb{N}$.

It follows from Lemma 2.3 that

$$\frac{1}{\sqrt{\pi}\left(2n+\frac{5}{2}\right)^{1/2}}<\frac{\Gamma\left(2n+\frac{5}{2}\right)}{\sqrt{\pi}\Gamma(2n+3)}<\frac{1}{\sqrt{\pi}(2n+2)^{1/2}}. \tag{2.11}$$

From Lemma 2.2, Lemma 2.5 and (2.8)-(2.11) we know that $g(t)$ is strictly decreasing on $(0,\infty)$ and

$$\lim_{t\to\infty}g(t)=\lim_{n\to\infty}\frac{c_n}{d_n}=0. \tag{2.12}$$

Therefore, Lemma 2.9 follows from (2.7), (2.9), (2.12) and the monotonicity of the function $g(t)$ on the interval $(0,\infty)$. □

Lemma 2.10. *The function* $h(t)=[\sinh(t)I_0(t)-t]/[t\cosh(2t)-t]$ *is strictly decreasing from* $(0,\infty)$ *onto* $(0,5/24)$.

Proof. Let $n\in\mathbb{N}$, u_n and v_n be defined by

$$u_n=\frac{(4n+6)!}{2^{2n+3}[(2n+3)!]^3},\qquad v_n=\frac{2^{2n+2}}{(2n+2)!}. \tag{2.13}$$

Then from Lemma 2.1, Lemma 2.7 and (2.13) one has

$$\frac{u_0}{v_0}=\frac{5}{24}, \tag{2.14}$$

$$\frac{u_n}{v_n}=\frac{(4n+5)!}{2^{4n+4}[(2n+3)!]^2}=\frac{(4n+5)\Gamma\left(2n+\frac{5}{2}\right)}{\sqrt{\pi}(2n+3)^2\Gamma(2n+3)}, \tag{2.15}$$

$$h(t) = \frac{\sum_{n=0}^{\infty} \frac{(4n+2)!}{2^{2n+1}[(2n+1)!]^3} t^{2n+1} - t}{t\sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} t^{2n} - t} = \frac{\sum_{n=0}^{\infty} u_n t^{2n}}{\sum_{n=0}^{\infty} v_n t^{2n}}, \tag{2.16}$$

$$\frac{u_{n+1}}{v_{n+1}} - \frac{u_n}{v_n} = \frac{(4n+9)!}{2^{4n+8}[(2n+5)!]^2} - \frac{(4n+5)!}{2^{4n+4}[(2n+3)!]^2} \tag{2.17}$$
$$= -\frac{(n+2)(48n^2+202n+211)(4n+5)!}{2^{4n+5}[(2n+5)!]^2} < 0$$

for all $n \in \mathbb{N}$.

It follows from Lemma 2.3 that

$$\frac{(4n+5)}{\sqrt{\pi}(2n+3)^2\left(2n+\frac{5}{2}\right)^{1/2}} < \frac{(4n+5)\Gamma\left(2n+\frac{5}{2}\right)}{\sqrt{\pi}(2n+3)^2\Gamma(2n+3)} < \frac{(4n+5)}{\sqrt{\pi}(2n+3)^2(2n+2)^{1/2}}. \tag{2.18}$$

From Lemma 2.2, Lemma 2.5 and (2.15)-(2.18) we clearly see that $h(t)$ is strictly decreasing on $(0,\infty)$ and

$$\lim_{t\to\infty} h(t) = \lim_{n\to\infty} \frac{u_n}{v_n} = 0. \tag{2.19}$$

Therefore, Lemma 2.10 follows easily from (2.14), (2.16), (2.19) and the monotonicity of $h(t)$ on the interval $(0,\infty)$. □

Lemma 2.11. *The function* $\lambda(t) = [tI_0^2(t) - t]/[\sinh(2t) - 2t]$ *is strictly decreasing from* $(0,\infty)$ *onto* $(1/\pi, 3/8)$.

Proof. Let $n \in \mathbb{N}$, σ_n and τ_n be defined by

$$\sigma_n = \frac{(2n+2)!}{2^{2n+2}[(n+1)!]^4}, \qquad \tau_n = \frac{2^{2n+3}}{(2n+3)!}. \tag{2.20}$$

Then from Lemma 2.1, Lemma 2.3, Lemma 2.6, (2.20) one has

$$\frac{\sigma_0}{\tau_0} = \frac{3}{8}, \tag{2.21}$$

$$I_0^2(t) = \frac{(2n)!}{2^{2n}(n!)^4} t^{2n},$$

$$\lambda(t) = \frac{\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^4} t^{2n+1} - t}{\sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} t^{2n+1} - 2t} = \frac{\sum_{n=0}^{\infty} \sigma_n t^{2n}}{\sum_{n=0}^{\infty} \tau_n t^{2n}}, \tag{2.22}$$

$$\frac{\sigma_n}{\tau_n} = \frac{(2n+3)[(2n+2)!]^2}{2^{4n+5}[(n+1)!]^4}$$

$$= \frac{\left(n+\frac{3}{2}\right)}{\Gamma^2(n+2)}\left[\frac{(2n+2)!}{2^{2n+2}(n+1)!}\right]^2 = \frac{n+\frac{3}{2}}{\pi}\left[\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma(n+2)}\right]^2,$$

$$\frac{1}{\pi} < \frac{\sigma_n}{\tau_n} < \frac{n+\frac{3}{2}}{\pi(n+1)},$$

$$\lim_{n\to\infty} \frac{\sigma_n}{\tau_n} = \frac{1}{\pi}, \tag{2.23}$$

$$\frac{\sigma_{n+1}}{\tau_{n+1}} - \frac{\sigma_n}{\tau_n} = -\frac{(2n+3)(n+2)^2[(2n+2)!]^2}{2^{4n+7}[(n+2)!]^4} < 0 \tag{2.24}$$

for all $n \in \mathbb{N}$.

It follows from Lemma 2.2, Lemma 2.5 and (2.22)-(2.24) that $\lambda(t)$ is strictly decreasing on $(0,\infty)$ and

$$\lim_{t\to\infty}\lambda(t)=\frac{1}{\pi}. \tag{2.25}$$

Therefore, Lemma 2.11 follows easily from (2.21), (2.22), (2.25) and the monotonicity of the function $\lambda(t)$ on the interval $(0,\infty)$. □

3. Main Results

Theorem 3.1. *The double inequalities*

$$A^{\alpha_1}(a,b)H^{1-\alpha_1}(a,b)<TQ(a,b)<\beta_1A(a,b)+(1-\beta_1)H(a,b),$$

$$\frac{[\alpha_2A(a,b)+(1-\alpha_2)H(a,b)]A(a,b)}{L(a,b)}<TQ(a,b)$$
$$<\frac{[\beta_2A(a,b)+(1-\beta_2)H(a,b)]A(a,b)}{L(a,b)},$$

$$\sqrt{[\alpha_3L(a,b)+(1-\alpha_3)H(a,b)]A(a,b)}<TQ(a,b)$$
$$<\sqrt{[\beta_3L(a,b)+(1-\beta_3)H(a,b)]A(a,b)}$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leq 3/4$, $\beta_1\geq 3/4$, $\alpha_2\leq 0$, $\beta_2\geq 5/12$, $\alpha_3\leq 2/\pi$ and $\beta_3\geq 3/4$.

Proof. Since $H(a,b)$, $L(a,b)$, $A(a,b)$ and $TQ(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $b>a>0$. Let $t=\log\sqrt{b/a}>0$, then from (1.1)-(1.3) one has

$$H(a,b)=\frac{\sqrt{ab}}{\cosh(t)},\quad L(a,b)=\sqrt{ab}\frac{\sinh(t)}{t}, \tag{3.1}$$

$$TQ(a,b)=\sqrt{ab}I_0(t),\quad A(a,b)=\sqrt{ab}\cosh(t), \tag{3.2}$$

$$\frac{\log TQ(a,b)-\log H(a,b)}{\log A(a,b)-\log H(a,b)} \tag{3.3}$$
$$=\frac{\log I_0(t)+\log\cosh(t)}{2\log\cosh(t)}=\frac{1}{2}f(t)+\frac{1}{2},$$

$$\frac{TQ(a,b)-H(a,b)}{A(a,b)-H(a,b)}=\frac{I_0(t)\cosh(t)-1}{\cosh^2(t)-1}=2g(t), \tag{3.4}$$

$$\frac{TQ(a,b)L(a,b)-H(a,b)A(a,b)}{A^2(a,b)-H(a,b)A(a,b)} \tag{3.5}$$
$$=\frac{\sinh(t)I_0(t)-t}{t[\cosh^2(t)-1]}=2h(t),$$

$$\frac{TQ^2(a,b)-H(a,b)A(a,b)}{L(a,b)A(a,b)-H(a,b)A(a,b)} \tag{3.6}$$
$$=\frac{t[I_0^2(t)-1]}{\sinh(t)\cosh(t)-t}=2\lambda(t),$$

where, $f(t)$, $g(t)$, $h(t)$ and $\lambda(t)$ are given by Lemma 2.8, Lemma 2.9, Lemma 2.10 and Lemma 2.11, respectively.

Therefore, Theorem 3.1 follows easily from (3.3)-(3.6), and Lemma 2.8, Lemma 2.9, Lemma 2.10 and Lemma 2.11. □

From Theorem 3.1, (3.1) and (3.2), we get Corollary 3.2 immediately.

Corollary 3.2. *The double inequalities*

$$\cosh^{1/2}(t) < I_0(t) < \frac{3\cosh^2(t)+1}{4\cosh(t)},$$

$$\frac{t}{\sinh(t)} < I_0(t) < \frac{[5\cosh^2(t)+7]t}{12\sinh(t)},$$

$$\sqrt{\frac{\sinh(2t)}{\pi t}+1-\frac{2}{\pi}} < I_0(t) < \sqrt{\frac{3\sinh(2t)}{8t}+\frac{1}{4}}$$

for all $t > 0$.

References

[1] H.-Z. Xu, Y.-M. Chu and W.-M. Qian, Sharp bounds for the Sándor-Yang means in terms of arithmetic and contra-harmonic means, J. Inequal. Appl., 2018, **2018**, Article 127, 13 pages.

[2] W.-F. Xia, W. Janous and Y.-M. Chu, The optimal convex combination bounds of arithmetic and harmonic means in terms of power mean, J. Math. Inequal., 2012, **6**(2), 241–248.

[3] W.-M. Gong, Y.-Q. Song, M.-K. Wang and Y.-M. Chu, A sharp double inequality between Seiffert, arithmetic, and geometric means, Abstr. Appl. Anal., 2012, **2012**, Article ID 684834, 7 pages.

[4] Y.-M. Chu and M.-K. Wang, Inequalities between arithmetic-geometric, Gini, and Toader means, Abstr. Appl. Anal., 2012, **2012**, Article ID 830585, 11 pages.

[5] Y.-M. Chu, Y.-F. Qiu, M.-K. Wang and G.-D. Wang, The optimal convex combination bounds of arithmetic and harmonic means for the Seiffert's mean, J. Inequal. Appl., 2010, **2010**, Article ID 436457, 7 pages.

[6] X.-M. Zhang and Y.-M. Chu, Convexity of the integral arithmetic mean of a convex function, Rocky Mountain J. Math., 2010, **40**(3), 1061–1068.

[7] W.-F. Xia, Y.-M. Chu and G.-D. Wang, The optimal upper and lower power mean bounds for a convex combiantion of the arithmetic and logarithmic means, Abstr. Appl. Anal., 2010, **2010**, Article ID 604804, 9 pages.

[8] Y.-M. Chu, Y.-M. Li, W.-F. Xia and X.-H. Zhang, Best possible inequalities for the harmonic mean of error function, J. Inequal. Appl., 2014, **2014**, Article 525, 9 pages.

[9] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, Best possible inequalities among harmonic, geometric, logarithmic and Seiffert means, Math. Inequal. Appl., 2012, **15**(2), 415–422.

[10] Y.-M. Chu, W.-F. Xia and X.-H. Zhang, The Schur concavity, Schur multiplicative and harmonic convexities of the second dual form of the Hamy symmetric function with applications, J. Multivariate Anal., 2012, **105**, 412–421.

[11] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, A best-possible double inequality between Seiffert and harmonic means, J. Inequal. Appl., 2011, **2011**, Article 94, 7 pages.

[12] Y.-M. Chu and M.-K. Wang, Optimal inequalities betwen harmonic, geometric, logarithmic, and arithmetic-geometric means, J. Appl. Math., 2011, **2011**, Article ID 618929, 9 pages.

[13] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, A sharp double inequality between harmonic and identric means, Abstr. Appl. Anal., 2011, **2011**, Article ID 657935, 7 pages.

[14] Y.-M. Chu, S.-S. Wang and C. Zong, Optimal lower power mean bound for the convex combination of harmonic and logarithmic means, Abstr. Appl. Anal., 2011, **2011**, Article ID 520648, 9 pages.

[15] Y.-M. Chu, G.-D. Wang and X.-H. Zhang, The Schur multiplicative and harmonic convexities of the complete symmetric function, Math. Nachr., 2011, **284**(5-6), 653–663.

[16] Y.-M. Chu and T.-C. Sun, The Schur harmonic convexity for a class of symmetric functions, Acta Math. Sci., 2010, **30B**(5), 1501–1506.

[17] W.-M. Qian and Y.-M. Chu, Best possible bounds for Yang mean using generalized logarithmic mean, Math. Probl. Eng., 2016, **2016**, Article ID 8901258, 7 pages.

[18] Y.-M. Li, B.-Y. Long and Y.-M. Chu, Sharp bounds for the Neuman-Sándor mean in terms of generalized logarithmic mean, J. Math. Inequal., 2012, **6**(4), 567–577.

[19] Y.-M. Chu, M.-K. Wang and G.-D. Wang, The optimal generalized logarithmic mean bounds for Seiffert's mean, Acta Math. Sci., 2012, **32B**(4), 1619–1626.

[20] Y.-F. Qiu, M.-K. Wang, Y.-M. Chu and G.-D. Wang, Two sharp inequalities for Lehmer mean, identric mean and logarithmic mean, J. Math. Inequal., 2011, **5**(3), 301–306.

[21] Y.-M. Chu and W.-F. Xia, Two double inequalities between power mean and logarithmic mean, Comput. Math. Appl., 2010, **60**(1), 83–89.

[22] Y.-M. Chu and B.-Y. Long, Best possible inequalities between generalized logarithmic mean and classical means, Abstr. Appl. Anal., 2010, **2010**, Article ID 303286, 13 pages.

[23] Gh. Toader, Some mean values related to the arithmetic-geometric mean, J. Math. Anal. Appl., 1998, **218**(2), 358–368.

[24] F. Qi, X.-T. Shi, F.-F. Liu and Zh.-H. Yang, A double inequality for an integral mean in terms of the exponential and logarithmic mean, Perod. Math. Hungar., 2017, **75**(2), 180–189.

[25] A. O. Pittenger, Inequalities between arithmetic and logarithmic means, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., 1980, **678-715**, 15–18.

[26] W. Fechner, On some functional inequalities related to the logarithmic mean, Acta Math. Hungar., 2010, **128**(1-2), 36–45.

[27] Y.-M. Chu, M.-K. Wang and S.-L. Qiu, Optimal combinations bounds of root-square and arithmetic means for Toader mean, Proc. Indian Acad. Sci. Math. Sci., 2012, **122**(1), 41–51.

[28] Zh.-H. Yang, New bounds for identric mean in terms of logarithmic mean and arithmetic mean, J. Math. Inequal., 2012, **6**(4), 533–543.

[29] E. Neuman, Sharp inequalities involving Neuman-Sándor and logarithmic mean, J. Math. Inequal., 2013, **7**(3), 413–419.

[30] L. Matejíčka, Optimal convex combinations bounds of centroidal and harmonic means for weighted geometric mean of logarithmic and identric means, J. Math. Inequal., 2014, **8**(4), 939–945.

[31] J. Sándor and B. A. Bhayo, On some inequalities for the identric, logarithmci and related means, J. Math. Inequal., 2015, **9**(3), 889–896.

[32] B.-N. Gu and F. Qi, On inequalities for the exponential and logarithmic functions and means, Malays. J. Math. Sci., 2016, **10**(1), 23–34.

[33] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind, J. Math. Anal. Appl., 2018, **462**(2), 1714–1726.

[34] Zh.-H. Yang, W.-M. Qian and Y.-M. Chu, Monotonicity properties and bounds involving the complete elliptic integrals of the first kind, Math. Inequal. Appl., 2018, **21**(4), 1185–1199.

[35] T.-H. Zhao, M.-K. Wang, W. Zhang and Y.-M. Chu, Quadratic transformation inequalities for Gaussian hypergeometric function, J. Inequal. Appl., 2018, **2018**, Article 251, 15 pages.

[36] T.-R. Huang, S.-Y. Tan, X.-Y. Ma and Y.-M. Chu, Monotonicity properties and bounds for the complete p-elliptic integrals, J. Inequal. Appl., 2018, **2018**, Article 239, 11 pages.

[37] M.-K. Wang, S.-L. Qiu and Y.-M. Chu, Infinite series formula for Hübner upper bound function with applications to Hersch-Pfluger distortion function, Math. Inequal. Appl., 2018, **21**(3), 629–648.

[38] M.-K. Wang and Y.-M. Chu, Landen inequalities for a class of hypergeometric functions with applications, Math. Inequal. Appl., 2018, **21**(2), 521–537.

[39] W.-M. Qian and Y.-M. Chu, Sharp bounds for a special quasi-arithmetic mean in terms of arithmetic and geometric means with two parameters, J. Inequal. Appl., 2017, **2017**, Article 274, 10 pages.

[40] Zh.-H. Yang and Y.-M. Chu, A monotonicity properties involving the generalized elliptic integral of the first kind, 2017, **20**(3), 729–735.

[41] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, Monotonicity rule for the quotient of two function and its applications, J. Inequal. Appl., 2017, **2017**, Article 106, 13 pages.

[42] Y.-M. Chu and M.-K. Wang, Optimal Lehmer mean bounds for the Toader mean, Results Math., 2012, **61**(3-4), 223–229.

[43] M.-K. Wang, Y.-M. Chu and Y.-Q. Song, Asymptotical formulas for Gaussian and generalized hypergeometric functions, Appl. Math. Comput., 2016, **276**, 44–60.

[44] M.-K. Wang, Y.-M. Chu and Y.-P. Jiang, Ramanujan's cubic transformaation inequalities for zero-balanced hypergeometric functions, Rocky Mountain J. Math., 2016, **46**(2), 679–691.
[45] M.-K. Wang, Y.-M. Chu, Y.-F. Qiu and S.-L. Qiu, An optimal power mean inequality for the complete elliptic integrals, Appl. Math. Lett., 2011, **24**(6), 887–890.
[46] Y.-M. Chu, M.-K. Wang and Y.-F. Qiu, On Alzer and Qiu's conjecture for complete elliptic integral and inverse hyperbolic tangent function, Abstr. Appl. Anal., 2011, **2011**, Article ID 697547, 7 pages.
[47] M.-K. Wang and Y.-M. Chu, Asymptotical bounds for complete elliptic integrals of the second kind, J. Math. Anal. Appl., 2013, **402**(1), 119–126.
[48] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, Hölder mean inequalities for the complete elliptic integrals, Integral Transforms Spec. Funct., 2012, **23**(7), 521–527.
[49] Y.-M. Chu, M.-K. Wang, S.-L. Qiu and Y.-P. Jiang, Bounds for complete elliptic integrals of the second kind with applications, Comput. Math. Appl., 2012, **63**(7), 1177–1184.
[51] M.-K. Wang, Y.-M. Chu, S.-L. Qiu and Y.-P. Jiang, Convexity of the complete elliptic integrals of the first kind with respect to Hölder means, J. Math. Anal. Appl., 2012, **388**(2), 1141–1146.
[52] M.-K. Wang, S.-L. Qiu, Y.-M. Chu and Y.-P. Jiang, Generalized Hersch-Pfluger distortion function and complete elliptic integrals, J. Math. Anal. Appl., 2012, **385**(1), 221–229.
[53] A. Iqbal, M. Adil Khan, S. Ullah, Y.-M. Chu and A. Kashuri, Hermite-Hadamard type inequalities pertaining conformable fractional integrals and their applications, AIP Advances, 2018, **8**, Article ID 075101, 18 pages, DOI: 10.1063/1.5031954.
[54] Y.-M. Chu, M. Adil Khan, T. Ali and S. S. Dragomir, Inequalities for α-fractional differentiable functions, J. Inequal. Appl., 2017, **2017**, Article 93, 12 pages.
[55] M. Adil Khan, S. Begum, Y. Khurshid and Y.-M. Chu, Ostrowski type inequalities involving conformable fractional integrals, J. Inequal. Appl., 2018, **2018**, Article 70, 14 pages.
[56] M. Adil Khan, Y. Khurshid, T.-S. Du and Y.-M. Chu, Generalized of Hermite-Hadamard type inequalities via conformable fractional integrals, J. Funct. Spaces, 2018, **2018**, Article ID 5357463, 12 pages.
[57] M. Adil Khan, Y.-M. Chu, T. U. Khan and J. Khan, Some new inequalities of Hermite-Hadamard type for s-convex functions with applications, Open Math., 2017, **15**, 1414–1430.
[58] M. Adil Khan, Z. M. Al-sahwi and Y.-M. Chu, New estimations for Shannon and Zipf-Mandelbrot entropies, Entropy, 2018, **20**, Article 608, 10 pages, DOI: 10.3390/e2008608.
[59] M. Adil Khan, Y.-M. Chu, A. Kashuri and R. Liko, Hermite-Hadamard type fractional integral inequalities for $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions, J. Comput. Anal. Appl., 2019, **26**(8), 1487–1503.
[60] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, U. S. Government Printing Office, Washington, 1964.
[61] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Cambrigeg University Press, New York, 1962.
[62] M.-K. Wang, Y.-M. Li and Y.-M. Chu, Inequalities and infinite product formula for Ramanujan generalized modular equation function, Ramanujan J., 2018, **46**(1), 189–200.
[63] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On rational bounds for the gamma function, J. Inequal. Appl., 2017, **2017**, Article 210, 17 pages.
[64] T.-H. Zhao and Y.-M. Chu, A class of logarithmically completely monotonic functions associated with gamma function, J. Inequal. Appl., 2010, **2010**, Article ID 392431, 11 pages.
[65] T.-H. Zhao, Y.-M. Chu and H. Wang, Logarithmically complete monotonicity properties relating to the gamma function, Abstr. Appl. Anal., 2011, **2011**, Article ID 896483, 13 pages.
[66] T.-R. Huang, B.-W. Han, X.-Y. Ma and Y.-M. Chu, Optimal bounds for the generalized Euler-Mascheronic constant, J. Inequal. Appl., 2018, **2018**, Article 118, 9 pages.
[67] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the error function, Math. Inequal. Appl., 2018, **21**(2), 469–479.
[68] C. C. Maican, Integral Evaluations Using the Gamma and Beta Functions and Elliptic Integrals in Engineering, International Press, Cambridge, 2005.
[69] Zh.-H. Yang, W. Zhang and Y.-M. Chu, Sharp Gautschi inequality for parameter $0 < p < 1$ with applications, Math. Inequal. Appl., 2017, **20**(4), 1107–1120.
[70] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, An optimal double inequality between Seiffert and geoemtric means, J. Appl. Math., 2011, **2011**, Article ID 261237, 6 pages.

[71] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, Sharp power mean bounds for the combination of Seiffert and geoemtric means, Abstr. Appl. Anal., 2010, **2010**, Article ID 108920, 12 pages.
[72] B.-Y. Long and Y.-M. Chu, Optimal power mean bounds for the weighted geometric mean of classical means, J. Inequal. Appl., 2010, **2010**, Article ID 905697, 6 pages.
[73] Y.-M. Chu, B.-Y. Long and B.-Y. Liu, Bounds of the Neuman-Sándor mean using power and identric means, Abstr. Appl. Anal., 2013, **2013**, Article ID 832591, 6 pages.
[74] M.-K. Wang, Y.-M. Chu and Y.-F. Qiu, Some comparison inequalities for generalized Muirhead and identric means, J. Inequal. Appl., 2010, **2010**, Article ID 295620, 10 pages.
[75] M.-K. Wang, Z.-K. Wang and Y.-M. Chu, An double inequality between geometric and identric means, Appl. Math. Lett., 2012, **25**(3), 471–475.
[76] Zh.-H. Yang and Y.-M. Chu, On approximating the modified Bessel function of the first kind and Toader-Qi mean, J. Inequal. Appl., 2016, **2016**, Article 40, 21 pages.
[77] Zh.-H. Yang and Y.-M. Chu, A sharp lower bound for Toader-Qi mean with applications, J. Funct. Spaces, 2016, **2016**, Article ID 4165601, 5 pages.
[78] Zh.-H. Yang, Y.-M. Chu and Y.-Q. Song, Sharp bounds for Toader-Qi mean in terms of logarithmic and identric means, Math. Inequal. Appl., 2016, **19**(2), 721–730.
[79] W.-M. Qian, X.-H. Zhang and Y.-M. Chu, Sharp bounds for the Toader-Qi mean in terms of harmonic and geometric means, J. Math. Inequal., 2017, **11**(1), 121–127.
[80] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Conformal Invariants, Inequalities, and Quasiconformal Maps, John Wiley & Sons, New York, 1997.
[81] G. Pólya and G. Szegő, Problems and Theorems in Analysis I, Springer-Verlag, Berlin, 1998.
[82] Zh.-H. Yang, Y.-M. Chu and W. Zhang, Accurate approximations for the complete elliptic of the second kind, J. Math. Anal. Appl., 2016, **438**(2), 875–888.
[83] M. Biernacki and J. Krzyż, On the monotonicity of certain functionals in the theory of analytic functions, Ann. Univ. Mariae Curie-Skłodowska. Sect. A, 1955, **9**, 135–147.
[84] S. Ponnusamy and M. Vuorinen, Asymptotic expansions and inequalities for hypergeometric functions, Mathematika, 1997, **44**(2), 278–301.
[85] V. R. Thiruvenkatachar and T. S. Nanjundiah, Inequalities concerning Bessel functions and orghogonal polynomials, Proc. Indian Acad. Sci., Sect. A, 1951, **33**, 373–384.

Wen-Mao Qian, [1]College of Science, Hunan City University, Yiyang 413000, Hunan, China; [2]School of Continuing Education, Huzhou Vocational and Technological College, Huzhou 313000, Zhejiang, China
E-mail address: qwm661977@126.com

Wen Zhang, [3]Friedman Brain Institute, Icahn School of Medicine at Mount Sinai, New York, NY 10029, USA
E-mail address: zhang.wen81@gmail.com

Yu-Ming Chu (Corresponding author),[4]Department of Mathematics, Huzhou University, Huzhou 413000, Zhejiang, China
E-mail address: chuyuming2005@126.com

Symmetric identities for Dirichlet-type multiple twisted (h, q)-l-function and higher-order generalized twisted (h, q)-Euler polynomials

C. S. Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Korea

Abstract : In this paper we investigate some interesting symmetric identities for multiple twisted (h, q)-l-function and higher-order generalized twisted (h, q)-Euler polynomials in complex field.

Key words : Symmetric properties, power sums, Euler numbers and polynomials, multiple twisted (h, q)-l-function, higher-order generalized twisted (h, q)-Euler numbers and polynomials.

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics, mathematical physics and statistical physics. Many mathematicians have studied in the area of the q- extension of Euler numbers and polynomials(see [1-10]). Y. He studied several identities of symmetry for Carlitz's q-Bernoulli numbers and polynomials in complex field(see [2]). D. Kim *et al.*[3] derived some identities of symmetry for (h, q)-extension of higher-order Euler numbers and polynomials. D. V. Dolgy *et al.*[1] derived some identities of symmetry for higher-order generalized q-Euler polynomials. In this paper, we present a systemic study of the generalized twisted (h, q)-Euler numbers and polynomials of higher-order by using the multiple twisted(h, q)-l-function. Throughout this paper, the notations $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ denote the sets of positive integers, integers, real numbers, and complex numbers, respectively, and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We assume that $q \in \mathbb{C}$ with $|q| < 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q} \text{ (cf. [1, 2, 3, 5]) }.$$

Note that $\lim_{q\to 1}[x] = x$. Let χ be a Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$ and ε be the p^N-th root of unity(see [8, 9, 10]). T. Kim introduced the multiple q-Euler zeta function which interpolates higher-order q-Euler polynomials at negative integers as follows(see [4, 5]):

$$\zeta_{q,r}(s,x) = [2]_q^r \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(-1)^{\sum_{j=1}^r m_j} q^{\sum_{j=1}^r m_j}}{[m_1+\cdots+m_r+x]_q^s}, \tag{1}$$

where $s \in \mathbb{C}$ and $x \in \mathbb{R}$, with $x \neq 0, -1, -2, \ldots$.

Recently, D. V. Dolgy *et al.*[1] considered some symmetric identities for higher-order generalized q-Euler polynomials. The generalized Euler polynomials of order $r \in \mathbb{N}$ attached to χ are also defined by the generating function:

$$\left(2\sum_{l=0}^{d-1} \frac{\chi(l)(-1)^l e^{(x+l)t}}{e^{dt}+1}\right)^r = \sum_{m=0}^{\infty} E_{m,\chi}^{(r)}(x)\frac{t^m}{m!}. \tag{2}$$

When $x = 0, E_{n,\chi}^{(r)} = E_{n,\chi}^{(r)}(0)$ are called the generalized Euler numbers $E_{n,\chi}^{(r)}$ attached to χ.

For $h \in \mathbb{Z}, \alpha, k \in \mathbb{N}$, and $n \in \mathbb{Z}_+$, we introduced the higher order twisted q-Euler polynomials with weight α as follows(see [7]):

$$\widetilde{E}_{n,q,\varepsilon}^{(\alpha)}(h,k|x) = \frac{[2]_q^k}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{(1+\varepsilon q^{\alpha l+h}) \cdots (1+\varepsilon q^{\alpha l+h-k+1})}.$$

In the special case, $x = 0$, $\widetilde{E}_{n,q,\varepsilon}^{(\alpha)}(h,k|0) = \widetilde{E}_{n,q,\varepsilon}^{(\alpha)}(h,k)$ are called the higher-order twisted q-Euler numbers with weight α.

We consider the higher order generalized q-Euler polynomials of order r attached to χ twisted by ramified roots of unity as follows(see [8]):

$$\sum_{n=0}^{\infty} E_{n,\chi,\zeta,q}^{(r)}(x) \frac{t^n}{n!} = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-\zeta)^{\sum_{j=0}^{r} m_j} \left(\prod_{i=1}^{r} \chi(m_i) \right) e^{[x+\sum_{j=1}^{r} m_j]_q t}.$$

In the special case $x = 0$, the sequence $E_{n,\chi,\zeta,q}^{(r)}(0) = E_{n,\chi,\zeta,q}^{(r)}$ are called the n-th generalized q-Euler numbers of order r attached to χ twisted by ramified roots of unity.

As is well known, the higher-order generalized twisted (h,q)-Euler polynomials $E_{n,\chi,q,\varepsilon}^{(h,k)}(x)$ attached to χ are defined by the following generating function to be

$$\begin{aligned}
\widetilde{F}_{\chi,q,\varepsilon}^{(h,k)}(t,x) = [2]_q^k & \sum_{m_1,\cdots,m_k=0}^{\infty} (-1)^{m_1+\cdots+m_k} q^{\sum_{j=1}^{k}(h-j+1)m_j} \varepsilon^{m_1+\cdots+m_k} \\
& \times \left(\prod_{j=1}^{k} \chi(m_j) \right) e^{[m_1+\cdots+m_k+x]_q t} \\
= \sum_{n=0}^{\infty} & E_{n,\chi,q,\varepsilon}^{(h,k)}(x) \frac{t^n}{n!},
\end{aligned} \tag{3}$$

where $h \in \mathbb{Z}$ and $k \in \mathbb{N}$. When $x = 0, E_{n,\chi,q,\varepsilon}^{(h,k)} = E_{n,\chi,q,\varepsilon}^{(h,k)}(0)$ are called the higher-order generalized twisted (h,q)-Euler numbers $E_{n,\chi,q,\varepsilon}^{(h,k)}$ attached to χ. Observe that if $q \to 1, \varepsilon \to 1$, then $E_{n,\chi,q,\varepsilon}^{(h,k)} \to E_{n,\chi}^{(k)}$ and $E_{n,\chi,q,\varepsilon}^{(h,k)}(x) \to E_{n,\chi}^{(k)}(x)$. By using (3) and Cauchy product, we have

$$E_{n,\chi,q,\varepsilon}^{(h,k)}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} E_{l,\chi,q,\varepsilon}^{(h,k)} [x]_q^{n-l} = (q^x E_{\chi,q,\varepsilon}^{(h,k)} + [x]_q)^n, \tag{4}$$

with the usual convention about replacing $(E_{\chi,q,\varepsilon}^{(h,k)})^n$ by $E_{n,\chi,q,\varepsilon}^{(h,k)}$.

By using complex integral and (3), we can also obtain the Dirichlet-type multiple twisted (h,q)-l-function as follows:

$$\begin{aligned}
l_{\chi,q,\varepsilon}^{(h,k)}(s,x) &= \frac{1}{\Gamma(s)} \int_0^\infty \widetilde{F}_{\chi,q,\varepsilon}^{(h,k)}(-t,x) t^{s-1} dt \\
&= [2]_q^k \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^{k} m_j} \left(\prod_{j=1}^{k} \chi(m_j) \right) q^{\sum_{j=1}^{k}(h-j+1)m_j} \varepsilon^{\sum_{j=1}^{k} m_j}}{[m_1+\cdots+m_k+x]_q^s},
\end{aligned} \tag{5}$$

where $s \in \mathbb{C}$ and $x \in \mathbb{R}$, with $x \neq 0, -1, -2, \ldots$.

By using Cauchy residue theorem, the value of Dirichlet-type multiple twisted (h,q)-l-function at negative integers is given explicitly by the following theorem:

Theorem 1. Let $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. We obtain

$$l_{\chi,q,\varepsilon}^{(h,k)}(-n,x) = E_{n,\chi,q,\varepsilon}^{(h,k)}(x).$$

The purpose of this paper is to obtain some interesting identities of the power sums and the higher-order generalized twisted (h,q)-Euler polynomials $E^{(h,k)}_{n,\chi,q,\varepsilon}(x)$ attached to χ using the symmetric properties for Dirichlet-type multiple twisted (h,q)-l-function. In this paper, if we take $\chi^0=1, \varepsilon=1$, then [3] is the special case of this paper. If we take $\varepsilon=1$ in all equations of this article, then [1] are the special case of our results.

2. Symmetry identities for Dirichlet-type multiple twisted (h,q)-l-function

In this section, by using the similar method of [1, 2, 3], expect for obvious modifications, we investigate some symmetric identities for higher-order generalized twisted (h,q)-Euler polynomials $E^{(h,k)}_{n,\chi,q,\varepsilon}(x)$ attached to χ using the symmetric properties for Dirichlet-type multiple twisted (h,q)-l-function. We assume that χ is a Dirichlet character with conductor $d\in\mathbb{N}$ with $d\equiv 1 \pmod 2$ and ε be the p^N-th root of unity. Let $w_1,w_2\in\mathbb{N}$ with $w_1\equiv 1 \pmod 2$, $w_2\equiv 1 \pmod 2$. For $h\in\mathbb{Z}, k\in\mathbb{N}$ and $n\in\mathbb{Z}_+$, we obtain certain symmetry identities for Dirichlet-type multiple twisted (h,q)-l-function.

Observe that $[xy]_q=[x]_{q^y}[y]_q$ for any $x,y\in\mathbb{C}$. In (5), we derive next result by substitute $w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)$ for x in and replace q and ε by q^{w_1} and ε^{w_1}, respectively.

$$\begin{aligned}
&\frac{1}{[2]^k_{q^{w_1}}} l^{(h,k)}_{\chi,q^{w_1},\varepsilon^{w_1}}(s, w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k))\\
&=\sum_{m_1,\cdots,m_k=0}^{\infty}\frac{(-1)^{\sum_{j=1}^k m_j}\left(\prod_{j=1}^k\chi(m_j)\right)q^{w_1\sum_{j=1}^k(h-j+1)m_j}\varepsilon^{w_1\sum_{j=1}^k m_j}}{\left[\frac{w_1(m_1+\cdots+m_k)+w_1w_2x+w_2(j_1+\cdots+j_k)}{w_1}\right]^s_{q^{w_1}}}\\
&=\sum_{m_1,\cdots,m_k=0}^{\infty}\frac{(-1)^{\sum_{j=1}^k m_j}\left(\prod_{j=1}^k\chi(m_j)\right)q^{w_1\sum_{j=1}^k(h-j+1)m_j}\varepsilon^{w_1\sum_{j=1}^k m_j}}{\frac{[w_1(m_1+\cdots+m_k)+w_1w_2x+w_2(j_1+\cdots+j_k)]^s_q}{[w_1]^s_q}}\\
&=[w_1]^s_q\sum_{m_1,\cdots,m_k=0}^{\infty}\sum_{i_1,\cdots,i_k=0}^{dw_2-1}\frac{(-1)^{\sum_{j=1}^k m_j}\left(\prod_{j=1}^k\chi(m_j)\right)q^{w_1\sum_{j=1}^k(h-j+1)m_j}\varepsilon^{w_1\sum_{j=1}^k m_j}}{[w_1(m_1+\cdots+m_k)+w_1w_2x+w_2(j_1+\cdots+j_k)]^s_q}\\
&=[w_1]^s_q\sum_{m_1,\cdots,m_k=0}^{\infty}\sum_{i_1,\cdots,i_k=0}^{dw_2-1}(-1)^{\sum_{j=1}^k m_j}(-1)^{\sum_{j=1}^k i_j}\left(\prod_{j=1}^k\chi(i_j)\right)\\
&\quad\times q^{dw_1w_2\sum_{j=1}^k(h-j+1)m_j}q^{w_1\sum_{j=1}^k(h-j+1)i_j}\varepsilon^{dw_1w_2\sum_{j=1}^k m_j}\varepsilon^{w_1\sum_{j=1}^k i_j}\\
&\quad\times\left([w_1w_2(x+dm_1+\cdots+dm_k)+w_1(i_1+\cdots+i_k)+w_2(j_1+\cdots+j_k)]^s_q\right)^{-1}
\end{aligned}\tag{6}$$

Thus, from (6), we can derive the following equation.

$$\begin{aligned}
&\frac{[w_2]^s_q}{[2]^k_{q^{w_1}}}\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}\\
&\qquad\times l^{(h,k)}_{\chi,q^{w_1},\varepsilon^{w_1}}(s, w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k))\\
&=[w_1]^s_q[w_2]^s_q\sum_{m_1,\cdots,m_k=0}^{\infty}\sum_{i_1,\cdots,i_k=0}^{dw_2-1}\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k(j_l+i_l+m_l)}\left(\prod_{l=1}^k\chi(j_l)\right)\left(\prod_{l=1}^k\chi(i_l)\right)\\
&\quad\times q^{dw_1w_2\sum_{l=1}^k(h-l+1)m_l}q^{w_1\sum_{l=1}^k(h-l+1)i_l}q^{w_2\sum_{l=1}^k(h-l+1)j_l}\\
&\quad\times\varepsilon^{dw_1w_2\sum_{l=1}^k m_l}\varepsilon^{w_1\sum_{l=1}^k i_l}\varepsilon^{w_2\sum_{l=1}^k j_l}\\
&\quad\times\left([w_1w_2(x+dm_1+\cdots+dm_k)+w_1(i_1+\cdots+i_k)+w_2(j_1+\cdots+j_k)]^s_q\right)^{-1}
\end{aligned}\tag{7}$$

By using the same method as (7), we have

$$\begin{aligned}
&\frac{[w_1]_q^s}{[2]_{q^{w_2}}^k}\sum_{j_1,\cdots,j_k=0}^{dw_2-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_1\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_1\sum_{l=1}^k j_l}\\
&\qquad\times l^{(h,k)}_{\chi,q^{w_2},\varepsilon^{w_2}}(s,w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k))\\
&=[w_1]_q^s[w_2]_q^s\sum_{m_1,\cdots,m_k=0}^{\infty}\sum_{j_1,\cdots,j_k=0}^{dw_2-1}\sum_{i_1,\cdots,i_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k(j_l+i_l+m_l)}\left(\prod_{l=1}^k\chi(j_l)\right)\left(\prod_{l=1}^k\chi(i_l)\right)\\
&\quad\times q^{dw_1w_2\sum_{l=1}^k(h-l+1)m_l}q^{w_2\sum_{l=1}^k(h-l+1)i_l}q^{w_1\sum_{l=1}^k(h-l+1)j_l}\\
&\quad\times\varepsilon^{dw_1w_2\sum_{l=1}^k m_l}\varepsilon^{w_2\sum_{l=1}^k i_l}\varepsilon^{w_1\sum_{l=1}^k j_l}\\
&\quad\times\left([w_1w_2(x+dm_1+\cdots+dm_k)+w_1(j_1+\cdots+j_k)+w_2(i_1+\cdots+i_k)]_q^s\right)^{-1}
\end{aligned}\tag{8}$$

Therefore, by (7) and (8), we have the following theorem.

Theorem 2. Let $w_1,w_2\in\mathbb{N}$ with $w_1\equiv 1\pmod 2$, $w_2\equiv 1\pmod 2$. For $h\in\mathbb{Z}$, we obtain

$$\begin{aligned}
&[w_2]_q^s[2]_{q^{w_2}}^k\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}\\
&\qquad\times l^{(h,k)}_{\chi,q^{w_1},\varepsilon^{w_1}}\left(s,w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\
&[w_1]_q^s[2]_{q^{w_1}}^k\sum_{j_1,\cdots,j_k=0}^{dw_2-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_1\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_1\sum_{l=1}^k j_l}\\
&\qquad\times l^{(h,k)}_{\chi,q^{w_2},\varepsilon^{w_2}}\left(s,w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k)\right)
\end{aligned}\tag{9}$$

By (9) and Theorem 1, we obtain the following theorem.

Theorem 3. Let $w_1,w_2\in\mathbb{N}$ with $w_1\equiv 1\pmod 2$, $w_2\equiv 1\pmod 2$. For $h\in\mathbb{Z},k\in\mathbb{N}$ and $n\in\mathbb{Z}_+$, we obtain

$$\begin{aligned}
&[w_2]_q^s[2]_{q^{w_2}}^k\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}\\
&\qquad\times E^{(h,k)}_{n,\chi,q^{w_1},\varepsilon^{w_1}}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\
&=[w_1]_q^s[2]_{q^{w_1}}^k\sum_{j_1,\cdots,j_k=0}^{dw_2-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_1\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_1\sum_{l=1}^k j_l}\\
&\qquad\times E^{(h,k)}_{n,\chi,q^{w_2},\varepsilon^{w_2}}\left(w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k)\right).
\end{aligned}\tag{10}$$

From (4), we note that

$$E^{(h,k)}_{n,\chi,q,\varepsilon}(x+y)=(q^{x+y}E^{(h,k)}_{n,\chi,q,\varepsilon}+[x+y]_q)^n=\sum_{i=0}^n\binom{n}{i}q^{xi}E^{(h,k)}_{i,\chi,q,\varepsilon}(y)[x]_q^{n-i}.\tag{11}$$

with the usual convention about replacing $(E^{(h,k)}_{\chi,q,\varepsilon})^n$ by $E^{(h,k)}_{n,\chi,q,\varepsilon}$.

By (11), we have

$$\begin{aligned}
&\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}E_{n,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\
&=\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}\\
&\times\sum_{i=0}^n\binom{n}{i}q^{w_2i(j_1+\cdots+j_k)}E_{i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(w_2x)\left[\frac{w_2}{w_1}(j_1+\cdots+j_k)\right]_{q^{w_1}}^{n-i}\\
&=\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}\\
&\times\sum_{i=0}^n\binom{n}{i}q^{w_2(n-i)\sum_{l=1}^k j_l}E_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(w_2x)\left[\frac{w_2}{w_1}(j_1+\cdots+j_k)\right]_{q^{w_1}}^{i}
\end{aligned}\tag{12}$$

Hence we have the following theorem.

Theorem 4. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}
&\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}E_{n,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\
&=\sum_{i=0}^n\binom{n}{i}[w_2]_q^i[w_1]_q^{-i}E_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(w_2x)\\
&\times\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(n+h-l-i+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}[j_1\cdots+j_k]_{q^{w_2}}^i.
\end{aligned}$$

For each integer $n \geq 0$, let

$$\mathcal{S}_{n,i,\chi,q,\varepsilon}^{(h,k)}(w)=\sum_{j_1,\cdots,j_k=0}^{w-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{\sum_{l=1}^k(n+h-l-i+1)j_l}\varepsilon^{\sum_{l=1}^k j_l}[j_1\cdots+j_k]_q^i.$$

The above sum $\mathcal{S}_{n,i,\chi,q,\varepsilon}^{(h,k)}(w)$ is called the alternating generalized (h,q)-power sums.

By Theorem 4, we have

$$\begin{aligned}
&[2]_{q^{w_2}}^k[w_1]_q^n\sum_{j_1,\cdots,j_k=0}^{dw_1-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_2\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_2\sum_{l=1}^k j_l}\\
&\qquad\times E_{n,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}\left(w_2x+\frac{w_2}{w_1}(j_1+\cdots+j_k)\right)\\
&=[2]_{q^{w_2}}^k\sum_{i=0}^n\binom{n}{i}[w_2]_q^i[w_1]_q^{n-i}E_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(w_2x)\mathcal{S}_{n,i,\chi,q^{w_2},\varepsilon^{w_2}}^{(h,k)}(dw_1)
\end{aligned}\tag{13}$$

By using the same method as in (13), we obtain

$$\begin{aligned}
&[2]_{q^{w_1}}^k[w_2]_q^n\sum_{j_1,\cdots,j_k=0}^{dw_2-1}(-1)^{\sum_{l=1}^k j_l}\left(\prod_{l=1}^k\chi(j_l)\right)q^{w_1\sum_{l=1}^k(h-l+1)j_l}\varepsilon^{w_1\sum_{l=1}^k j_l}\\
&\qquad\times E_{n,\chi,q^{w_2},\varepsilon^{w_2}}^{(h,k)}\left(w_1x+\frac{w_1}{w_2}(j_1+\cdots+j_k)\right)\\
&=[2]_{q^{w_1}}^k\sum_{i=0}^n\binom{n}{i}[w_1]_q^i[w_2]_q^{n-i}E_{n-i,\chi,q^{w_2},\varepsilon^{w_2}}^{(h,k)}(w_1x)\mathcal{S}_{n,i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(dw_2)
\end{aligned}\tag{14}$$

Therefore, by (13), (14), and Theorem 3, we have the following theorem.

Theorem 5. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}&[2]_{q^{w_2}}^k \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} E_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(w_2 x) \mathcal{S}_{n,i,\chi,q^{w_2},\varepsilon^{w_2}}^{(h,k)}(dw_1)\\ &= [2]_{q^{w_1}}^k \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} E_{n-i,\chi,q^{w_2},\varepsilon^{w_2}}^{(h,k)}(w_1 x) \mathcal{S}_{n,i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(dw_2).\end{aligned}$$

By Theorem 5, we obtain the interesting symmetric identity for the higher-order generalized twisted (h,q)-Euler numbers $E_{n,\chi,q,\varepsilon}^{(h,k)}$ in complex field.

Corollary 6. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}&[2]_{q^{w_2}}^k \sum_{i=0}^{n} \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} \mathcal{S}_{n,i,\chi,q^{w_2},\varepsilon^{w_2}}^{(h,k)}(dw_1) E_{n-i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}\\ &= [2]_{q^{w_1}}^k \sum_{i=0}^{n} \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} \mathcal{S}_{n,i,\chi,q^{w_1},\varepsilon^{w_1}}^{(h,k)}(dw_2) E_{n-i,\chi,q^{w_2},\varepsilon^{w_2}}^{(h,k)}.\end{aligned}$$

REFERENCES

1. D. V. Dolgy, D.S. Kim, T.G. Kim, J.J. Seo, Identities of Symmetry for Higher-Order Generalized q-Euler Polynomials, Abstract and Applied Analysis, 2014(2014), Article ID 286239, 6 pages.
2. Yuan He, Symmetric identities for Carlitz's q-Bernoulli numbers and polynomials, Adv. Difference Equ., 246(2013), 10 pages.
3. D. Kim, T. Kim, J.-J. Seo, Identities of symmetric for (h,q)-extension of higher-order Euler polynomials, Applied Mathemtical Sciences 8 (2014), 3799-3808.
4. T. Kim, New approach to q-Euler polynomials of higher order, Russ. J. Math. Phys. 17(2010), 218-225.
5. T. Kim, Barnes type multiple q-zeta function and q-Euler polynomials, J. phys. A : Math. Theor. 43(2010) 255201(11pp).
6. H. Y. Lee, N. S. Jung, J. Y. Kang, C. S. Ryoo, Some identities on the higher-order-twisted q-Euler numbers and polynomials with weight α, Adv. Difference Equ., 2012:21(2012), 10pp.
7. E.-J. Moon, S.-H. Rim, J.-H. Jin, S.-J. Lee, On the symmetric properties of higher-order twisted q-Euler numbers and polynomials, Adv. Difference Equ., 2010, Art ID 765259, 8pp.
8. C. S. Ryoo, On the generalized Barnes type multiple q-Euler polynomials twisted by ramified roots of unity, Proc. Jangjeon Math. Soc. 13(2010), 255-263.
9. C. S. Ryoo, A note on the weighted q-Euler numbers and polynomials, Adv. Stud. Contemp. Math., 21(2011), 47-54.
10. Y. Simsek, q-analogue of twisted l-series and q-twisted Euler numbers, Journal of Number Theory, 110(2005), 267-278.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

An efficient m-step Levenberg-Marquardt method for systems of nonlinear equations*

Liang Chen†

School of Mathematical Sciences,
Huaibei Normal University, Huaibei 235000, P.R. China

Yanfang Ma

School of Computer Science and Technology,
Huaibei Normal University, Huaibei, Anhui 235000, PR China

Abstract

In this paper, we propose an efficient m-step Levenberg-Marquardt method for systems of nonlinear equations. At every iteration, the efficient m-step LM method computes not only the classical LM step, but also $m-1$ approximate LM steps with frozen $(J_k^T J_k+\lambda_k I)^{-1}J_k^T$. Also, we employ $m-1$ line searches for $m-1$ approximate LM steps for better numerical performance. Under the local error bound condition which is weaker than nonsingularity, the efficient m-step LM method has been proved to have $(m+1)$th convergence order. The global convergence has also been given by trust region technique. Numerical results show that the efficient m-step LM method is efficient and could save many calculations of the Jacobian especially for large scale problems.

Keywords: Unconstrained optimization; Systems of nonlinear equations; Levenberg-Marquardt method; Trust region

MSC2010: 65K05; 90C30

1 Introduction

It's a well-known problem in science and engineering that is to find the solutions of systems of nonlinear equations

$$F(x)=0, \tag{1}$$

where $F: D\subset \mathrm{R}^n \to \mathrm{R}^n$ is continuously differentiable function. Due to the nonlinearity of $F(x)$, (1) may have no solutions. Throughout the paper, we let that the solution set of (1) is nonempty and denote it by X^*, and in all cases $\|\cdot\|$ refers to the 2-norm.

There are many numerical methods to approximate the solutions of (1) because the exact solutions is difficult to find. A classical numerical method is Newton method which computes the trial step

$$d_k^N=-J_k^{-1}F_k$$

at every iteration, where $F_k=F(x_k)$ and $J_k=F'(x_k)$ is the Jacobian. And the Newton method has quadratic rate of convergence under the condition that $J(x)$ is Lipschitz continuous and nonsingular at the solution of (1). However, the Newton method will be failed when J_k is singular or near singular. To overcome these disadvantages, a large number of researchers have presented many modifications of Newton

*The work is supported by the Anhui Provincial Natural Science Foundation (1508085MA14, 1708085MF159), the Natural Science Foundation of the Anhui Higher Education Institutions (KJ2017A375) and the Major Teaching Reform Project of Anhui Higher Education Revitalization Plan (2014ZDJY058).

†Corresponding author. Email: clmyf2@163.com, chenliang1977@gmail.com. Tel: +86 157 5613 7533

method [1]. One of them is the Levenberg-Marquardt method (LM) [2, 3], which is a famous numerical method with computing the linear equation

$$\left(J_k^T J_k + \lambda_k I\right) d = -J_k^T F_k \tag{2}$$

to obtain the LM trail step

$$d_k = -\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_k \tag{3}$$

at every iteration, where $\lambda_k \geqslant 0$ is the LM parameter. It is well-known that the LM method has quadratic convergence as the Newton method if the Jacobian matrix is nonsingular and Lipschitz continuous at the solution. A large number of researchers have focused on this system and many efficient solution techniques are available [4–7].

As we all known, the cost of Jacobian computations is expensive when $F(x)$ is complicated or n is quite large. Recently, to save Jacobian calculations and achieve a fast convergence rate, Fan [8] presented a modified Levenberg-Marquardt method (MLM) with cubic convergence. At every iteration, the MLM method solves not only the linear equations (2) to obtain the LM step (3), but also the linear equations

$$\left(J_k^T J_k + \lambda_k I\right) d = -J_k^T F_{k,1}$$

to obtain the approximate LM step

$$d_{k,1} = -\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_{k,1} \tag{4}$$

with $F_{k,1} = F\left(x_{k,1}\right)$, $x_{k,1} = x_k + d_k$, $\lambda_k = \mu_k \left\|F_k\right\|^\delta$, $\mu_k > 0$ and $\delta \in [1, 2]$, and the trial step is

$$s_k^{MLM} = d_k + d_{k,1}.$$

Fan use $\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T$ in stead of

$$\left(J\left(x_{k,1}\right)^T J\left(x_{k,1}\right) + \mu_{k+1} \left\|F\left(x_{k,1}\right)\right\|^\delta I\right)^{-1} J\left(x_{k,1}\right)^T \tag{5}$$

in (4), which does not involve the calculation of $J\left(x_{k,1}\right)$. Since J_k has been used in (3), the cost of Jacobian calculations will be saved.

Similarly, to save more Jacobian calculations, based on the MLM method, Yang [9] presented a high-order Levenberg-Marquardt method (HLM) with biquadratic convergence by solving another linear equations

$$\left(J_k^T J_k + \lambda_k I\right) d = -J_k^T F_{k,2} \tag{6}$$

to obtain another approximate LM step

$$d_{k,2} = -\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_{k,2} \tag{7}$$

with $F_{k,2} = F\left(x_{k,2}\right)$, $x_{k,2} = x_{k,1} + d_{k,1}$, $\lambda_k = \mu_k \left\|F_k\right\|^\delta$, $\mu_k > 0$ and $\delta \in [1, 2]$. Yang still use $\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T$ in stead of (5) in (4), $\left(J\left(x_{k,2}\right)^T J\left(x_{k,2}\right) + \mu_{k+2} \left\|F\left(x_{k,2}\right)\right\|^\delta I\right)^{-1} J\left(x_{k,2}\right)^T$ in (7) respectively, which does not need to compute $J\left(x_{k,1}\right)$ and $J\left(x_{k,2}\right)$. The trial step of the HLM method is

$$s_k^{HLM} = d_k + d_{k,1} + d_{k,2}.$$

Furthermore, to save more Jacobian calculations and achieve a faster convergence rate, Fan [10] presented a Shamanskii-like Levenberg-Marquardt (SLM) method with $(m+1)$th convergence by solving $m-1$ linear equations

$$\left(J_k^T J_k + \lambda_k I\right) d = -J_k^T F_{k,i} \quad \text{with} \quad i = 1, \cdots, m-1 \tag{8}$$

to obtain $m-1$ approximate LM steps

$$d_{k,i} = -\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_{k,i} \tag{9}$$

where $F_{k,i} = F(x_{k,i})$, $x_{k,i} = x_{k,i-1} + d_{k,i-1}$ with $x_{k,0} = x_k$, $d_{k,0} = d_k$, $\lambda_k = \mu_k \|F_k\|^\delta$, $\mu_k > 0$ and $\delta \in [1, 2]$. Fan still use $\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T$ in stead of $\left(J(x_{k,i})^T J(x_{k,i}) + \mu_{k+i} \|F(x_{k,i})\|^\delta I\right)^{-1} J(x_{k,i})^T$ in (9), which does not need to compute $J(x_{k,i})$ $(i = 1, 2, \cdots, m-1)$. The trial step of the SLM method is

$$s_k^{SLM} = d_{k,0} + d_{k,1} + \cdots + d_{k,m-1} = \sum_{i=0}^{m-1} d_{k,i}. \tag{10}$$

If we consider the MLM method as two-step Levenberg-Marquardt method and the HLM method as three-step Levenberg-Marquardt method respectively, then, the Shamanskii-like Levenberg-Marquardt method can be considered as m-step Levenberg-Marquardt method. Also, it is easy to see that $\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T$ is computed in all of the classical LM step (3) and the approximate LM step (4), (7), (9) respectively. So, we can consider $\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T$ is frozen in the two-step LM method, three-step LM method and m-step LM method.

To accelerate the MLM method and for better numerical performance, Fan [11] proposed an accelerated version of the MLM (AMLM) method by employing a line search for the approximate LM step $d_{k,1}$ and computed the trial step by

$$s_k^{AMLM} = d_{k,0} + \alpha_{k,1} d_{k,1}, \tag{11}$$

where $\alpha_{k,1} \in [1, \hat{\alpha}_1]$ is step size with $\hat{\alpha}_1 > 1$ is a positive constant. For the same purpose, based on the AMLM method, Chen [12] compute the linear equation (6) with $x_{k,2} = x_{k,1} + \alpha_{k,1} d_{k,1}$ to obtain an approximate LM step $\bar{d}_{k,2}$. By employing another line search for the approximate LM step $\bar{d}_{k,2}$, Chen presented a new modified Levenberg-Marquardt (NMLM) method. The trial step of the NMLM method is

$$s_k^{NMLM} = d_{k,0} + \alpha_{k,1} d_{k,1} + \alpha_{k,2} \bar{d}_{k,2}, \tag{12}$$

where $\alpha_{k,2} \in [1, \hat{\alpha}_2]$ is step size with $\hat{\alpha}_2 > 1$ is a positive constant.

Now, motivated by (10), (11) and (12), we will employ $m-1$ line searches for approximate LM step $d_{k,i}$ by solving linear equation (8) with $x_{k,i} = x_{k,i-1} + \alpha_{k,i-1} d_{k,i-1}$ and present an efficient m-step Levenberg-Marquardt method with trial step as

$$s_k = d_{k,0} + \alpha_{k,1} d_{k,1} + \cdots + \alpha_{k,m-1} d_{k,m-1}, \tag{13}$$

where $\alpha_{k,i} \in [1, \hat{\alpha}]$ are step size with $\hat{\alpha} > 1$ $(i = 1, \cdots, m-1)$ is a positive constants. It is quite clear that the above new LM method will reduce to the classical Levenberg-Marquardt method while $m = 1$, the AMLM method while $m = 2$ and the NMLM method while $m = 3$ respectively.

We will organize the rest of this paper as follow: In Section 2, we first give the new modified Levenberg-Marquardt method which is called efficient m-step Levenberg-Marquardt algorithm. In Section 3, we derive the global convergence of the new algorithm by using trust region technique. Then we derive the convergence order of the algorithm under the local error bound condition in Section 4. Finally, some numerical results of the new algorithm are given in Section 5.

2 The efficient m-step Levenberg-Marquardt algorithm

In this section, we first present the efficient m-step Levenberg-Marquardt algorithm by using trust region technique, then prove the global convergence.

2.1 The motivation

We take

$$\Phi(x) = \|F(x)\|^2 \tag{14}$$

as the merit function for (1). It is easy to see that $d_{k,i}$ $(i = 0, \cdots, m-1)$ is not only the minimizer of the convex minimization problem

$$\min_{d \in \mathrm{R}^n} \|F_{k,i} + J_k d\|^2 + \lambda_k \|d\|^2 \triangleq \varphi_{k,i}(d), \tag{15}$$

but also a solution of the trust region problem

$$\begin{aligned} &\min_{d\in\mathrm{R}^n} \|F_{k,i}+J_kd\|^2, \\ &s.t. \quad \|d\| \leqslant \Delta_{k,i}, \end{aligned} \tag{16}$$

where $\Delta_{k,i}=\|d_{k,i}\|=\left\|-\left(J_k^TJ_k+\lambda_kI\right)^{-1}J_k^TF_{k,i}\right\|$. From the result given by Powell in [13], we have

$$\|F_{k,i}\|^2-\|F_{k,i}+J_kd_{k,i}\|^2 \geqslant \left\|J_k^TF_{k,i}\right\| \min\left\{\|d_{k,i}\|, \frac{\left\|J_k^TF_{k,i}\right\|}{\left\|J_k^TJ_k\right\|}\right\}. \tag{17}$$

Moreover, similar to Fan proposed in [11], if $d_{k,i}$ is a descent direction of the merit function $\Phi(x)$ at $x_{k,i}$, then more reduction of $\Phi(x)$ at $x_{k,i}$ could be expected. So we may perform many line searches at $x_{k,i}$ along $d_{k,i}$ by solving the problem

$$\min_{\alpha>0} \|F(x_{k,i}+\alpha d_{k,i})\|^2.$$

By Taylor extension, replace $J(x_{k,i})$ with J_k for save Jacobian calculations, the above problem could be approximated by

$$\min_{\alpha>0} \|F(x_{k,i})+\alpha J_kd_{k,i}\|^2.$$

The above problem is equivalent to

$$\max_{\alpha>0} \|F_{k,i}\|^2-\|F_{k,i}+\alpha J_kd_{k,i}\|^2 \triangleq \phi(\alpha), \tag{18}$$

where

$$\phi(\alpha)=-d_{k,i}^TJ_k^TJ_kd_{k,i}\alpha^2+2d_{k,i}^T\left(J_k^TJ_k+\lambda_kI\right)d_{k,i}\alpha$$

is a quadratic function of α, and attains its maximum at

$$\tilde{\alpha}_{k,i}=\frac{d_{k,i}^T\left(J_k^TJ_k+\lambda_kI\right)d_{k,i}}{d_{k,i}^TJ_k^TJ_kd_{k,i}}=1+\frac{\lambda_kd_{k,i}^Td_{k,i}}{d_{k,i}^TJ_k^TJ_kd_{k,i}},$$

provided that $J_kd_{k,i}\neq 0$. We bound $\tilde{\alpha}_{k,i}\in[1,\hat{\alpha}]$ with $\hat{\alpha}>1$ is a positive constant because of $\tilde{\alpha}_{k,i}$ may be very large if $J_kd_{k,i}$ is close to 0. The problem (18) now is equivalent to

$$\max_{\alpha\in[1,\hat{\alpha}]} \|F_{k,i}\|^2-\|F_{k,i}+\alpha J_kd_{k,i}\|^2 \triangleq \phi(\alpha). \tag{19}$$

And we have

$$\|F_{k,i}\|^2-\|F_{k,i}+\alpha_{k,i}J_kd_{k,i}\|^2 \geqslant \|F_{k,i}\|^2-\|F_{k,i}+J_kd_{k,i}\|^2 \tag{20}$$

2.2 The algorithm

Now, we define the actual reduction of $\Phi(x)$ at the kth iteration as

$$\mathrm{Ared}_k=\|F_k\|^2-\|F(x_k+d_{k,0}+\alpha_{k,1}d_{k,1}+\cdots+\alpha_{k,m-1}d_{k,m-1})\|^2. \tag{21}$$

where $d_{k,i}$ are computed by (9). Note that the predicted reduction cannot be defined as usual definition $\|F_k\|^2-\|F_k+J_k(d_{k,0}+\alpha_{k,1}d_{k,1}+\cdots+\alpha_{k,m-1}d_{k,m-1})\|^2$, because it cannot be proven to be nonnegative, which is required for the global convergence in the trust region method. Hence, we define the new modified predicted reduction as

$$\mathrm{Pred}_k=\sum_{i=0}^{m-1}\left(\|F_{k,i}\|^2-\|F_{k,i}+\alpha_{k,i}J_kd_{k,i}\|^2\right), \tag{22}$$

with $\alpha_{k,0}=1$.

Lemma 2.1. *Let the predicted reduction is defined by (22), then*

$$\mathrm{Pred}_k \geqslant \left\|J_k^T F_{k,0}\right\| \min\left\{\left\|d_{k,0}\right\|, \frac{\left\|J_k^T F_{k,0}\right\|}{\left\|J_k^T J_k\right\|}\right\}, \tag{23}$$

where $m \geqslant 1$.

Proof. From (17) and (20), we have

$$\begin{aligned}
\mathrm{Pred} &= \sum_{i=0}^{m-1}\left(\left\|F_{k,i}\right\|^2 - \left\|F_{k,i} + \alpha_{k,i} J_k d_{k,i}\right\|^2\right) \\
&\geqslant \sum_{i=0}^{m-1}\left(\left\|F_{k,i}\right\|^2 - \left\|F_{k,i} + J_k d_{k,i}\right\|^2\right) \\
&\geqslant \sum_{i=0}^{m-1}\left(\left\|J_k^T F_{k,i}\right\| \min\left\{\left\|d_{k,i}\right\|, \frac{\left\|J_k^T F_{k,i}\right\|}{\left\|J_k^T J_k\right\|}\right\}\right) \\
&\geqslant \left\|J_k^T F_{k,0}\right\| \min\left\{\left\|d_{k,0}\right\|, \frac{\left\|J_k^T F_{k,0}\right\|}{\left\|J_k^T J_k\right\|}\right\}.
\end{aligned}$$

Then (23) holds. The proof is completed. □

Now, we present the efficient m-step Levenberg-Marquardt algorithm.

Algorithm 2.2 (The efficient m-step Levenberg-Marquardt algorithm)**.**
Input: Given $x_0 \in \mathrm{R}^n$, $\mu_1 > \mu > 0$, $0 < p_0 \leqslant p_1 \leqslant p_2 < 1$, $1 \leqslant \delta \leqslant 2$, $\varepsilon > 0$, $\hat{\alpha} > 1$ **and** $m \geqslant 1$.

Step 1. Set $x_{k,0} = x_k$, $d_{k,0} = d_k$ **and** $k := 0$.

Step 2. Compute $F_k = F_{k,0} = F(x_{k,0})$, $J_k = J(x_{k,0})$. **If** $\left\|J_k^T F_k\right\| < \varepsilon$, **then stop. Otherwise compute**

$$\left(J_k^T J_k + \lambda_k I\right) d = -J_k^T F_{k,i} \quad \textbf{with} \quad \lambda_k = \mu_k \left\|F_k\right\|^\delta, \tag{24}$$

where $x_{k,i} = x_{k,i-1} + \alpha_{k,i-1} d_{k,i-1}$ **to obtain** $d_{k,i}$, $i = 0, 1, \cdots, m-1$. **Set**

$$s_k = \sum_{i=0}^{m-1} \alpha_{k,i} d_{k,i}, \tag{25}$$

where $\alpha_{k,0} = 1$, $\alpha_{k,i}$ $(i = 1, \cdots, m-1)$ **is the step size obtained by solving (19).**

Step 3. Compute $r_k = \mathrm{Ared}_k / \mathrm{Pred}_k$. **Set**

$$x_{k+1} = \begin{cases} x_k + s_k, & \textbf{if } r_k \geqslant p_0, \\ x_k, & \textbf{otherwise}. \end{cases} \tag{26}$$

Step 4. Update μ_{k+1} **as**

$$\mu_{k+1} = \begin{cases} 4\mu_k, & \textbf{if } r_k < p_1, \\ \mu_k, & \textbf{if } r_k \in [p_1, p_2], \\ \max\left\{\frac{\mu_k}{4}, \mu\right\}, & \textbf{if } r_k > p_2. \end{cases} \tag{27}$$

Step 5. Set $k = k + 1$**, and go to Step 2.**

Remark 2.3. *(a) Notice that,* μ_k *should be no less than a positive constant* μ *to prevent the steps from being too large when the sequence* $\{x_k\}$ *is near the solution.*

(b) Fan set $\delta \in (0, 2]$ *in [11], but here, we still set* $\delta \in [1, 2]$ *as usual in [8–10, 12] for stable and preferable.*

3 The global convergence

To study the global convergence of Algorithm 2.2, we need the following assumptions.

Assumption 3.1. *Let $F(x)$ is continuously differentiable, and both $F(x)$ and its Jacobian $J(x)$ are Lipschitz continuous, i.e., there exist positive constant L_1 and L_2 such that*

$$\|J(y) - J(x)\| \leqslant L_1 \|y - x\|, \quad \forall x, y \in \mathrm{R}^n \tag{28}$$

and

$$\|F(y) - F(x)\| \leqslant L_2 \|y - x\|, \quad \forall x, y \in \mathrm{R}^n. \tag{29}$$

By the Lipschitzness of the Jacobian proposed by (28), we have

$$\|F(y) - F(x) - J(x)(y - x)\| \leqslant L_1 \|y - x\|^2, \quad \forall x, y \in \mathrm{R}^n. \tag{30}$$

Theorem 3.2. *Under the conditions of Assumption 3.1, Algorithm 2.2 will terminates in finite iterations or satisfies*

$$\lim_{k\to\infty} \left\|J_k^T F_k\right\| = 0. \tag{31}$$

Proof. By contradiction, suppose there exist a positive τ and infinite many k such that

$$\left\|J_k^T F_k\right\| \geqslant \tau. \tag{32}$$

Let T_1, T_2 be the sets of the indices as follow:

$$\begin{aligned} T_1 &= \left\{k \mid \left\|J_k^T F_k\right\| \geqslant \tau\right\}, \\ T_2 &= \left\{k \mid \left\|J_k^T F_k\right\| \geqslant \frac{\tau}{2} \quad \text{and} \quad x_{k+1} \neq x_k\right\}. \end{aligned}$$

It is easy to see that T_1 is infinite. In the following, we will derive the contradictions whether T_2 is finite or infinite.

Case 1: T_2 is finite. Then the set

$$T_3 = \left\{k \mid \left\|J_k^T F_k\right\| \geqslant \tau \quad \text{and} \quad x_{k+1} \neq x_k\right\}$$

is also finite. Let $\tilde{k}$ be the largest index of T_3. Then it is easy to see that $x_{k+1} = x_k$ holds for all $k \in \left\{k > \tilde{k} \mid k \in T_1\right\}$. Define the indices set

$$T_4 = \left\{k > \tilde{k} \mid \left\|J_k^T F_k\right\| \geqslant \tau \quad \text{and} \quad x_{k+1} = x_k\right\}.$$

If $k \in T_4$, we can deduce that $\left\|J_{k+1}^T F_{k+1}\right\| \geqslant \tau$ and $x_{k+2} = x_{k+1}$. Hence, we have $x_{k+1} \in T_4$. By induction, we know that $\left\|J_k^T F_k\right\| \geqslant \tau$ and $x_{k+1} = x_k$ hold for all $k > \tilde{k}$, which means $r_k < p_0$. Now, we obtain

$$\lambda_k \to +\infty \quad \text{and} \quad \mu_k \to +\infty \tag{33}$$

and, due to (24), (25) and (27),

$$d_{k,0} = \left\|-\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_{k,0}\right\| \to 0.$$

Moreover, it follows from (29) and (30) that

$$
\begin{aligned}
\|d_{k,i}\| &= \left\| -\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_{k,i} \right\| \\
&\leqslant \left\| \left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_{k,0} \right\| + \left\| \left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T J_k \sum_{j=0}^{i-1} \alpha_{k,j} d_{k,j} \right\| \\
&\quad + L_1 \left\| \left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T \right\| \left\| \sum_{j=0}^{i-1} \alpha_{k,j} d_{k,j} \right\|^2 \\
&\leqslant \|d_{k,0}\| + \left\| \sum_{j=0}^{i-1} \alpha_{k,j} d_{k,j} \right\| + \frac{L_1 L_2}{\lambda_k} \left\| \sum_{j=0}^{i-1} \alpha_{k,j} d_{k,j} \right\|^2 \\
&\leqslant \|d_{k,0}\| + \sum_{j=0}^{i-1} \alpha_{k,j} \|d_{k,j}\| + \frac{L_1 L_2}{\lambda_k} \left(\sum_{j=0}^{i-1} \alpha_{k,j} \|d_{k,j}\| \right)^2
\end{aligned}
$$

with $i = 1, \cdots, m-1$. Hence, by induction, we obtain

$$
\|d_{k,i}\| \leqslant O\left(\|d_{k,0}\|\right). \tag{34}
$$

Note that

$$
\begin{aligned}
&\|F_{k,i+1}\|^2 - \|F_{k,i} + \alpha_{k,i} J_k d_{k,i}\|^2 \\
&\quad = \left(\|F_{k,i+1}\| + \|F_{k,i} + \alpha_{k,i} J_k d_{k,i}\|\right)\left(\|F_{k,i+1}\| - \|F_{k,i} + \alpha_{k,i} J_k d_{k,i}\|\right) \\
&\quad \leqslant \left(2 \left\| F_{k,0} + J_k \sum_{j=0}^{i} \alpha_{k,j} d_{k,j} \right\| + L_1 \left\| \sum_{j=0}^{i} \alpha_{k,j} d_{k,j} \right\|^2 + L_1 \left\| \sum_{j=0}^{i-1} \alpha_{k,j} d_{k,j} \right\|^2 \right) \\
&\qquad \times \left(L_1 \left\| \sum_{j=0}^{i} \alpha_{k,j} d_{k,j} \right\|^2 + L_1 \left\| \sum_{j=0}^{i-1} \alpha_{k,j} d_{k,j} \right\|^2 \right)
\end{aligned} \tag{35}
$$

with $i = 0, 1, \cdots, m-1$. It's clear that while $i = 0$ and $\alpha_{k,0} = 1$,

$$
\|F_{k,1}\|^2 - \|F_{k,0} + J_k d_{k,0}\|^2 \leqslant 2L_1 \|F_{k,0} + J_k d_{k,0}\| \|d_{k,0}\|^2 + L_1^2 \|d_{k,0}\|^4 = O\left(\|d_{k,0}\|^2\right).
$$

It follows from (21), (22), (29), (35) and Lemma 2.1 that

$$
\begin{aligned}
|r_k - 1| &= \left| \frac{\mathrm{Ared}_k - \mathrm{Pred}_k}{\mathrm{Pred}_k} \right| \\
&\leqslant \left| \frac{\sum_{i=0}^{m-1} \left(\|F_{k,i+1}\|^2 - \|F_{k,i} + \alpha_{k,i} J_k d_{k,i}\|^2 \right)}{\sum_{i=0}^{m-1} \left(\|F_{k,i}\|^2 - \|F_{k,i} + \alpha_{k,i} J_k d_{k,i}\|^2 \right)} \right| \\
&\leqslant \frac{O\left(\|d_{k,0}\|^2\right)}{\left\|J_k^T F_{k,0}\right\| \min\left\{ \|d_{k,0}\|, \frac{\|J_k^T F_{k,0}\|}{\|J_k^T J_k\|} \right\}} \to 0,
\end{aligned}
$$

which implies that $r_k \to 1$. In view of the updating rule of μ_k, we know that there exists a positive constant $\bar{\mu} > \mu$ such that $\mu_k < \bar{\mu}$ holds for all sufficiently large k, which is a contradiction to (33) .

Case 2: T_2 is infinite. It follows from (23) and (29) that

$$\begin{aligned}\|F_1\|^2 &\geqslant \sum_k \left(\|F_k\|^2 - \|F_{k+1}\|^2\right) \geqslant \sum_{k\in T_2} \left(\|F_k\|^2 - \|F_{k+1}\|^2\right)\\ &\geqslant \sum_{k\in T_2} p_0 \mathrm{Pred}_k \geqslant \sum_{k\in T_2} p_0 \left\|J_k^T F_{k,0}\right\| \min\left\{\|d_{k,0}\|, \frac{\left\|J_k^T F_{k,0}\right\|}{\left\|J_k^T J_k\right\|}\right\}\\ &\geqslant \sum_{k\in T_2} \frac{p_0\tau}{2} \min\left\{\|d_{k,0}\|, \frac{\tau}{2L_2^2}\right\}. \end{aligned} \tag{36}$$

which implies

$$\|d_{k,0}\| \to 0, \quad k\in T_2. \tag{37}$$

Then by the definition of $d_{k,0}$, we have

$$\mu_k \to +\infty, \quad k \in T_2. \tag{38}$$

Moreover, it follows from (28), (29), (34) and (36) that

$$\begin{aligned}&\sum_{k\in T_2} \left|\left\|J_k^T F_k\right\| - \left\|J_{k+1}^T F_{k+1}\right\|\right|\\ &\quad\leqslant \sum_{k\in T_2} \left|\left(\left\|J_k^T F_k\right\| - \left\|J_k^T F_{k+1}\right\|\right) - \left(\left\|J_{k+1}^T F_{k+1}\right\| - \left\|J_k^T F_{k+1}\right\|\right)\right|\\ &\quad\leqslant \sum_{k\in T_2} \left|L_2 \left\|J_k^T\right\| \|s_k\| - L_1 \|F_{k+1}\| \|s_k\|\right|\\ &\quad\leqslant L_1 L_2 \dot{c} \sum_{k\in T_2} \|d_{k,0}\| < +\infty,\end{aligned}$$

with some constants $\dot{c} > 0$, which together with (32) implies there exists a sufficiently large $\hat{k}$ such that

$$\left\|J_k^T F_k\right\| \geqslant \tau \quad \text{and} \quad \sum_{k\in T_2} \left|\left\|J_k^T F_k\right\| - \left\|J_{k+1}^T F_{k+1}\right\|\right| < \frac{\tau}{2}.$$

Hence we can derive that $\left\|J_k^T F_k\right\| \geqslant \frac{\tau}{2}$ for all $k \geqslant \hat{k}$. Combining (37) with (38), we have

$$\|d_{k,0}\| \to 0 \quad \text{and} \quad \mu_k \to +\infty. \tag{39}$$

In the same way as proved in Case 1, we can also obtain that

$$r_k \to 1.$$

Hence, there exists a positive constant $\bar{\mu}$ such that $\mu_k < \bar{\mu}$ holds for all sufficiently large k, which is contradicted to (39). The proof is completed. □

4 The local convergence

In this section, we assume that $x_k \to x^* \in X^*$ and the sequence $\{x_k\}$ lies on some neighbourhood of x^*, i.e., there exist a positive constant $b_1 < 1$ such that $x \in N(x^*, b_1)$. We give some assumptions which the local convergence theory required.

Assumption 4.1. *(a) $F(x)$ is continuously differentiable, and Jacobian $J(x)$ is Lipschitz continuous on $N(x^*, b_1)$, i.e., there exist a positive constant L_1 such that*

$$\|J(y) - J(x)\| \leqslant L_1 \|y - x\|, \quad \forall x, y \in N(x^*, b_1) = \{x \mid \|x - x^*\| \leqslant b_1\}. \tag{40}$$

(b) $\|F(x)\|$ *provides a local error bound on some neighborhood of* $x^* \in X^*$*, i.e., there exist a positive constant* $c > 0$ *such that*

$$\|F(x)\| \geqslant c \ \mathrm{dist}\left(x, X^*\right), \quad \forall x \in N\left(x^*, b_1\right). \tag{41}$$

Since the condition of nonsingularity of $J(x)$ is too strong, the Assumption 4.1 (b) provides a weak local error bound condition, which implies that the converse is not necessarily true [4].

By (40), we have

$$\|F(y) - F(x) - J(x)\left(y - x\right)\| \leqslant L_1 \left\|y - x\right\|^2, \quad \forall x, y \in N\left(x^*, b_1\right), \tag{42}$$

and

$$\|F(y) - F(x)\| \leqslant L_2 \left\|y - x\right\|, \quad \forall x, y \in N\left(x^*, b_1\right), \tag{43}$$

where L_2 is a positive constant.

There exists a positive constant $\omega > 0$ if $F(x)$ provides a local error bound which proposed by Behling and Iusem in [14], then

$$\mathrm{rank}\left(J\left(\tilde{x}\right)\right) = \mathrm{rank}\left(J\left(x^*\right)\right), \quad \forall \tilde{x} \in N\left(x^*, \omega\right) \cap X^*.$$

Let $b \in (0,1)$ and $b_1 = \min\{\omega, b\}$. Without loss of generality, we further assume that $x_{k,i}$, $i = 0, 1, \cdots, m-1$ lie in $N\left(x^*, \frac{b_1}{2}\right)$.

In the following, we denote $\bar{x}_k \in X^*$ such that

$$\|\bar{x}_k - x_k\| = \mathrm{dist}\left(x_k, X^*\right) = \inf_{y \in X^*} \|y - x_k\|.$$

Hence, we have

$$\|\bar{x}_k - x^*\| \leqslant \|\bar{x}_k - x_k\| \leqslant \| + \|x_k - x^*\| \leqslant 2\left\|x_k - x^*\right\| \leqslant b_1,$$

which implies that $\bar{x}_k \in N\left(x^*, b_1\right)$.

Lemma 4.2. *Let Assumption 4.1 hold, then*

$$\left\|\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T\right\| \leqslant O\left(\|\bar{x}_k - x_k\|^{-\frac{\delta}{2}}\right). \tag{44}$$

Proof. Suppose $\mathrm{rank}\left(J\left(\bar{x}_k\right)\right) = r$ for all $\bar{x}_k \in N\left(x^*, b_1\right) \cap X^*$ and the SVD of $J\left(\bar{x}_k\right)$ is

$$J\left(\bar{x}_k\right) = \bar{U}_k \bar{\Sigma}_k \bar{V}_k^T = \left(\bar{U}_{k,1}, \bar{U}_{k,2}\right) \begin{pmatrix} \bar{\Sigma}_{k,1} & \\ & 0 \end{pmatrix} \begin{pmatrix} \bar{V}_{k,1}^T \\ \bar{V}_{k,2}^T \end{pmatrix} = \bar{U}_{k,1} \bar{\Sigma}_{k,1} \bar{V}_{k,1}^T,$$

where $\bar{\Sigma}_{k,1} = \mathrm{diag}\left(\bar{\sigma}_{k,1}, \bar{\sigma}_{k,2}, \cdots, \bar{\sigma}_{k,r}\right)$ with $\bar{\sigma}_{k,1} \geqslant \bar{\sigma}_{k,2} \geqslant \cdots \geqslant \bar{\sigma}_{k,r} > 0$. The corresponding SVD of J_k is

$$\begin{aligned} J_k = & U_k \Sigma_k V_k^T = \left(U_{k,1}, U_{k,2}, U_{k,3}\right) \begin{pmatrix} \Sigma_{k,1} & & \\ & \Sigma_{k,2} & \\ & & 0 \end{pmatrix} \begin{pmatrix} V_{k,1}^T \\ V_{k,2}^T \\ V_{k,3}^T \end{pmatrix} \\ = & U_{k,1} \Sigma_{k,1} V_{k,1}^T + U_{k,2} \Sigma_{k,2} V_{k,2}^T, \end{aligned}$$

where $\Sigma_{k,1} = \mathrm{diag}\left(\sigma_{k,1}, \sigma_{k,2}, \cdots, \sigma_{k,r}\right)$ with $\sigma_{k,1} \geqslant \sigma_{k,2} \geqslant \cdots \geqslant \sigma_{k,r} > 0$, and $\Sigma_{k,2} = \mathrm{diag}(\sigma_{k,r+1}, \sigma_{k,r+2}, \cdots, \sigma_{k,r+q})$ with $\sigma_{k,r+1} \geqslant \sigma_{k,r+2} \geqslant \cdots \geqslant \sigma_{k,r+q} > 0$. We will neglect the subscript k if the context is clear in the following, and write J_k as

$$J_k = U_1 \Sigma_1 V_1^T + U_2 \Sigma_2 V_2^T. \tag{45}$$

By the theory of matrix perturbation [15] and the Lipschitzness of J_k, we have

$$\left\|\mathrm{diag}\left(\Sigma_1 - \bar{\Sigma}_1, \Sigma_2, 0\right)\right\| \leqslant \left\|J_k - \bar{J}_k\right\| \leqslant L_1 \left\|\bar{x}_k - x_k\right\|, \tag{46}$$

which yields

$$\left\|\Sigma_1 - \bar{\Sigma}_1\right\| \leqslant L_1 \left\|\bar{x}_k - x_k\right\| \quad \text{and} \quad \left\|\Sigma_2\right\| \leqslant L_1 \left\|\bar{x}_k - x_k\right\|. \tag{47}$$

Hence

$$\left\|\lambda_k^{-1} \Sigma_2\right\| = \frac{\|\Sigma_2\|}{\mu_k \left\|F_k\right\|^\delta} \leqslant \frac{L_1 \left\|\bar{x}_k - x_k\right\|}{m c^\delta \left\|\bar{x}_k - x_k\right\|^\delta} = L_1 m^{-1} c^{-\delta} \left\|\bar{x}_k - x_k\right\|^{1-\delta} \tag{48}$$

Since for any positive σ_i $(i = 1, 2, \cdots, r)$, we have

$$\frac{\sigma_i}{\sigma_i^2 + \lambda_k} \leqslant \frac{\sigma_i}{2\sigma_i\sqrt{\lambda_k}} = \frac{1}{2\sqrt{\lambda_k}},$$

which implies

$$\left\| \left(\Sigma_1^2 + \lambda_k I\right)^{-1} \Sigma_1 \right\| \leqslant \frac{1}{2\sqrt{\mu_k \left\|F_k\right\|^{\delta}}} \leqslant \frac{1}{2} m^{-\frac{1}{2}} c^{-\frac{\delta}{2}} \left\|\bar{x}_k - x_k\right\|^{-\frac{\delta}{2}}. \tag{49}$$

Combining (48) and (49) with $\delta \in [1, 2]$, we have

$$\begin{aligned} \left\| \left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T \right\| &= \left\| (V_1, V_2, V_3) \begin{pmatrix} \left(\Sigma_1^2 + \lambda_k I\right)^{-1} \Sigma_1 & & \\ & \left(\Sigma_2^2 + \lambda_k I\right)^{-1} \Sigma_2 & \\ & & 0 \end{pmatrix} \begin{pmatrix} U_1^T \\ U_2^T \\ U_3^T \end{pmatrix} \right\| \\ &\leqslant \left\| \begin{pmatrix} \left(\Sigma_1^2 + \lambda_k I\right)^{-1} \Sigma_1 & & \\ & \left(\Sigma_2^2 + \lambda_k I\right)^{-1} \Sigma_2 & \\ & & 0 \end{pmatrix} \right\| \\ &\leqslant \left\| \left(\Sigma_1^2 + \lambda_k I\right)^{-1} \Sigma_1 \right\| + \left\| \lambda_k^{-1} \Sigma_2 \right\| \\ &\leqslant \frac{1}{2} m^{-\frac{1}{2}} c^{-\frac{\delta}{2}} \left\|\bar{x}_k - x_k\right\|^{-\frac{\delta}{2}} + L_1 m^{-1} c^{-\delta} \left\|\bar{x}_k - x_k\right\|^{1-\delta} \\ &\leqslant O\left(\left\|\bar{x}_k - x_k\right\|^{-\frac{\delta}{2}} \right). \end{aligned}$$

The proof is completed. □

4.1 Properties of the trial step

Firstly, we investigate the properties of $d_{k,i}$, and hence s_k.

Lemma 4.3. *Under the condition of Assumption 4.1, for sufficiently large k, we have*

$$\left\|d_{k,i}\right\| \leqslant c_i \operatorname{dist}\left(x_k, X^*\right), \quad i = 0, 1, \cdots, m-1,$$

where c_i are some positive constants.

Proof. The proof of $d_{k,0}$ can be found in Lemma 1 of [11], thus

$$\left\|d_{k,0}\right\| \leqslant c_0 \operatorname{dist}\left(x_k, X^*\right). \tag{50}$$

Now we prove $i \geqslant 1$. From (24), (42), (44) and (50), we obtain

$$\begin{aligned} \left\|d_{k,i}\right\| &= \left\| -\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_{k,i} \right\| \\ &\leqslant \left\| \left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T F_{k,0} \right\| + \left\| \left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T J_k \sum_{j=0}^{i-1} \alpha_{k,i} d_{k,i} \right\| \\ &\quad + L_1 \left\| \left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T \right\| \left\| \sum_{j=0}^{i-1} \alpha_{k,i} d_{k,i} \right\|^2 \\ &\leqslant \left\|d_{k,0}\right\| + \sum_{j=0}^{i-1} \alpha_{k,i} \left\|d_{k,i}\right\| + L_1 \left(\sum_{j=0}^{i-1} \alpha_{k,i} \|d_{k,i}\| \right)^2 O\left(\left\|\bar{x}_k - x_k\right\|^{-\frac{\delta}{2}} \right) \\ &\leqslant c_i \operatorname{dist}\left(x_k, X^*\right), \end{aligned}$$

with $i = 1, \cdots, m-1$, for some positive constant c_i. The proof is completed. □

Lemma 4.3 indicates that the trail step

$$\|s_k\| = \left\| \sum_{i=0}^{m-1} \alpha_{k,i} d_{k,i} \right\| \leqslant \sum_{i=0}^{m-1} \alpha_{k,i} \|d_{k,i}\| \leqslant \ddot{c} \operatorname{dist}(x_k, X^*),$$

for some positive constants $\ddot{c}$.

4.2 Boundedness of the LM parameter

Lemma 4.4. *Under the conditions of Assumption 4.1, there exists a positive $\bar{\mu} > \mu$ such that $\mu_k \leqslant \bar{\mu}$ holds for all sufficiently large k.*

Proof. Following the result given in [10, Lemma 2], we have the following inequalities for all sufficiently large k,

$$\|F_{k,i}\|^2 - \|F_{k,i} + J_k d_{k,i}\|^2 \geqslant \bar{c}_i \|F_{k,i}\| \min\{\|d_{k,i}\|, \|\bar{x}_{k,i} - x_{k,i}\|\},$$

where $\bar{c}_i$ are some positive constants, $i = 0, 1, \cdots, m-1$.

In fact, if $\|\bar{x}_{k,i} - x_{k,i}\| \leqslant \|d_{k,i}\|$, by (41), (42) and the fact that $d_{k,i}$ is the solution of (16), we have

$$\begin{aligned}
&\|F(x_{k,i})\| - \|F_{k,i} + J_k d_{k,i}\| \\
&\qquad \geqslant \|F_{k,i}\| - \|F_{k,i} + J_k(\bar{x}_{k,i} - x_{k,i})\| \\
&\qquad \geqslant \|F_{k,i}\| - \|F_{k,i} + J_{k,i}(\bar{x}_{k,i} - x_{k,i})\| - \|J_k - J_{k,i}\| \|\bar{x}_{k,i} - x_{k,i}\| \\
&\qquad \geqslant c\|\bar{x}_{k,i} - x_{k,i}\| - L_1 \|\bar{x}_{k,i} - x_{k,i}\|^2 - L_1 \|\bar{x}_{k,i} - x_{k,i}\| \sum_{j=0}^{i-1} \alpha_{k,j} \|d_{k,j}\| \\
&\qquad \geqslant \bar{c}_i \|\bar{x}_{k,i} - x_{k,i}\|,
\end{aligned} \tag{51}$$

for some $\bar{c}_i > 0$ when k is sufficiently large. In the other case when $\|\bar{x}_{k,i} - x_{k,i}\| > \|d_{k,i}\|$, we have

$$\begin{aligned}
\|F_{k,i}\| - \|F_{k,i} + J_k d_{k,i}\| &\geqslant \|F_{k,i}\| - \left\| F_{k,i} + \frac{\|d_{k,i}\|}{\|\bar{x}_{k,i} - x_{k,i}\|} J_k(\bar{x}_{k,i} - x_{k,i}) \right\| \\
&\geqslant \frac{\|d_{k,i}\|}{\|\bar{x}_{k,i} - x_{k,i}\|} (\|F_{k,i}\| - \|F_{k,i} + J_k(\bar{x}_{k,i} - x_{k,i})\|) \\
&\geqslant \frac{\|d_{k,i}\|}{\|\bar{x}_{k,i} - x_{k,i}\|} \bar{c}_i \|\bar{x}_{k,i} - x_{k,i}\| \\
&\geqslant \bar{c}_i \|d_{k,i}\|.
\end{aligned} \tag{52}$$

Combining (51) with (52), we obtain

$$\begin{aligned}
\|F_{k,i}\|^2 - \|F_{k,i} + J_k d_{k,i}\|^2 &= (\|F_{k,i}\| + \|F_{k,i} + J_k d_{k,i}\|)(\|F_{k,i}\| - \|F_{k,i} + J_k d_{k,i}\|) \\
&\geqslant \bar{c}_i \|F_{k,i}\| \min\{\|d_{k,i}\|, \|\bar{x}_{k,i} - x_{k,i}\|\}.
\end{aligned}$$

Together with (20), we have

$$\begin{aligned}
\|F_{k,i}\|^2 - \|F_{k,i} + \alpha_{k,i} J_k d_{k,i}\|^2 &\geqslant \|F_{k,i}\|^2 - \|F_{k,i} + J_k d_{k,i}\|^2 \\
&\geqslant \bar{c}_i \|F_{k,i}\| \min\{\|d_{k,i}\|, \|\bar{x}_{k,i} - x_{k,i}\|\}.
\end{aligned} \tag{53}$$

Hence, it follows from (22) and Lemma 4.3, we have

$$\mathrm{Pred}_k \geqslant O(\|\bar{x}_k - x_k\| \|d_{k,0}\|).$$

Since $d_{k,0}$ is a minimizer of (15), we have the following results from (43) and Lemma 4.3 that

$$\begin{aligned}
&\|F_{k,0} + J_k(\alpha_{k,0} d_{k,0} + \cdots + \alpha_{k,i} d_{k,i})\| \\
&\qquad \leqslant \|F_{k,0} + \alpha_{k,0} J_k d_{k,0}\| + \|J_k\| (\alpha_{k,1} \|d_{k,1}\| + \cdots + \alpha_{k,i} \|d_{k,i}\|) \\
&\qquad \leqslant \tilde{c}_i \|\bar{x}_k - x_k\|,
\end{aligned}$$

with $i = 1, \cdots, m-1$ for some positive constants $\tilde{c}_i > 0$. Also, follows from (35), we have

$$\|F_{k,i+1}\|^2 - \|F(x_{k,i}) + \alpha_{k,i} J_k d_{k,i}\|^2 \leqslant O\left(\|\bar{x}_k - x_k\| \|d_{k,0}\|^2\right)$$

which implies that

$$|r_k - 1| = \left|\frac{\mathrm{Ared}_k - \mathrm{Pred}_k}{\mathrm{Pred}_k}\right| \leqslant \frac{O\left(\|\bar{x}_k - x_k\| \|d_{k,0}\|^2\right)}{O\left(\|\bar{x}_k - x_k\| \|d_{k,0}\|\right)} \to 0$$

holds for sufficiently large k. Hence

$$r_k \to 1.$$

Therefore there exists a positive $\bar{\mu} > \mu$ such that $\mu_k \leqslant \bar{\mu}$ holds for all sufficiently large k. The proof is completed. □

4.3 Convergence order of m-step Levenberg-Marquardt algorithm

We now prove the convergence order of m-step LM algorithm based on the results obtained in the above two subsections.

By the SVD of J_k proposed in (45), we have

$$d_{k,i} = -V_1\left(\Sigma_1^2 + \lambda_k I\right)^{-1} \Sigma_1 U_1^T F_{k,i} - V_2\left(\Sigma_2^2 + \lambda_k I\right)^{-1} \Sigma_2 U_2^T F_{k,i}, \tag{54}$$

$$\begin{aligned} &F(x_{k,i}) + J_k d_{k,i} \\ &= F_{k,i} - U_1\Sigma_1\left(\Sigma_1^2 + \lambda_k I\right)^{-1} \Sigma_1 U_1^T F_{k,i} - U_2\Sigma_2\left(\Sigma_2^2 + \lambda_k I\right)^{-1} \Sigma_2 U_2^T F_{k,i} \\ &= \lambda_k U_1\left(\Sigma_1^2 + \lambda_k I\right)^{-1} U_1^T F_{k,i} + \lambda_k U_2\left(\Sigma_2^2 + \lambda_k I\right)^{-1} U_2^T F_{k,i} + U_3 U_3^T F_{k,i}, \end{aligned} \tag{55}$$

with $i = 0, \cdots, m-1$.

Lemma 4.5. *Under the condition of Assumption 4.1, if $x_{k,i} \in N(x^*, b_1/2)$, then we have*

(a) $\left\|U_1 U_1^T F_{k,i}\right\| \leqslant O\left(\|\bar{x}_k - x_k\|^{i+1}\right)$;

(b) $\left\|U_2 U_2^T F_{k,i}\right\| \leqslant O\left(\|\bar{x}_k - x_k\|^{i+2}\right)$;

(c) $\left\|U_3 U_3^T F_{k,i}\right\| \leqslant O\left(\|\bar{x}_k - x_k\|^{i+2}\right)$;

with $i = 0, \cdots, m-1$.

Proof. We will prove this lemma by an induction process.

For $i = 1, 2$, the results have been shown by Fan and Chen respectively (see [11, 12]), and we have

$$\begin{aligned} \|d_{k,1}\| &\leqslant O\left(\|\bar{x}_k - x_k\|^2\right), \quad \|F_{k,1} + J_k d_{k,1}\| \leqslant O\left(\|\bar{x}_k - x_k\|^3\right), \\ \|d_{k,2}\| &\leqslant O\left(\|\bar{x}_k - x_k\|^3\right), \quad \|F_{k,2} + J_k d_{k,2}\| \leqslant O\left(\|\bar{x}_k - x_k\|^4\right). \end{aligned}$$

Assuming the truth for some $i-1$, we obtain the induction hypothesis:

$$\|d_{k,i-1}\| \leqslant O\left(\|\bar{x}_k - x_k\|^i\right), \quad \|F(x_{k,i-1}) + J_k d_{k,i-1}\| \leqslant O\left(\|\bar{x}_k - x_k\|^{i+1}\right).$$

Turning now to the case for i. It follows from above induction hypothesis that

$$\begin{aligned}
\|F_{k,i}\| &= \|F(x_{k,i-1}+\alpha_{k,i-1}d_{k,i-1})\| \\
&\leqslant \|F_{k,i-1}+\alpha_{k,i-1}J_{k,i-1}d_{k,i-1}\| + L_1\alpha_{k,i-1}^2\|d_{k,i-1}\|^2 \\
&\leqslant \|F_{k,i-1}+J_{k,i-1}d_{k,i-1}\| + L_1\alpha_{k,i-1}^2\|d_{k,i-1}\|^2 \\
&\leqslant \|F_{k,i-1}+J_kd_{k,i-1}\| + \|J_{k,i-1}-J_k\|\|d_{k,i-1}\| + L_1\alpha_{k,i-1}^2\|d_{k,i-1}\|^2 \\
&\leqslant \|F_{k,i-1}+J_kd_{k,i-1}\| + L_1\left\|\sum_{j=0}^{i-1}\alpha_{k,j}d_{k,j}\right\|\|d_{k,i-1}\| + L_1\alpha_{k,i-1}^2\|d_{k,i-1}\|^2 \\
&\leqslant O\left(\|\bar{x}_k-x_k\|^{i+1}\right) + L_1\|\bar{x}_k-x_k\|O\left(\|\bar{x}_k-x_k\|^{i}\right) \\
&\quad + L_1\alpha_{k,i-1}^2O\left(\|\bar{x}_k-x_k\|^{2i}\right) \\
&\leqslant O\left(\|\bar{x}_k-x_k\|^{i+1}\right).
\end{aligned}$$

So, we have

$$\left\|U_1U_1^TF_{k,i}\right\| \leqslant \|F_{k,i}\| \leqslant O\left(\|\bar{x}_k-x_k\|^{i+1}\right).$$

Moreover, the local error bound condition implies that

$$\|\bar{x}_{k,i}-x_{k,i}\| \leqslant c^{-1}\|F_{k,i}\| \leqslant O\left(\|\bar{x}_k-x_k\|^{i+1}\right). \tag{56}$$

Let $\bar{q}_k = -J_k^+F_{k,i}$. Then $\bar{q}_k$ is the least squares solution of $\|\min F_{k,i}+J_kq\|$. It follows from (40), (42), (56) and Lemma 4.3 that

$$\begin{aligned}
\left\|U_3U_3^TF_{k,i}\right\| &= \|F_{k,i}+J_k\bar{q}_k\| \leqslant \|F_{k,i}+J_k(\bar{x}_{k,i}-x_{k,i})\| \\
&\leqslant \|F_{k,i}+J_{k,i}(\bar{x}_{k,i}-x_{k,i})\| + \|(J_{k,i}-J_k)(\bar{x}_{k,i}-x_{k,i})\| \\
&\leqslant L_1\|\bar{x}_{k,i}-x_{k,i}\|^2 + L_1\left\|\sum_{j=0}^{i-1}\alpha_{k,j}d_{k,j}\right\|\|\bar{x}_{k,i}-x_{k,i}\| \\
&\leqslant O\left(\|\bar{x}_k-x_k\|^{2i+2}\right) + O\left(\|\bar{x}_k-x_k\|^{i+2}\right) \\
&= O\left(\|\bar{x}_k-x_k\|^{i+2}\right).
\end{aligned} \tag{57}$$

Let $\tilde{J}_k = U_1\Sigma_1V_1^T$ and $\tilde{q}_k = -\tilde{J}_k^+F_{k,i}$. Since $\tilde{q}_k$ is the least squares solution of $\left\|\min F_{k,i}+\tilde{J}_kq\right\|$, deducing from (40), (42),(47), (56) and Lemma 4.3 that

$$\begin{aligned}
&\left\|\left(U_2U_2^T+U_3U_3^T\right)F_{k,i}\right\| \\
=&\left\|F_{k,i}+\tilde{J}_k\tilde{q}_k\right\| \leqslant \left\|F_{k,i}+\tilde{J}_k(\bar{x}_{k,i}-x_{k,i})\right\| \\
\leqslant&\|F_{k,i}+J_{k,i}(\bar{x}_{k,i}-x_{k,i})\| + \left\|\left(\tilde{J}_k-J_{k,i}\right)(\bar{x}_{k,i}-x_{k,i})\right\| \\
\leqslant&L_1\|\bar{x}_{k,i}-x_{k,i}\|^2 + \left\|\left(J_k-J_{k,i}-U_2\Sigma_2V_2^T\right)(\bar{x}_{k,i}-x_{k,i})\right\| \\
\leqslant&L_1\|\bar{x}_{k,i}-x_{k,i}\|^2 + \|(J_k-J_{k,i})(\bar{x}_{k,i}-x_{k,i})\| + \left\|U_2\Sigma_2V_2^T(\bar{x}_{k,i}-x_{k,i})\right\| \\
\leqslant&L_1\|\bar{x}_{k,i}-x_{k,i}\|^2 + L_1\left\|\sum_{j=0}^{i-1}\alpha_{k,j}d_{k,j}\right\|\|\bar{x}_{k,i}-x_{k,i}\| + L_1\|\bar{x}_k-x_k\|\|\bar{x}_{k,i}-x_{k,i}\| \\
\leqslant&O\left(\|\bar{x}_k-x_k\|^{2i+2}\right) + O\left(\|\bar{x}_k-x_k\|^{i+2}\right) + O\left(\|\bar{x}_k-x_k\|^{i+2}\right) \\
\leqslant&O\left(\|\bar{x}_k-x_k\|^{i+2}\right).
\end{aligned} \tag{58}$$

Due to the orthogonality of U_2 and U_3, combining (57) and (58), we know that

$$\left\|U_2 U_2^T F_{k,i}\right\| \leqslant O\left(\|\bar{x}_k - x_k\|^{i+2}\right).$$

The proof is completed. ☐

Now, we are ready to give the estimations of $d_{k,m-1}$ and $\|F(x_{k,m-1}) + J_k d_{k,m-1}\|$.

Lemma 4.6. *Under the condition of Assumption 4.1, for sufficiently large k, we have*

(a) $\|d_{k,m-1}\| \leqslant O\left(\|\bar{x}_k - x_k\|^m\right)$;

(b) $\|F(x_{k,m-1}) + J_k d_{k,m-1}\| \leqslant O\left(\|\bar{x}_k - x_k\|^{m+1}\right)$.

Proof. By (46), we have

$$\|\Sigma_1\|^{-1} = \left|\frac{1}{\sigma_r}\right| \leqslant \left|\frac{1}{\bar{\sigma}_r - L_1\|\bar{x}_k - x_k\|}\right|,$$

which implies

$$\|\Sigma_1\|^{-1} \leqslant \frac{2}{\bar{\sigma}_r}.$$

When $\delta \in [1,2]$, it then follows from Lemma 4.4, Lemma 4.5, (48), (54) and (55) that

$$\begin{aligned}
\|d_{k,m-1}\| &= \left\|-V_1\left(\Sigma_1^2 + \lambda_k I\right)^{-1}\Sigma_1 U_1^T F(x_{k,m-1}) - V_2\left(\Sigma_2^2 + \lambda_k I\right)^{-1}\Sigma_2 U_2^T F(x_{k,m-1})\right\| \\
&\leqslant \|\Sigma_1\|^{-1}\left\|U_1^T F(x_{k,m-1})\right\| + \left\|\lambda_k^{-1}\Sigma_2\right\|\left\|U_2^T F(x_{k,m-1})\right\| \\
&\leqslant O\left(\|\bar{x}_k - x_k\|^m\right) + O\left(\|\bar{x}_k - x_k\|^{m+2-\delta}\right) \\
&= O\left(\|\bar{x}_k - x_k\|^m\right),
\end{aligned}$$

and

$$\begin{aligned}
&\|F(x_{k,m-1}) + J_k d_{k,m-1}\| \\
&= \left\|\lambda_k U_1\left(\Sigma_1^2 + \lambda_k I\right)^{-1} U_1^T F(x_{k,m-1}) + \lambda_k U_2\left(\Sigma_2^2 + \lambda_k I\right)^{-1} U_2^T F(x_{k,m-1}) + U_3 U_3^T F(x_{k,m-1})\right\| \\
&\leqslant \lambda_k\left\|\Sigma_1^2\right\|^{-1}\left\|U_1^T F(x_{k,m-1})\right\| + \left\|U_2^T F(x_{k,m-1})\right\| + \left\|U_3^T F(x_{k,m-1})\right\| \\
&\leqslant O\left(\|\bar{x}_k - x_k\|^{m+\delta}\right) + O\left(\|\bar{x}_k - x_k\|^{m+1}\right) + O\left(\|\bar{x}_k - x_k\|^{m+1}\right) \\
&\leqslant O\left(\|\bar{x}_k - x_k\|^{m+1}\right).
\end{aligned}$$

The proof is completed. ☐

Based on the results above, we obtain the convergence rate of Algorithm 2.2.

Theorem 4.7. *Under the conditions of Assumptions 4.1, the convergence rate of Algorithm 2.2 is $(m+1)$th.*

Proof. It follows from Lemma 4.3 and Lemma 4.6 that

$$\begin{aligned}
&c\|\bar{x}_{k+1} - x_{k+1}\| \\
&\leqslant \|F(x_{k+1})\| = \|F(x_k + s_k)\| = \|F(x_{k,m-1} + \alpha_{k,m-1} d_{k,m-1})\| \\
&\leqslant \|F(x_{k,m-1}) + \alpha_{k,m-1} J(x_{k,m-1}) d_{k,m-1}\| + L_1\alpha_{k,m-1}^2\|d_{k,m-1}\|^2 \\
&\leqslant \|F(x_{k,m-1}) + J(x_{k,m-1}) d_{k,m-1}\| + L_1\alpha_{k,m-1}^2\|d_{k,m-1}\|^2 \\
&\leqslant \|F(x_{k,m-1}) + J_k d_{k,m-1}\| + \|(J(x_{k,m-1}) - J_k) d_{k,m-1}\| + L_1\alpha_{k,m-1}^2\|d_{k,m-1}\|^2 \\
&\leqslant \|F(x_{k,m-1}) + J_k d_{k,m-1}\| + L_1\left\|\sum_{j=0}^{m-2}\alpha_{k,j} d_{k,j}\right\|\|d_{k,m-1}\| + L_1\alpha_{k,m-1}^2\|d_{k,m-1}\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \|F(x_{k,m-1}) + J_k d_{k,m-1}\| + L_1 \sum_{j=0}^{m-2} \alpha_{k,j} \|d_{k,j}\| \|d_{k,m-1}\| + L_1 \alpha_{k,m-1}^2 \|d_{k,m-1}\|^2 \\
&\leqslant O\left(\|\bar{x}_k - x_k\|^{m+1}\right) + O\left(\|\bar{x}_k - x_k\|^{m+1}\right) + O\left(\|\bar{x}_k - x_k\|^{2m}\right) \\
&\leqslant O\left(\|\bar{x}_k - x_k\|^{m+1}\right),
\end{aligned}
$$

with $m \geqslant 1$. Hence we have

$$
\|\bar{x}_{k+1} - x_{k+1}\| \leqslant O\left(\|\bar{x}_k - x_k\|^{m+1}\right), \tag{59}
$$

which means that $\{x_k\}$ generated by m-step LM method converges to the solution set X^* with $(m+1)$th order. The proof is completed. □

Since

$$
\|\bar{x}_k - x_k\| \leqslant \|\bar{x}_{k+1} - x_{k+1}\| + \|s_k\|,
$$

we obtain from (59) that

$$
\|\bar{x}_k - x_k\| \leqslant 2\|s_k\|
$$

holds for sufficiently large k. By Lemma 4.3, we finally have

$$
\|s_{k+1}\| \leqslant O\left(\|s_k\|^{m+1}\right),
$$

which indicates that $\{x_k\}$ converges to some solution of (1) with Q-order $m+1$. This result is stronger than the convergence to the solution set.

5 Numerical results

We will compute some singular problems, which come from [16] with the same forms as in [17], to test Algorithm 2.2, and compare it with the general LM algorithm (LM), the SLM method which has presented in [10] with $m = 4$.

We compute these test problems with different initial points and different size,

$$
\hat{F}(x) = F(x) - J(x^*) A \left(A^T A\right)^{-1} A^T (x - x^*),
$$

where $F(x)$ is the standard nonsingular test function, x^* is its root, and $A \in R^{n\times k}$ has full column rank with $1 \leqslant k \leqslant n$. Obviously, $\hat{F}(x^*) = 0$ and

$$
\hat{J}(x^*) = J(x^*) \left(I - A\left(A^T A\right)^{-1} A^T\right)
$$

has rank $n-k$. A disadvantage of these problems is that $\hat{F}(x)$ may have roots that are not roots of $F(x)$. We chose the rank of $\hat{J}(x^*)$ to be $n-1$ and $n-2$, respectively, by using

$$
A \in R^{n\times 1}, \quad A^T = (1, 1, \cdots, 1)
$$

and

$$
A \in R^{n\times 2}, \quad A^T = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & -1 & \cdots & \pm 1 \end{pmatrix}.
$$

Set $p_0 = 0.0001$, $p_1 = 0.25$, $p_2 = 0.75$, $\tilde{m} = 10^{-8}$, $\mu_1 = 1$, $\delta = 1$ for all the tests. The stopping criteria for the Algorithm is $\left\|J_k^T F_k\right\| < 10^{-5}$ or the iteration number exceeds $100(n+1)$. The points x_0, $10x_0$, $100x_0$ in the third column of the tables are the starting points, where x_0 was suggested by Moré et. al in [16]. “NF” and “NJ” represent the number of function calculations and Jacobian calculations, respectively. If the method failed to find the solution in $100(n+1)$ iterations, we denoted it by the sign “-”, and if the iterations had underflows or overflows, we denoted it by “OF”. We also denote ”TIME” represents the running time of the problem. All codes are written in MATLAB R2012 programming environment on a personal PC with Inter(R) Core(TM) i5-4590 CPU, 3.30GHz, 4GB RAM, using Windows 7 operation system.

Table 1: Results on the first singular test set with rank($F'(x^*)$)$= n-1$

			Algorith LM	Algorithm SLM with $m=4$	Algorithm 2.2 with $m=4$
Problem	n	x_0	NF/NJ/F/TIME	NF/NJ/F/TIME	NF/NJ/F/TIME
8	3000	1	9/9/1.7993e-05/16.0927	17/5/6.0835e-06/27.9715	17/5/1.6373e-05/45.5972
9	3000	1	1/1/6.9349e-06/0.39072	1/1/6.9349e-06/0.38856	1/1/6.9349e-06/0.38503
		10	3/3/4.9157e-03/4.6299	9/3/1.8731e-03/14.9589	9/3/1.468e-03/16.2261
		100	5/5/2.9136e-02/8.8327	13/4/2.8697e-02/22.3504	13/4/2.3369e-02/24.1051
10	3000	1	7/7/1.8841e-05/113.7354	13/4/1.5023e-05/90.6983	13/4/1.7169e-05/96.5679
		10	9/9/1.1683e-05/146.8708	17/5/1.2381e-05/115.8011	21/6/7.057e-06/155.4833
		100	10/10/9.479e-09/163.7237	21/6/1.4677e-10/142.4809	21/6/1.0401e-13/159.7523
11	3000	1	20/10/2.2123e-04/34.464	77/6/2.3676e-04/128.9774	161/15/1.9047e-04/297.2913
		10	38/26/1.9482e-03/68.8421	161/17/1.3971e-03/269.0105	165/17/2.8922e-03/340.8817
		100	37/22/3.0277e-03/65.9771	145/16/2.45e-04/236.8702	129/12/4.9012e-04/272.922
13	3000	1	9/9/1.4397e-04/16.3563	17/5/8.5893e-05/27.4254	17/5/1.8655e-04/44.4526
		10	14/14/1.4123e-04/26.2765	25/7/2.604e-04/41.0561	29/8/5.5618e-05/79.057
		100	17/17/2.5192e-04/32.2734	33/9/8.9702e-05/54.5243	37/10/2.8248e-05/102.0578
14	3000	1	12/12/3.6595e-05/22.8178	25/7/4.4361e-06/41.4525	25/7/1.3946e-05/65.4345
		10	18/18/4.3039e-05/35.2549	37/10/4.9713e-06/62.3313	37/10/2.4413e-05/98.7529
		100	24/24/2.5066e-05/47.6055	49/13/2.9067e-06/82.7621	49/13/2.1752e-05/132.2612

Table 2: Results on the first singular test set with rank($F'(x^*)$)$= n-2$

			Algorith LM	Algorithm SLM with $m=4$	Algorithm 2.2 with $m=4$
Problem	n	x_0	NF/NJ/F/TIME	NF/NJ/F/TIME	NF/NJ/F/TIME
8	3000	1	9/9/1.7993e-05/15.758	17/5/6.0835e-06/27.7172	17/5/1.6373e-05/44.3831
9	3000	1	1/1/6.9349e-06/0.38195	1/1/6.9349e-06/0.38875	1/1/6.9349e-06/0.38878
		10	3/3/4.9157e-03/4.5209	9/3/1.8731e-03/14.8385	9/3/1.468e-03/16.9752
		100	5/5/2.9136e-02/8.6748	13/4/2.8697e-02/21.854	13/4/2.3369e-02/25.3346
10	3000	1	7/7/1.8841e-05/113.166	13/4/1.5023e-05/89.9889	13/4/1.7169e-05/96.959
		10	9/9/1.1683e-05/145.9313	17/5/1.2381e-05/115.1672	21/6/7.057e-06/155.429
		100	17/12/5.585e-06/210.7712	21/6/5.259e-06/141.4457	21/6/5.2587e-06/156.9107
11	3000	1	20/10/2.2124e-04/33.8705	77/6/2.3676e-04/125.7451	149/14/1.9077e-04/288.4942
		10	37/24/2.3572e-03/65.5213	149/16/1.9236e-03/245.8551	165/17/2.8922e-03/353.1612
		100	46/26/3.0342e-03/80.6403	137/15/2.5525e-04/227.3232	129/13/4.9734e-04/280.1359
13	3000	1	9/9/1.4397e-04/16.2237	17/5/8.5893e-05/27.5697	17/5/1.8655e-004/43.5593
		10	14/14/1.4123e-04/26.2398	25/7/2.604e-04/41.4618	29/8/5.5618e-05/77.8504
		100	17/17/2.5192e-04/32.1537	33/9/8.9702e-05/55.034	37/10/2.8248e-05/100.1965
14	3000	1	12/12/3.6595e-05/22.7153	25/7/4.4361e-06/41.5594	25/7/1.3946e-05/67.8233
		10	18/18/4.3039e-05/34.929	37/10/4.9713e-06/62.0526	37/10/2.4413e-05/102.6231
		100	24/24/2.5066e-05/47.1261	49/13/2.9067e-06/82.8124	49/13/2.1752e-05/135.4006

From table 1 and table 2, we can see that, though Algorithm 2.2 take more running time than the SLM method to compute step size $\alpha_{k,i}$, Algorithm 2.2 still almost always outperforms or at least performs as well as the SLM method whether on the first singular test set or on the second test set, which indicate that the line search really makes the method more efficient and contributes a lot to the numerical performance. That would be great helpful for the real application of the method and especially useful for the large scale problems.

6 Conclusions

In this work, to save more Jacobian calculations, we presented the efficient m-step LM method for systems of nonlinear equations. At every iteration, we compute $m-1$ approximate LM steps with frozen $\left(J_k^T J_k + \lambda_k I\right)^{-1} J_k^T$ and employ $m-1$ line search for better numerical performance. The efficient m-step LM method have been proved to have $(m+1)$th convergence order under the local error bound condition

which is weaker than nonsingularity. Numerical results show that the efficient m-step LM method saved more Jacobian calculations although the calculations of function are more.

References

[1] W. Sun, Y. Yuan, Optimization theory and methods: Nonlinear programming, springer, 2006.

[2] K. Levenberg, A method for the solution of certain nonlinear problems in least squares, Quarterly of Applied Mathematics 2 (1944) 164–168.

[3] D. W. Marquardt, An algorithm for least-squares estimation of nonlinear parameters, Journal of the Society for Industrial and Applied Mathematics 11 (1963) 431–441.

[4] N. Yamashita, M. Fukushima, On the Rate of Convergence of the Levenberg-Marquardt Method, Springer Vienna, Vienna, 2001, pp. 239–249.

[5] A. Fischer, P. K. Shukla, M. Wang, On the inexactness level of robust Levenberg-Marquardt methods, Optimization 59 (2) (2010) 273–287, 65H10 (90C30).

[6] K. Amini, F. Rostami, Three-steps modified levenberg-marquardt method with a new line search for systems of nonlinear equations, Journal of Computational and Applied Mathematics 300 (2016) 30–42.

[7] P. Kazemi, R. J. Renka, A Levenberg-Marquardt method based on sobolev gradients, Nonlinear Analysis: Theory, Methods & Applications 75 (16) (2012) 6170–6179.

[8] J. Fan, The modified Levenberg-Marquardt method for nonlinear equations with cubic convergence, Mathematics of Computation 81 (2012) 447–466.

[9] X. Yang, A higher-order Levenberg-Marquardt method for nonlinear equations, Applied Mathematics and Computation 219 (22) (2013) 10682–10694, 65H10.

[10] J. Fan, A shamanskii-like Levenberg-Marquardt method for nonlinear equations, Computational Optimization and Applications 56 (1) (2013) 63–80.

[11] J. Fan, Accelerating the modified Levenberg-Marquardt method for nonlinear equations, Mathematics of Computation 83 (2014) 1173–1187.

[12] L. Chen, A modified levenberg–marquardt method with line search for nonlinear equations, Computational Optimization and Applications (2016) 1–27.

[13] M. Powell, Convergence properties of a class of minimization algorithms, in: O. Mangasarian, , R. Meyer, , S. Robinson (Eds.), Nonlinear Programming 2, Academic Press, 1975, pp. 1 – 27.

[14] R. Behling, A. Iusem, The effect of calmness on the solution set of systems of nonlinear equations, Mathematical Programming 137 (1-2) (2013) 155–165.

[15] G. W. Stewart, J. Sun, Matrix Perturbation Theory, Computer Science and Scientific Computing, Academic Press Inc., Boston, MA, 1990.

[16] J. J. Moré, B. S. Garbow, K. H. Hillstrom, Testing unconstrained optimization software, ACM Trans. Math. Software 7 (1981) 17–41.

[17] R. B. Schnabel, P. D. Frank, Tensor methods for nonlinear equations, SIAM Journal on Numerical Analysis 21 (5) (1984) 815–843.

OPTIMAL BOUNDS FOR A TOADER TYPE MEAN USING ARITHMETIC AND GEOMETRIC MEANS*

WEI-MAO QIAN[1,2], WEN ZHANG[3], AND YU-MING CHU[4,**]

ABSTRACT. In the aritcle, we prove that the double inequalities $\alpha A(a,b)+(1-\alpha)G(a,b) < T[A(a,b),G(a,b)] < \beta A(a,b)+(1-\beta)G(a,b)$ and $G[\lambda a+(1-\lambda)b, \lambda b+(1-\lambda)a] < T[A(a,b),G(a,b)] < G[\mu a+(1-\mu)b, \mu b+(1-\mu)a]$ hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha \leq 1/2$, $\beta \geq 2/\pi$, $\lambda \leq (1-\sqrt{1-4/\pi^2})/2$ and $\mu \geq 1/2-\sqrt{2}/4$ if $\alpha,\beta \in \mathbb{R}$ and $\lambda,\mu \in (0,1/2)$, and find new bounds for the complete elliptic integral $\mathcal{E}(r)=\int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}d\theta$ $(0<r<1)$ of the second kind, where $G(a,b)=\sqrt{ab}$, $T(a,b)=2\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}d\theta/\pi$ and $A(a,b)=(a+b)/2$ are respectively the geometric, Toader and arithmetic means of a and b.

1. INTRODUCTION

Let $r\in(0,1)$ and $a,b>0$. Then the complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ [1-24] of the first and second kind, Toader mean $T(a,b)$ [25-34], geometric mean $G(a,b)$ [35-41] and arithmetic mean $A(a,b)$ [42-50] are respectively given by

$$\mathcal{K}(r)=\int_0^{\pi/2}(1-r^2\sin^2\theta)^{-1/2}d\theta, \quad \mathcal{E}(r)=\int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}d\theta,$$

$$T(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}d\theta, \tag{1.1}$$

$$G(a,b)=\sqrt{ab}, \quad A(a,b)=\frac{a+b}{2}. \tag{1.2}$$

It is well known that

$$\mathcal{K}(0^+)=\mathcal{E}(0^+)=\pi/2, \quad \mathcal{K}(1^-)=+\infty, \quad \mathcal{E}(1^-)=1,$$

$\mathcal{K}(r)$ is strictly increasing and $\mathcal{E}(r)$ is strictly decreasing on $(0,1)$, $\mathcal{K}(r)$ and $\mathcal{E}(r)$ satisfy the derivatives formulas [51, Appendix E, p. 474-475]

$$\frac{d\mathcal{K}(r)}{dr}=\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r(1-r^2)}, \quad \frac{d\mathcal{E}(r)}{dr}=\frac{\mathcal{E}(r)-\mathcal{K}(r)}{r},$$

and $T(a,b)$ can be rewritten as

$$T(a,b)=\begin{cases}\frac{2a}{\pi}\mathcal{E}\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right), & a>b,\\ a, & a=b,\\ \frac{2b}{\pi}\mathcal{E}\left(\sqrt{1-\left(\frac{a}{b}\right)^2}\right), & a<b.\end{cases} \tag{1.3}$$

2010 *Mathematics Subject Classification.* Primary: 26E60; Secondary: 33E05.

Key words and phrases. Toader mean, arithmetic mean, geometric mean.

*The research was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11301127, 11701176, 11626101, 11601485), the Science and Technology Research Program of Zhejiang Educational Committee (Grant No. Y201635325) and the Natural Science Foundation of Huzhou City (Grant No. 2018YZ07).

**Corresponding author: Yu-Ming Chu, Email: chuyuming2005@126.com.

Recently, the bounds for the Toader mean $T(a,b)$ have attracted the attention of several researchers. Barnard et. al. [52], and Alzer and Qiu [53] proved that the double inequality

$$M_{p_1}(a,b) < T(a,b) < M_{p_2}(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $p_1\leq 3/2$ and $p_2\geq \log 2/(\log\pi-\log 2)=1.5349\cdots$, where $M_p(a,b)=[(a^p+b^p)/2]^{1/p}$ $(p\neq 0)$ and $M_0(a,b)=\sqrt{ab}$ is the pth power mean.

Very recently, Song et. al. [54] proved that the double inequality

$$M_{q_1}(a,b) < T[A(a,b),Q(a,b)] < M_{q_2}(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $q_1\leq 2\log 2/[2\log\pi-\log 2-2\log\mathcal{E}(\sqrt{2}/2)]=1.3930\cdots$ and $q_2\geq 3/2$, where $Q(a,b)=\sqrt{(a^2+b^2)/2}$ is the quadratic mean of a and b.

Let $a,b>0$ with $a\neq b$. Then from (1.1) and (1.2) together with $G(a,b)<A(a,b)$ we clearly see that the function $\lambda\to R(\lambda)=G[\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a]$ is continuous and strictly increasing on $[0,1/2]$, and

$$R(0)=G(a,b)<T[A(a,b),G(a,b)]<A(a,b)=R\left(\frac{1}{2}\right).$$

It is the aim of this article to find the best possible parameters $\alpha,\beta\in\mathbb{R}$ and $\lambda,\mu\in(0,1/2)$ such that the double inequalities

$$\alpha A(a,b)+(1-\alpha)G(a,b) < T[A(a,b),G(a,b)] < \beta A(a,b)+(1-\beta)G(a,b),$$

$$G[\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a] < T[A(a,b),G(a,b)] < G[\mu a+(1-\mu)b,\mu b+(1-\mu)a]$$

hold for all $a,b>0$ with $a\neq b$ holds for all $a,b>0$ with $a\neq b$.

2. Lemmas

Lemma 2.1. *(See [51, Theorem 3.21(1)]) The function $r\mapsto[\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)]/r^2$ is strictly increasing from $(0,1)$ onto $(\pi/4,1)$.*

Lemma 2.2. *(See [51, Exercise 3.43(11)]) The function $r\mapsto[\mathcal{K}(r)-\mathcal{E}(r)]/r^2$ is strictly increasing from $(0,1)$ onto $(\pi/4,+\infty)$.*

Lemma 2.3. *(See [51, Theorem 3.21(7)]) The function $r\mapsto(1-r^2)^{\lambda}\mathcal{K}(r)$ is strictly decreasing from $(0,1)$ onto $(0,\pi/2)$ if $\lambda\geq 1/4$.*

Lemma 2.4. *(See [51, Theorem 1.25]) Let $a,b\in\mathbb{R}$ with $a<b$, $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)},\quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.5. *The function $r\mapsto\sqrt{1-r^2}[\mathcal{E}(r)-\mathcal{K}(r)]/r^2$ is strictly increasing from $(0,1)$ onto $(-\pi/4,0)$.*

Proof. Let

$$f(r)=\frac{\sqrt{1-r^2}[\mathcal{E}(r)-\mathcal{K}(r)]}{r^2}, \tag{2.1}$$

$$g(r)=[\mathcal{K}(r)-\mathcal{E}(r)]-[\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)]. \tag{2.2}$$

Then it follows from (2.1), (2.2), L'Hôpital rule, and Lemmas 2.1 and 2.3 that

$$f(1^-)=0,\quad f(0^+)=\lim_{r\to 0^+}\frac{[\sqrt{1-r^2}(\mathcal{E}(r)-\mathcal{K}(r))]'}{2r}=\lim_{r\to 0^+}\frac{\mathcal{K}(r)-2\mathcal{E}(r)}{2\sqrt{1-r^2}}=-\frac{\pi}{4}, \tag{2.3}$$

$$f'(r)=\frac{1}{r^3\sqrt{1-r^2}}g(r), \tag{2.4}$$

$$g(0^+)=0, \tag{2.5}$$

$$g'(r)=\frac{r^3}{1-r^2}\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r^2}>0 \tag{2.6}$$

for $r\in(0,1)$.

Therefore, Lemma 2.5 follows easily from (2.3)-(2.6). □

Lemma 2.6. *The function $r\mapsto \mathcal{E}(r)[\mathcal{K}(r)-\mathcal{E}(r)]/r^2$ is strictly increasing from $(0,1)$ onto $(\pi^2/8,+\infty)$.*

Proof. Let

$$h(r)=\frac{\mathcal{E}(r)[\mathcal{K}(r)-\mathcal{E}(r)]}{r^2},\quad h_1(r)=\mathcal{E}(r)-\sqrt{1-r^2}\mathcal{K}(r). \tag{2.7}$$

Then from Lemma 2.2 and (2.7) we clearly see that

$$h(0^+)=\frac{\pi^2}{8},\quad h(1^-)=+\infty,\quad h_1(0^+)=0, \tag{2.8}$$

$$h'(r)=\frac{\mathcal{E}(r)+\sqrt{1-r^2}\mathcal{K}(r)}{r^3(1-r^2)}h_1(r), \tag{2.9}$$

$$h_1'(r)=\frac{r(1-\sqrt{1-r^2})}{\sqrt{1-r^2}}\frac{\mathcal{K}(r)-\mathcal{E}(r)}{r^2}>0 \tag{2.10}$$

for $r\in(0,1)$.

Therefore, Lemma 2.6 follows easily from (2.8)-(2.10). □

3. MAIN RESULTS

Theorem 3.1. *The double inequality*

$$\alpha A(a,b)+(1-\alpha)G(a,b)<T[A(a,b),G(a,b)]<\beta A(a,b)+(1-\beta)G(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq 1/2$ and $\beta\geq 2/\pi=0.6366\cdots$.

Proof. Since $A(a,b)$, $T(a,b)$ and $G(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a>b>0$ and $r=(a-b)/(a+b)\in(0,1)$. Then (1.2) and (1.3) lead to

$$T[A(a,b),G(a,b)]=\frac{2}{\pi}A(a,b)\mathcal{E}(r),\quad G(a,b)=A(a,b)\sqrt{1-r^2},$$

$$\frac{T[A(a,b),G(a,b)]-G(a,b)}{A(a,b)-G(a,b)}=\frac{\frac{2}{\pi}\mathcal{E}(r)-\sqrt{1-r^2}}{1-\sqrt{1-r^2}}. \tag{3.1}$$

Let

$$F_1(r)=\frac{2}{\pi}\mathcal{E}(r)-\sqrt{1-r^2},\quad F_2(r)=1-\sqrt{1-r^2}, \tag{3.2}$$

$$F(r)=\frac{F_1(r)}{F_2(r)}=\frac{\frac{2}{\pi}\mathcal{E}(r)-\sqrt{1-r^2}}{1-\sqrt{1-r^2}}. \tag{3.3}$$

Then Lemma 2.5, (3.2) and (3.3) lead to

$$F_1(0^+)=F_2(0^+)=0, \tag{3.4}$$

$$\frac{F_1'(r)}{F_2'(r)}=\frac{2}{\pi}\frac{\sqrt{1-r^2}[\mathcal{E}(r)-\mathcal{K}(r)]}{r^2}+1, \tag{3.5}$$

$$F(0^+)=\lim_{r\to 0^+}\frac{F_1'(r)}{F_2'(r)}=\frac{1}{2},\quad F(1^-)=\frac{2}{\pi}. \tag{3.6}$$

It follows from Lemmas 2.4 and 2.5 together with (3.3)-(3.5) that $F(r)$ is strictly increasing on $(0,1)$. Therefore, Theorem 3.1 follows from (3.1), (3.3) and (3.6) together with the monotonicity of $F(r)$. □

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.3, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Theorem 3.2. *Let $\lambda, \mu \in (0, 1/2)$. Then the double inequality*

$$G[\lambda a + (1-\lambda)b, \lambda b + (1-\lambda a)] < T[A(a,b), G(a,b)] < G[\mu a + (1-\mu)b, \mu b + (1-\mu a)]$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda \leq (1 - \sqrt{1 - 4/\pi^2})/2 = 0.1144\cdots$ and $\mu \geq 1/2 - \sqrt{2}/4 = 0.1464\cdots$.

Proof. We assume that $a > b > 0$, $r = (a-b)/(a+b) \in (0,1)$ and $p \in (0, 1/2)$. Then (1.2) and (1.3) lead to

$$\begin{aligned} &G[pa + (1-p)b, pb + (1-pa)] - T[A(a,b), G(a,b)] \qquad (3.7)\\ &= A(a,b)\left[\sqrt{1-(1-2p)^2r^2} - \frac{2}{\pi}\mathcal{E}(r)\right]\\ &= \frac{A(a,b)}{\sqrt{1-(1-2p)^2r^2} + \frac{2}{\pi}\mathcal{E}(r)}H(r), \end{aligned}$$

where

$$H(r) = 1 - (1-2p)^2r^2 - \frac{4}{\pi^2}\mathcal{E}^2(r),$$

$$H(0^+) = 0, \tag{3.8}$$

$$H(1^-) = 4p(1-p) - \frac{4}{\pi^2}, \tag{3.9}$$

$$H'(r) = 2rH_1(r), \tag{3.10}$$

where

$$H_1(r) = \frac{4}{\pi^2}\frac{\mathcal{E}(r)[\mathcal{K}(r) - \mathcal{E}(r)]}{r^2} - (1-2p)^2. \tag{3.11}$$

It follows from Lemma 2.6 and (3.11) that

$$H_1(0^+) = \frac{1}{2} - (1-2p)^2, \tag{3.12}$$

$$H_1(1^-) = +\infty. \tag{3.13}$$

We divide the proof into four cases.

Case 1 $p = \mu_0 = 1/2 - \sqrt{2}/4$. Then (3.12) becomes

$$H_1(0^+) = 0. \tag{3.14}$$

From Lemma 2.6, (3.11) and (3.14) we clearly see that

$$H_1(r) > 0 \tag{3.15}$$

for all $r \in (0,1)$. Therefore,

$$T[A(a,b), G(a,b)] < G[\mu_0 a + (1-\mu_0)b, \mu_0 b + (1-\mu_0)a]$$

follows from (3.7), (3.8), (3.10) and (3.15).

Case 2 $p = \lambda_0 = (1 - \sqrt{1-4/\pi^2})/2$. Then (3.9) and (3.12) lead to

$$H(1^-) = 0, \tag{3.16}$$

$$H_1(0^+) = -\frac{\pi^2 - 8}{2\pi^2} < 0. \tag{3.17}$$

From Lemma 2.6, (3.10), (3.11), (3.13) and (3.17) we know that there exists $r_0 \in (0,1)$ such that $H(r)$ is strictly decreasing on $(0, r_0)$ and strictly increasing on $(r_0, 1)$. Therefore,

$$T[A(a,b), G(a,b)] > G[\lambda_0 a + (1-\lambda_0)b, \lambda_0 b + (1-\lambda_0)a]$$

follows from (3.7), (3.8) and (3.16) together with the piecewise monotonicity of $H(r)$.

Case 3 $0 < p = \mu^* < 1/2 - \sqrt{2}/4$. Then (3.12) leads to

$$H_1(0^+) < 0. \tag{3.18}$$

Equations (3.7), (3.8) and (3.10) together with inequality (3.18) imply that there exists small enough $\delta_0 \in (0,1)$ such that

$$T[A(a,b), G(a,b)] > G[\mu^* a + (1-\mu^*)b, \mu^* b + (1-\mu^*)a]$$

for all $a > b > 0$ with $(a-b)/(a+b) \in (0, \delta_0)$.

Case 4 $(1-\sqrt{1-4/\pi^2})/2 < p = \lambda^* < 1/2$. Then (3.9) leads to

$$H(1^-) > 0. \tag{3.19}$$

Equation (3.7) and inequality (3.19) imply that there exists small enough $\delta_1 \in (0,1)$ such that

$$T[A(a,b), G(a,b)] < G[\lambda^* a + (1-\lambda^*)b, \lambda^* b + (1-\lambda^*)a]$$

for all $a > b > 0$ with $(a-b)/(a+b) \in (1-\delta_1, 1)$. □

From Theorems 3.1 and 3.2 we get Corollary 3.3 immediately.

Corollary 3.3. *The double inequality*

$$\max\left\{\frac{\pi}{4}\left(1+\sqrt{1-r^2}\right), \frac{\pi}{2}\sqrt{1+\left(\frac{4}{\pi^2}-1\right)r^2}\right\} < \mathcal{E}(r)$$
$$< \min\left\{1+\left(\frac{\pi}{2}-1\right)\sqrt{1-r^2}, \frac{\sqrt{2}\pi}{4}\left(2-r^2\right)\right\}$$

holds for all $r \in (0,1)$.

References

[1] W.-M. Qian and Y.-M. Chu, Sharp bounds for a special quasi-arithmetic mean in terms of arithmetic and geometric means with two parameters, J. Inequal. Appl., 2017, **2017**, Article 274, 10 pages.
[2] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, Hölder mean inequalities for the complete elliptic integrals, Integral Transforms Spec. Funct., 2012, **23**(7), 521–527.
[3] Y.-M. Chu, M.-K. Wang, Y.-P. Jiang and S.-L. Qiu, Concavity of the complete elliptic integrals of the second kind with respect to Hölder means, J. Math. Anal. Appl., 2012, **395**(2), 637–642.
[4] Y.-M. Chu, M.-K. Wang and Y.-F. Qiu, On Alzer and Qiu's conjecture for complete elliptic integral and inverse hyperbolic tangent function, Abstr. Appl. Anal., 2011, **2011**, Article ID 697547, 7 pages.
[5] Y.-M. Chu, M.-K. Wang, S.-L. Qiu and Y.-P. Jiang, Bounds for complete elliptic integrals of the second kind with applications, Comput. Math. Appl., 2012, **63**(7), 1177–1184.
[6] Y.-M. Chu and T.-H. Zhao, Convexity and concavity of the complete elliptic integrals with respect to Lehmer mean, J. Inequal. Appl., 2015, **2015**, Article 396, 6 pages.
[7] T.-R. Huang, S.-Y. Tan, X.-Y. Ma and Y.-M. Chu, Monotonicity properties and bounds for the complete p-elliptic integrals, J. Inequal. Appl., 2018, **2018**, Article 239, 11 pages.
[8] M.-K. Wang and Y.-M. Chu, Asymptotical bounds for complete elliptic integrals of the second kind, J. Math. Anal. Appl., 2013, **402**(1), 119–126.
[9] M.-K. Wang and Y.-M. Chu, Refinements of transformation inequalities for zero-balanced hypergeometric function, Acta Math. Sci., 2017, **37B**(3), 607–622.
[10] M.-K. Wang and Y.-M. Chu, Landen inequalities for a class of hypergeometric functions with applications, Math. Inequal. Appl., 2018, **21**(2), 521–537.
[11] M.-K. Wang, Y.-M. Chu and Y.-P. Jiang, Ramanujan's cubic transformation inequalities for zero-balanced hypergeometric functions, Rocky Mountain J. Math., 2016, **46**(2), 679–691.
[12] M.-K. Wang, Y.-M. Chu and S.-L. Qiu, Some monotonicity properties of generalized elliptic integrals with applications, Math. Inequal. Appl., 2013, **16**(3), 671–677.
[13] M.-K. Wang, Y.-M. Chu, S.-L. Qiu and Y.-P. Jiang, Convexity of the complete elliptic integrals of the first kind with respect to Hölder means, J. Math. Anal. Appl., 2012, **388**(2), 1141–1146.
[14] M.-K. Wang, Y.-M. Chu and Y.-Q. Song, Asymptotical formulas for Gaussian and generalized hypergeometric functions, Appl. Math. Comput., 2016, **276**, 44–60.
[15] M.-K. Wang, Y.-M. Li and Y.-M. Chu, Inequalities and infinite product formula for Ramanujan generalized modular equation function, Ramanujan J., 2018, **46**(1), 189–200.
[16] M.-K. Wang, S.-L. Qiu and Y.-M. Chu, Infinite series formula for Hübner upper bound function with applications to Hersch-Pfluger distortion function, Math. Inequal. Appl., 2018, **21**(3), 629–648.
[17] M.-K. Wang, S.-L. Qiu, Y.-M. Chu and Y.-P. Jiang, Generalized Hersch-Pfluger distortion function and complete elliptic integrals, J. Math. Anal. Appl., 2012, **385**(1), 221–229.
[18] Zh.-H. Yang and Y.-M. Chu, A monotonicity property involving the generalized elliptic integral of the first kind, Math. Inequal. Appl., 2017, **20**(3), 729–735.
[19] Zh.-H. Yang, Y.-M. Chu and M.-K. Wang, Monotonicity criterion for the quotient of power series with applications, J. Math. Anal. Appl., 2015, **428**(1), 587–604.
[20] Zh.-H. Yang, Y.-M. Chu and W. Zhang, Monotonicity of the ratio for the complete elliptic integral and Stolarsky mean, J. Inequal. Appl., 2016, **2016**, Article 176, 10 pages.
[21] Zh.-H. Yang, Y.-M. Chu and W. Zhang, Accurate approximations for the complete elliptic integral of the second kind, J. Math. Anal. Appl., 2016, **438**(2), 875–888.

[22] Zh.-H. Yang, W.-M. Qian and Y.-M. Chu, Monotonicity properties and bounds involving the complete elliptic integrals of the first kind, Math. Inequal. Appl., 2018, **21**(4), 1185–1199.

[23] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind, J. Math. Anal. Appl., 2018, **462**(2), 1714–1726.

[24] T.-H. Zhao, M.-K. Wang, W. Zhang and Y.-M. Chu, Quadratic transformation inequalities for Gaussian hypergeometric function, J. Inequal. Appl., 2018, **2018**, Article 251, 15 pages.

[25] Gh. Toader, Some mean values related to the arithmetic-geometric mean, J. Math. Anal. Appl., 1998, **218**(2), 358–368.

[26] Y.-M. Chu, M.-K. Wang, S.-L. Qiu and Y.-F. Qiu, Sharp generalized Seiffert mean bounds for Toader mean, Abstr. Appl. Anal., 2011, **2011**, Article ID 605259, 8 pages.

[27] Y.-M. Chu and M.-K. Wang, Inequalities between arithmetic-geometric, Gini, and Toader means, Abstr. Appl. Anal., 2012, **2012**, Article ID 830585, 11 pages.

[28] W.-M. Qian, Z.-H. Zhang and Y.-M. Chu, Sharp bounds for Toader-Qi mean in terms of harmonic and geometric means, J. Math. Inequal., 2017, **11**(1), 121–127.

[29] Y.-M. Chu, M.-K. Wang and S.-L. Qiu, Optimal combinations bounds of root-square and arithmetic means for Toader mean, Proc. Indian Acad. Sci. Math. Sci., 2012, **122**(1), 41–51.

[30] Y.-M. Chu and M.-K. Wang, Optimal Lehmer mean bouns for the Toader mean, Results Math., 2012, **61**(3-4), 223–229.

[31] J.-F. Li, W.-M. Qian and Y.-M. Chu, Sharp bounds for Toader mean in terms of arithmetic, quadratic, and Neuman means, J. Inequal. Appl., 2015, Article 277, 9 pages.

[32] Y.-Q. Song, W.-D. Jiang, Y.-M. Chu and D.-D. Yan, Optimal bounds for Toader mean in terms of arithmetic and contraharmonic means, J. Math. Inequal., 2013, **7**(4), 751–757.

[33] W.-H. Li and M.-M. Zheng, Some inequalities for bounding Toader mean, J. Funct. Spaces Appl., 2013, Article ID 394194, 5 pages.

[34] Y.-M. Chu, M.-K. Wang and X.-Y. Ma, Sharp bounds for Toader mean in terms of contraharmonic mean with applications, J. Math. Inequal., 2013, **7**(2), 161–166.

[35] W.-M. Gong, Y.-Q. Song, Y.-M. Chu and M.-K. Wang, A sharp double inequality between Seiffert, arithmetic, and geoemtric means, Abstr. Appl. Anal., 2012, **2012**, Article ID 684834, 7 pages.

[36] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, Best possible inequalities among harmonic, geometric, logarithmic and Seiffert means, Math. Inequal. Appl., 2012, **15**(2), 415–422.

[37] M.-K. Wang, Z.-K. Wang and Y.-M. Chu, An optimal double inequality between geometric and identric means, Appl. Math. Lett., 2012, **25**(3), 471–475.

[38] Y.-M. Chu, C. Zong and G.-D. Wang, Optimal convex combination bounds of Seiffert and geometric means for arithmetic mean, J. Math. Inequal., 2011, **5**(2), 429–434.

[39] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, An optimal double inequality between Seiffert and geometric means, J. Appl. Math., 2011, **2011**, Article ID 261237, 6 pages.

[40] Y.-M. Chu and M.-K. Wang, Optimal inequalities between harmonic, geometric, logarithmic, and arithmetic-geometric means, J. Appl. Math., 2011, **2011**, Article ID 618929, 9 pages

[41] B.-Y. Long and Y.-M. Chu, Optimal inequalities for generalized logarithmic, arithmetic, and geometric means, J. Inequal. Appl., 2010, **2010**, Article ID 806825, 10 pages.

[42] A. Iqbal, M. Adil Khan, S. Ullah, Y.-M. Chu and A. Kashuri, Hermite-Hadamard type inequalities pertaining conformable fractional integrals and their applications, AIP Advances, 2018, **8**, Article ID 075101, 18 pages, DOI: 10.1063/1.5031954.

[43] Y.-M. Chu, M. Adil Khan, T. Ali and S. S. Dragomir, Inequalities for α-fractional differentiable functions, J. Inequal. Appl., 2017, **2017**, Article 93, 12 pages.

[44] M. Adil Khan, S. Begum, Y. Khurshid and Y.-M. Chu, Ostrowski type inequalities involving conformable fractional integrals, J. Inequal. Appl., 2018, **2018**, Article 70, 14 pages.

[45] M. Adil Khan, Y. Khurshid, T.-S. Du and Y.-M. Chu, Generalized of Hermite-Hadamard type inequalities via conformable fractional integrals, J. Funct. Spaces, 2018, **2018**, Article ID 5357463, 12 pages.

[46] M. Adil Khan, Y.-M. Chu, T. U. Khan and J. Khan, Some new inequalities of Hermite-Hadamard type for s-convex functions with applications, Open Math., 2017, **15**, 1414–1430.

[47] M. Adil Khan, Z. M. Al-sahwi and Y.-M. Chu, New estimations for Shannon and Zipf-Mandelbrot entropies, Entropy, 2018, **20**, Article 608, 10 pages, DOI: 10.3390/e2008608.

[48] M. Adil Khan, Y.-M. Chu, A. Kashuri and R. Liko, Hermite-Hadamard type fractional integral inequalities for $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions, J. Comput. Anal. Appl., 2019, **26**(8), 1487–1503.

[49] Y.-M. Chu, Y.-F. Qiu, M.-K. Wang and G.-D. Wang, The optimal convex combination bounds of arithmetic and harmonic means for the Seiffert mean, J. Inequal. Appl., 2010, **2010**, Article ID 436457, 7 pages.

[50] X.-M. Zhang and Y.-M. Chu, Convexity of the integral arithmetic mean of a convex function, Rocky Mountain J. Math., 2010, **40**(3), 1061–1068.

[51] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Conformal Invariants, Inequalities, and Quasiconformal Maps, John Wiley & Song, New York, 1997.

[52] R. W. Barnard, K. Pearce and K. C. Richards, An inequality involving the generalized hypergeometric function and the arc length of an ellipse, SIAM J. Math. Anal., 2000, **31**(3),693–699.

[53] H. Alzer and S.-L. Qiu, Monotonicity theorems and inequalities for the complete elliptic integrals, J. Comput. Appl. Math., 2004, **172**(2), 289–312.

[54] Y.-Q. Song, T.-H. Zhao, Y.-M. Chu and X.-H. Zhang, Optimal evaluation of a Toader-type mean by power mean, J. Inequal. Appl., 2015, Article 408, 12 pages.

WEI-MAO QIAN, [1]COLLEGE OF SCIENCE, HUNAN CITY UNIVERSITY, YIYANG 413000, HUNAN, CHINA; [2]SCHOOL OF CONTINUING EDUCATION, HUZHOU VOCATIONAL AND TECHNOLOGICAL COLLEGE, HUZHOU 313000, ZHEJIANG, CHINA

E-mail address: qwm661977@126.com

WEN ZHANG, [3]FRIEDMAN BRAIN INSTITUTE, ICAHN SCHOOL OF MEDICINE AT MOUNT SINAI, NEW YORK, NY 10029, USA

E-mail address: zhang.wen81@gmail.com

YU-MING CHU (CORRESPONDING AUTHOR), [4]DEPARTMENT OF MATHEMATICS, HUZHOU UNIVERSITY, HUZHOU 313000, CHINA

E-mail address: chuyuming2005@126.com

Addition Theorem For Exton's q–Exponential Functions

Mahmoud Jafari Shah Belaghi

Bahçeşehir University, Istanbul, Turkey
mahmoud.belaghi@eng.bau.edu.tr

Nuri Kuruoğlu

İstanbul Gelişim University, Istanbul, Turkey
nkuruoglu@gelisim.edu.tr

Abstract. In this paper, we study about the q-exponential function which was introduced by Exton. We propose the addition theorem for this q–exponential function and also Continued fraction representation for this q–exponential function is given.
Keywords. Exton's q-Exponential Function, Symmetric q–derivative, Symmetric q-Binomial.
Mathematics Subject Classification. 11B65, 33D05.

1 Introduction

The $\widetilde{q}$–derivative (or symmetric q–derivative) of a function $f(x)$ is defined [3] as

$$\widetilde{D}_q\ f(x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}$$

where $q \neq \pm 1$. This $\widetilde{q}$–derivative is invariant under inversion of basis.
For any number α, the $\widetilde{q}$–derivative of powers of x are given by

$$\widetilde{D}_q\ x^{\alpha} = [\alpha]_{\widetilde{q}}\ x^{\alpha-1}$$

where $[\alpha]_{\widetilde{q}} = \frac{q^{\alpha} - q^{-\alpha}}{q - q^{-1}}$ and it is called symmetric q–number. In the case, if α is a positive integer we have

$$[\alpha]_{\widetilde{q}} = \frac{q^{\alpha} - q^{-\alpha}}{q - q^{-1}} = q^{1-\alpha}(1 + q^2 + q^4 + \cdots + q^{2\alpha-2}).$$

Relation between q–number and symmetric q–number is

$$[\alpha]_{\widetilde{q}} = \frac{q^{\alpha} - q^{-\alpha}}{q - q^{-1}} = q^{1-\alpha}[\alpha]_{q^2} \tag{1}$$

where $[\alpha]_q = \frac{q^{\alpha}-1}{q-1}$ is called q–number.
With easy calculation, one can see [5] that

$$[\alpha]_{\widetilde{\frac{1}{q}}} = [\alpha]_{\widetilde{q}}. \tag{2}$$

$$[-\alpha]_{\widetilde{q}} = -[\alpha]_{\widetilde{q}}. \tag{3}$$

$$[\alpha + \beta]_{\widetilde{q}} = q^{\beta}[\alpha]_{\widetilde{q}} + q^{-\alpha}[\beta]_{\widetilde{q}}. \tag{4}$$

Furthermore, the $\widetilde{q}$–analogue of factorial, denoted by $[n]_{\widetilde{q}}\ !$, is defined [1] as

$$[n]_{\widetilde{q}}\ ! = \begin{cases} 1 & \text{if } n = 0, \\ [n]_{\widetilde{q}} \times [n-1]_{\widetilde{q}} \times \cdots \times [1]_{\widetilde{q}} & \text{if } n = 1, 2, \ldots \end{cases} \tag{5}$$

and by using (1), we may also write the $\widetilde{q}$–factorial as follows

$$[n]_{\widetilde{q}}\ ! = [n]_{q^2}!\ q^{-\binom{n}{2}} \tag{6}$$

where $[n]_q! = [n]_q \times [n-1]_q \times \cdots \times [1]_q$ for $n = 1, 2, \ldots$.
The $\widetilde{q}$–analogue of $(a - x)^n$, denoted by $(a - x)^n_{\widetilde{q}}$, is defined [3] as

$$(a - x)^n_{\widetilde{q}} = \begin{cases} 1 & n = 0, \\ \prod_{i=0}^{n-1} \left(a - xq^{1-n+2i}\right) & n = 1, 2, \ldots. \end{cases} \tag{7}$$

The $\widetilde{q}$–analogue in (7) is invariant under inversion of basis and one can see that

$$(a-x)_{\widetilde{q}}^{n}=(-1)^{n}(x-a)_{\widetilde{q}}^{n}. \tag{8}$$

The $\widetilde{q}$–derivative of $(x-a)_{\widetilde{q}}^{n}$ is founded [3] as

$$\widetilde{D}_q\ (x-a)_{\widetilde{q}}^{n}=[n]_{\widetilde{q}}\ (x-a)_{\widetilde{q}}^{n-1}. \tag{9}$$

The $\widetilde{q}$–Taylor series expansion of $(a+x)_{\widetilde{q}}^{n}$ about $x=0$ is

$$(a+x)_{\widetilde{q}}^{n}=\sum_{k=0}^{n}\binom{n}{k}_{\widetilde{q}}a^{n-k}x^{k} \tag{10}$$

where $\binom{n}{k}_{\widetilde{q}}=\frac{[n]_{\widetilde{q}}!}{[k]_{\widetilde{q}}!\ [n-k]_{\widetilde{q}}!}$ are called symmetric q–binomial coefficients. Formula (10) is called Gauss's $\widetilde{q}$–binomial formula (see [3], p. 100).

The object of study in this paper is the q–exponential function which was introduced by Exton (see [6] or [4], p. 128) as

$$E(q,x)=\sum_{n=0}^{\infty}\frac{1}{[n]_q!}x^{n}q^{\frac{1}{2}\binom{n}{2}},\qquad x\in\mathbb{C} \tag{11}$$

where $[n]_q=\frac{q^n-1}{q-1}$. This q–exponential function is invariant under inversion of basis and unfortunately, there is no known addition theorem for it. Our goal is to give the addition theorem for this q–exponential function and also represent it as a continued fractions.

2 Some Identities

Definition 1. *For any number α, we define*

$$(a+x)_{\widetilde{q}}^{\alpha}=\frac{(a+q^{1-\alpha}x)_{q^2}^{\infty}}{(a+q^{1+\alpha}x)_{q^2}^{\infty}} \tag{12}$$

where $(a+x)_q^{\infty}:=\lim_{n\to\infty}\prod_{j=0}^{n}(a+q^jx)$.

Theorem 1. *For any numbers α and β,*

$$(a+x)_{\widetilde{q}}^{\alpha+\beta}=(a+q^{-\beta}x)_{\widetilde{q}}^{\alpha}\ (a+q^{\alpha}x)_{\widetilde{q}}^{\beta}.$$

Proof. The result will be obtained directly by using the definition of $(a+x)_{\widetilde{q}}^{\alpha}$, which is given in (12). ☐

Corollary 1. *For any number α,*

$$(a+x)_{\widetilde{q}}^{-\alpha}=\frac{1}{(a+x)_{\widetilde{q}}^{\alpha}}$$

Proof. The result will be obtained by using (12).

☐

Proposition 1. *For $1\le j\le n-1$, the $\widetilde{q}$–Pascal rule is*

$$\binom{n}{j}_{\widetilde{q}}=q^{n-j}\binom{n-1}{j-1}_{\widetilde{q}}+q^{-j}\binom{n-1}{j}_{\widetilde{q}}$$

Proof. Let us expand the symmetric q–binomial coefficient $\binom{n}{j}_{\widetilde{q}}$, then we have

$$\begin{aligned}\binom{n}{j}_{\widetilde{q}}&=\frac{[n]_{\widetilde{q}}!}{[j]_{\widetilde{q}}!\ [n-j]_{\widetilde{q}}!}\\&=\frac{[n-1]_{\widetilde{q}}!\ [n]_{\widetilde{q}}}{[j]_{\widetilde{q}}!\ [n-j]_{\widetilde{q}}!}\\&=\frac{[n-1]_{\widetilde{q}}!\ (q^{n-j}[j]_{\widetilde{q}}+q^{-j}[n-j]_{\widetilde{q}})}{[j]_{\widetilde{q}}!\ [n-j]_{\widetilde{q}}!}\\&=q^{n-j}\binom{n-1}{j-1}_{\widetilde{q}}+q^{-j}\binom{n-1}{j}_{\widetilde{q}}\end{aligned}$$

which completes the proof. We used (4) in the third line. ☐

Lemma 1. *For any number x and positive integer r,*

$$\binom{-x}{r}_{\widetilde{q}} = (-1)^r \binom{x+r-1}{r}_{\widetilde{q}}$$

Proof. To prove the lemma we make a use of (5) and (3), then we may write

$$\begin{aligned}\binom{-x}{r}_{\widetilde{q}} &= \frac{[-x]_{\widetilde{q}}!}{[r]_{\widetilde{q}}!\ [-x-r]_{\widetilde{q}}!}\\ &= (-1)^r\ \frac{[x]_{\widetilde{q}}\ [x+1]_{\widetilde{q}}\ \dots\ [x+r-1]_{\widetilde{q}}}{[r]_{\widetilde{q}}!}\\ &= (-1)^r\ \frac{[x+r-1]_{\widetilde{q}}!}{[x-1]_{\widetilde{q}}!\ [r]_{\widetilde{q}}!} = (-1)^r \binom{x+r-1}{r}_{\widetilde{q}}\end{aligned}$$

which completes the proof. □

The following theorem is a symmetric version of Heine's q–binomial formula.

Theorem 2. *For any number x and positive integer n, the following equation holds*

$$\frac{1}{(1-x)_{\widetilde{q}}^{n}} = \sum_{j=0}^{\infty} \binom{n+j-1}{j}_{\widetilde{q}} x^j.$$

Proof. To prove the Theorem we make a use of Corollary 1 and Lemma 1, then we may write

$$\frac{1}{(1-x)_{\widetilde{q}}^{n}} = (1-x)_{\widetilde{q}}^{-n} = \sum_{j=0}^{\infty} \binom{-n}{j}_{\widetilde{q}} (-x)^j = \sum_{j=0}^{\infty} \binom{n+j-1}{j}_{\widetilde{q}} x^j$$

which completes the proof. □

In the next theorem, $\widetilde{q}$–analogue of Vandermonde's identity is given.

Theorem 3. *For any $m, n, r \in \mathbb{N}_0$*

$$\binom{m+n}{r}_{\widetilde{q}} = q^{mr} \sum_{k=0}^{r} \binom{m}{k}_{\widetilde{q}} \binom{n}{r-k}_{\widetilde{q}} q^{-(m+n)k}$$

Proof. We make a use of Theorem 1 to write that

$$(1+x)_{\widetilde{q}}^{m+n} = (1+q^{-n}x)_{\widetilde{q}}^{m}\ (1+q^{m}x)_{\widetilde{q}}^{n}.$$

Using the $\widetilde{q}$–binomial formula in (10) for both sides of the above formula, and then we obtain

$$\begin{aligned}\sum_{r=0}^{m+n} \binom{m+n}{r}_{\widetilde{q}} x^r &= \sum_{r=0}^{m} \binom{m}{r}_{\widetilde{q}} (q^{-n}x)^r\ \sum_{r=0}^{n} \binom{n}{r}_{\widetilde{q}} (q^{m}x)^r\\ &= \sum_{r=0}^{m+n} \left(q^{mr} \sum_{k=0}^{r} \binom{m}{k}_{\widetilde{q}} \binom{n}{r-k}_{\widetilde{q}} q^{-(m+n)k} \right) x^r.\end{aligned}$$

The proof is complete by comparing coefficients of x^r. □

The following corollary is the special case of Vandermonde's identity.

Corollary 2. *For any positive integer n,*

$$\sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} q^{n(n-2k)} = \binom{2n}{n}_{\widetilde{q}} \tag{13}$$

Proof. Take $m = r = n$ in Theorem 3 and make a use of the identity $\binom{n}{k}_{\widetilde{q}} = \binom{n}{n-k}_{\widetilde{q}}$ to prove the corollary. □

Corollary 3. *For any positive integer n,*

$$\sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} [2k]_{\widetilde{q^n}} = [n]_{\widetilde{q^n}} \binom{2n}{n}_{\widetilde{q}}. \tag{14}$$

Proof. Let us change the base q to q^{-1} in Corollary 2 to obtain

$$\sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} q^{-n(n-2k)} = \binom{2n}{n}_{\widetilde{q}} \tag{15}$$

because of the identity $\binom{n}{k}_{\widetilde{\frac{1}{q}}} = \binom{n}{k}_{\widetilde{q}}$. Now by comparing equations (13) and (15) we can write

$$\sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} \left(q^{n(n-2k)} - q^{-n(n-2k)}\right) = 0 \tag{16}$$

and also

$$\sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} [n-2k]_{\widetilde{q^n}} = 0 \tag{17}$$

since $[\alpha]_{\widetilde{q}} = \frac{q^{\alpha} - q^{-\alpha}}{q - q^{-1}}$. Then we make a use of equations (3) and (4) to rewrite the equation (17) as

$$\sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} \left(q^{2nk}[n]_{\widetilde{q^n}} + q^{n^2}[-2k]_{\widetilde{q^n}}\right) = 0,$$
$$\sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} \left(q^{2nk}[n]_{\widetilde{q^n}} - q^{n^2}[2k]_{\widetilde{q^n}}\right) = 0,$$
$$\sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} [2k]_{\widetilde{q^n}} = [n]_{\widetilde{q^n}} \sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}}^{2} q^{-n^2+2nk}.$$

The proof will be complete if we apply the identity in (15) to the right side of the last equation. □

3 $\widetilde{q}$–Exponential Functions

In this section, we study about the q–exponential functions (11) which was introduced by Exton. Let us consider $E(q^2, x)$, then we have

$$E(q^2, x) = \sum_{n=0}^{\infty} \frac{1}{[n]_{q^2}!} x^n q^{\binom{n}{2}}, \qquad x \in \mathbb{C}. \tag{18}$$

Now we make a use of (6) to rewrite the above formula as follows

$$E(q^2, x) = \sum_{n=0}^{\infty} \frac{1}{[n]_{\widetilde{q}}!} x^n, \qquad x \in \mathbb{C}. \tag{19}$$

We use a different notation for the Exton's q–exponential function as

$$e_{\widetilde{q}}^{x} := E(q^2, x) = \sum_{n=0}^{\infty} \frac{1}{[n]_{\widetilde{q}}!} x^n, \qquad x \in \mathbb{C}. \tag{20}$$

One can see that this $\widetilde{q}$–exponential function (20) is invariant under inversion of basis and its $\widetilde{q}$–derivative is equal to itself, that means

$$e_{\widetilde{\frac{1}{q}}}^{x} = e_{\widetilde{q}}^{x} \tag{21}$$

$$\widetilde{D}_q \, e_{\widetilde{q}}^{x} = e_{\widetilde{q}}^{x} \tag{22}$$

The next theorem is about the product of two $\widetilde{q}$–exponential functions.

Theorem 4. *For any x and y, the following equation holds*

$$e_{\widetilde{q}}^{x}\ e_{\widetilde{q}}^{y} = e_{\widetilde{q}}^{(x+y)_{\widetilde{q}}} \tag{23}$$

where $(x+y)_{\widetilde{q}}^{n}$ is defined in (10).

Proof. We use (20) to expand both $e_{\widetilde{q}}^{x}$ and $e_{\widetilde{q}}^{y}$, therefore we obtain

$$\begin{aligned}
e_{\widetilde{q}}^{x}\ e_{\widetilde{q}}^{y} &= \sum_{n=0}^{\infty} \frac{1}{[n]_{\widetilde{q}}!} x^n \ \sum_{n=0}^{\infty} \frac{1}{[n]_{\widetilde{q}}!} y^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{[k]_{\widetilde{q}}!\ [n-k]_{\widetilde{q}}!} x^k y^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{[n]_{\widetilde{q}}!} \sum_{k=0}^{n} \binom{n}{k}_{\widetilde{q}} x^k y^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{[n]_{\widetilde{q}}!} (x+y)_{\widetilde{q}}^{n} \\
&= e_{\widetilde{q}}^{(x+y)_{\widetilde{q}}}
\end{aligned}$$

and the proof is complete. □

4 Continued Fractions

A continued fraction is an expression of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}},$$

where $a_0, a_1, a_2, \ldots$ and $b_1, b_2, b_3, \ldots$ are two sequences of real or complex numbers. We use the following symbol for the above continued fraction

$$a_0 + \mathop{\mathrm{K}}_{n=1}^{\infty} \left[\frac{b_n}{a_n}\right]. \tag{24}$$

The following theorem is the convergent theorem of continued fractions (See [7], p. 126).

Theorem 5. *If $a_n > 0$ for $n > 1$ then the continued fraction $K_{n=1}^{\infty}\left[\frac{1}{a_n}\right]$ converges if and only if the series $\sum_{n=1}^{\infty} a_n$ diverges.*

4.1 Continued Fraction Representation of $\widetilde{q}$–Exponential Functions

The q–exponential functions e_q^x and E_q^x can be written as infinite product form as follows

$$e_q^x = \frac{1}{(1-(1-q)x))_q^{\infty}}, \qquad E_q^x = (1+(1-q)x))_q^{\infty}.$$

In this section, we want to show that the $\widetilde{q}$–exponential function also can be written as infinite product form.
Let us consider the $\widetilde{q}$–derivative of a function $f(x)$ which is defined as

$$\widetilde{D}_q f(x) = \frac{f(qx) - f(q^{-1}x)}{(q-q^{-1})x}. \tag{25}$$

Take $f(x) = e_{\widetilde{q}}^{qx}$, therefore we can write (25) as follows

$$q\ e_{\widetilde{q}}^{qx} = \frac{e_{\widetilde{q}}^{q^2 x} - e_{\widetilde{q}}^{x}}{(q-q^{-1})x}, \tag{26}$$

because $\widetilde{D}_q\, e_{\widetilde{q}}^{qx} = q\, e_{\widetilde{q}}^{qx}$. Now by easy manipulation, we may write (26) as

$$\frac{e_{\widetilde{q}}^{x}}{e_{\widetilde{q}}^{qx}} - \frac{e_{\widetilde{q}}^{q^2x}}{e_{\widetilde{q}}^{qx}} = (1-q^2)x. \tag{27}$$

Let us define $g(x) := \frac{e_{\widetilde{q}}^{x}}{e_{\widetilde{q}}^{qx}}$ and then we may write (27) as

$$g(x) = (1-q^2)x + \frac{1}{g(qx)}. \tag{28}$$

Iterating the formula in (28) infinity many times to obtain

$$g(x) = (1-q^2)x + \cfrac{1}{(1-q^2)qx + \cfrac{1}{(1-q^2)q^2x + \cfrac{1}{(1-q^2)q^3x + \cdots}}}. \tag{29}$$

Now by using continued fractions symbol which is defined in (24), we may rewrite (29) as follows

$$g(x) = (1-q^2)x + \mathop{\mathrm{K}}_{n=1}^{\infty}\left[\frac{1}{(1-q^2)q^nx}\right] \tag{30}$$

or

$$\frac{1}{g(x)} = \mathop{\mathrm{K}}_{n=0}^{\infty}\left[\frac{1}{(1-q^2)q^nx}\right]. \tag{31}$$

By using Theorem 5, one can see that the continued fraction in the right hand side of equation (31) is converge, if $x<0$ and $q>1$.
Substitute x with $q^{-1}x$ in the equation (31) and then replace $g(x) = \frac{e_{\widetilde{q}}^{x}}{e_{\widetilde{q}}^{qx}}$ to obtain

$$e_{\widetilde{q}}^{x} = \mathop{\mathrm{K}}_{n=0}^{\infty}\left[\frac{1}{(1-q^2)q^{n-1}x}\right]\, e_{\widetilde{q}}^{q^{-1}x}. \tag{32}$$

Iterating this formula k times to obtain

$$e_{\widetilde{q}}^{x} = \prod_{j=1}^{k}\mathop{\mathrm{K}}_{n=0}^{\infty}\left[\frac{1}{(1-q^2)q^{n-j}x}\right]\, e_{\widetilde{q}}^{q^{-k}x}. \tag{33}$$

In the case, if $k\to\infty$, we have

$$e_{\widetilde{q}}^{x} = \prod_{j=1}^{\infty}\mathop{\mathrm{K}}_{n=0}^{\infty}\left[\frac{1}{(1-q^2)q^{n-j}x}\right] \tag{34}$$

because if $q>1$, then we have $\lim_{k\to\infty} e_{\widetilde{q}}^{q^{-k}x} = 1$.

References

[1] Ernst, T., *A Comprehensive Treatment of Q-calculus*, Springer, 2012.

[2] Gasper, G., Rahman, M., *Basic hypergeometric series*, Vol. 96 Cambridge university press, 2004.

[3] Kac, V., Cheung, P., *Quantum Calculus*, Springer, 2002.

[4] Exton, H., *q-Hypergeometric Functions and ApplicationsEllis*, Horwood, Chichester, 1983.

[5] McAnally, D. S., q-exponential and q-gamma functions. I. q-exponential functions. *Journal of Mathematical Physics*, Vol. 36, no. 1, pp. 546–573, 1995.

[6] Exton, H. Basic circular functions. In *Indagationes Mathematicae (Proceedings)*, Vol. 84, no. 2, pp. 165–171, North-Holland. 1981.

[7] Khrushchev, S.V.: Orthogonal Polynomials and Continued Fractions from Euler's Point of View, Encyclopedia of Mathematics and Its Applications, Vol. 122. Cambridge. Cambridge University Press 2008.

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO. 3, 2020

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO. 3, 2020

(continued)

Volume 28, Number 4 August 2020
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(six times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, St.Martin Univ.,Olympia,WA,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Editorial Board
Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Jong Kyu Kim
Department of Mathematics
Kyungnam University
Masan Kyungnam,631-701,Korea
Tel 82-(55)-249-2211
Fax 82-(55)-243-8609
jongkyuk@kyungnam.ac.kr
Nonlinear Functional Analysis,
Variational Inequalities, Nonlinear
Ergodic Theory, ODE, PDE,
Functional Equations.

Robert Kozma
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
rkozma@memphis.edu
Neural Networks, Reproducing Kernel
Hilbert Spaces,
Neural Percolation Theory

Mustafa Kulenovic
Department of Mathematics
University of Rhode Island
Kingston, RI 02881,USA
kulenm@math.uri.edu
Differential and Difference
Equations

Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional
Analysis, lasiecka@memphis.edu

Burkhard Lenze
Fachbereich Informatik
Fachhochschule Dortmund
University of Applied Sciences
Postfach 105018
D-44047 Dortmund, Germany
e-mail: lenze@fh-dortmund.de
Real Networks, Fourier Analysis,
Approximation Theory

Hrushikesh N. Mhaskar
Department Of Mathematics
California State University
Los Angeles, CA 90032
626-914-7002
e-mail: hmhaska@gmail.com
Orthogonal Polynomials,
Approximation Theory, Splines,
Wavelets, Neural Networks

Ram N. Mohapatra
Department of Mathematics
University of Central Florida
Orlando, FL 32816-1364
tel.407-823-5080
ram.mohapatra@ucf.edu
Real and Complex Analysis,
Approximation Th., Fourier
Analysis, Fuzzy Sets and Systems

Gaston M. N'Guerekata
Department of Mathematics
Morgan State University
Baltimore, MD 21251, USA
tel: 1-443-885-4373
Fax 1-443-885-8216
Gaston.N'Guerekata@morgan.edu
nguerekata@aol.com
Nonlinear Evolution Equations,
Abstract Harmonic Analysis,
Fractional Differential Equations,
Almost Periodicity & Almost
Automorphy

M.Zuhair Nashed
Department Of Mathematics
University of Central Florida
PO Box 161364
Orlando, FL 32816-1364
e-mail: znashed@mail.ucf.edu
Inverse and Ill-Posed problems,
Numerical Functional Analysis,
Integral Equations, Optimization,
Signal Analysis

Mubenga N. Nkashama
Department OF Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
205-934-2154
e-mail: nkashama@math.uab.edu
Ordinary Differential Equations,
Partial Differential Equations

Vassilis Papanicolaou
Department of Mathematics
National Technical University of
Athens
Zografou campus, 157 80
Athens, Greece
tel:: +30(210) 772 1722
Fax +30(210) 772 1775
papanico@math.ntua.gr
Partial Differential Equations,
Probability

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics, Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

T. E. Simos
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

Fractional: Differential Equations-Operators-Fourier Transforms, Special functions, Approximations, and Applications

Ram Verma
International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis, Numerical Analysis, Variational Inequalities, Optimization Theory, Computational Mathematics, Operator Theory

Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Xiao-Jun Yang
State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou 221116, China
Local Fractional Calculus and Applications, Fractional Calculus and Applications, General Fractional Calculus and Applications, Variable-order Calculus and Applications, Viscoelasticity and Computational methods for Mathematical Physics. dyangxiaojun@163.com

Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

Ding-Xuan Zhou
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

Xin-long Zhou
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Jessada Tariboon
Department o Mathemati s
King Mongkut s niversity o
Te hnology . angkok
Pra harat Rd. ongsa ang
angsue angkok Thailand
jessada.t@sci.kmutnb.ac.th, Time scales, Di erential Di eren e uations
ra tional Di erential uations

Instructions to Contributors

Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

<u>Journal Article</u>

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

<u>Book</u>

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

<u>Contribution to a Book</u>

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus <u>homepage</u>.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

HERMITE-HADAMARD TYPE INEQUALITIES INVOLVING CONFORMABLE FRACTIONAL INTEGRALS*

YOUSAF KHURSHID[1,2], MUHAMMAD ADIL KHAN[2], AND YU-MING CHU[3,**]

Abstract. In the article, we establish an identity and several new Hermite-Hadamard type inequalities for conformable fractional integrals. As applications, we provide some inequalities for certain bivariate means and present the error estimations for the trapezoidal formula. The given results are the generalization of previously results.

1. Introduction

A real-valued function $f : I \to \mathbb{R}$ is said to be convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$. If inequality (1.1) holds in the reverse direction, then we say that f is a concave function on the interval I.

The word "convexity" is one of the most important, natural and fundamental notations in mathematics. Convex functions were presented by Johan Jensen over 100 years ago. Over the past few years, many generalizations and extensions have been made for convexity. These extensions and generalizations in the theory of inequalities have made valuable contributions in many areas of mathematics. Some new generalized concepts in this point of view are quasi-convex [1], strongly convex [2], approximately convex [3], logarithmically convex [4], midconvex functions [5], pseudo-convex [6], φ-convex [7], λ-convex [8], h-convex [9], delta-convex [10], Schur convex [11-17] and and others [18-24].

Let $I \subseteq \mathbb{R}$ be an interval and $h : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function. Then the well-known Hermite-Hadamard inequality [25-33] for convex functions states that the double inequality

$$h\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} h(x)dx \leq \frac{h(a_1) + h(a_2)}{2} \tag{1.2}$$

holds for all $a_1, a_2 \in I$ with $a_1 \neq a_2$. If the function h is concave on I, then both the inequalities in (1.2) hold in the reverse direction. It gives an estimate from both sides of the mean value of a convex function and also ensure the integrability of convex function. It is also a matter of great interest and one has to note that some

2010 *Mathematics Subject Classification.* Primary: 26D15, 26A33; Secondary: 26A51, 26A42.
Key words and phrases. Hermite-Hadamard inequality, convex function, conformable fractional integral, mean, midpoint formula.
*The research was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11301127, 11701176, 11626101, 11601485) and the Science and Technology Research Program of Zhejiang Educational Committee (Grant no. Y201635325).
**Corresponding author: Yu-Ming Chu, Email: chuyuming2005@126.com.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

of the classical inequalities for means can be obtained from Hadamard's inequality under the utility of peculiar convex functions h. These inequalities for convex functions play a crucial role in analysis and as well as in other areas of pure and applied mathematics. In the last 60 years, many efforts have gone on generalizations extensions and variants of Hermite-Hadamard's inequality (see [34-36]).

Recently, the authors in [37] defined the conformable fractional derivative as follows: for a function $h:[0,\infty)\to\mathbb{R}$ the (conformable) fractional derivative of order $0<\alpha\leq 1$ of h at $s>0$ was defined by

$$D_\alpha(h)(s)=\lim_{\epsilon\to 0}\frac{h(s+\epsilon s^{1-\alpha})-f(s)}{\epsilon},$$

if the conformable fractional derivative of h of order α exists, then we say that h is α-differentiable. The fractional derivative at 0 is defined as $h^\alpha(0)=\lim_{s\to 0^+}h^\alpha(s)$.

Now we recall some results for the conformable fractional derivative.

Theorem 1.1. *Let $\alpha\in(0,1]$ and h_1,h_2 be α-differentiable at a point $s>0$. Then*

$$\frac{d_\alpha}{d_\alpha s}\left(s^n\right)=ns^{n-\alpha}$$

for all $n\in\mathbb{R}$;

$$\frac{d_\alpha}{d_\alpha s}(c)=0$$

for all constant $c\in\mathbb{R}$;

$$\frac{d_\alpha}{d_\alpha s}\left(a_1h_1(s)+a_2h_2(s)\right)=a_1\frac{d_\alpha}{d_\alpha s}(h_1(s))+a_2\frac{d_\alpha}{d_\alpha s}(h_2(s))$$

for all constants $a_1,a_2\in\mathbb{R}$;

$$\frac{d_\alpha}{d_\alpha s}\left(h_1(s)h_2(s)\right)=h_1(s)\frac{d_\alpha}{d_\alpha s}(h_2(s))+h_2(s)\frac{d_\alpha}{d_\alpha s}(h_1(s));$$

$$\frac{d_\alpha}{d_\alpha s}\left(\frac{h_1(s)}{h_2(s)}\right)=\frac{h_2(s)\frac{d_\alpha}{d_\alpha s}(h_1(s))-h_1(s)\frac{d_\alpha}{d_\alpha s}(h_2(s))}{(h_2(s))^2};$$

$$\frac{d_\alpha}{d_\alpha s}\left(h_1(h_2)(s)\right)=h_1'(h_2(s))\frac{d_\alpha}{d_\alpha s}(h_2(s))$$

if h_1 differentiable at $h_2(s)$.

If in addition h_1 is differentiable, then one has

$$\frac{d_\alpha}{d_\alpha s}\left(h_1(s)\right)=s^{1-\alpha}\frac{d}{ds}(h_1(s)).$$

Definition 1.2. ***(Conformable fractional integral)*** *Let $\alpha\in(0,1]$ and $0\leq a_1<a_2$. Then the function $h_1:[a_1,a_2]\to\mathbb{R}$ is said to be α-fractional integrable on $[a_1,a_2]$ if the integral*

$$\int_{a_1}^{a_2}h_1(x)d_\alpha x:=\int_{a_1}^{a_2}h_1(x)x^{\alpha-1}dx$$

exists and is finite. All α-fractional integrable functions on $[a_1,a_2]$ is indicated by $L^1_\alpha([a_1,a_2])$.

Remark 1. *Let $\alpha \in (0,1]$. Then*

$$I_\alpha^{a_1}(h_1)(s) = I_1^{a_1}(s^{\alpha-1}h_1) = \int_{a_1}^{s} \frac{h_1(x)}{x^{1-\alpha}}dx,$$

where the integral is the usual Riemann improper integral.

Anderson [38] established the conformable integral version of Hermite-Hadamard inequality as follows:

Theorem 1.3. *If $\alpha \in (0,1]$ and $h : [a_1, a_2] \to \mathbb{R}$ is an α-fractional differentiable function such that $D_\alpha h$ is increasing, then one has*

$$\frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} h(x)d_\alpha x \leq \frac{h(a_1)+h(a_2)}{2}.$$

Moreover, if the function h is decreasing on $[a_1, a_2]$, then we have

$$h\left(\frac{a_1+a_2}{2}\right) \leq \frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} h(x)d_\alpha x.$$

In particular, if $\alpha = 1$, then this reduces to the classical Hermite-Hadamard inequality.

Due to the great importance of Hermite-Hadamard inequality, in recent years many mathematician have shown their interest for generalizations, extensions and variants for this inequality. In the article, we deal with the conformable integral version of Hermite-Hadamard inequality investigated by Anderson [38]. We shall establish an identity for the left side of the inequality and discuss their particular case. By applying Jensen's inequality, power mean inequality and the convexity of the functions $x^{\alpha-1}$ and $-x^\alpha$ $(x > 0, \alpha \in (0,1])$ in the identity, we obtain inequalities for conformable integral version of Hermite-Hadamard inequality. By using particular classes of convex functions we find new inequalities for special bivariate means. We also apply the results for error estimations of the mid point formula, for some related results (see [39-42]).

2. Main Results

We begin this section with the following Lemma 2.1, which is needed for the establishment of our main results.

Lemma 2.1. *Let $\alpha \in (0,1]$, $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$ and $h : [a_1, a_2] \to \mathbb{R}$ be an α-fractional differentiable function. Then the identity*

$$h\left(\frac{a_1+a_2}{2}\right) - \frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} h(x)d_\alpha x \tag{2.1}$$
$$= \frac{(a_2-a_1)}{2(a_2^\alpha - a_1^\alpha)}\Bigg[\int_0^1 \left(\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)^{2\alpha-1} - a_1^\alpha\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)^{\alpha-1}\right)$$
$$\times D_\alpha(h)\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)s^{1-\alpha}d_\alpha s + \int_0^1 \left(\left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)^{2\alpha-1}\right.$$
$$\left. - a_2^\alpha\left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)^{\alpha-1}\right) \times D_\alpha(h)\left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)s^{1-\alpha}d_\alpha s\Bigg].$$

holds if $D_\alpha(h) \in L_\alpha^1([a_1, a_2])$.

Proof. Integrating by parts, we have

$$\int_0^1 \left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{2\alpha-1} - a_1^\alpha\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha-1}\right)\times D_\alpha(h)\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)ds$$

$$+\int_0^1 \left(\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{2\alpha-1} - a_2^\alpha\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha-1}\right)$$

$$\times D_\alpha(h)\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)ds$$

$$=\int_0^1 \left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha} - a_1^\alpha\right)h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)ds$$

$$+\int_0^1 \left(\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha} - a_2^\alpha\right)h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)ds$$

$$=\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha} - a_1^\alpha\right)\frac{h\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)}{\frac{a_2-a_1}{2}}\Bigg|_0^1$$

$$-\int_0^1 \alpha\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha-1}\left(\frac{a_2-a_1}{2}\right)\frac{h\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)}{\frac{a_2-a_1}{2}}ds$$

$$+\left(\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha} - a_2^\alpha\right)\frac{h\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)}{\frac{a_2-a_1}{2}}\Bigg|_0^1$$

$$-\int_0^1 \alpha\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha-1}\left(\frac{a_2-a_1}{2}\right)\frac{h\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)}{\frac{a_2-a_1}{2}}ds$$

$$=\frac{2}{a_2-a_1}\left[\left(\left(\frac{a_1+a_2}{2}\right)^{\alpha}-a_1^\alpha\right)h\left(\frac{a_1+a_2}{2}\right)-\alpha\int_a^{\frac{a_1+a_2}{2}}h(s)d_\alpha s\right]$$

$$+\frac{2}{a_2-a_1}\left[\left(a_2^\alpha-\left(\frac{a_1+a_2}{2}\right)^{\alpha}\right)h\left(\frac{a_1+a_2}{2}\right)-\alpha\int_{\frac{a_1+a_2}{2}}^{a_2}h(s)d_\alpha s\right]$$

$$=\frac{2(a_2^\alpha-a_1^\alpha)}{a_2-a_1}h\left(\frac{a_1+a_2}{2}\right)-\frac{2\alpha}{a_2-a_1}\int_{a_1}^{a_2}h(x)d_\alpha x,$$

where we have used the change of variable $x=(1-s)a_1+sa_2$ and then multiplying both sides by $\frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}$ to get the desired result in (2.1). □

Remark 2. *By putting $\alpha=1$ in (2.1), we get the identity*

$$h\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}h(x)dx$$

$$=\frac{a_2-a_1}{4}\left[\int_0^1 sh'\left(\frac{sa_2}{2}+\frac{2-s}{2}a_1\right)ds-\int_0^1(1-s)h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)ds\right].$$

Theorem 2.2. *Let $\alpha \in (0,1]$, $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$ and $h : [a_1, a_2] \to \mathbb{R}$ be an α-differentiable function on (a_1, a_2). Then the inequality*

$$\left|h\left(\frac{a_1+a_2}{2}\right) - \frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} h(x)d_\alpha x\right| \tag{2.2}$$

$$\leq \frac{a_2-a_1}{2(a_2^\alpha - a_1^\alpha)}\left[\frac{|h'(a_1)|}{96}\left[13a_2^\alpha - 19a_1^\alpha\right] + \frac{|h'(a_2)|}{96}\left[19a_2^\alpha - 21a_1^\alpha\right]\right.$$

$$\left. -a_1^\alpha a_2^{\alpha-1}\left[\frac{2|h'(a_1)| + |h'(a_2)|}{12}\right] + (a_1 a_2^{\alpha-1} + a_1^{\alpha-1}a_2)\left[\frac{11|h'(a_1)| + 5|h'(a_2)|}{96}\right]\right]$$

holds if $D_\alpha(h) \in L_\alpha^1([a_1, a_2])$ and $|h'|$ is convex on $[a_1, a_2]$.

Proof. Let $\varphi_1 = x^{\alpha-1}$ and $\varphi_2 = -x^\alpha$ $(x > 0, \alpha \in (0,1])$. Then we clearly see that the functions φ_1 and φ_2 are convex. Now using Lemma 2.1 and the convexity of φ_1, φ_2 and $|h'|$, we have

$$\left|h\left(\frac{a_1+a_2}{2}\right) - \frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} h(x)d_\alpha x\right|$$

$$\leq \frac{a_2-a_1}{2(a_2^\alpha - a_1^\alpha)}\left[\int_0^1 \left(\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)^\alpha - a_1^\alpha\right)\left|h'\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)\right|ds\right.$$

$$\left. + \int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)^\alpha\right)\left|h'\left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)\right|ds\right]$$

$$= \frac{a_2-a_1}{2(a_2^\alpha - a_1^\alpha)}\left[\int_0^1 \left(\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)^{\alpha+1-1} - a_1^\alpha\right)\left|h'\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)\right|ds\right.$$

$$\left. + \int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)^\alpha\right)\left|h'\left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)\right|ds\right]$$

$$\leq \frac{a_2-a_1}{2(a_2^\alpha - a_1^\alpha)}\left[\int_0^1 \left(\left(\frac{2-s}{2}a_1 + \frac{s}{2}a_2\right)^{\alpha-1}\left(\frac{2-s}{2}a_1 + \frac{s}{2}a_2\right) - a_1^\alpha\right)\left|h'\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)\right|ds\right.$$

$$\left. + \int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1^\alpha + \frac{1+s}{2}a_2^\alpha\right)\right)\left|h'\left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)\right|ds\right]$$

$$\leq \frac{a_2-a_1}{2(a_2^\alpha - a_1^\alpha)}\left[\int_0^1 \left(\left(\frac{2-s}{2}a_1^{\alpha-1} + \frac{s}{2}a_2^{\alpha-1}\right)\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right) - a_1^\alpha\right)\left|h'\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)\right|ds\right.$$

$$\left. + \int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1^\alpha + \frac{1+s}{2}a_2^\alpha\right)\right)\left|h'\left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)\right|ds\right]$$

$$\leq \frac{b-a}{2(a_2^\alpha - a_1^\alpha)}\left[\int_0^1 \left(\left(\frac{2-s}{2}a_1^{\alpha-1} + \frac{s}{2}a_2^{\alpha-1}\right)\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right) - a_1^\alpha\right)\left[\frac{2-s}{2}|h'(a_1)|\right.\right.$$

$$\left.\left. + \frac{s}{2}|h'(a_2)|\right]ds + \int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1^\alpha + \frac{1+s}{2}a_2^\alpha\right)\right)\left[\frac{1-s}{2}|h'(a_1)| + \frac{1+s}{2}|h'(a_2)|\right]ds\right].$$

Evaluating all the above integrals, we have the following

$$\frac{a_2-a_1}{2(a_2^\alpha - a_1^\alpha)}\left[\int_0^1 \left(\left(\frac{2-s}{2}a_1^{\alpha-1} + \frac{s}{2}a_2^{\alpha-1}\right)\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right) - a_1^\alpha\right)\left[\frac{2-s}{2}|h'(a_1)|\right.\right.$$

$$+\frac{s}{2}|h'(a_2)|\Bigg]ds+\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1^\alpha+\frac{1+s}{2}a_2^\alpha\right)\right)\left[\frac{1-s}{2}|h'(a_1)|+\frac{1+s}{2}|h'(a_2))|\right]ds$$

$$=\frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\Bigg[\frac{15}{32}a_1^\alpha|h'(a_1)|+\frac{11}{96}a_1^{\alpha-1}a_2|h'(a_1)|-\frac{7}{12}a_1^\alpha|h'(a_1)|+\frac{11}{96}a_1a_2^{\alpha-1}|h'(a_1)|$$
$$+\frac{5}{96}a_2^\alpha|h'(a_1)|-\frac{1}{6}a_1^\alpha a_2^{\alpha-1}|h'(a_1)|+\frac{11}{96}a_1^\alpha|h'(a_2)|+\frac{5}{96}a_1^{\alpha-1}a_2|h'(a_2)|-\frac{1}{6}a_1^\alpha|h'(a_2)|$$
$$+\frac{5}{96}a_1a_2^{\alpha-1}|h'(a_2)|+\frac{1}{32}a_2^\alpha|h'(a_2)|-\frac{1}{12}a_1^\alpha a_2^{\alpha-1}|h'(a_2)|+\frac{1}{4}a_2^\alpha|h'(a_1)|-\frac{1}{12}a_1^\alpha|h'(a_1)|$$
$$-\frac{1}{6}a_2^\alpha|h'(a_1)|+\frac{3}{4}a_2^\alpha|h'(a_2)|-\frac{1}{6}a_1^\alpha|h'(a_2)|-\frac{7}{12}a_2^\alpha|h'(a_2)|\Bigg]$$

$$=\frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\Bigg[\frac{|h'(a_1)|}{96}\left[13a_2^\alpha-19a_1^\alpha\right]+\frac{|h'(a_2)|}{96}\left[19a_2^\alpha-21a_1^\alpha\right]$$
$$-a_1^\alpha a_2^{\alpha-1}\left[\frac{2|h'(a_1)|+|h'(a_2)|}{12}\right]+\left(a_1a_2^{\alpha-1}+a_1^{\alpha-1}a_2\right)\left[\frac{11|h'(a_1)|+5|h'(a_2)|}{96}\right]\Bigg].$$

□

Remark 3. *By putting $\alpha=1$ in (2.2), we obtain the inequality which is proved by Kirmaci in [43]*

$$\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}h(x)dx\right|\le\frac{(a_2-a_1)(|h'(a_1)|+|h'(a_2)|)}{8}.$$

Theorem 2.3. *Let $q>1$, $\alpha\in(0,1]$, $a_1,a_2\in\mathbb{R}^+$ with $a_1<a_2$ and $h:[a_1,a_2]\to\mathbb{R}$ be an α-differentiable function on (a_1,a_2). Then the inequality*

$$\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}h(x)d_\alpha x\right| \tag{2.3}$$
$$\le\frac{(a_2-a_1)}{2(a_2^\alpha-a_1^\alpha)}\Bigg[(A_1(\alpha))^{1-\frac{1}{q}}\left\{A_2(\alpha)|h'(a_1)|^q+A_3(\alpha)|h'(a_2)|^q\right\}^{\frac{1}{q}}$$
$$+(B_1(\alpha))^{1-\frac{1}{q}}\left\{B_2(\alpha)|h'(a_1)|^q+B_3(\alpha)|h'(a_2)|^q\right\}^{\frac{1}{q}}\Bigg]$$

holds if $D_\alpha(h)\in L^1_\alpha([a_1,a_2])$ and $|h'|^q$ is convex on $[a_1,a_2]$, where

$$A_1(\alpha)=\left[\frac{(a_1+a_2)^{\alpha+1}-(2a_1)^{\alpha+1}}{2^\alpha(\alpha+1)(a_2-a_1)}\right]-a_1^\alpha,$$

$$B_1(\alpha)=a_2^\alpha-\left[\frac{(2a_2)^{\alpha+1}-(a_1+a_2)^{\alpha+1}}{2^\alpha(\alpha+1)(a_2-a_1)}\right],$$

$$A_2(\alpha)=\frac{(a_1+a_2)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(a_2-a_1)(\alpha+2)}\right]$$
$$-\frac{2a_1^{\alpha+1}}{(a_2-a_1)(\alpha+1)}\left[\frac{(a_2-a_1)(\alpha+2)+a_1}{(\alpha+2)(a_2-a_1)}\right]-\frac{3a_1^\alpha}{4},$$

$$B_2(\alpha)=\frac{(a_1+a_2)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(a_2-a_1)(\alpha+2)}\right]$$

$$-\frac{a_2^{\alpha+2}}{(a_2-a_1)^2(\alpha+1)(\alpha+2)}+\frac{a_2^{\alpha}}{4},$$

$$A_3(\alpha)=\frac{(a_1+a_2)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\left[\frac{(a_2-a_1)(\alpha+2)-(a_1+a_2)}{(a_2-a_1)(\alpha+2)}\right]$$
$$-\frac{2a_1^{\alpha+2}}{(a_2-a_1)^2(\alpha+1)(\alpha+2)}-\frac{a_1^{\alpha}}{4},$$

$$B_3(\alpha)=\frac{3a_2^{\alpha}}{4}+\frac{(a_1+a_2)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(a_2-a_1)(\alpha+2)}\right]$$
$$-\frac{2a_2^{\alpha+1}}{(a_2-a_1)(\alpha+1)}\left[\frac{(\alpha+2)(a_2-a_1)+a_2}{(\alpha+2)(a_2-a_1)}\right].$$

Proof. It follows from Lemma 2.1 that

$$\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{\alpha}{a_2^{\alpha}-a_1^{\alpha}}\int_{a_1}^{a_2}h(x)d_{\alpha}x\right|$$
$$=\left|\frac{(a_2-a_1)}{2(a_2^{\alpha}-a_1^{\alpha})}\left[\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{2\alpha-1}-a_1^{\alpha}\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha-1}\right)D_{\alpha}(h)\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)ds\right.\right.$$
$$\left.\left.+\int_0^1\left(\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{2\alpha-1}-a_2^{\alpha}\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha-1}\right)\right]D_{\alpha}(h)\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)ds\right|$$
$$\leq\frac{(a_2-a_1)}{2(a_2^{\alpha}-a_1^{\alpha})}\left[\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha}-a_1^{\alpha}\right)\left|h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)\right|ds\right.$$
$$\left.+\int_0^1\left(a_2^{\alpha}-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha}\right)\left|h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)\right|ds\right].$$

From the power-mean inequality and the convexity $|h'|^q$ we get

$$\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha}-a_1^{\alpha}\right)\left|h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)\right|ds$$
$$\leq\left(\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha}-a_1^{\alpha}\right)ds\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left(\left(\frac{2-s}{2}a+\frac{sa_2}{2}\right)^{\alpha}-a_1^{\alpha}\right)\left|h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)\right|^q ds\right)^{\frac{1}{q}},$$

$$\int_0^1\left(a_2^{\alpha}-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha}\right)\left|h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)\right|ds$$
$$\leq\left(\int_0^1\left(a_2^{\alpha}-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha}\right)ds\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left(a_2^{\alpha}-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^{\alpha}\right)\left|h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)\right|^q ds\right)^{\frac{1}{q}},$$

$$\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^{\alpha}-a_1^{\alpha}\right)\left|h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)\right|^q ds$$

$$\leq \int_0^1 \left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha - a_1^\alpha\right)\left[\frac{2-s}{2}|h'(a_1)|^q+\frac{s}{2}|h'(a_2)|^q\right]ds$$
$$= |h'(a_1)|^q\int_0^1 \left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha - a_1^\alpha\right)\frac{2-s}{2}ds$$
$$+|h'(a_2)|^q\int_0^1 \left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha - a_1^\alpha\right)\frac{s}{2}ds$$
$$= |h'(a_1)|^q\Bigg(\frac{(a_1+a_2)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(a_2-a_1)(\alpha+2)}\right]$$
$$-\frac{2a_1^{\alpha+1}}{(a_2-a_1)(\alpha+1)}\left[\frac{(a_2-a_1)(\alpha+2)+a_1}{(\alpha+2)(a_2-a_1)}\right]-\frac{3a_1^\alpha}{4}\Bigg)$$
$$+|h'(a_2)|^q\Bigg(\frac{(a_1+a_2)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\left[\frac{(a_2-a_1)(\alpha+2)-(a_1+a_2)}{(a_2-a_1)(\alpha+2)}\right]$$
$$-\frac{2a_1^{\alpha+2}}{(a_2-a_1)^2(\alpha+1)(\alpha+2)}-\frac{a_1^\alpha}{4}\Bigg),$$

$$\int_0^1 \left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)\left|h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)\right|^q ds$$
$$\leq \int_0^1 \left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)\left[\frac{1-s}{2}|h'(a_1)|^q+\frac{1+s}{2}|h'(a_2)|^q\right]ds$$
$$= |h'(a_1)|^q\int_0^1 \left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)\frac{1-s}{2}ds$$
$$+|h'(a_2)|^q\int_0^1 \left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)\frac{1+s}{2}ds$$
$$= |h'(a_1)|^q\Bigg(\frac{(a_1+a_2)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(a_2-a_1)(\alpha+2)}\right]$$
$$-\frac{a_2^{\alpha+2}}{(a_2-a_1)^2(\alpha+1)(\alpha+2)}+\frac{a_2^\alpha}{4}\Bigg)$$
$$+|h'(a_2)|^q\Bigg(\frac{3a_2^\alpha}{4}+\frac{(a_1+a_2)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(a_2-a_1)(\alpha+2)}\right]$$
$$-\frac{2a_2^{\alpha+1}}{(a_2-a_1)(\alpha+1)}\left[\frac{(\alpha+2)(a_2-a_1)+a_2}{(\alpha+2)(a_2-a_1)}\right]\Bigg),$$

where we have used the facts that

$$\int_0^1 \left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha - a_1^\alpha\right)ds = \left[\frac{(a_1+a_2)^{\alpha+1}-(2a_1)^{\alpha+1}}{2^\alpha(\alpha+1)(a_2-a_1)}\right]-a_1^\alpha,$$

$$\int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)^\alpha\right)ds = a_2^\alpha - \left[\frac{(2a_2)^{\alpha+1} - (a_1+a_2)^{\alpha+1}}{2^\alpha(\alpha+1)(a_2-a_1)}\right].$$

Hence, we get the desired inequality (2.3). □

Remark 4. *Let $\alpha = 1$. Then inequality (2.3) leads to*

$$\left|h\left(\frac{a_1+a_2}{2}\right) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} h(x)dx\right|$$
$$\leq \frac{1}{2}\left(\frac{a_2-a_1}{4}\right)^{1-\frac{1}{q}}\left[\{A_2(1)|h'(a_1)|^q + A_3(1)|h'(a_2)|^q\}^{\frac{1}{q}}\right.$$
$$\left. + \{B_2(1)|h'(a_1)|^q + B_3(1)|h'(a_2)|^q\}^{\frac{1}{q}}\right],$$

where

$$A_2(1) = \frac{(a_1+a_2)^2(4a_2-2a_1) - 8a_1^2(3a_2-2a_1) - 18a_1(a_2-a_1)^2}{24(a_2-a_1)^2},$$
$$B_2(1) = \frac{(a_1+a_2)^2(4a_2-2a_1) - 4a_2^3 - 6a_2(a_2-a_1)^2}{24(a_2-a_1)^2},$$
$$A_3(1) = \frac{(a_1+a_2)^2(2a_2-4a_1) - 8a_1^3 - 6a_1(a_2-a_1)^2}{24(a_2-a_1)^2},$$
$$B_3(1) = \frac{(a_1+a_2)^2(4a_2-2a_1) - 8a_2^2(4a_2-3a_1) + 18a_2(a_2-a_1)^2}{24(a_2-a_1)^2}.$$

Theorem 2.4. *Let $q > 1$, $\alpha \in (0,1]$, $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$ and $h : [a_1,a_2] \to \mathbb{R}$ be an α-differentiable function on (a_1,a_2). Then the inequality*

$$\left|h\left(\frac{a_1+a_2}{2}\right) - \frac{\alpha}{a_2^\alpha - a_1^\alpha}\int_{a_1}^{a_2} h(x)d_\alpha x\right| \tag{2.4}$$
$$\leq \frac{(a_2-a_1)}{2(a_2^\alpha - a_1^\alpha)}\left[A_1(\alpha)h'\left(\frac{C_1(\alpha)}{A_1(\alpha)}\right) + B_1(\alpha)h'\left(\frac{C_2(\alpha)}{B_1(\alpha)}\right)\right]$$

holds if $D_\alpha(h) \in L_\alpha^1([a_1,a_2])$ and $|h'|^q$ is concave on $[a_1,a_2]$, where

$$A_1(\alpha) = \left[\frac{(a_1+a_2)^{\alpha+1} - (2a_1)^{\alpha+1}}{2^\alpha(\alpha+1)(a_2-a_1)}\right] - a_1^\alpha,$$
$$B_1(\alpha) = a_2^\alpha - \left[\frac{(2a_2)^{\alpha+1} - (a_1+a_2)^{\alpha+1}}{2^\alpha(\alpha+1)(a_2-a_1)}\right],$$
$$C_1(\alpha) = (a_1+a_2)^{\alpha+2}\left[\frac{(\alpha+2)-1}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\right]$$
$$-\frac{2a_1^{\alpha+2}}{(\alpha+2)(a_2-a_1)^2}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(\alpha+1)}\right] + \frac{a_1^\alpha}{4}(3a_1+a_2),$$
$$C_2(\alpha) = \frac{(a_1+a_2)^{\alpha+2}}{2^{\alpha+1}(a_2-a_1)(\alpha+1)}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(\alpha+2)(a_2-a_1)}\right]$$
$$-a_2^{\alpha+1}\left[\frac{(a_1+a_2)+2(\alpha+2)(a_2-a_1)}{(\alpha+2)(a_2-a_1)^2(\alpha+1)}\right] + \frac{a_2^\alpha}{4}(a_1+3a_2).$$

Proof. It follows from [44] and the concavity of $|h'|^q$ that $|h'|$ is also concave. Making use of Lemma 2.1 and Jensen's integral inequality we get

$$\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}h(x)d_\alpha x\right|$$

$$=\left|\frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\left[\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)ds\right.\right.$$
$$\left.+\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)ds\right|$$
$$\leq\frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\left[\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)\left|h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)\right|ds\right.$$
$$\left.+\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)\left|h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)\right|ds\right],$$

$$\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)\left|h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)\right|ds$$
$$\leq\left(\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)\right)$$
$$h'\left(\frac{\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)ds}{\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)ds}\right)$$
$$=A_1(\alpha)h'\left(\frac{C_1(\alpha)}{A_1(\alpha)}\right),$$

$$\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)\left|h'\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)\right|ds$$
$$\leq\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)$$
$$h'\left(\frac{\int_0^1\left(\left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)\right)\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)ds}{\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)ds}\right)$$
$$=B_1(\alpha)h'\left(\frac{C_2(\alpha)}{B_1(\alpha)}\right),$$

where we used the facts that

$$\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)ds=A_1(\alpha)=\left[\frac{(a_1+a_2)^{\alpha+1}-(2a_1)^{\alpha+1}}{2^\alpha(\alpha+1)(a_2-a_1)}\right]-a_1^\alpha,$$

$$\int_0^1\left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)ds=B_1(\alpha)=a_2^\alpha-\left[\frac{(2a_2)^{\alpha+1}-(a_1+a_2)^{\alpha+1}}{2^\alpha(\alpha+1)(a_2-a_1)}\right],$$

$$\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)ds$$

$$= C_1(\alpha) = (a_1+a_2)^{\alpha+2}\left[\frac{(\alpha+2)-1}{2^{\alpha+1}(\alpha+1)(a_2-a_1)}\right]$$
$$-\frac{2a_1^{\alpha+2}}{(\alpha+2)(a_2-a_1)^2}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(\alpha+1)}\right]+\frac{a_1^\alpha}{4}(3a_1+a_2)$$

and

$$\int_0^1\left(\left(a_2^\alpha-\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)^\alpha\right)\right)\left(\frac{1-s}{2}a_1+\frac{1+s}{2}a_2\right)ds$$
$$= C_2(\alpha) = \frac{(a_1+a_2)^{\alpha+2}}{2^{\alpha+1}(a_2-a_1)(\alpha+1)}\left[\frac{(a_2-a_1)(\alpha+2)+(a_1+a_2)}{(\alpha+2)(a_2-a_1)}\right]$$
$$-a_2^{\alpha+1}\left[\frac{(a_1+a_2)+2(\alpha+2)(a_2-a_1)}{(\alpha+2)(a_2-a_1)^2(\alpha+1)}\right]+\frac{a_2^\alpha}{4}(a_1+3a_2).$$

□

Remark 5. *Let $\alpha=1$. Then inequality (2.4) becomes*

$$\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}h(x)dx\right|$$
$$\leq\frac{(a_2-a_1)}{8}\left[h'\left(\frac{(a_1+a_2)^3(a_2-a_1)-2a_1^3(4a_2-2a_1)+3a_1(3a_1+a_2)(a_2-a_1)^2}{3(a_2-a_1)}\right)\right.$$
$$\left.+h'\left(\frac{(a_1+a_2)^3(a_2-a_1)(2a_2-a_1)-2a_2^2(7a_2-5a_1)+3a_2(a_1+3a_2)(a_2-a_1)^2}{3(a_2-a_1)}\right)\right].$$

Theorem 2.5. *Let $q>1$, $\alpha\in(0,1]$, $a_1,a_2\in\mathbb{R}^+$ with $a_1<a_2$ and $h:[a_1,a_2]\to\mathbb{R}$ be an α-differentiable function. Then the inequality*

$$\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}h(x)d_\alpha x\right| \tag{2.5}$$
$$\leq\frac{(a_2-a_1)}{2(a_2^\alpha-a_1^\alpha)}\left[D_1(\alpha)h'\left(\frac{F_1(\alpha)}{D_1(\alpha)}\right)+E_1(\alpha)h'\left(\frac{F_2(\alpha)}{E_1(\alpha)}\right)\right]$$

holds if $D_\alpha(h)\in L^1_\alpha([a_1,a_2])$ and $|h'|^q$ is concave on $[a_1,a_2]$, where

$$D_1(\alpha)=\frac{-5a_1^\alpha+2a_1^{\alpha-1}a_2+2a_2^{\alpha-1}a_1+a_2^\alpha}{12},$$
$$E_1(\alpha)=\frac{a_2^\alpha-a_1^\alpha}{4},$$
$$F_1(\alpha)=\frac{-27a_1^{\alpha+1}-2a_1^\alpha a_2+11a_1^2a_2^{\alpha-1}+5a_1a_2^\alpha+5a_1^{\alpha-1}a_2^2+3a_2^{\alpha+1}}{96},$$
$$F_2(\alpha)=\frac{a_1a_2^\alpha-a_1^{\alpha+1}+2a_2^{\alpha+1}-2a_1^\alpha a_2}{12}.$$

Proof. It follows from [44] and the concavity of $|h'|^q$ that $|h'|$ is also concave. Making use of Lemma 2.1 and Jensen's integral inequality one has

$$\left|h\left(\frac{a_1+a_2}{2}\right)-\frac{\alpha}{a_2^\alpha-a_1^\alpha}\int_{a_1}^{a_2}h(x)d_\alpha x\right|$$
$$\leq\frac{a_2-a_1}{2(a_2^\alpha-a_1^\alpha)}\left[\int_0^1\left(\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)^\alpha-a_1^\alpha\right)\left|h'\left(\frac{2-s}{2}a_1+\frac{sa_2}{2}\right)\right|ds\right.$$

$$+\int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)^\alpha\right)\left|h'\left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)\right|ds\Bigg]$$

$$\le \frac{a_2-a_1}{2(a_2^\alpha - a_1^\alpha)}\Bigg[\int_0^1 \left(\left(\frac{2-s}{2}a_1^\alpha + \frac{sa_2^\alpha}{2}\right) - a_1^\alpha\right)\left|h'\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)\right|ds$$

$$+\int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1^\alpha + \frac{1+s}{2}a_2^\alpha\right)\right)\left|h'\left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)\right|ds\Bigg]$$

$$\le \frac{a_2-a_1}{2(a_2^\alpha - a_1^\alpha)}\Bigg[\int_0^1 \left(\left(\frac{2-s}{2}a_1^{\alpha-1} + \frac{s}{2}a_2^{\alpha-1}\right)\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right) - a_1^\alpha\right)\left|h'\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)\right|ds$$

$$+\int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1^\alpha + \frac{1+s}{2}a_2^\alpha\right)\right)\left|h'\left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)\right|ds\Bigg],$$

$$\int_0^1 \left(\left(\frac{2-s}{2}a_1^{\alpha-1} + \frac{s}{2}a_2^{\alpha-1}\right)\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right) - a_1^\alpha\right)\left|h'\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)\right|ds$$

$$\le \left(\int_0^1 \left(\left(\frac{2-s}{2}a_1^{\alpha-1} + \frac{s}{2}a_2^{\alpha-1}\right)\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right) - a_1^\alpha\right)\right)$$

$$h'\left(\frac{\int_0^1 \left(\left(\frac{2-s}{2}a_1^{\alpha-1} + \frac{s}{2}a_2^{\alpha-1}\right)\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right) - a_1^\alpha\right)\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)ds}{\int_0^1 \left(\left(\frac{2-s}{2}a_1^{\alpha-1} + \frac{s}{2}a_2^{\alpha-1}\right)\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right) - a_1^\alpha\right)ds}\right)$$

$$= D_1(\alpha)h'\left(\frac{F_1(\alpha)}{D_1(\alpha)}\right),$$

$$\int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1^\alpha + \frac{1+s}{2}a_2^\alpha\right)\right)\left|h'\left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)\right|ds$$

$$\le \int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1^\alpha + \frac{1+s}{2}a_2^\alpha\right)\right)$$

$$h'\left(\frac{\int_0^1 \left(\left(a_2^\alpha - \left(\frac{1-s}{2}a_1^\alpha + \frac{1+s}{2}a_2^\alpha\right)\right)\right)\left(\frac{1-s}{2}a_1^\alpha + \frac{1+s}{2}a_2^\alpha\right)ds}{\int_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1^\alpha + \frac{1+s}{2}a_2^\alpha\right)\right)ds}\right)$$

$$= E_1(\alpha)f'\left(\frac{F_2(\alpha)}{E_1(\alpha)}\right),$$

where we have used the facts that

$$\int_0^1 \left(\left(\frac{2-s}{2}a_1^{\alpha-1} + \frac{s}{2}a_2^{\alpha-1}\right)\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right) - a_1^\alpha\right)ds$$

$$= D_1(\alpha) = \frac{-5a_1^\alpha + 2a_1^{\alpha-1}a_2 + 2a_2^{\alpha-1}a_1 + a_2^\alpha}{12},$$

$$\int\limits_0^1 \left(a_2^\alpha - \left(\frac{1-s}{2}a_1^\alpha + \frac{1+s}{2}a_2^\alpha\right)\right)ds = E_1(\alpha) = \frac{a_2^\alpha - a_1^\alpha}{4},$$

$$\int_0^1 \left(\left(\frac{2-s}{2}a_1^{\alpha-1} + \frac{s}{2}a_2^{\alpha-1}\right)\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right) - a_1^\alpha\right)\left(\frac{2-s}{2}a_1 + \frac{sa_2}{2}\right)ds = F_1(\alpha)$$

$$= \frac{-27a_1^{\alpha+1} - 2a_1^{\alpha}a_2 + 11a_1^2a_2^{\alpha-1} + 5a_1a_2^{\alpha} + 5a_1^{\alpha-1}a_2^2 + 3a_2^{\alpha+1}}{96}$$

and

$$\int_0^1 \left(\left(a_2^{\alpha} - \left(\frac{1-s}{2}a_1^{\alpha} + \frac{1+s}{2}a_2^{\alpha}\right)\right)\right)\left(\frac{1-s}{2}a_1 + \frac{1+s}{2}a_2\right)ds$$

$$= F_2(\alpha) = \frac{a_1a_2^{\alpha} - a_1^{\alpha+1} + 2a_2^{\alpha+1} - 2a_1^{\alpha}a_2}{12}.$$

□

Remark 6. *Let $\alpha = 1$. Then inequality (2.5) leads to*

$$h\left(\frac{a_1+a_2}{2}\right) - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} h(x)dx \tag{2.6}$$

$$\leq \frac{a_2-a_1}{8}\left[\left|h'\left(\frac{a_2+2a_1}{3}\right)\right| + \left|h'\left(\frac{a_1+2a_2}{3}\right)\right|\right].$$

Note that inequality (2.6) is an improvement of the inequality obtained by Pearce and Pečarić in [44] due to $|h'|$ is concave on $[a_1, a_2]$ and

$$\frac{a_2-a_1}{8}\left[\left|h'\left(\frac{a_2+2a_1}{3}\right)\right| + \left|h'\left(\frac{a_1+2a_2}{3}\right)\right|\right]$$

$$= \frac{a_2-a_1}{4}\left[\frac{1}{2}\left|h'\left(\frac{a_2+2a_1}{3}\right)\right| + \frac{1}{2}\left|h'\left(\frac{a_1+2a_2}{3}\right)\right|\right] \leq \frac{a_2-a_1}{4}\left|h'\left(\frac{a_1+a_2}{2}\right)\right|.$$

3. Applications to Special Bivariate Means

Let $a, b > 0$ with $a \neq b$. Then the arithmetic mean $A(a,b)$ [45-50], logarithmic mean $L(a,b)$ [51-55] and (α, r)-th generalized logarithmic mean $L_{(\alpha,r)}(a,b)$ [56-59] are defined by

$$A(a,b) = \frac{a+b}{2}, \quad L(a,b) = \frac{b-a}{\log b - \log a}, \quad L_{(\alpha,r)}(a,b) = \left[\frac{\alpha(b^{r+\alpha} - a^{r+\alpha})}{(r+\alpha)(b^{\alpha} - a^{\alpha})}\right]^{1/r},$$

respectively. Recently, the bivariate means have been the subject of intensive research [60-74] and many remarkable inequalities for the bivariate means and related special functions can be found in the literature [75-97].

Making use of Theorems 2.2 and 2.3 together with the convexity of the functions x^r and $1/x$ $(x > 0)$ we get some new inequalities for the arithmetic, logarithmic and generalized means immediately.

Theorem 3.1. *Let $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$. Then the inequality*

$$\left|A^r(a_1,a_2) - L^r_{(\alpha,r)}(a_1,a_2)\right| \tag{3.1}$$

$$\leq \frac{r(a_2-a_1)}{2(a_2^{\alpha} - a_1^{\alpha})}\left[\frac{|a_1|^{r-1}}{96}\left[13a_2^{\alpha} - 19a_1^{\alpha}\right] + \frac{|a_2|^{r-1}}{96}\left[19a_2^{\alpha} - 21a_1^{\alpha}\right]\right.$$

$$\left. - a_1^{\alpha}a_2^{\alpha-1}\left\{\frac{2|a_1|^{r-1} + |a_2|^{r-1}}{12}\right\}\right.$$

$$+(a_1a_2^{\alpha-1}+a_1^{\alpha-1}a_2)\left\{\frac{11|a_1|^{r-1}+5|a_2|^{r-1}}{192}\right\}\Bigg]$$

holds for all $r>1$ and $\alpha\in(0,1]$.

Remark 7. *Let $\alpha=1$. Then inequality (3.1) leads to*

$$|A^r(a_1,a_2)-L_r^r(a_1,a_2)|\leq\frac{r(a_2-a_1)}{4}A(|a_1|^{r-1},|a_2|^{r-1}),$$

which was proved by Kirmaci in [43].

Theorem 3.2. *Let $a_1,a_2\in\mathbb{R}^+$ with $a_1<a_2$ and $r>1$. Then the inequality*

$$\left|A^r(a_1,a_2)-L^r_{(\alpha,r)}(a_1,a_2))\right|$$

$$\leq\frac{r(a_2-a_1)}{2(a_2^\alpha-a_1^\alpha)}\Bigg[(A_1(\alpha))^{1-\frac{1}{q}}\left\{A_2(\alpha)|a_1|^{(r-1)q}+A_3(\alpha)|a_2|^{(r-1)q}\right\}^{\frac{1}{q}}$$

$$+(B_1(\alpha))^{1-\frac{1}{q}}\left\{B_2(\alpha)|a_1|^{(r-1)q}+B_3(\alpha)|a_2|^{(r-1)q}\right\}^{\frac{1}{q}}\Bigg]$$

holds for all $q>1$ and $\alpha\in(0,1]$.

Theorem 3.3. *Let $a_1,a_2\in\mathbb{R}^+$ with $a_1<a_2$. The the inequality*

$$\left|A^r(a_1,a_2)-L^r_{(\alpha,r)}(a_1,a_2)\right| \tag{3.2}$$

$$\leq\frac{(a_2-a_1)}{2(a_2^\alpha-a_1^\alpha)}\Bigg[\frac{|a_1|^{-2}}{96}\left[13a_2^\alpha-19a_1^\alpha\right]+\frac{|a_2|^{-2}}{96}\left[19b^\alpha-21a_1^\alpha\right]$$

$$-a_1^\alpha a_2^{\alpha-1}\left\{\frac{2|a_1|^{-2}+|a_2|^{-2}}{12}\right\}+(a_1a_2^{\alpha-1}+a_1^{\alpha-1}a_2)\left\{\frac{11|a_1|^{-2}+5|a_2|^{-2}}{192}\right\}\Bigg]$$

holds for all $\alpha\in(0,1]$.

Remark 8. *Let $\alpha=1$. Then inequality (3.2) reduces to*

$$|A^r(a_1,a_2)-L_r^r(a_1,a_2)|\leq\frac{(a_2-a_1)}{4}A(|a_1|^{-2},|a_2|^{-2}),$$

which was proved by Kirmaci in [43].

Theorem 3.4. *Let $a_1,a_2\in\mathbb{R}^+$ with $a_1<a_2$. Then the inequality*

$$\left|A^r(a_1,a_2)-L^r_{(\alpha,r)}(a_1,a_2))\right|$$

$$\leq\frac{(a_2-a_1)}{2(b^\alpha-a_1^\alpha)}\Bigg[(A_1(\alpha))^{1-\frac{1}{q}}\left\{A_2(\alpha)|a_1|^{-2q}+A_3(\alpha)|a_2|^{-2q}\right\}^{\frac{1}{q}}$$

$$+(B_1(\alpha))^{1-\frac{1}{q}}\left\{B_2(\alpha)|a_1|^{-2q}+B_3(\alpha)|a_2|^{-2q}\right\}^{\frac{1}{q}}\Bigg]$$

holds for all $q>1$ and $\alpha\in(0,1]$.

4. Applications to Mid-Point Formula

Let P be the partition of the points $a_1 = y_0 < y_1 < ... < y_{n-1} < y_n = a_2$ of the interval $[a_1, a_2]$ and consider the quadrature formula

$$\int_a^b h(x) d_\alpha x = T_\alpha(h, P) + E_\alpha(h, P),$$

where

$$T_\alpha(h, P) = \sum_{i=0}^{n-1} h\left(\frac{y_i + y_{i+1}}{2}\right) \frac{(y_{i+1}^\alpha - y_i^\alpha)}{\alpha},$$

is the midpoint version and $E_\alpha(h, P)$ denotes the associated approximation error.

In this section, we shall present some new estimates for the midpoint formula.

Theorem 4.1. *Let $\alpha \in (0, 1]$, $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$ and $h : [a_1, a_2] \to \mathbb{R}$ be an α-differentiable function. Then the inequality*

$$|E_\alpha(h, P)| \leq \sum_{i=0}^{n-1} \frac{(y_{i+1} - y_i)}{2\alpha} \left[\frac{|h'(y_i)|}{96} \left[13y_{i+1}^\alpha - 19y_i^\alpha\right] + \frac{|h'(y_{i+1})|}{96} \left[19y_{i+1}^\alpha - 21y_i^\alpha\right] \right.$$

$$\left. -y_i^\alpha y_{i+1}^{\alpha-1} \left[\frac{2|h'(y_i)| + |h'(y_{i+1})|}{12}\right] + (y_i y_{i+1}^{\alpha-1} + y_i^{\alpha-1} y_{i+1}) \left[\frac{11|h'(y_i)| + 5|h'(y_{i+1})|}{12}\right] \right]$$

holds if $D_\alpha(h) \in L_\alpha^1([a_1, a_2])$ and $|h'|$ is convex on $[a_1, a_2]$.

Proof. Applying Theorem 2.2 on the subinterval $[y_i, y_{i+1}]$ $(i = 0, 1, ..., n-1)$ of the partition P, we have

$$\left| h\left(\frac{y_i + y_{i+1}}{2}\right) \frac{(y_{i+1}^\alpha - y_i^\alpha)}{\alpha} - \int_{y_i}^{y_{i+1}} h(x) d_\alpha x \right|$$

$$\leq \frac{(y_{i+1} - y_i)}{2\alpha} \left[\frac{|h'(y_i)|}{96} \left[13y_{i+1}^\alpha - 19y_i^\alpha\right] + \frac{|h'(y_{i+1})|}{96} \left[19y_{i+1}^\alpha - 21y_i^\alpha\right] \right.$$

$$\left. -y_i^\alpha y_{i+1}^{\alpha-1} \left[\frac{2|h'(y_i)| + |h'(y_{i+1})|}{12}\right] + (y_i y_{i+1}^{\alpha-1} + y_i^{\alpha-1} y_{i+1}) \left[\frac{11|h'(y_i)| + 5|h'(y_{i+1})|}{12}\right] \right],$$

$$\left| \int_{a_1}^{a_2} h(x) d_\alpha x - T_\alpha(h, P) \right|$$

$$= \left| \sum_{i=0}^{n-1} \left\{ \int_{y_i}^{y_{i+1}} h(x) d_\alpha x - h\left(\frac{y_i + y_{i+1}}{2}\right) \frac{(y_{i+1}^\alpha - y_i^\alpha)}{\alpha} \right\} \right|$$

$$\leq \sum_{i=0}^{n-1} \left| \left\{ \int_{y_i}^{y_{i+1}} h(x) d_\alpha x - h\left(\frac{y_i + y_{i+1}}{2}\right) \frac{(y_{i+1}^\alpha - y_i^\alpha)}{\alpha} \right\} \right|$$

$$\leq \sum_{i=0}^{n-1} \frac{(y_{i+1} - y_i)}{2\alpha} \left[\frac{|h'(y_i)|}{96} \left[13y_{i+1}^\alpha - 19y_i^\alpha\right] + \frac{|h'(y_{i+1})|}{96} \left[19y_{i+1}^\alpha - 21y_i^\alpha\right] \right.$$

$$\left. -y_i^\alpha y_{i+1}^{\alpha-1} \left[\frac{2|h'(y_i)| + |h'(y_{i+1})|}{12}\right] + (y_i y_{i+1}^{\alpha-1} + y_i^{\alpha-1} y_{i+1}) \left[\frac{11|h'(y_i)| + 5|h'(y_{i+1})|}{12}\right] \right].$$

□

Theorem 4.2. *Let $q > 1$, $\alpha \in (0, 1]$, $a_1, a_2 \in \mathbb{R}^+$ with $a_1 < a_2$ and $h : [a_1, a_2] \to \mathbb{R}$ be an α-differentiable function. Then the inequality*

$$|E_\alpha(h,P)| \leq \sum_{i=0}^{n-1} \frac{(y_{i+1}-y_i)}{2\alpha} \left[(A_1(\alpha))^{1-\frac{1}{q}} \left\{ A_2(\alpha)|h'(y_i)|^q + A_3(\alpha)|h'(y_{i+1})|^q \right\}^{\frac{1}{q}} \right.$$

$$\left. + (B_1(\alpha))^{1-\frac{1}{q}} \left\{ B_2(\alpha)|h'(y_i)|^q + B_3(\alpha)|h'(y_{i+1})|^q \right\}^{\frac{1}{q}} \right]$$

holds if $D_\alpha(h) \in L^1_\alpha([a_1, a_2])$ and $|h'|^q$ is convex on $[a_1, a_2]$.

Proof. The proof is analogous to that of Theorem 4.1 only by using Theorem 2.3. □

References

[1] M. Alomari, M. Darus and S. S. Dragomir, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are quasi-convex, Tamkang J. Math., 2010, **41**(4), 353–359.

[2] S. S. Dragomir and K. Nikodem, Jensen's and Hermite-Hadamard's type inequalities for lower and strongly convex functions on normed spaces, Bull. Iranian Math. Soc., 2018, **44**(5), 1337–1349.

[3] J. Makó and Z. Páles, Approximate Hermite-Hadamard type inequalities for approximately convex functions, Math. Inequal. Appl., 2013, **16**(2), 507–526.

[4] J. Park, Some Hermite-Hadamard-like type inequalities for logarithmically convex functions, Int. J. Math. Anal., 2013, **7**(45-48), 2217–2233.

[5] P. Spurek and Ja. Tabor, Numerical verification of condition for approximately midconvex functions, Aequationes Math., 2012, **83**(3), 223–237.

[6] P. Mereau and J.-G. Paquet, Second order conditions for pseudo-convex functions, SIAM J. Anal. Math., 1974, **27**, 131–137.

[7] S. S. Dragomir, Inequalities of Hermite-Hadamard type for φ-convex functions, Konuralp J. Math., 2016, **4**(1), 54–67.

[8] M. Adamek, A characterization of λ-convex functions, JIPAM. J. Inequal. Pure Appl. Math., 2004, **5**(3), Article 71, 5 pages.

[9] M. Bombardelli and S. Varošanec, Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities, Comput. Math. Appl., 2009, **58**(9), 1869–1877.

[10] T. Rajba, On strong delta-convexity and Hermite-Hadamard type inequalities for delta-convex functions of higher order, Math. Inequal. Appl., 2015, **18**(1), 267–293.

[11] Y.-M. Chu and T.-C. Sun, The Schur harmonic convexity for a class of symmetric functions, Acta Math. Sci., 2010, **30B**(5), 1501–1506.

[12] Y.-M. Chu, W.-F. Xia and T.-H. Zhao, Schur convexity for a class of symmetric functions, Sci. China Math., 2010, **53**(2), 465–474.

[13] W.-F. Xia and Y.-M. Chu, On Schur convexity of some symmetric functions, J. Inequal. Appl., 2010, **2010**, Article ID 543250, 12 pages.

[14] W.-F. Xia and Y.-M. Chu, Necessary and sufficient conditions for the Schur harmonic convexity or concavity of the extended mean values, Rev. Un. Mat. Argentina, 2011, **52**(1), 121–132.

[15] W.-F. Xia and Y.-M. Chu, The Schur convexity of Gini mean values in the sense of harmonic mean, Acta Math. Sci., 2011, **31B**(3), 1103–1112.

[16] Y.-M. Chu, W.-F. Xia and X.-H. Zhang, The schur concavity, Schur multiplicative and harmonic convexities of the second dual form of the Hamy symmetric function with applications, J. Multivariate Anal., 2012, **105**, 412–421.

[17] Y.-M. Chu, G.-D. Wang and X.-H. Zhang, The Schur multiplicative and harmonic convexities of the complete symmetric function, Math. Nachr., 2011, **284**(5-6), 653–663.

[18] W.-D. Jiang, M.-K. Wang, Y.-M. Chu, Y.-P. Jiang and F. Qi, Convexity of the generalized sine function and the generalized hyperbolic sine function, J. Approx. Theory, 2013, **174**, 1–9.

[19] M.-K. Wang, Y.-M. Chu, S.-L. Qiu and Y.-P. Jiang, Convexity of the complete elliptic integrals of the first kind with respect to Hölder means, J. Math. Anal. Appl., 2012, **388**(2), 1141–1146.
[20] Y.-M. Chu, M.-K. Wang, Y.-P. Jiang and S.-L. Qiu, Monotonicity, convexity, and inequalities invoving the Agard distortion function, Abstr. Appl. Anal., 2011, **2011**, Article ID 671765, 8 pages.
[21] X.-M. Zhang and Y.-M. Chu, Convexity of the integral arithmetic mean of a convex function, Rocky Mountain J. Math., 2010, **40**(3), 1061–1068.
[22] Y.-M. Chu and T.-H. Zhao, Concavity of the error functions with respect to Hölder means, Math. Inequal. Appl., 2016, **19**(2), 589–595.
[23] Y.-M. Chu, M.-K. Wang, Y.-P. Jiang and S.-L. Qiu, Concavity of the complete elliptic integrals of the second kind with respect to Hölder means, J. Math. Anal. Appl., 2012, **395**(2), 637–642.
[24] Y.-M. Chu and X.-M. Zhang, Multiplicative concavity of the integral of multiplicatively concave functions, J. Inequal. Appl., 2010, **2010**, Article ID 845930, 8 pages.
[25] J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d̀une fonction considérée par Riemann, J. Math. Pures Appl., **58**, (1893) 171–215.
[26] X.-M. Zhang, Y.-M. Chu and X.-H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its applications, J. Inequal. Appl., 2010, **2010**, Article ID 507560, 11 pages.
[27] Y.-M. Chu, G.-D. Wang and X.-H. Zhang, Schur convexity and Hadamard's inequality, Math. Inequal. Appl., 2010, **13**(4), 725–731.
[28] M. Adil Khan, A. Iqbal, M. Suleman and Y.-M. Chu, Hermite-Hadamard typ inequalities for fractional integrals via Green's function, J. Inequal. Appl., 2018, **2018**, Article 161, 15 pages.
[29] M. Adil Khan, Y.-M. Chu, A. Kashuri, R. Liko and G. Ali, Conformable fractional integrals version of Hermite-Hadamard inequalities and their applications, J. Funct. Spaces, 2018, **2018**, Article ID 6928130, 9 pages.
[30] M. Adil Khan, Y. Khurshid, T.-S. Du and Y.-M. Chu, Generalization of Hermite-Hadamard type inequalities via conformable fractional integrals, J. Funct. Spaces, 2018, **2018**, Article ID 5357463, 12 pages.
[31] A. Iqbal, M. Adil Khan, S. Ullah, Y.-M. Chu and A. Kashuri, Hermite-Hadamard type inequalities pertaining conformable fractional integrals and their applications, AIP Advances, 2018, **8**, Article ID 075101, 18 pages, DOI: 10.1063/1.5031954.
[32] M. Adil Khan, Z. M. Al-sahwi and Y.-M. Chu, New estimations for Shannon and Zipf-Mandelbrot entropies, Entropy, 2018, **20**, Article 608, 10 pages, DOI: 10.3390/e2008608.
[33] M. Adil Khan, Y.-M. Chu, A. Kashuri and R. Liko, Hermite-Hadamard type fractional integral inequalities for $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions, J. Comput. Anal. Appl., 2019, **26**(8), 1487–1503.
[34] M. Z. Sarikaya, E. Set, H. Yaldiz and N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Math. Comput. Modelling, 2013, **57**(9-10), 2403–2407.
[35] Eze R. Nwaeze, Time scale version of the Hermite-Hadamard inequality for functions convex on the coordinates, Adv. Dyn. Syst. Appl., 2017, **12**(2), 159–171.
[36] M. Adil Khan, Y.-M. Chu, T. U. Khan and J. Khan, Some new inequalities of Hermite-Hadamard type for s-convex functions with applications, Open Math., 2017, **15**, 1414–1430.
[37] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, A new definitionof fractional derivative, J. Comput. Appl. Math., 2014, **264**, 65–70.
[38] D. R. Anderson, Taylor's formula and integral inequalities for conformable fractional derivatives, Contributions in Mathematics and Engineering, 25–43, Springer, [Cham], 2016.
[39] Y.-M. Chu, M. Adil khan, T. Ali and S. S. Dragomir, Inequalities for α-fractional differentiable functions, J. Inequal. Appl, 2017, **2017**, Article 93, 12 pages.
[40] M. Adil khan, T. Ali, S. S. Dragomir and M. Z. Sarikaya, Hermite-Hadamard type inequalities for conformable fractional integrals, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, 2018, **112**(4), 1033–1048.
[41] Y.-Q. Song, M. Adil Khan, S. Zaheer Ullah and Y.-M. Chu, Integral inequalities involving strongly convex functions, J. Funct. Spaces, 2018, **2018**, Article ID 6595921, 8 pages.
[42] M. Adil Khan, S. Begum, Y. Khurshid and Y.-M. Chu, Ostrowski type inequalities involving conformable fractional integrals, J. Inequal. Appl., 2018, **2018**, Article 70, 14 pages.

[43] U. S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comput., 2004, **147**(1), 137–146.

[44] C. E. M. Pearce and J. E. Pečarić, Inequalities for diffrentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett., 2000, **13**(2), 51–55.

[45] H.-Z. Xu, Y.-M. Chu and W.-M. Qian, Sharp bounds for the Sándor-Yang means in terms of arithmetic and contra-harmonic means, J. Inequal. Appl., 2018, **2018**, Article 127, 13 pages.

[46] W.-F. Xia, W. Janous and Y.-M. Chu, The optimal convex combination bounds of arithmetic and harmonic means in terms of power mean, J. Math. Inequal., 2012, **6**(2), 241–248.

[47] W.-M. Gong, Y.-Q. Song, M.-K. Wang and Y.-M. Chu, A sharp double inequality between Seiffert, arithmetic, and geometric means, Abstr. Appl. Anal., 2012, **2012**, Article ID 684834, 7 pages.

[48] Y.-M. Chu and M.-K. Wang, Inequalities between arithmetic-geometric, Gini, and Toader means, Abstr. Appl. Anal., 2012, **2012**, Article ID 830585, 11 pages.

[49] Y.-M. Chu, Y.-F. Qiu, M.-K. Wang and G.-D. Wang, The optimal convex combination bounds of arithmetic and harmonic means for the Seiffert's mean, J. Inequal. Appl., 2010, **2010**, Article ID 436457, 7 pages.

[50] W.-F. Xia, Y.-M. Chu and G.-D. Wang, The optimal upper and lower power mean bounds for a convex combiantion of the arithmetic and logarithmic means, Abstr. Appl. Anal., 2010, **2010**, Article ID 604804, 9 pages.

[51] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, Best possible inequalities among harmonic, geometric, logarithmic and Seiffert means, Math. Inequal. Appl., 2012, **15**(2), 415–422.

[52] Y.-M. Chu and M.-K. Wang, Optimal inequalities betwen harmonic, geometric, logarithmic, and arithmetic-geometric means, J. Appl. Math., 2011, **2011**, Article ID 618929, 9 pages.

[53] Y.-M. Chu, S.-S. Wang and C. Zong, Optimal lower power mean bound for the convex combination of harmonic and logarithmic means, Abstr. Appl. Anal., 2011, **2011**, Article ID 520648, 9 pages.

[54] Y.-F. Qiu, M.-K. Wang, Y.-M. Chu and G.-D. Wang, Two sharp inequalities for Lehmer mean, identric mean and logarithmic mean, J. Math. Inequal., 2011, **5**(3), 301–306.

[55] Y.-M. Chu and W.-F. Xia, Two double inequalities between power mean and logarithmic mean, Comput. Math. Appl., 2010, **60**(1), 83–89.

[56] Y.-M. Chu and B.-Y. Long, Best possible inequalities between generalized logarithmic mean and classical means, Abstr. Appl. Anal., 2010, **2010**, Article ID 303286, 13 pages.

[57] W.-M. Qian and Y.-M. Chu, Best possible bounds for Yang mean using generalized logarithmic mean, Math. Probl. Eng., 2016, **2016**, Article ID 8901258, 7 pages.

[58] Y.-M. Li, B.-Y. Long and Y.-M. Chu, Sharp bounds for the Neuman-Sándor mean in terms of generalized logarithmic mean, J. Math. Inequal., 2012, **6**(4), 567–577.

[59] Y.-M. Chu, M.-K. Wang and G.-D. Wang, The optimal generalized logarithmic mean bounds for Seiffert's mean, Acta Math. Sci., 2012, **32B**(4), 1619–1626.

[60] Y.-M. Chu, Y.-M. Li, W.-F. Xia and X.-H. Zhang, Best possible inequalities for the harmonic mean of error function, J. Inequal. Appl., 2014, **2014**, Article 525, 9 pages.

[61] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, A best-possible double inequality between Seiffert and harmonic means, J. Inequal. Appl., 2011, **2011**, Article 94, 7 pages.

[62] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, A sharp double inequality between harmonic and identric means, Abstr. Appl. Anal., 2011, **2011**, Article ID 657935, 7 pages.

[63] Y.-M. Chu, M.-K. Wang and S.-L. Qiu, Optimal combinations bounds of root-square and arithmetic means for Toader mean, Proc. Indian Acad. Sci. Math. Sci., 2012, **122**(1), 41–51.

[64] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind, J. Math. Anal. Appl., 2018, **462**(2), 1714–1726.

[65] W.-M. Qian and Y.-M. Chu, Sharp bounds for a special quasi-arithmetic mean in terms of arithmetic and geometric means with two parameters, J. Inequal. Appl., 2017, **2017**, Article 274, 10 pages.

[66] Y.-M. Chu and M.-K. Wang, Optimal Lehmer mean bounds for the Toader mean, Results Math., 2012, **61**(3-4), 223–229.

[67] M.-K. Wang, Y.-M. Chu, Y.-F. Qiu and S.-L. Qiu, An optimal power mean inequality for the complete elliptic integrals, Appl. Math. Lett., 2011, **24**(6), 887–890.

[68] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, An optimal double inequality between Seiffert and geoemtric means, J. Appl. Math., 2011, **2011**, Article ID 261237, 6 pages.

[69] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, Sharp power mean bounds for the combination of Seiffert and geoemtric means, Abstr. Appl. Anal., 2010, **2010**, Article ID 108920, 12 pages.
[70] B.-Y. Long and Y.-M. Chu, Optimal power mean bounds for the weighted geometric mean of classical means, J. Inequal. Appl., 2010, **2010**, Article ID 905697, 6 pages.
[71] Y.-M. Chu, B.-Y. Long and B.-Y. Liu, Bounds of the Neuman-Sándor mean using power and identric means, Abstr. Appl. Anal., 2013, **2013**, Article ID 832591, 6 pages.
[72] M.-K. Wang, Y.-M. Chu and Y.-F. Qiu, Some comparison inequalities for generalized Muirhead and identric means, J. Inequal. Appl., 2010, **2010**, Article ID 295620, 10 pages.
[73] M.-K. Wang, Z.-K. Wang and Y.-M. Chu, An double inequality between geometric and identric means, Appl. Math. Lett., 2012, **25**(3), 471–475.
[74] W.-M. Qian, X.-H. Zhang and Y.-M. Chu, Sharp bounds for the Toader-Qi mean in terms of harmonic and geometric means, J. Math. Inequal., 2017, **11**(1), 121–127.
[75] Zh.-H. Yang, W.-M. Qian and Y.-M. Chu, Monotonicity properties and bounds involving the complete elliptic integrals of the first kind, Math. Inequal. Appl., 2018, **21**(4), 1185–1199.
[76] T.-H. Zhao, M.-K. Wang, W. Zhang and Y.-M. Chu, Quadratic transformation inequalities for Gaussian hypergeometric function, J. Inequal. Appl., 2018, **2018**, Article 251, 15 pages.
[77] T.-R. Huang, S.-Y. Tan, X.-Y. Ma and Y.-M. Chu, Monotonicity properties and bounds for the complete p-elliptic integrals, J. Inequal. Appl., 2018, **2018**, Article 239, 11 pages.
[78] M.-K. Wang, S.-L. Qiu and Y.-M. Chu, Infinite series formula for Hübner upper bound function with applications to Hersch-Pfluger distortion function, Math. Inequal. Appl., 2018, **21**(3), 629–648.
[79] M.-K. Wang and Y.-M. Chu, Landen inequalities for a class of hypergeometric functions with applications, Math. Inequal. Appl., 2018, **21**(2), 521–537.
[80] Zh.-H. Yang and Y.-M. Chu, A monotonicity property involving the generalized elliptic integral of the first kind, Math. Inequal. Appl., 2017, **20**(3), 729–735.
[81] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, Monotonicity rule for the quotient of two function and its applications, J. Inequal. Appl., 2017, **2017**, Article 106, 13 pages.
[82] M.-K. Wang, Y.-M. Chu and Y.-Q. Song, Asymptotical formulas for Gaussian and generalized hypergeometric functions, Appl. Math. Comput., 2016, **276**, 44–60.
[83] M.-K. Wang, Y.-M. Chu and Y.-P. Jiang, Ramanujan's cubic transformaation inequalities for zero-balanced hypergeometric functions, Rocky Mountain J. Math., 2016, **46**(2), 679–691.
[84] Y.-M. Chu, M.-K. Wang and Y.-F. Qiu, On Alzer and Qiu's conjecture for complete elliptic integral and inverse hyperbolic tangent function, Abstr. Appl. Anal., 2011, **2011**, Article ID 697547, 7 pages.
[85] M.-K. Wang and Y.-M. Chu, Asymptotical bounds for complete elliptic integrals of the second kind, J. Math. Anal. Appl., 2013, **402**(1), 119–126.
[86] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, Hölder mean inequalities for the complete elliptic integrals, Integral Transforms Spec. Funct., 2012, **23**(7), 521–527.
[87] Y.-M. Chu, M.-K. Wang, S.-L. Qiu and Y.-P. Jiang, Bounds for complete elliptic integrals of the second kind with applications, Comput. Math. Appl., 2012, **63**(7), 1177–1184.
[88] M.-K. Wang, S.-L. Qiu, Y.-M. Chu and Y.-P. Jiang, Generalized Hersch-Pfluger distortion function and complete elliptic integrals, J. Math. Anal. Appl., 2012, **385**(1), 221–229.
[89] M.-K. Wang, Y.-M. Li and Y.-M. Chu, Inequalities and infinite product formula for Ramanujan generalized modular equation function, Ramanujan J., 2018, **46**(1), 189–200.
[90] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On rational bounds for the gamma function, J. Inequal. Appl., 2017, **2017**, Article 210, 17 pages.
[91] T.-H. Zhao and Y.-M. Chu, A class of logarithmically completely monotonic functions associated with gamma function, J. Inequal. Appl., 2010, **2010**, Article ID 392431, 11 pages.
[92] T.-H. Zhao, Y.-M. Chu and H. Wang, Logarithmically complete monotonicity properties relating to the gamma function, Abstr. Appl. Anal., 2011, **2011**, Article ID 896483, 13 pages.
[93] T.-R. Huang, B.-W. Han, X.-Y. Ma and Y.-M. Chu, Optimal bounds for the generalized Euler-Mascheronic constant, J. Inequal. Appl., 2018, **2018**, Article 118, 9 pages.
[94] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the error function, Math. Inequal. Appl., 2018, **21**(2), 469–479.
[95] Y.-Q. Song, P.-G. Zhou and Y.-M. Chu, Inequalities for the Gaussian hypergeometric function, Sci. China Math., 2014, **57**(11), 2369–2380.

[96] Zh.-H. Yang, W. Zhang and Y.-M. Chu, Sharp Gautschi inequality for parameter $0 < p < 1$ with applications, Math. Inequal. Appl., 2017, **20**(4), 1107–1120.
[97] Zh.-H. Yang, Y.-M. Chu and W. Zhang, Accurate approximations for the complete elliptic of the second kind, J. Math. Anal. Appl., 2016, **438**(2), 875–888.

Yousaf Khurshid, [1]College of Science, Hunan City University, Yiyang 413000, Hunan, China; [2]Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan
E-mail address: yousafkhurshid90@gmail.com

Muhammad Adil Khan, [2]Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan
E-mail address: adilswati@gmail.com

Yu-Ming Chu (Corresponding author), [3]Department of Mathematics, Huzhou University, Huzhou 313000, Zhejiang, China
E-mail address: chuyuming2005@126.com

Neutrosophic BCC-ideals in BCC-algebras

Sun Shin Ahn

Department of Mathematics Education, Dongguk University, Seoul 04620, Korea

Abstract. The notions of a neutrosophic subalgebra and a neutrosohic ideal of a BCC-algebra are introduced and consider characterizations of a neutrosophic subalgebra and a neutrosophic ideal. We define the notion of a neutrosophic BCC-ideal of a BCC-algebra, and investigated some properties of it.

1. Introduction

Y. Kormori [8] introduced a notion of a BCC-algebras, and W. A. Dudek [4] redefined the notion of BCC-algebras by using a dual from of the ordinary definition of Y. Kormori. In [6], J. Hao introduced the notion of ideals in a BCC-algebra and studied some related properties. W. A. Dudek and X. Zhang [5] introdued a BCC-ideals in a BCC-algebra and described connections between such BCC-ideals and congruences. S. S. Ahn and S. H. Kwon [2] defined a topological BCC-algebra and investigated some properties of it.

Zadeh [10] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [3] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality (i) as independent component in 1995 (published in 1998) and defined the neutrosophic set on three components (t, i, f) = (truth, indeterminacy, falsehood). Jun et. al [7] introduced the notions of a neutrosophic $\mathscr{N}$-subalgebras and a (closed) neutrosophic $\mathscr{N}$-ideal in a BCK/BCI-algebras and investigated some related properties. subalgebras

In this paper, we introduce the notions of a neutrosophic subalgebra and a neutrosohic ideal of a BCC-algebra and consider characterizations of a neutrosophic subalgebra and a neutrosophic ideal. We define the notion of a neutrosophic BCC-ideal of a BCC-algebra, and investigate some properties of it.

2. Preliminaries

By a *BCC-algebra* [4] we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions: for all $x, y, z \in X$,

(a1) $((x * y) * (z * y)) * (x * z) = 0$,

(a2) $0 * x = 0$,

(a3) $x * 0 = x$,

(a4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

For brevity, we also call X a *BCC-algebra.* In X, we can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$. Then $\leq$ is a partial order on X.

---

[0] **2010 Mathematics Subject Classification**: 06F35; 03G25; 03B52.

[0] **Keywords**: BCC-algebra; ; (BCC-)ideal; neutrosophic subalgebra; neutrosophic (BCC-)ideal.

[0]**E-mail**: `sunshine@dongguk.edu`

A BCC-algebra X has the following properties: for any $x, y \in X$,

(b1) $x * x = 0$,

(b2) $(x * y) * x = 0$,

(b3) $x \leq y \Rightarrow\ x * z \leq y * z$ and $z * y \leq z * x$.

Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebra [4]. Note that a BCC-algebra is a BCK-algebra if and only if it satisfies:

(b4) $(x * y) * z = (x * z) * y$, for all $x, y, z \in X$.

Let $(X, *, 0_X)$ and $(Y, *, 0_Y)$ be BCC-algebras. A mapping $\varphi : X \to Y$ is called a *homomorphism* if $\varphi(x *_X y) = \varphi(x) *_Y \varphi(y)$ for all $x, y \in X$. A non-empty subset S of a BCC-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A non-empty subset I of a BCI-algebra X is called an *ideal* [6] of X if it satisfies:

(c1) $0 \in I$,

(c2) $x * y, y \in I\ \Rightarrow x \in I$ for all $x, y \in X$.

I is called an *BCC-ideal* [5] of X if it satisfies (c1) and

(c3) $(x * y) * z, y \in I\ \Rightarrow\ x * z \in I$, for all $x, y, z \in X$.

Theorem 2.1. [6] *In a BCC-algebra, an ideal is a subalgebra.*

Theorem 2.2. [5] *In a BCC-algebra, a BCC-ideal is an ideal.*

Corollary 2.3. [5] *Any BCC-ideal of a BCC-algebra is a subalgebra.*

Definition 2.4. Let X be a space of points (objects) with generic elements in X denoted by x. A simple valued neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. Then a simple valued neutrosophic set A can be denoted by

$$A := \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\},$$

where $T_A(x), I_A(x), F_A(x) \in [0, 1]$ for each point x in X. Therefore the sum of $T_A(x), I_A(x)$, and $F_A(x)$ satisfies the condition $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

For convenience, "simple valued neutrosophic set" is abbreviated to "neutrosophic set" later.

Definition 2.5. Let A be a neutrosophic set in a B-algebra X and $\alpha, \beta, \gamma \in [0, 1]$ with $0 \leq \alpha + \beta + \gamma \leq 3$ and an (α, β, γ)-level set of X denoted by $A^{(\alpha,\beta,\gamma)}$ is defined as

$$A^{(\alpha,\beta,\gamma)} = \{x \in X | T_A(x) \leq \alpha, I_A(x) \geq \beta, F_A(x) \leq \gamma\}.$$

For any family $\{a_i | i \in \Lambda\}$, we define

$$\bigvee\{a_i | i \in \Lambda\} := \begin{cases} \max\{a_i | i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i | i \in \Lambda\} & \text{otherwise} \end{cases}$$

and

$$\bigwedge\{a_i|i\in\Lambda\}:=\begin{cases}\min\{a_i|i\in\Lambda\} & \text{if } \Lambda \text{ is finite,}\\ \inf\{a_i|i\in\Lambda\} & \text{otherwise.}\end{cases}$$

3. Neutrosophic BCC-ideals

In what follows, let X be a BCC-algebra unless otherwise specified.

Definition 3.1. A neutrosophic set A in a BCC-algebra X is called a *neutrosophic subalgebra* of X if it satisfies:

(NSS) $T_A(x*y)\leq\max\{T_A(x),T_A(y)\}, I_A(x*y)\geq\min\{I_A(x),I_A(y)\},$ and $F_A(x*y)\leq\max\{F_A(x),F_A(y)\}$, for any $x,y\in X$.

Proposition 3.2. *Every neutrosophic subalgebra of a BCC-algebra X satisfies the following conditions:*

(3.1) $T_A(0)\leq T_A(x), I_A(0)\geq I_A(x)$, *and* $F_A(0)\leq F_A(x)$ *for any* $x\in X$.

Proof. Straightforward. □

Example 3.3. Let $X:=\{0,1,2,3\}$ be a BCC-algebra [6] with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	1
3	3	3	3	0

Define a neutrosophic set A in X as follows:

$$T_A: X\to[0,1],\ x\mapsto\begin{cases}0.12 & \text{if } x\in\{0,1,2\}\\ 0.83 & \text{if } x=3,\end{cases}$$

$$I_A: X\to[0,1],\ x\mapsto\begin{cases}0.81 & \text{if } x\in\{0,1,2\}\\ 0.14 & \text{if } x=3,\end{cases}$$

and

$$F_A: X\to[0,1],\ x\mapsto\begin{cases}0.12 & \text{if } x\in\{0,1,2\}\\ 0.83 & \text{if } x=3.\end{cases}$$

It is easy to check that A is a neutrosophic subalgebra of X.

Theorem 3.4. *Let A be a neutrosophic set in a BCC-algebra X and let $\alpha,\beta,\gamma\in[0,1]$ with $0\leq\alpha+\beta+\gamma\leq 3$. Then A is a neutrosophic subalgebra of X if and only if all of (α,β,γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)}\neq\emptyset$.*

Proof. Assume that A is a neutrosophic subalgebra of X. Let $\alpha,\beta,\gamma\in[0,1]$ be such that $0\leq\alpha+\beta+\gamma\leq 3$ and $A^{(\alpha,\beta,\gamma)}\neq\emptyset$. Let $x,y\in A^{(\alpha,\beta,\gamma)}$. Then $T_A(x)\leq\alpha, T_A(y)\leq\alpha, I_A(x)\geq\beta, I_A(y)\geq\beta$ and $F_A(x)\leq\gamma, F_A(y)\leq\gamma$. Using (NSS), we have $T_A(x*y)\leq\max\{T_A(x),T_A(y)\}\leq\alpha$, $I_A(x*y)\geq\min\{I_A(x),I_A(y)\}\geq\beta$, and $F_A(x*y)\leq\max\{F_A(x),F_A(y)\}\leq\gamma$. Hence $x*y\in A^{(\alpha,\beta,\gamma)}$. Therefore $A^{(\alpha,\beta,\gamma)}$ is a subalgebra of X.

Conversely, all of (α,β,γ)-level set $A^{(\alpha,\beta,\gamma)}$ are subalgebras of X when $A^{(\alpha,\beta,\gamma)}\neq\emptyset$. Assume that there exist $a_t,b_t,a_i,b_i\in X$ and $a_f,b_f\in X$ such that $T_A(a_t*b_t)>\max\{T_A(a_t),T_A(b_t)\}, I_A(a_i*b_i)<\min\{I_A(a_i),I_A(b_i)\}$

and $F_A(a_f * b_f) > \max\{F_A(a_f), F_A(b_f)\}$. Then $T_A(a_t * b_t) > \alpha_1 \geq \max\{T_A(a_t), T_A(b_t)\}, I_A(a_i * b_i) < \beta_1 \leq \min\{I_A(a_i), I_A(b_i)\}$ and $F_A(a_f * b_f) > \gamma_1 \geq \max\{F_A(a_f), F_A(b_f)\}$ for some $\alpha_1, \gamma_1 \in [0,1)$ and $\beta_1 \in (0,1]$. Hence $a_t, b_t, a_i, b_i \in A^{(\alpha_1,\beta_1,\gamma_1)}$, and $a_f, b_f \in A^{(\alpha_1,\beta_1,\gamma_1)}$. But $a_t * b_t, a_i * b_i \notin A^{(\alpha_1,\beta_1,\gamma_1)}$, and $a_f * b_f \notin A^{(\alpha_1,\beta_1,\gamma_1)}$, which is a contradiction. Hence $T_A(x*y) \leq \max\{T_A(x), T_A(y)\}, I_A(x*y) \geq \min\{I_A(x), I_A(y)\}$, and $F_A(x*y) \leq \max\{T_A(x), T_A(y)\}$, for any $x, y \in X$. Therefore A is a neutrosophic subalgebra of X. $\square$

Since $[0,1]$ is a completely distributive lattice with respect to the usual ordering, we have the following theorem.

Theorem 3.5. *If $\{A_i | i \in \mathbb{N}\}$ is a family of neutrosopic subalgebras of a BCC-algebra X, then $(\{A_i | i \in \mathbb{N}\}, \subseteq)$ forms a complete distributive lattice.*

Theorem 3.6. *Let A be a neutrosophic subalgebra of a BCC-algebra X. If there exists a sequence $\{a_n\}$ in X such that $\lim_{n\to\infty} T_A(a_n) = 0, \lim_{n\to\infty} I_A(a_n) = 1$, and $\lim_{n\to\infty} F_A(a_n) = 0$, then $T_A(0) = 0, I_A(0) = 1$, and $F_A(0) = 0$.*

Proof. By Proposition 3.2, we have $T_A(0) \leq T_A(x), I_A(0) \geq I_A(x)$, and $F_A(0) \leq F_A(x)$ for all $x \in X$. Hence we have $T_A(0) \leq T_A(a_n), I_A(0) \geq I_A(a_n)$, and $F_A(0) \leq F_A(a_n)$ for every positive integer n. Therefore $0 \leq T_A(0) \leq \lim_{n\to\infty} T_A(a_n) = 0, 1 = \lim_{n\to\infty} I_A(a_n) \leq I_A(0) \leq 1$, and $0 \leq F_A(0) \leq \lim_{n\to\infty} F_A(a_n) = 0$. Thus we have $T_A(0) = 0, I_A(0) = 1$, and $F_A(0) = 0$. $\square$

Proposition 3.7. *If every neutrosophic subalgebra A of a BCC-algebra X satisfies the condition*

(3.2) $T_A(x*y) \leq T_A(y), I_A(x*y) \geq I_A(y), F_A(x*y) \leq F_A(y)$, *for any $x, y \in X$,*

then T_A, I_A, and F_A are constant functions.

Proof. It follows from (3.2) that $T_A(x) = T_A(x*0) \leq T_A(0), I_A(x) = I_A(x*0) \geq I_A(0)$, and $F_A(x) = F_A(x*0) \leq F_A(0)$ for any $x \in X$. By Proposition 3.2, we have $T_A(x) = T_A(0), I_A(x) = I_A(0)$, and $F_A(x) = F_A(0)$ for any $x \in X$. Hence T_A, I_A, and F_A are constant functions. $\square$

Theorem 3.8. *Every subalgebra of a BCC-algebra X can be represented as an (α,β,γ)-level set of a neutrosophic subalgebra A of X.*

Proof. Let S be a subalgebra of a BCC-algebra X and let A be a neutrosophic subalgebra of X. Define a neutrosophic set A in X as follows:

$$T_A : X \to [0,1],\ x \mapsto \begin{cases} \alpha_1 & \text{if } x \in S \\ \alpha_2 & \text{otherwise,} \end{cases}$$

$$I_A : X \to [0,1],\ x \mapsto \begin{cases} \beta_1 & \text{if } x \in S \\ \beta_2 & \text{otherwise,} \end{cases}$$

$$F_A : X \to [0,1],\ x \mapsto \begin{cases} \gamma_1 & \text{if } x \in S \\ \gamma_2 & \text{otherwise,} \end{cases}$$

where $\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in [0,1)$ and $\beta_1, \beta_2 \in (0,1]$ with $\alpha_1 < \alpha_2, \beta_1 > \beta_2, \gamma_1 < \gamma_2$, and $0 \leq \alpha_1 + \beta_1 + \gamma_1 \leq 3, 0 \leq \alpha_2 + \beta_2 + \gamma_2 \leq 3$. Obviously, $S = A^{(\alpha_1,\beta_1,\gamma_1)}$. We now prove that A is a neutrosophic subalgebra of X. Let $x, y \in X$. If $x, y \in S$, then $x * y \in S$ because S is a subalgebra of X. Hence $T_A(x) = T_A(y) = T_A(x*y) = \alpha_1$,

$I_A(x) = I_A(y) = I_A(x*y) = \beta_1$, $F_A(x) = F_A(y) = F_A(x*y) = \gamma_1$ and so $T_A(x*y) \leq \max\{T_A(x), T_A(y)\}$, $I_A(x*y) \geq \min\{I_A(x), I_A(y)\}$, $F_A(x*y) \leq \max\{F_A(x), F_A(y)\}$. If $x \in S$ and $y \notin S$, then $T_A(x) = \alpha_1, T_A(y) = \alpha_2$, $I_A(x) = \beta_1, I_A(y) = \beta_2$, $F_A(x) = \gamma_1, F_A(y) = \gamma_2$ and so $T_A(x*y) \leq \max\{T_A(x), T_A(y)\} = \alpha_2$, $I_A(x*y) \geq \min\{I_A(x), I_A(y)\} = \beta_2$, $F_A(x*y) \leq \max\{F_A(x), F_A(y)\} = \gamma_2$. Obviously, if $x \notin A$ and $y \notin A$, then $T_A(x*y) \leq \max\{T_A(x), T_A(y)\} = \alpha_2$, $I_A(x*y) \geq \min\{I_A(x), I_A(y)\} = \beta_2$, $F_A(x*y) \leq \max\{F_A(x), F_A(y)\} = \gamma_2$. Therefore A is a neutrosophic subalgebra of X. □

Definition 3.9. A neutrosophic set A in a BCC-algebra X is said to be *neutrosophic ideal* of X if it satisfies:

(NSI1) $T_A(0) \leq T_A(x), I_A(0) \geq I_A(x)$, and $F_A(0) \leq F_A(x)$ for any $x \in X$;

(NSI2) $T_A(x) \leq \max\{T_A(x*y), T_A(y)\}, I_A(x) \geq \min\{I_A(x*y), I_A(y)\}$, and $F_A(x) \leq \max\{F_A(x*y), F_A(y)\}$, for any $x, y \in X$.

Proposition 3.10. *Every neutrosophic ideal of a BCC-algebra X is a neutrosophic subalgebra of X.*

Proof. Let A be a neutrosophic ideal of X. Put $x := x*y$ and $y := x$ in (NSI2). Then we have $T_A(x*y) \leq \max\{T_A((x*y)*x), T_A(x)\}, I_A(x*y) \geq \min\{I_A((x*y)*x), I_A(x)\}$, and $F_A(x*y) \leq \max\{F_A((x*y)*x), F_A(x)\}$. It follows from (b2) and (NSI1) that $T_A(x*y) \leq \max\{T_A((x*y)*x), T_A(x)\} = \max\{T_A(0), T_A(x)\} \leq \max\{T_A(x), T_A(y)\}, I_A(x*y) \geq \min\{I_A((x*y)*x), I_A(x)\} = \max\{I_A(0), I_A(x)\} \geq \max\{I_A(x), I_A(y)\}$, and $F_A(x*y) \leq \max\{F_A((x*y)*x), F_A(x)\} = \max\{F_A(0), F_A(x)\} \leq \max\{F_A(x), F_A(y)\}$. Thus A is a neutrosophic subalgebra of X. □

Theorem 3.11. *Let A be a neutrosophic set in a BCC-algebra X and let $\alpha, \beta, \gamma \in [0,1]$ with $0 \leq \alpha+\beta+\gamma \leq 3$. Then A is a neutrosophic ideal of X if and only if all of (α, β, γ)-level set $A^{(\alpha,\beta,\gamma)}$ are ideals of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.*

Proof. Assume that A is a neutrosophic ideal of X. Let $\alpha, \beta, \gamma \in [0,1]$ be such that $0 \leq \alpha+\beta+\gamma \leq 3$ and $A^{(\alpha,\beta,\gamma)} \neq \emptyset$. Let $x, y \in X$ be such that $x*y, y \in A^{(\alpha,\beta,\gamma)}$. Then $T_A(x*y) \leq \alpha, T_A(y) \leq \alpha, I_A(x*y) \geq \beta, I_A(y) \geq \beta$, and $F_A(x*y) \leq \gamma, F_A(y) \leq \gamma$. By Definition 3.9, we have $T_A(0) \leq T_A(x) \leq \max\{T_A(x*y), T_A(y)\} \leq \alpha, I_A(0) \geq I_A(x) \geq \min\{I_A(x*y)), I_A(y)\} \geq \beta$, and $F_A(0) \leq F_A(x) \leq \max\{F_A(x*y), T_A(y)\} \leq \gamma$. Hence $0, x \in A^{(\alpha,\beta,\gamma)}$. Therefore $A^{(\alpha,\beta,\gamma)}$ is an ideal of X.

Conversely, suppose that there exist $a, b, c \in X$ such that $T_A(0) > T_A(a), I_A(0) < I_A(b)$, and $F_A(0) > F_A(c)$. Then there exist $a_t, c_t \in [0,1)$ and $b_t \in (0,1]$ such that $T_A(0) > a_t \geq T_A(a), I_A(0) < b_t \leq I_A(b)$ and $F_A(0) > c_t \geq F_A(c)$. Hence $0 \notin A^{(a_t,b_t,c_t)}$, which is a contradiction. Therefore $T_A(0) \leq T_A(x), I_A(0) \geq I_A(x)$ and $F_A(0) \leq F_A(x)$ for all $x \in X$. Assume that there exist $a_t, b_t, a_i, b_i, a_f, b_f \in X$ such that $T_A(a_t) > \max\{T_A(a_t * b_t), T_A(b_t)\}, I_A(a_i) < \min\{I_A(a_i*b_i), I_A(b_i)\}$, and $F_A(a_f) > \max\{T_A(a_f*b_f), T_A(b_f)\}$. Then there exist $s_t, s_f \in [0,1)$ and $s_i \in (0,1]$ such that $T_A(a_t) > s_t \geq \max\{T_A(a_t*b_t), T_A(b_t)\}, I_A(a_i) < s_i \leq \min\{I_A(a_i*b_i), I_A(b_i)\}$, and $F_A(a_f) > s_f \geq \max\{T_A(a_f*b_f), T_A(b_f)\}$. Hence $a_t*b_t, b_t, a_i*b_i, a_f*b_f \in A^{(s_t,s_i,s_f)}$, and $b_t, b_i, b_f \in A^{(s_t,s_i,s_f)}$. But $a_t, a_i \notin A^{(s_t,s_i,s_f)}$ and $a_f \notin A^{(s_t,s_i,s_f)}$. This is a contradiction. Therefore $T_A(x) \leq \max\{T_A(x*y), T_A(y)\}, I_A(x) \geq \min\{I_A(x*y)), I_A(y)\}$ and $F_A(x) \leq \max\{F_A(x*y), F_A(y)\}$, for any $x, y \in X$. Therefore A is a neutrosophic ideal of X □

Proposition 3.12. *Every neutrosophic ideal A of a BCC-algebra X satisfies the following properties:*

(i) $(\forall x, y \in X)(x \le y \Rightarrow T_A(x) \le T_A(y), I_A(x) \ge I_A(y), F_A(x) \le F_A(y))$,

(ii) $(\forall x, y, z \in X)(x*y \le z \Rightarrow T_A(x) \le \max\{T_A(y), T_A(z)\}, I_A(x) \ge \min\{I_A(y), I_A(z)\}, F_A(x) \le \max\{F_A(y), F_A(z)\})$.

Proof. (i) Let $x, y \in X$ be such that $x \le y$. Then $x * y = 0$. Using (NSI2) and (NSI1), we have $T_A(x) \le \max\{T_A(x*y), T_A(y)\} = \max\{T_A(0), T_A(y)\} = T_A(y), I_A(y) \ge \min\{I_A(x*y), I_A(y)\} = \min\{I_A(0), I_A(y)\} = I_A(y)$, and $F_A(x) \le \max\{F_A(x*y), F_A(y)\} = \max\{F_A(0), F_A(y)\} = F_A(y)$.
(ii) Let $x, y, z \in X$ be such that $x * y \le z$. By (NSI2) and (NSI1). we get $T_A(x*y) \le \max\{T_A((x*y)*z), T_A(z)\} = \max\{T_A(0), T_A(z)\} = T_A(z), I_A(x*y) \ge \min\{I_A((x*y)*z), I_A(z)\} = \min\{I_A(0), I_A(z)\} = I_A(z)$, and $F_A(x*y) \le \max\{F_A((x*y)*z), F_A(z)\} = \max\{F_A(0), F_A(z)\} = F_A(z)$. Hence $T_A(x) \le \max\{T_A(x*y), T_A(y)\} \le \max\{T_A(y), T_A(z)\}, I_A(x) \ge \min\{I_A(x*y), I_A(y)\} \ge \min\{I_A(y), I_A(z)\}$, and $F_A(x) \le \max\{F_A(x*y), F_A(y)\} \le \max\{F_A(y), F_A(z)\}$. □

The following corollary is easily proved by induction.

Corollary 3.13. *Every neutrosophic ideal A of a BCC-algebra X satisfies the following property:*

(3.3) $(\cdots(x*a_1)*\cdots)*a_n = 0 \Rightarrow T_A(x) \le \bigvee_{k=1}^{n} T_A(a_k), I_A(x) \ge \bigwedge_{k=1}^{n} I_A(a_k), F_A(x) \le \bigvee_{k=1}^{n} F_A(a_k)$, *for all* $x, a_1, \cdots, a_n \in X$.

Definition 3.14. Let A and B be neutrosophic sets of a set X. The *union* of A and B is defined to be a neutrosophic set

$$A \tilde{\cup} B := \{\langle x, T_{A\cup B}(x), I_{A\cup B}(x), F_{A\cup B}(x)\rangle | x \in X\},$$

where $T_{A\cup B}(x) = \min\{T_A(x), T_B(x)\}, I_{A\cup B}(x) = \max\{I_A(x), I_B(x)\}, F_{A\cup B}(x) = \min\{F_A(x), F_B(x)\}$, for all $x \in X$. The *intersection* of A and B is defined to be a neutrosophic set

$$A \tilde{\cap} B := \{\langle x, T_{A\cap B}(x), I_{A\cap B}(x), F_{A\cap B}(x)\rangle | x \in X\},$$

where $T_{A\cap B}(x) = \max\{T_A(x), T_B(x)\}, I_{A\cap B}(x) = \min\{I_A(x), I_B(x)\}, F_{A\cap B}(x) = \max\{F_A(x), F_B(x)\}$, for all $x \in X$.

Theorem 3.15. *The intersection of two neutrosophic ideals of a BCC-algebra X is a also a neutrosophic ideal of X.*

Proof. Let A and B be neutrosophic ideals of X. For any $x \in X$, we have $T_{A\cap B}(0) = \max\{T_A(0), T_B(0)\} \le \max\{T_A(x), T_B(x)\} = T_{A\cap B}(x), I_{A\cap B}(0) = \min\{T_A(0), T_B(0)\} \ge \min\{I_A(x), I_B(x)\} = I_{A\cap B}(x)$, and $F_{A\cap B}(0) = \max\{F_A(0), F_B(0)\} \le \max\{F_A(x), F_B(x)\} = F_{A\cap B}(x)$. Let $x, y \in X$. Then we have

$$\begin{aligned} T_{A\cap B}(x) &= \max\{T_A(x), T_B(x)\} \\ &\le \max\{\max\{T_A(x*y), T_A(y)\}, \max\{T_B(x*y), T_B(y)\}\} \\ &= \max\{\max\{T_A(x*y), T_B(x*y)\}, \max\{T_A(y), T_B(y)\}\} \\ &= \max\{T_{A\cap B}(x*y), T_{A\cap B}(y)\}, \end{aligned}$$

$$\begin{aligned} I_{A\cap B}(x) &= \min\{I_A(x), I_B(x)\} \\ &\geq \min\{\min\{I_A(x*y), I_A(y)\}, \min\{I_B(x*y), I_B(y)\}\} \\ &= \min\{\min\{I_A(x*y), I_B(x*y)\}, \min\{I_A(y), I_B(y)\}\} \\ &= \min\{I_{A\cap B}(x*y), I_{A\cap B}(y)\}, \end{aligned}$$

and

$$\begin{aligned} F_{A\cap B}(x) &= \max\{F_A(x), F_B(x)\} \\ &\leq \max\{\max\{F_A(x*y), F_A(y)\}, \max\{F_B(x*y), F_B(y)\}\} \\ &= \max\{\max\{F_A(x*y), F_B(x*y)\}, \max\{F_A(y), F_B(y)\}\} \\ &= \max\{F_{A\cap B}(x*y), F_{A\cap B}(y)\}. \end{aligned}$$

Hence $A\tilde{\cap}B$ is a neutrosophic ideal of X. □

Corollary 3.16. *If $\{A_i | i \in \mathbb{N}\}$ is a family of neutrosophic ideals of a BCC-algebra X, then so is $\tilde{\cap}_{i\in\mathbb{N}} A_i$.*

The union of any set of neutrosophic ideals of a BCC-algebra X need not be a neutrosophic ideal of X.

Example 3.17. Let $X = \{0, 1, 2, 3, 4\}$ be a BCC-algebra [5] with the following table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Define neutrosophic sets A and B of X as follows:

$$T_A : X \to [0,1],\ x \mapsto \begin{cases} 0.12, & \text{if } x \in \{0,1\} \\ 0.74 & \text{otherwise,} \end{cases}$$

$$I_A : X \to [0,1],\ x \mapsto \begin{cases} 0.63, & \text{if } x \in \{0,1\} \\ 0.11 & \text{otherwise,} \end{cases}$$

$$F_A : X \to [0,1],\ x \mapsto \begin{cases} 0.12, & \text{if } x \in \{0,1\} \\ 0.74 & \text{otherwise,} \end{cases}$$

$$T_B : X \to [0,1],\ x \mapsto \begin{cases} 0.13, & \text{if } x \in \{0,2\} \\ 0.63 & \text{otherwise,} \end{cases}$$

$$I_B : X \to [0,1],\ x \mapsto \begin{cases} 0.75, & \text{if } x \in \{0,2\} \\ 0.14 & \text{otherwise,} \end{cases}$$

and

$$F_B : X \to [0,1],\ x \mapsto \begin{cases} 0.13, & \text{if } x \in \{0,2\} \\ 0.63 & \text{otherwise.} \end{cases}$$

It is easy to check that A and B are neutrosophic ideals of X. But $A\tilde{\cup}B$ is not a neutrosophic ideal of X, since $T_{A\cup B}(3) = \min\{T_A(3), T_B(3)\} = 0.63 \not\leq \max\{T_{A\cup B}(3*2), T_{A\cup B}(2)\} = \max\{T_{A\cup B}(1), T_{A\cup B}(2)\} = \max\{\min\{T_A(1), T_B(1)\}, \min\{T_A(2), T_B(2)\}\} = \max\{0.12, 0.13\} = 0.13$.

Definition 3.18. A neutrosophic set A in a BCC-algebra X is said to be a *neutrosophic BCC-ideal* of X if it satisfies (NSI1) and

(NSI3) $T_A(x * z) \leq \max\{T_A((x * y) * z), T_A(y)\}, I_A(x * z) \geq \min\{I_A((x * y) * z), I_A(y)\}$, and $F_A(x * z) \leq \max\{F_A((x * y) * z), F_A(y)\}$, for any $x, y, z \in X$.

Lemma 3.19. *Every neutrosophic BCC-ideal of a BCC-algebra X is a neutrosophic ideal of X.*

Proof. Let A be a neutrosophic BCC-ideal of a BCC-algebra X. Put $z := 0$ in (NSI3). By (a3), we have $T_A(x * 0) = T_A(x) \leq \max\{T_A((x * y) * 0), T_A(y)\} = \max\{T_A(x * y), T_A(y)\}, I_A(x * 0) = I_A(x) \geq \min\{I_A((x * y) * 0), I_A(y)\} = \min\{I_A(x*y), I_A(y)\}$, and $F_A(x*0) = F_A(x) \leq \max\{F_A((x*y)*0), F_A(y)\} = \max\{F_A(x*y), F_A(y)\}$, for any $x, y \in X$. Hence A is a neutrosophic ideal of X. □

Corollary 3.20. *Every neutrosophic BCC-ideal of a BCC-algebra X is a neutrosophic subalgebra of X.*

The converse of Proposition 3.10 and Lemma 3.19 need not be true in general (see Example 3.21).

Example 3.21. Let $X = \{0, 1, 2, 3, 4\}$ be a BCC-algebra as in Example 3.17. Define a neutrosophic set A of X as follows:

$$T_A : X \to [0, 1],\ x \mapsto \begin{cases} 0.13 & \text{if } x \in \{0, 1, 2, 3\} \\ 0.83 & \text{if } x = 4, \end{cases}$$

$$I_A : X \to [0, 1],\ x \mapsto \begin{cases} 0.82 & \text{if } x \in \{0, 1, 2, 3\} \\ 0.11 & \text{if } x = 4, \end{cases}$$

and

$$F_A : X \to [0, 1],\ x \mapsto \begin{cases} 0.13 & \text{if } x \in \{0, 1, 2, 3\} \\ 0.83 & \text{if } x = 4, \end{cases}$$

It is easy to check that A is a neutrosophic subalgebra of X, but not a neutrosophic ideal of X, since $T_A(4) = 0.83 \not\leq \max\{T_A(4*3), T_A(3)\} = \max\{T_A(3), T_A(3)\} = 0.13$. Consider a neutrosophic set B of X which is given by

$$T_B : X \to [0, 1],\ x \mapsto \begin{cases} 0.14 & \text{if } x \in \{0, 1\}, \\ 0.84 & \text{if } x \in \{2, 3, 4\} \end{cases}$$

$$I_B : X \to [0, 1],\ x \mapsto \begin{cases} 0.85 & \text{if } x \in \{0, 1\} \\ 0.12 & \text{if } x \in \{2, 3, 4\}, \end{cases}$$

and

$$F_B : X \to [0, 1],\ x \mapsto \begin{cases} 0.14 & \text{if } x \in \{0, 1\} \\ 0.84 & \text{if } x \in \{2, 3, 4\}. \end{cases}$$

It is easy to show that B is a neutrosophic ideal of X, but not a neutrosophic BCC-ideal of X, since $T_B(4 * 3) = T_B(3) = 0.84 \not\leq \max\{T_B((4 * 1) * 3), T_B(1)\} = \max\{T_B(0), T_B(1)\} = 0.14$.

Example 3.22. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a BCC-algebra [5] with the following table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Define a neutrosophic set A of X as follows:

$$T_A : X \to [0,1],\ x \mapsto \begin{cases} 0.43 & \text{if } x \in \{0,1,2,3,4\} \\ 0.55 & \text{if } x = 5, \end{cases}$$

$$I_A : X \to [0,1],\ x \mapsto \begin{cases} 0.54 & \text{if } x \in \{0,1,2,3\,4\} \\ 0.42 & \text{if } x = 5, \end{cases}$$

and

$$F_A : X \to [0,1],\ x \mapsto \begin{cases} 0.43 & \text{if } x \in \{0,1,2,3,4\} \\ 0.55 & \text{if } x = 5. \end{cases}$$

It is easy to check that A is a neutrosophic BCC-ideal of X.

Theorem 3.23. *Let A be a neutrosophic set in a BCC-algebra X and let $\alpha, \beta, \gamma \in [0,1]$ with $0 \leq \alpha+\beta+\gamma \leq 3$. Then A is a neutrosophic BCC-ideal of X if and only if all of (α,β,γ)-level set $A^{(\alpha,\beta,\gamma)}$ are BCC-ideals of X when $A^{(\alpha,\beta,\gamma)} \neq \emptyset$.*

Proof. Similar to Theorem 3.11. □

Proposition 3.24. *Let A be a neutrosophic BCC-ideal of a BCC-algebra X. Then $X_T := \{x \in X | T_A(x) = T_A(0)\}, X_T := \{x \in X | I_A(x) = I_A(0)\}$, and $X_F := \{x \in X | F_A(x) = F_A(0)\}$ are BCC-ideals of X.*

Proof. Clearly, $0 \in X_T$. Let $(x*y)*z, y \in X_T$. Then $T_A((x*y)*z) = T_A(0)$ and $T_A(y) = T_A(0)$. It follows from (NSI3) that $T_A(x*z) \leq \max\{T_A((x*y)*z), T_A(y)\} = T_A(0)$. By (NSI1), we get $T_A(x*z) = T_A(0)$. Hence $x*z \in X_T$. Therefore X_T is a BCC-ideal of X. By a similar way, X_I and X_F are BCC-ideals of X. □

References

[1] S. S. Ahn, Applications of soft sets to BCC-ideals in BCC-algebras, J. Comput. Anal. Appl. (to appear).

[2] S. S. Ahn and S. H. Kwon, Toplogical properties in BCC-algerbras, Commun. Korean Math. Soc. **23(2)** (2008), 169-178.

[3] K. Atanassov, *Intuitionistic fuzzy sets,* Fuzzy sets and Systems 20 (1986), 87–96.

[4] W. A. Dudek, *On constructions of BCC-algebras,* Selected Papers on BCK- and BCI-algebras **1** (1992), 93-96.

[5] W. A. Dudek and X. Zhang, *On ideals and congruences in BCC-algeras,* Czecho Math. J. **48** (1998), 21-29.

[6] J. Hao, *Ideal Theory of BCC-algebras,* Sci. Math. Japo. **3** (1998), 373-381.

[7] Y. B. Jun, F. Smarandache and H. Bordbar, *Neutrosophic $\mathcal{N}$-structures applied to BCK/BCI-algebras,* Information, (to submit).

[8] Y. Kormori, *The class of BCC-algebras is not a varity,* Math. Japo. **29** (1984), 391-394.

[9] F. Smarandache, Neutrosophy, Neutrosophic Probablity, Sets, and Logic, Amer. Res. Press, Rehoboth, USA, 1998.

[10] L. A. Zadeh, Fuzzy sets, Inform. Control **8** (1965), 338-353.

Global Dynamics and Bifurcations of Two Second Order Difference Equations in Mathematical Biology

M. R. S. Kulenović[1 2] and Sarah Van Beaver
Department of Mathematics
University of Rhode Island, Kingston, Rhode Island 02881-0816, USA

Abstract. We investigate the global behavior of two difference equations with exponential nonlinearities

$$x_{n+1} = be^{-cx_n} + px_{n-1}, \quad n = 0, 1, \ldots$$

where the parameters b, c are positive real numbers and $p \in (0, 1)$ and

$$x_{n+1} = a + bx_{n-1}e^{-x_n}, \quad n = 0, 1, \ldots$$

where the parameters a, b are positive numbers. The the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers. The two equations are well known mathematical models in biology which behavior was studied by other authors and resulted in partial global dynamics behavior. In this paper, we complete the results of other authors and give the global dynamics of both equations. In order to obtain our results we will prove several results on global attractivity and boundedness and unboundedness for general second order difference equations

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots$$

which are of interest on their own.

Keywords. attractivity, difference equation, invariant sets, period doubling, periodic solutions, stable set .

AMS 2010 Mathematics Subject Classification: 39A20, 39A28, 39A30, 92D25

1 Introduction and Preliminaries

We investigate the global behavior of the system of difference equations

$$x_{n+1} = be^{-cx_n} + py_n, \quad y_{n+1} = x_n, \quad n = 0, 1, \ldots \tag{1}$$

where the parameters b and c are positive real numbers, $p \in (0, 1)$, and the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers. This system can be rewritten in the form of the second order difference equation

$$x_{n+1} = be^{-cx_n} + px_{n-1}, \quad n = 0, 1, \ldots \tag{2}$$

In [5], the authors originally studied this model to describe the synchrony of ovulation cycles of the Glaucous-winged Gulls. The model assumed that there is an infinite breeding season as well as the number of gulls available to breed is infinite. The value of c is a positive number representing the colony density. The parameter b is the number of birds per day ready to begin ovulating. The parameter p is the probability that a bird will begin to ovulate and $1 - e^{-cx_n}$ is the probability of delaying ovulation. In making the model, the authors assumed that the delay only occurs for birds entering the system, not birds switching between different segments of the cycle. Note the authors state that the bifurcation of two-cycle solutions is the same as ovulation synchrony with the value of c increasing. In [5], they used the local bifurcation theory to come to the conclusion that there exists a unique equilibrium such that for sufficiently small values of c, the equilibrium branch is locally asymptotically stable. Additionally, for large enough values of c, there exists a two-cycle branch that will be locally asymptotically stable. In this paper we will improve these results by making them global. Using the results of Camouzis and Ladas, see [2] and [6], we are able to find the global dynamics of (1), which was not completed in [5]. We will show that Equation (1) exhibits global period doubling bifurcation described by Theorem 5.1 in [11], which shows that global dynamics of Equation (1) changes from global asymptotic stability of the unique equilibrium solution to the global asymptotic stability of the minimal period-two solution within its basin of attraction, as the parameter passes through the critical value.

By using a similar method, we investigate the dynamics of

$$x_{n+1} = a + bx_{n-1}e^{-x_n}, \quad n = 0, 1, \ldots \tag{3}$$

where the parameters a, b are positive real numbers and the the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers. As it was mentioned in [8], Equation (3) could be considered as a mathematical model in biology where a represent the

[1]Corresponding author, *e-mail: mkulenovic@uri.edu*
[2]Partially supported by Maitland P. Simmons Foundation

constant immigration and b represent the population growth rate. In this paper we find a simpler equivalent condition to $\frac{-a+\sqrt{a^2+4a}}{a+\sqrt{a^2+4a}}e^{\frac{a+\sqrt{a^2+4a}}{2}} < b$ in [8] for the existence of a minimal period-two solution. We split the results into the two cases of $b \geq e^a$ and $b < e^a$. While using a similar method as in [9] to establish the existence of a period-two solution when $b < e^a$, we are able to find the global dynamics of Equation (3). By using new results for general second order difference equation we will prove the existence of unbounded solutions for the case when $b \geq e^a$. Similar as for Equation (1) we will show that Equation (3) exhibits global period doubling bifurcation described by Theorem 5.1 in [11]. In addition, we give the precise description of the basins of attractions of all attractors of both Equations (1) and (3).
The rest of the paper is organized as follows. In the rest of this section we introduce some known results about monotone systems in the plane needed for the proofs of the main results as well as some new results about the existence of unbounded solutions. Section 2 gives the global dynamics of Equation (1) and Section 3 gives the global dynamics of Equation (3).

The next result, which is combination of two theorems from [2] and [6], is important for the global dynamics of general second order difference equation.

Theorem 1 *Let I be a set of real numbers and $f : I \times I \to I$ be a function which is either non-increasing in the first variable and non-decreasing in the second variable or non-decreasing in both variables. Then, for every solution $\{x_n\}_{n=-1}^{\infty}$ of the equation*

$$x_{n+1} = f(x_n, x_{n-1}), \quad x_{-1}, x_0 \in I, \ n = 0, 1, \ldots \tag{4}$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ of even and odd terms of the solution are eventually monotonic.

We now give some basic notions about monotone maps in the plane.

Consider a partial ordering $\preceq$ on $\mathbb{R}^2$ where $x, y \in \mathbb{R}^2$ are said to be related if $x \preceq y$ or $y \preceq x$. Also, a strict inequality between points may be defined as $x \prec y$ if $x \preceq y$ and $x \neq y$. A stronger inequality may be defined as $x = (x_1, x_2) \ll y = (y_1, y_2)$ if $x \preceq y$ with $x_1 \neq y_1$ and $x_2 \neq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ is a continuous function $T : \mathcal{R} \to \mathcal{R}$. The map T is monotone if $x \preceq y$ implies $T(x) \preceq T(y)$ for all $x, y \in \mathcal{R}$, and it is strongly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \ll T(y)$ for all $x, y \in \mathcal{R}$. The map is strictly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \prec T(y)$ for all $x, y \in \mathcal{R}$.

Throughout this paper we shall use the *North-East ordering* (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the South-East (SE) ordering defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ which is monotone with respect to the North-East ordering is called *cooperative* and a map monotone with respect to the South-East ordering is called *competitive*.

If T is differentiable map on a nonempty set $\mathcal{R}$, a sufficient condition for T to be strongly monotone with respect to the SE ordering is that the Jacobian matrix at all points x has the sign configuration

$$\operatorname{sign}(J_T(\mathbf{x})) = \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \tag{5}$$

provided that $\mathcal{R}$ is open and convex.

For $x \in \mathbb{R}^2$, define $Q_\ell(x)$ for $\ell = 1, \ldots, 4$ to be the usual four quadrants based at x and numbered in a counterclockwise direction. *Basin of attraction* of a fixed point $(\bar{x}, \bar{y})$ of a map T, denoted as $\mathcal{B}((\bar{x}, \bar{y}))$, is defined as the set of all initial points (x_0, y_0) for which the sequence of iterates $T^n((x_0, y_0))$ converges to $(\bar{x}, \bar{y})$. Similarly, we define a basin of attraction of a periodic point of period p. The next five results, from [12, 11], are useful for determining basins of attraction of fixed points of competitive maps. Related results have been obtained by H. L. Smith in [14, 13].

Theorem 2 *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap int\,(Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}}))$ is nonempty (i.e., $\overline{\mathrm{x}}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true.*

a. The map T has a C^1 extension to a neighborhood of $\overline{\mathrm{x}}$.

b. The Jacobian $J_T(\overline{\mathrm{x}})$ of T at $\overline{\mathrm{x}}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{\mathrm{x}}$ that is invariant and a subset of the basin of attraction of $\overline{\mathrm{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{\mathrm{x}}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

We shall see in Theorem 4 that the situation where the endpoints of $\mathcal{C}$ are boundary points of $\mathcal{R}$ is of interest. The following result gives a sufficient condition for this case.

Theorem 3 *For the curve $\mathcal{C}$ of Theorem 2 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.*

i. The map T has no fixed points nor periodic points of minimal period-two in Δ.

ii. The map T has no fixed points in Δ, $\det J_T(\overline{\mathrm{x}}) > 0$, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.

iii. The map T has no points of minimal period-two in Δ, $\det J_T(\overline{\mathrm{x}}) < 0$, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.

For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 2 reduces just to $|\lambda| < 1$. This follows from a change of variables [14] that allows the Perron-Frobenius Theorem to be applied. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis.

The next result is useful for determining basins of attraction of fixed points of competitive maps.

Theorem 4 *(A) Assume the hypotheses of Theorem 2, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 2. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\mathcal{W}_- := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } \mathrm{x} \preceq_{se} y\} \quad \text{and} \quad \mathcal{W}_+ := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}, \tag{6}$$

such that the following statements are true.

(i) $\mathcal{W}_-$ is invariant, and $\mathrm{dist}(T^n(x), Q_2(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_-$.

(ii) $\mathcal{W}_+$ is invariant, and $\mathrm{dist}(T^n(x), Q_4(\overline{\mathrm{x}})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_+$.

(B) If, in addition to the hypotheses of part (A), $\overline{\mathrm{x}}$ is an interior point of $\mathcal{R}$ and T is C^2 and strongly competitive in a neighborhood of $\overline{\mathrm{x}}$, then T has no periodic points in the boundary of $Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}})$ except for $\overline{\mathrm{x}}$, and the following statements are true.

(iii) For every $x \in \mathcal{W}_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \operatorname{int} Q_2(\overline{\mathrm{x}})$ for $n \geq n_0$.

(iv) For every $x \in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \operatorname{int} Q_4(\overline{\mathrm{x}})$ for $n \geq n_0$.

If T is a map on a set $\mathcal{R}$ and if $\overline{\mathrm{x}}$ is a fixed point of T, the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set $\{x \in \mathcal{R} : T^n(x) \to \overline{\mathrm{x}}\}$ and unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set

$$\left\{ x \in \mathcal{R} : \text{there exists } \{x_n\}_{n=-\infty}^{0} \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1},\ x_0 = x, \text{ and } \lim_{n \to -\infty} x_n = \overline{\mathrm{x}} \right\}$$

When T is non-invertible, the set $\mathcal{W}^s(\overline{\mathrm{x}})$ may not be connected and made up of infinitely many curves, or $\mathcal{W}^u(\overline{\mathrm{x}})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on $\mathcal{R}$, the sets $\mathcal{W}^s(\overline{\mathrm{x}})$ and $\mathcal{W}^u(\overline{\mathrm{x}})$ are the stable and unstable manifolds of $\overline{x}$.

Theorem 5 *In addition to the hypotheses of part (B) of Theorem 4, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axis. If the curve $\mathcal{C}$ of Theorem 2 has endpoints in $\partial\mathcal{R}$, then $\mathcal{C}$ is the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$, and the unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{x}$ is a curve in $\mathcal{R}$ that is tangential to E^μ at $\overline{\mathrm{x}}$ and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\overline{\mathrm{x}})$ in $\mathcal{R}$ are fixed points of T.*

Remark 1 We say that $f(u, v)$ is strongly decreasing in the first argument and strongly increasing in the second argument if it is differentiable and has first partial derivative $D_1 f$ negative and first partial derivative $D_2 f$ positive in a considered set. The connection between the theory of monotone maps and the asymptotic behavior of Equation (4) follows from the fact that if f is strongly decreasing in the first argument and strongly increasing in the second argument, then the second iterate of a map associated to Equation (4) is a strictly competitive map on $I \times I$, see [11].

Set $x_{n-1} = u_n$ and $x_n = v_n$ in Equation (4) to obtain the equivalent system

$$\begin{array}{l} u_{n+1} = v_n \\ v_{n+1} = f(v_n, u_n) \end{array}, \quad n = 0, 1, \ldots.$$

Let $T(u, v) = (v, f(v, u))$. The second iterate T^2 is given by

$$T^2(u, v) = (f(v, u),\ f(f(v, u), v))$$

and it is strictly competitive on $I \times I$, see [12].

Remark 2 The characteristic equation of Equation (4) at an equilibrium point $(\bar{x}, \bar{x})$:

$$\lambda^2 - D_1 f(\bar{x}, \bar{x})\lambda - D_2 f(\bar{x}, \bar{x}) = 0, \tag{7}$$

has two real roots λ, μ which satisfy $\lambda < 0 < \mu$, and $|\lambda| < \mu$, whenever f is strictly decraesing in first and increasing in second variable. Thus the applicability of Theorems 2-5 depends on the existence or nonexistence of minimal period-two solutions.

We now present theorems relating to the existence of unbounded solutions of Equation (4). The original result was obtained in [4]. Here we give an improved version of Theorem 2.1 in [4] taking out the extraneous conditions of requiring a continuity of f and the existence of an equilibrium solution. Additionally, we have extended the results in [4] to obtain a theorem in which the function f is nondecreasing in both arguments.

Theorem 6 *Assume that the function $f : I \times I \to I$ is nonincreasing in the the first variable and nondecreasing in the second variable, where $I \subset R$ is an interval. Assume there exists numbers $L, U \in I$ such that $L < U$ which satisfy*

$$f(U, L) \leq L \tag{8}$$

and

$$f(L, U) \geq U, \tag{9}$$

where at least one inequality is strict. If $x_{-1} \leq L$ and $x_0 \geq U$, then the corresponding solution $\{x_n\}_{n=-1}^{\infty}$ satisfies

$$x_{2n-1} \leq L \quad \text{and} \quad x_{2n} \geq U, \quad n = 0, 1, \ldots$$

If, in addition, f is continuous and Equation (4) has no minimal period-two solution then,

$$\lim_{n\to\infty} x_{2n} = \infty \quad \text{and/or} \quad \lim_{n\to\infty} x_{2n-1} = -\infty.$$

Similarily, if $x_{-1} \geq U$ and $x_0 \leq L$, then the corresponding solution $\{x_n\}_{n=-1}^{\infty}$ satisfies

$$x_{2n-1} \geq U \quad \text{and} \quad x_{2n} \leq L, \quad n = 0, 1, \ldots$$

If, in addition, f is continuous and Equation (4) has no minimal period-two solution then,

$$\lim_{n\to\infty} x_{2n-1} = \infty \quad \text{and/or} \quad \lim_{n\to\infty} x_{2n} = -\infty.$$

Proof. Assume that $x_{-1} \leq L$ and $x_0 \geq U$. Then by using the monotonicity of f (nonincreasing in the first variable and nondecreasing in the second variable) and conditions (8) and (9) we obtain

$$x_1 = f(x_0, x_{-1}) \leq f(U, L) \leq L$$

and

$$x_2 = f(x_1, x_0) \geq f(L, U) \geq U.$$

By using induction it follows that $x_{2n-1} \leq L$ and $x_{2n} \geq U$ for all $n = 0, 1, \ldots$ where at least one inequality is strict. In view of Theorem 1 both sequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are eventually monotonic. Assume that f is a continuous function and there is no minimal period-two solution. We will consider a few cases based on the properties of the interval I. First suppose there exist $a \in \mathbb{R}$ such that $I = [a, \infty)$ and $a < L$. Then $\{x_{2n-1}\}_{n=0}^{\infty}$ will be convergent as the subsequence is bounded in $[a, L]$. If $\{x_{2n}\}_{n=0}^{\infty}$ converges, this would create a contradiction as there would exist a minimal period-two solution. Therefore,

$$\lim_{n\to\infty} x_{2n} = \infty.$$

Next suppose that for some $b \in \mathbb{R}$, both $I = (-\infty, b]$ and $U < b$. Here $\{x_{2n}\}_{n=0}^{\infty}$ will be convergent as the subsequence is bounded in the interval of $[U, b]$. So $\{x_{2n-1}\}_{n=0}^{\infty}$ cannot converge as there is no minimal period-two solution resulting in

$$\lim_{n\to\infty} x_{2n-1} = -\infty.$$

If $I = (-\infty, \infty)$, then similar to the two cases above, at most one subsequence can converge as there is no minimal period-two solution. So either

$$\lim_{n\to\infty} x_{2n} = \infty \quad \text{or} \quad \lim_{n\to\infty} x_{2n-1} = -\infty.$$

with the option of both occurring. Finally, we will prove that I cannot be $I = [a, b]$ where $a, b \in \mathbb{R}$. Suppose that $I = [a, b]$ such that $a < L < U < b$ and $a, b \in \mathbb{R}$. Since $x_n \in [a, b]$ for all n, both subsequences would be convergent. As $\lim_{n\to\infty} x_{2n-1} = p < \lim_{n\to\infty} x_{2n} = q$ for some $p, q \in \mathbb{R}$, there exists a period-two solution, which is a contradiction. The case when $x_{-1} \geq U$ and $x_0 \leq L$ will follow similarly to the proof used here. □

Many examples of the use of Theorem 6 are provided in [4].

Theorem 7 *Assume that $f : I \times I \to I$ is a function which is nondecreasing in both variables, where $I \subset R$ is an interval. Assume there exists numbers $L, U \in I$ such that $L < U$ where*

$$f(L, L) \leq L \tag{10}$$

and

$$f(U,U) \geq U \tag{11}$$

are satisfied, where at least one inequality is strict. If $x_{-1}, x_0 \leq L$, then the corresponding solution $\{x_n\}_{n=-1}^{\infty}$ of Equation (4) satisfies

$$x_n \leq L, \quad n = 0, 1, \ldots$$

If, in addition, f is continuous and Equation (4) has no minimal period-two solution, then either x_n converges to an equilibrium point or

$$\lim_{n\to\infty} x_{2n-1} = -\infty \quad \text{and/or} \quad \lim_{n\to\infty} x_{2n} = -\infty.$$

If $x_{-1}, x_0 \geq U$, then the corresponding solution $\{x_n\}_{n=-1}^{\infty}$ satisfies

$$x_n \geq U, \quad n = 0, 1, \ldots$$

If, in addition, f is continuous and Equation (4) has no period-two solution, then either x_n converges to an equilibrium point or

$$\lim_{n\to\infty} x_{2n-1} = \infty \quad \text{and/or} \quad \lim_{n\to\infty} x_{2n} = \infty.$$

Proof. Assume that $x_{-1}, x_0 \leq L$. Then by using the monotonicity of f (both variables are nondecreasing) and conditions (10) and (11) we obtain

$$x_1 = f(x_0, x_{-1}) \leq f(L,L) \leq L \quad \text{and} \quad x_2 = f(x_1, x_0) \leq f(L,L) \leq L.$$

By using induction it follows that $x_{2n-1}, x_{2n} \leq L$ for all $n = 0, 1, \ldots$ with at least one inequality being strict. In view of Theorem 1 both sequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are eventually monotonic. We can assume that f is continuous and that there is no minimal period-two solution. We can choose the value of L such that at most one equilibrium is included in the region. Note the subsequences may converge to the equilibrium point if present. We will break this proof into cases for different intervals I assuming that the subsequences do not converge to an equilibrium point. First suppose that either $I = [a, \infty)$ or $I = [a, b]$ for some $a, b \in \mathbb{R}$ such that $a < L < U < b$. As both subsequences are less than L, then $x_n \in [a, L]$ for every n. As a consequence, both subsequences will be convergent. Thus, $\lim_{n\to\infty} x_{2n-1} = p$ and $\lim_{n\to\infty} x_{2n} = q$. If $p = q$, we get a contradiction as the subsequences do not converge to an equilibrium point. Otherwise, $p \neq q$, so (p, q) is a period-two solution, which is a contradiction as well. Thus, for $I = [a, \infty)$ or $I = [a, b]$, there must be an equilibrium point present. Next suppose that either $I = (-\infty, a]$ or $I = (-\infty, \infty)$. Now $x_n \in (-\infty, L]$ for all n. At least one subsequence must be decreasing as the subsequences do not converge to an equilibrium point. Furthermore since there is no period-two solution, the subsequences cannot be bounded below resulting in either

$$\lim_{n\to\infty} x_{2n} = -\infty \quad \text{or} \quad \lim_{n\to\infty} x_{2n-1} = -\infty.$$

with the possibility of both options occurring.
Now assume that $x_{-1}, x_0 \geq U$. Then by using the monotonicity of f and conditions (10) and (11) we obtain

$$x_1 = f(x_0, x_{-1}) \geq f(U,U) \geq U$$

and

$$x_2 = f(x_1, x_0) \geq f(U,U) \geq U.$$

By using induction it follows that $x_{2n-1}, x_{2n} \geq U$ for all $n = 0, 1, \ldots$ with at least one inequality being strict. In view of Theorem 1 both sequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are eventually monotonic. Assume that f is continuous and that there is no minimal period-two solution. We can choose the value of U such that at most one equilibrium is included in the region. Note the subsequences may converge to the equilibrium point if present. We will break this proof into cases for different intervals I assuming that the subsequences do not converge to an equilibrium point. First suppose that either $I = (-\infty, b]$ or $I = [a, b]$ for some $a, b \in \mathbb{R}$ such that $a < L < U < b$. As both subsequences are greater than U, then $x_n \in [U, b]$ for every n. As a consequence, both subsequences will be convergent. Thus, $\lim_{n\to\infty} x_{2n-1} = p$ and $\lim_{n\to\infty} x_{2n} = q$. If $p = q$, we get a contradiction as the subsequences do not converge to an equilibrium point. Otherwise, $p \neq q$, so (p, q) is a period-two solution, which is a contradiction as well. Thus, for $I = (-\infty, b]$ or $I = [a, b]$, there must be an equilibrium point present. Next suppose that either $I = [a, \infty]$ or $I = (-\infty, \infty)$. Now $x_n \in [U, \infty)$ for all n. At least one subsequence must be increasing as the subsequences do not converge to an equilibrium point. Furthermore since there is no period-two solution, the subsequences cannot be bounded above resulting in either

$$\lim_{n\to\infty} x_{2n} = \infty \quad \text{or} \quad \lim_{n\to\infty} x_{2n-1} = \infty.$$

with the option of both occurring. □

Now we give few examples which illustrate all possible scenarios of Theorem 7.

Example 1 Consider the difference equation

$$x_{n+1} = x_n^2 + x_{n-1}^2, \quad n = 1, 2, \dots$$

where $x_{-1}, x_0 \in \mathbb{R}^+$, and $x_n \geq 0$ for $n = 1, 2, \dots$. Here $f(u, v) = u^2 + v^2$ is increasing in both variables. The equilibrium points are $\overline{x}_0 = 0$ and $\overline{x}_+ = 1/2$. The linearized difference equation is $z_{n+1} = 2\overline{x}z_n + 2\overline{x}z_{n-1}$ and the characteristic equation is $\lambda^2 = 2\overline{x}\lambda + 2\overline{x}$. The zero equilibrium $\overline{x}_0$ is locally asymptotically stable. For the equilibrium point $\overline{x}_+$, $\lambda^2 = \lambda + 1$, so that $\lambda_{1,2} = \frac{1\pm\sqrt{5}}{2}$. As $\frac{1+\sqrt{5}}{2} > 1$ and $\frac{1-\sqrt{5}}{2} \in (-1, 0)$, then $\overline{x}_+$ is a saddle point. There is no minimal period-two solution as

$$\phi = \psi^2 + \phi^2 \quad \text{and} \quad \psi = \phi^2 + \psi^2$$

implies $\phi = \psi$. Now we want to find a $L < U$ that satisfies the conditions (10) and (11). Condition (10) $f(L, L) \leq L$ implies $2L^2 \leq L$, which simplifies to $L \leq 1/2$. As well, $f(U, U) \geq U$ if $2U^2 \geq U$, which simplifies to $U \geq 1/2$. We can choose at least one of these inequalities to be strict. From Theorem 7, we can conclude that every solution with $x_1, x_0 \leq L$ converges to 0, while every solution with $x_{-1}, x_0 \geq U$ is eventually increasing and tends toward ∞. As $L < 1/2 < U$ are arbitrary this conclusion holds for every case where $x_{-1}, x_0 \leq L$ or $x_{-1}, x_0 \geq U$. These results do not give conclusions when $x_{-1} \leq L$ and $x_0 \geq U$ or $x_{-1} \geq U$ and $x_0 \leq L$. In this case one may use theory of monotone maps as in [3].

Example 2 Consider the difference equation

$$x_{n+1} = x_n^2 + x_{n-1}^2 + a, \quad n = 1, 2, \dots$$

where $a > 1/8$, $x_n \geq 0$, and $x_{-1}, x_0 \in \mathbb{R}$. Here $f(u, v) = u^2 + v^2 + a$ is increasing in both variables. There is no equilibrium points as the discriminant of the equilibrium equation $1 - 8a < 0$ and no minimal period-two solution exists as

$$\phi = \psi^2 + \phi^2 + a \quad \text{and} \quad \psi = \phi^2 + \psi^2 + a$$

implies $\phi = \psi$. We can find U that satisfies the conditions (10) and (11) of Theorem 7. As $f(U, U) \geq U$ simplifies to $2U^2 + a \geq U$, which always holds, every solution will be eventually increasing and tends to ∞.

Example 3 Consider the difference equation

$$x_{n+1} = x_n^5 + x_{n-1}^5, \quad n = 1, 2, \dots$$

where $x_{-1}, x_0 \in \mathbb{R}$. The function $f(u, v) = u^5 + v^5$ is increasing in both variables. The equilibrium points are $\overline{x}_0 = 0$ and $\overline{x}_\pm = \pm 1/\sqrt[4]{2}$. The characteristic equation at the equilibrium solution $\bar{x}$ is $\lambda^2 = 5\overline{x}^4\lambda + 5\overline{x}^4$. For the equilibrium point $\overline{x}_0$, $\lambda^2 = 0$ so that $\lambda_{1,2} = 0$ and $\overline{x}_0$ is locally asymptotically stable. For the equilibrium point $\overline{x}_\pm$, $\lambda^2 = 5/2\lambda + 5/2$, so that $\lambda_{1,2} = \frac{5\pm\sqrt{65}}{4}$. As $\frac{5+\sqrt{65}}{4} > 1$ and $\frac{5-\sqrt{65}}{4} \in (-1, 0)$, then the equilibrium points $\overline{x}_\pm$ are saddle points. There is no minimal period-two solution as

$$\phi = \psi^5 + \phi^5 \quad \text{and} \quad \psi = \phi^5 + \psi^5$$

implies $\phi = \psi$.

Now we want to find $L < U$ that satisfies the conditions of Theorem 7. Clearly $f(L, L) \leq L$ if $2L^5 \leq L$, which simplifies to $L \leq 1/\sqrt[4]{2}$ if $L > 0$ and to $L \leq -1/\sqrt[4]{2}$ if $L < 0$. As well, $f(U, U) \geq U$ if $2U^5 \geq U$, which simplifies to $U \geq 1/\sqrt[4]{2}$. We can choose at least one of these inequalities to be strict. From Theorem 7, we can conclude that every solution with $x_1, x_0 \leq L, L > 0$ converges to 0, while every solution with $x_{-1}, x_0 \geq U$ is eventually increasing and tends toward ∞. As $L < 1/\sqrt[4]{2} < U$ are arbitrary we conclude that

$$\lim_{n\to\infty} x_n = \begin{cases} 0 & \text{when} \quad \bar{x}_- < x_{-1}, x_0 < \bar{x}_+, \\ \infty & \text{when} \quad x_{-1}, x_0 > \bar{x}_+, \\ -\infty & \text{when} \quad x_{-1}, x_0 < \bar{x}_-. \end{cases}$$

Theorem 7 does not apply when $x_{-1} \leq L$ and $x_0 \geq U$ or $x_{-1} \geq U$ and $x_0 \leq L$. In this cases one can use the results from [3].

Example 4 Consider the difference equation

$$x_{n+1} = \frac{ax_n^2}{1 + x_n^2} + \frac{bx_{n-1}^2}{1 + x_{n-1}^2}, \quad n = 1, 2, \dots$$

where $a, b > 0$ and $x_{-1}, x_0 \in \mathbb{R}$. The function $f(u, v) = \frac{au^2}{1+u^2} + \frac{bv^2}{1+v^2}$ is increasing in both variables. One equilibrium point is $\overline{x}_0 = 0$. The non-zero equilibrium point satisfies the quadratic equation $1 + \overline{x}^2 - (a + b)\overline{x} = 0$ which has real solutions if $(a + b)^2 - 4 \geq 0$. If $a + b < 2$, then there only exist $\overline{x}_0$, if $a + b = 2$, then there exists $\overline{x}_0$ and $\overline{x}$, and if $a + b > 2$, then there exist

three equilibrium points $\overline{x}_0 < \overline{x}_- < \overline{x}_+$. The characteristic equation at the equilibrium solution $\bar{x}$ is $\lambda^2 = \frac{2a\overline{x}}{(1+\bar{x}^2)^2}\lambda + \frac{2b\overline{x}}{(1+\bar{x}^2)^2}$. For the equilibrium point $\overline{x}_0$, $\lambda^2 = 0$ so that $\lambda_{1,2} = 0$ and thus, $\overline{x}_0$ is locally asymptotically stable. The conditions for local stability of the equilibrium points $\bar{x}_\pm$ are quite involved and can be found in [1]. In particular $\overline{x}_-$ will either be a saddle point, repeller, or non-hyperbolic depending on whether $2a(a+b)+(a-b)\sqrt{(a+b)^2-4}$ is greater than, less than, or equal to 0, and the equilibrium point $\overline{x}_+$ is either locally asymptotically stable or non-hyperbolic when it exists.

Now we want to find a $L < U$ that satisfies the conditions (10) and (11) of Theorem 7. First $f(L,L) \le L$ if $\frac{(a+b)L^2}{1+L^2} \le L$, which simplifies to $0 \le 1 + L^2 - (a+b)L$. This will occur when $L < L_-$ or $L > L_+$ where we can set $L_- = \overline{x}_-$ and $L_+ = \overline{x}_+$. As well, $f(U,U) \ge U$ if $\frac{(a+b)U^2}{1+U^2} \ge U$, which simplifies to $0 \ge 1 + U^2 - (a+b)U$. This occurs when $U_- < U < U_+$ where we can set $U_- = \overline{x}_-$ and $U_+ = \overline{x}_+$. For both L and U to exist, we need $L < L_-$ to satisfy $L < U$. From Theorem 7, we can conclude that every solution with $x_1, x_0 \le L$ converges to 0, while every solution with $x_{-1}, x_0 \ge U$ converges to $\overline{x}_+$. Note that in the region where L and U exist, no minimal period-two solutions exists. All the period-two solutions are located in the region which is the union of the second and the fourth quadrant with respect to $\overline{x}_-$.

2 Global Dynamics of Equation (1)

In this section we present the global dynamics of Equation (1).

2.1 Local stability results

We begin by observing that the function $f(u,v) = be^{-cu} + pv$ is decreasing in the first variable and increasing in the second variable and so by Theorem 1, for every solution $\{x_n\}_{n=-1}^{\infty}$ of Equation (1) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are eventually monotonic.

Equation (1) has a unique positive equilibrium point $\overline{x}e^{c\overline{x}} = \frac{b}{1-p}$ where $0 < \overline{x} < \frac{b}{1-p}$. Note that $\frac{\partial f}{\partial u}(\overline{x},\overline{x}) = -cbe^{-c\overline{x}} = -c(1-p)\overline{x}$ and $\frac{\partial f}{\partial v}(\overline{x},\overline{x}) = p$. The characteristic equation of Equation (1) is

$$\lambda^2 + (1-p)c\overline{x}\lambda - p = 0.$$

Applying local stability test [10] we obtain

Lemma 1 *Equation* (1) *has a unique positive equilibrium solution* $\overline{x}e^{c\overline{x}} = \frac{b}{1-p}$.
i) If $\overline{x} < \frac{1}{c}$, *then the equilibrium point* $\overline{x}$ *is locally asymptotically stable.*
ii) If $\overline{x} > \frac{1}{c}$, *then the equilibrium point* $\overline{x}$ *is a saddle point.*
iii) If $\overline{x} = \frac{1}{c}$, *then the equilibrium point* $\overline{x}$ *is non-hyperbolic of the stable type (with eigenvalues* $\lambda_1 = -1$ *and* $\lambda_2 = p$*).*

Proof.
i) Equilibrium point $\overline{x}$ is locally asymptotically stable if

$$|(1-p)c\overline{x}| < 1-p < 2.$$

As $p \in (0,1)$ then $1-p < 2$ holds. As $(1-p)c\overline{x} > 0$, then $\overline{x}$ is stable if

$$(1-p)c\overline{x} < 1-p \Leftrightarrow c\overline{x} < 1 \Leftrightarrow \overline{x} < \frac{1}{c}.$$

Therefore, the equilibrium $\overline{x}$ is locally asymptotically stable if $\overline{x} < \frac{1}{c}$

ii) If $|(1-p)c\overline{x}| > |1-p|$, then the equilibrium point $\overline{x}$ is a saddle point. As $(1-p)c\overline{x}$ is positive, we obtain

$$(1-p)c\overline{x} > 1-p \Leftrightarrow c\overline{x} > 1 \Leftrightarrow \overline{x} > \frac{1}{c}.$$

So the equilibrium point $\overline{x}$ is a saddle point if $\overline{x} > \frac{1}{c}$.

iii) The equilibrium point $\overline{x}$ is non-hyperbolic if

$$|(1-p)c\overline{x}| = |1-p|.$$

We see that $c\overline{x} = 1 \Leftrightarrow \overline{x} = \frac{1}{c}$. The characteristic equation at the equilibrium becomes

$$\lambda^2 + (1-p)\lambda - p = 0,$$

with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = p$. $\square$

2.2 Periodic solutions

In this section we present results about existence and uniqueness of the minimal period-two solution of Equation (1).

Theorem 8 *If $\overline{x} > \frac{1}{c}$, then Equation* (1) *has a unique minimal period-two solution:*

$$\phi, \psi, \phi, \psi, \ldots (\phi \neq \psi, \phi > 0 \text{ and } \psi > 0).$$

Proof. Let $\{\phi, \psi\}$ be a minimal period-two solution of Equation (1), where ϕ and ψ are distinct positive real numbers. Then we have

$$\phi = be^{-c\psi} + p\phi, \quad \psi = be^{-c\phi} + p\psi, \tag{12}$$

where $\phi \neq \psi$. This implies

$$\psi = \frac{be^{-c\phi}}{1-p}, \quad \phi = be^{\frac{-cbe^{-c\phi}}{1-p}} + p\phi.$$

Let $F(\phi) = be^{\frac{-cbe^{-c\phi}}{1-p}} + (p-1)\phi$. The equilibrium point $\overline{x} = \frac{b}{1-p}e^{-c\overline{x}}$ will be a zero of F as

$$F(\overline{x}) = be^{\frac{-cbe^{-c\overline{x}}}{1-p}} + (p-1)\overline{x} = be^{-c\overline{x}} + (p-1)\overline{x} = 0.$$

Note that $F(0) = be^{\frac{-cb}{1-p}} > 0$ since $b > 0$. Additionally, as ϕ approaches ∞, then $F(\phi)$ approaches $-\infty$. Notice graphically, the the function F begins above the x-axis and ends approaching $-\infty$. As the function F crosses the x-axis at least once at $\overline{x}$, then F must cross the x-axis at least three times when $F'(\overline{x}) > 0$. This will result in the existence of a minimal period-two solution. We want to prove that $F'(\overline{x}) > 0$ holds true for some values of parameters. Observe that the derivative of F is

$$F'(\phi) = \frac{b^2c^2}{1-p}e^{-c\phi}e^{\frac{-cbe^{-c\phi}}{1-p}} + (p-1)$$

so that when $\overline{x}$ is substituted $F'(\overline{x}) = \overline{x}bc^2e^{-c\overline{x}} + (p-1)$. Then $F'(\overline{x}) > 0$ when $\overline{x} > \frac{1}{c}$ as

$$F'(\overline{x}) = \overline{x}bc^2e^{-c\overline{x}} + (p-1) > 0 \Leftrightarrow c^2\overline{x} > \frac{1-p}{b}e^{c\overline{x}} \Leftrightarrow c^2\overline{x} > \frac{1}{\overline{x}} \Leftrightarrow \overline{x} > \frac{1}{c}.$$

Thus when $\overline{x} > \frac{1}{c}$, there will be a minimal period-two solution.

Next we want to prove that the period-two solution is unique. Rewritting (12) we obtain

$$\phi e^{c\psi} = \frac{b}{1-p} = \psi e^{c\phi} \Leftrightarrow \phi e^{-c\phi} = \psi e^{-c\psi}.$$

Let $g(x) = xe^{-cx}$. As $g'(x) = e^{-cx}(1-cx)$, then the global maximum of g is attatined at $x = \frac{1}{c}$. For each y value there will be two corresponding x values when $g(x) < g(\frac{1}{c}) = \frac{1}{ce}$. This will happen when

$$xe^{-cx} < \frac{1}{ce} \Leftrightarrow e^{cx} - ecx > 0.$$

Let $G(x) = e^{cx} - ecx$ and notice that $G(0) = 1$. The derivative of G will be $G'(x) = c(e^{cx} - e)$. Notice $G'(x) \leq 0$ when $e^{cx} \leq e$ such that $x \leq \frac{1}{c}$, and $G'(x) > 0$ when $x > \frac{1}{c}$. Thus, $G(x) > 0$ on $[0, \frac{1}{c}) \cup (\frac{1}{c}, \infty)$ where $G(\frac{1}{c}) = 0$ is a global minimum. Thus when the period-two solution exists, it is unique. □

2.3 Global stability results

In view of Theorem 1 every bounded solution of Equation (1) converges to either an equilibrium solution or a minimal period-two solution.

Lemma 2 *The solutions of Equation* (1) *are bounded.*

Proof. Equation (1) implies

$$x_{n+1} = be^{-cx_n} + px_{n-1} \leq b + px_{n-1}, \quad n = 0, 1, \ldots.$$

Consider the difference equation of

$$u_{n+1} = b + pu_{n-1}, \quad n = 0, 1, \ldots. \tag{13}$$

The solution of Equation (13) is $u_n = \frac{b}{1-p} + C_1(\sqrt{p})^n + C_2(-\sqrt{p})^n$. As $n \to \infty$, then $u_n \to \frac{b}{1-p}$. In view of difference inequality result, see [7] $x_n \leq u_n \leq \frac{b}{1-p} + \epsilon = \mathcal{U}$ for $n = 0, 1, \ldots$ and some $+\epsilon > 0$ when $x_0 \leq u_0$. □

Theorem 9 *(i) If $\bar{x} > \frac{1}{c}$, then the equilibrium solution $\bar{x}$ is a saddle point and the minimal period-two solution $\{\phi, \psi\}, \phi < \psi$ is globally asymptotically stable within the basin of attraction $\mathcal{B}(\phi, \psi) = [0, \infty)^2 \setminus \mathcal{W}^s(\bar{x}, \bar{x})$, where $\mathcal{W}^s(\bar{x}, \bar{x})$ is the global stable manifold of $(\bar{x}, \bar{x})$.*
(ii) If $\bar{x} \leq \frac{1}{c}$, then the equilibrium solution $\bar{x}$ is globally asymptotically stable.

Proof. Using Theorem 1 every bounded solution of Equation (1) converges to an equilibrium solution or period-two solution. By Lemma 2, every solution of Equation (1) is bounded so that all solutions converge to either an equilibrium solution or to the unique period-two solution $\{\phi, \psi\}, \phi < \psi$. When $\overline{x} > \frac{1}{c}$, then $\overline{x}$ is a saddle point, by Lemma 1 part (ii), and has the global stable $\mathcal{W}^s(\overline{x}, \overline{x})$ and global unstable manifolds $\mathcal{W}^u(\overline{x}, \overline{x})$, where $\mathcal{W}^s(\overline{x}, \overline{x})$ is the graph of a non-decreasing function and $\mathcal{W}^u(\overline{x}, \overline{x})$ is the graph of a non-increasing function, which has endpoints at (ϕ, ψ) and (ψ, ϕ). Every initial point (x_{-1}, x_0) which starts south east of $\mathcal{W}^s(\overline{x}, \overline{x})$ is attracted to (ψ, ϕ), while every initial point (x_{-1}, x_0) which starts north west of $\mathcal{W}^s(\overline{x}, \overline{x})$ is attracted to (ϕ, ψ), see Theorems 2, 4. In this case in view of Theorem 1 global attractivity of period-two solution implies its local stability since the convergence is monotonic.

When $\overline{x} \leq \frac{1}{c}$, the equilibrium solution is locally and so globally asymptotically stable by Lemma 1 part (i) and part (iii) . □

Remark 3 For instance, case i) of Theorem 9 holds when $b = 1, p = .5, c = 2$, case ii) holds when $b = 1, p = .5, c = 1$ and when $b = 1, p = (e-1)/e, c = 1$.

3 Global Dynamics of Equation (3)

In this section we present global dynamics of Equation (3).

3.1 Local stability results

First, notice that the function $f(u, v) = a + bve^{-u}$ is decreasing in the first variable and increasing in the second variable. By Theorem 1, for all solutions $\{x_n\}_{n=-1}^{\infty}$ of Equation (3) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are eventually monotonic.

Equation (3) has a unique positive equilibrium point $\overline{x} = \frac{a}{1-be^{-\overline{x}}}$ where $a < \overline{x}$. Note that $\frac{\partial f}{\partial u}(\overline{x}, \overline{x}) = -b\overline{x}e^{-\overline{x}}$ and $\frac{\partial f}{\partial v}(\overline{x}, \overline{x}) = be^{-\overline{x}}$. The characteristic equation of Equation (3) is

$$\lambda^2 + b\overline{x}e^{-\overline{x}}\lambda - be^{-\overline{x}} = 0.$$

Lemma 3 *Equation (3) has a unique positive equilibrium solution $\overline{x} = \frac{a}{1-be^{-\overline{x}}}$.*
i) If $\overline{x} < \frac{a+\sqrt{a^2+4a}}{2}$, then the equilibrium solution $\overline{x}$ is locally asymptotically stable.
ii) If $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, then the equilibrium solution $\overline{x}$ is a saddle point.
iii) If $\overline{x} = \frac{a+\sqrt{a^2+4a}}{2}$, then the equilibrium solution $\overline{x}$ is non-hyperbolic of stable type (with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = be^{-\overline{x}}$).

Proof.
i) The equilibrium point $\overline{x}$ is locally asymptotically stable if

$$\left|b\overline{x}e^{-\overline{x}}\right| < 1 - be^{-\overline{x}} < 2.$$

As $be^{-\overline{x}} > 0$, then $1 - be^{-\overline{x}} < 2$ holds true. So rearranging the other inequality we obtain

$$b\overline{x}e^{-\overline{x}} < 1 - be^{-\overline{x}} \Leftrightarrow be^{-\overline{x}}(\overline{x}+1) < 1 \Leftrightarrow \overline{x} + 1 < \frac{1}{b}e^{\overline{x}} \Leftrightarrow \overline{x} < \frac{e^{\overline{x}}}{b} - 1.$$

Therefore, the equilibrium $\overline{x}$ is locally asymptotically stable if $\overline{x} < \frac{e^{\overline{x}}}{b} - 1$. As $\overline{x} = a + b\overline{x}e^{-\overline{x}}$ we have

$$e^{\overline{x}} = \frac{b\overline{x}}{\overline{x} - a}. \tag{14}$$

Then we can equivalently write the condition to be locally asymptotically stable as

$$\overline{x} < \frac{e^{\overline{x}}}{b} - 1 \Leftrightarrow \overline{x} < \frac{\frac{b\overline{x}}{\overline{x}-a}}{b} - 1 \Leftrightarrow \overline{x} < \frac{\overline{x}}{\overline{x}-a} - 1$$
$$\Leftrightarrow \overline{x}^2 - \overline{x}a - a < 0 \Leftrightarrow \overline{x} < \frac{a + \sqrt{a^2+4a}}{2}.$$

ii) If

$$\left|b\overline{x}e^{-\overline{x}}\right| > \left|1 - be^{-\overline{x}}\right|,$$

then the equilibrium solution $\overline{x}$ is a saddle point. Note that $be^{-\overline{x}} < 1$ since

$$be^{-\overline{x}} < 1 \Leftrightarrow \frac{\overline{x} - a}{\overline{x}} < 1 \Leftrightarrow \frac{-a}{\overline{x}} < 0$$

always holds as $a > 0$. The condition for $\overline{x}$ to be a saddle point yields

$$b\overline{x}e^{-\overline{x}} > 1 - be^{-\overline{x}} \Leftrightarrow be^{-\overline{x}}(\overline{x}+1) > 1 \Leftrightarrow \overline{x}+1 > \frac{1}{b}e^{\overline{x}} \Leftrightarrow \overline{x} > \frac{e^{\overline{x}}}{b} - 1.$$

So the equilibrium point $\overline{x}$ is a saddle point if $\overline{x} > \frac{e^{\overline{x}}}{b} - 1$. By using (14), the inequality can then equivalently be written as

$$\overline{x} > \frac{e^{\overline{x}}}{b} - 1 \Leftrightarrow \overline{x} > \frac{\frac{b\overline{x}}{\overline{x}-a}}{b} - 1 \Leftrightarrow \overline{x} > \frac{\overline{x}}{\overline{x}-a} - 1 \Leftrightarrow \overline{x}^2 - \overline{x}a - a > 0 \Leftrightarrow \overline{x} > \frac{a + \sqrt{a^2+4a}}{2}.$$

iii) The equilibrium point $\overline{x}$ is non-hyperbolic point if

$$\left|b\overline{x}e^{-\overline{x}}\right| = \left|1 - be^{-\overline{x}}\right|.$$

We see that

$$b\overline{x}e^{-\overline{x}} = 1 - be^{-\overline{x}} \Leftrightarrow be^{-\overline{x}}(\overline{x}+1) = 1 \Leftrightarrow \overline{x}+1 = \frac{1}{b}e^{\overline{x}} \Leftrightarrow \overline{x} = \frac{e^{\overline{x}}}{b} - 1.$$

In view of (14) this can be rewritten as

$$\overline{x} = \frac{e^{\overline{x}}}{b} - 1 \Leftrightarrow \overline{x} = \frac{\frac{b\overline{x}}{\overline{x}-a}}{b} - 1 \Leftrightarrow \overline{x} = \frac{\overline{x}}{\overline{x}-a} - 1 \Leftrightarrow \overline{x}^2 - \overline{x}a - a = 0 \Leftrightarrow \overline{x} = \frac{a + \sqrt{a^2+4a}}{2}.$$

The characteristic equation at the equilibrium point will become

$$\lambda^2 + (1 - be^{-\overline{x}})\lambda - be^{-\overline{x}} = 0,$$

with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = be^{-\overline{x}} \in (0,1)$. □

3.2 Periodic solutions

In this section we present results about existence and uniqueness of minimal period-two solutions of Equation (3).

Theorem 10 *Assume that $b < e^a$. If $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, then Equation (3) has minimal period-two solution:*

$$\phi, \psi, \phi, \psi, \ldots \quad (\phi \neq \psi \text{ and } \phi > 0, \psi > 0).$$

Proof. We want to find for which values of $\overline{x}$ there exists a minimal period-two solution (ϕ, ψ) where ϕ and ψ are distinct positive real numbers. A period-two solution satisfies

$$\phi = a + b\phi e^{-\psi}, \quad \psi = a + b\psi e^{-\phi}, \tag{15}$$

where ϕ and ψ are distinct real numbers. Rewritting ψ and then substituting into ϕ we obtain

$$\psi = \frac{a}{1 - be^{-\phi}}, \quad \phi = a + b\phi e^{-\frac{a}{1-be^{-\phi}}}. \tag{16}$$

Let $F(\phi) = a + \phi(be^{-\frac{a}{1-be^{-\phi}}} - 1)$. The equilibrium point $\overline{x} = \frac{e^{\overline{x}}(\overline{x}-a)}{b}$ will be a zero of F as

$$F(\overline{x}) = a + \overline{x}(be^{-\frac{a}{1-be^{-\overline{x}}}} - 1) = a + \overline{x}(be^{-\overline{x}} - 1) = 0.$$

Now

$$F(a) = a + a(be^{-\frac{a}{1-be^{-a}}} - 1) = abe^{-\frac{a}{1-be^{-a}}}$$

is positive as a and b are positive constants. As ϕ approaches ∞, then F approaches $-\infty$ assuming that $b < e^a$. When $F'(\overline{x}) > 0$ then F will cross the $x-$axis at least three times resulting in a minimal period-two solution. Thus, we want to prove when $F'(\overline{x}) > 0$ holds. Taking the derivative of F we have

$$F'(\phi) = (be^{-\frac{a}{1-be^{-\phi}}} - 1) + \frac{\phi ab^2 e^{-\phi}}{(1-be^{-\phi})^2} e^{-\frac{a}{1-be^{-\phi}}}$$

so that $F'(\overline{x}) = \frac{-a}{\overline{x}} + \frac{\overline{x}^3 b^2 e^{-2\overline{x}}}{a}$. Then $F'(\overline{x}) > 0$ hold true when

$$\frac{-a}{\overline{x}} + \frac{\overline{x}^3 b^2 e^{-2\overline{x}}}{a} > 0 \Leftrightarrow \overline{x}^4 b^2 e^{-2\overline{x}} > a^2 \Leftrightarrow \overline{x}^2 b e^{-\overline{x}} > a \Leftrightarrow \overline{x}^2 b > \frac{a\overline{x}b}{\overline{x}-a} \Leftrightarrow \overline{x}(\overline{x}-a) > a\overline{x}^2 - \overline{x}a - a > 0.$$

Thus, when $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, there will be a minimal period-two solution.

When $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, then

$$\frac{a}{1-be^{-\overline{x}}} > \frac{a+\sqrt{a^2+4a}}{2} \Leftrightarrow \frac{2a}{a+\sqrt{a^2+4a}} > 1 - be^{-\overline{x}}$$

$$\Leftrightarrow \frac{-a+\sqrt{a^2+4a}}{a+\sqrt{a^2+4a}} e^{\overline{x}} < b \Leftrightarrow \frac{-a+\sqrt{a^2+4a}}{a+\sqrt{a^2+4a}} e^{\frac{a+\sqrt{a^2+4a}}{2}} < b.$$

Next we want to prove that the minimal period-two solution is unique. By rewriting (15) we find that

$$\phi(1-be^{-\psi}) = a = \psi(1-be^{-\phi}) \Leftrightarrow \frac{\phi}{1-be^{-\phi}} = \frac{\psi}{1-be^{-\psi}}.$$

Let $g(x) = \frac{x}{1-be^{-x}}$. Using $g'(x) = \frac{1-be^{-x}(x+1)}{(1-be^{-x})^2}$ to find the critical points we get that $1 - be^{-x}(x+1) = 0 \Leftrightarrow e^x = b(x+1)$. There exists a unique value of m where $\frac{1}{m+1} = be^{-m}$ for which this holds. Using the first-derivative theorem we can check that m is a local minima. Note it suffices to check the numerator of $g'(m-1)$ as the denominator is always positive. Using the fact that $\frac{1}{m+1} = be^{-m}$

$$1 - be^{-(m-1)}m < 0 \Leftrightarrow \frac{1}{m} < be^{-(m-1)} \Leftrightarrow \frac{1}{m} < \frac{e}{m+1} \Leftrightarrow \frac{m+1}{m} < e.$$

This proves that $g'(m-1) < 0$. Next using the same method taking the numerator of $g'(m+1)$ we see that

$$1 - be^{-(m+1)}(m+2) > 0 \Leftrightarrow \frac{1}{m+2} > be^{-(m+1)} \Leftrightarrow \frac{1}{m+2} > \frac{e^{-1}}{m+1} \Leftrightarrow \frac{m+1}{m+2} > e^{-1}.$$

This proves that $g'(m+1) > 0$. As the derivative changes from negative to positive around the critical point, it will be a local minima. Note that $g(a) > 0$ and as x approaches ∞, $g(x)$ approaches ∞. Since m is the only critical point, each y value will have two x values with the exception at m. This results in the fact that there can only be one period-two solution. □

Proposition 1 *If $b \geq e^a$, there are no minimal period-two solutions.*

Proof. Assume that $\{\phi, \psi\}$ is a period-two solution. Then $\{\phi, \psi\}$ satisfies (15) and so it satisfies (16) as well.

Let $F(\phi) = a + \phi(be^{-\frac{a}{1-be^{-\phi}}} - 1)$. The equilibrium point $\overline{x} = \frac{e^{\overline{x}}(\overline{x}-a)}{b}$ will be a zero of F as

$$F(\overline{x}) = a + \overline{x}(be^{-\frac{a}{1-be^{-\overline{x}}}} - 1) = a + \overline{x}(be^{-\overline{x}} - 1) = 0.$$

We see that

$$F(a) = a + a(be^{-\frac{a}{1-be^{-a}}} - 1) = abe^{-\frac{a}{1-be^{-a}}}$$

which is a positive value as a and b are positive constants. As ϕ approaches ∞, then F approaches ∞ as $b \geq e^a$. As the function begins above the x-axis at a and approaches ∞, F will cross the x-axis an even number of times. Since $F(\overline{x}) = 0$ is one of the points that lie on the x-axis and the only equilibrium point, there cannot be a minimal period-two solution. □

3.3 Global stability results

By Theorem 1 every bounded solution of Equation (1) converges to either an equilibrium solution or a minimal period-two solution.

Lemma 4 *The solutions of Equation* (3) *are bounded if* $b < e^a$.

Proof. By Equation (3),

$$x_{n+1} = a + bx_{n-1}e^{-x_n} \le a + bx_{n-1}, \quad n = 0, 1, ...$$

Consider the difference equation of

$$u_{n+1} = a + bu_{n-1}, \quad n = 0, 1, ... \tag{17}$$

Suppose that $b < e^a$. The solution of Equation (17) is $u_n = \frac{a}{1-b} + C_1(\sqrt{b})^n + C_2(-\sqrt{b})^n$. As $n \to \infty$, then $u_n \to \frac{a}{1-b}$. In view of difference inequality result, see [7] $x_n \le u_n \le \frac{a}{1-b} + \epsilon = \mathcal{U}$ for $n = 0, 1, ...$ when $x_0 \le u_0$, where $\epsilon > 0$. □

Theorem 11 *Consider Equation* (3).

(i) If $b < e^a$ and $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, then there exists a period-two solution that is locally asymptotically stable and the equilibrium point, $\overline{x}$, that is is a saddle point. The unique period-two solution attracts all solutions which start off the global stable manifold of $\mathcal{W}^s(E(\overline{x}, \overline{x}))$.

(ii) If $b < e^a$ and $\overline{x} < \frac{a+\sqrt{a^2+4a}}{2}$, then the equilibrium solution, $\overline{x}$, is globally asymptotically stable.

(iii) If $b < e^a$ and $\overline{x} = \frac{a+\sqrt{a^2+4a}}{2}$, then the equilibrium solution, $\overline{x}$, is non-hyperbolic of the stable type and is global attractor.

Proof.

(i) Using Theorem 1 every bounded solution of Equation (3) converges to an equilibrium solution or period-two solution. By Lemma 4, when $b < e^a$ every solution of Equation (3) is bounded such that all solutions will converge to either an equilibrium solution or period-two solution. If $b < e^a$ and $\overline{x} > \frac{a+\sqrt{a^2+4a}}{2}$, then $\overline{x}$ will be a saddle point by Lemma 3 part (ii), and there will be a minimal period-two solution by Theorem 10. In view of Theorems 2, 4 there exist the global stable manifold $\mathcal{W}^s(\overline{x}, \overline{x})$ and global unstable manifold $\mathcal{W}^u(\overline{x}, \overline{x})$, where $\mathcal{W}^s(\overline{x}, \overline{x})$ is the graph of a non-decreasing function and $\mathcal{W}^u(\overline{x}, \overline{x})$ is the graph of a non-increasing function, which has endpoints at (ϕ, ψ) and (ψ, ϕ). Every initial point (x_{-1}, x_0) which starts south east of $\mathcal{W}^s(\overline{x}, \overline{x})$ is attracted to (ψ, ϕ), while every initial point (x_{-1}, x_0) which starts north west of $\mathcal{W}^s(\overline{x}, \overline{x})$ is attracted to (ϕ, ψ).

(ii) When $b < e^a$ and $\overline{x} < \frac{a+\sqrt{a^2+4a}}{2}$, then $\overline{x}$ is locally asymptotically stable by Lemma 3 part (i). Since $[a, U]^2$ is invariant box and $(\overline{x}, \overline{x})$ is the only fixed point then, by Theorem 2.1 in [11] is global attractor and so globally asymptotically stable.

(iii) Moreover, when $b < e^a$ and $\overline{x} = \frac{a+\sqrt{a^2+4a}}{2}$, $\overline{x}$ will be non-hyperbolic of the stable type by Lemma 3 part (iii). Since $[a, U]^2$ is invariant box and $(\overline{x}, \overline{x})$ is the only fixed point then, by Theorem 2.1 in [11] is global attractor and so globally asymptotically stable.

□

Theorem 12 *If $b \ge e^a$, then Equation* (3) *has unbounded solutions.*

Proof. We will use Theorem 6 to prove this theorem. The conditions of (8) and (9) of Theorem 6 become

$$f(U, L) = a + bLe^{-U} \le L \quad \text{and} \quad f(L, U) = a + bUe^{-L} \ge U.$$

These inequalities can be reduced to

$$a \le L(1 - be^{-U}) \quad \text{and} \quad a \ge U(1 - be^{-L}).$$

Any value of L and U such that $\frac{U}{1-be^{-U}} \le \frac{L}{1-be^{-L}}$ will satisfy the theorem. Let $G(x) = \frac{x}{1-be^{-x}}$. There is a vertical asymptote at $1 - be^{-x} = 0$ that is at $x = \ln(b)$. In interval $(\ln(b), \infty)$ we can find L and U that satisfies these inequalities. As $b \ge e^a$ then $\ln(b) \ge a$ so that $(\ln(b), \infty)$ is part of the domain of difference equation (3). An example of where this holds is when $L = a + \epsilon$. Using the fact that $b \ge e^a$ and ϵ is small, then $b \ge e^{a+\epsilon}$. By condition (9) the inequality holds true as

$$a + bUe^{-(a+\epsilon)} \ge U \Leftrightarrow e^{a+\epsilon} \le \frac{bU}{U - a}.$$

We will use condition (8) and $b \geq e^a$ to find the criteria for U based on our L. Thus,

$$a + b(a+\epsilon)e^{-U} \leq (a+\epsilon) \Leftrightarrow e^U \geq \frac{b(a+\epsilon)}{\epsilon} \Leftrightarrow e^U \geq \frac{e^a(a+\epsilon)}{\epsilon} \Leftrightarrow U \geq a + \ln\left(\frac{a+\epsilon}{\epsilon}\right).$$

Let U be such that $U > a + \ln\left(\frac{a+\epsilon}{\epsilon}\right)$. It holds that $U \geq L$. Overall, as f is continuous and there is no minimal period-two solution by Proposition 1, using Theorem (6) some solutions will approach ∞. □

Remark 4 For instance, case i) of Theorem 11 holds when $a = 1, b = 2$, case ii) holds when $a = 4, b = 2$ and case iii) holds when $a = 2, b = \frac{\sqrt{3}-1}{\sqrt{3}+1}e^{1+\sqrt{3}}$, and the conditions of Theorem 12 holds when $a = .5, b = 2$.

In conclusion, Equations (1) and (3) exhibit the global period doubling bifurcation described by Theorem 5.1 in [11]. Checking the conditions of Theorem 5.1 in [11] is exactly the content of Lemmas 1-3 and Theorems 10-12.

References

[1] A. Bilgin, M. R. S. Kulenović, and E. Pilav, Basins of Attraction of Period-two Solutions of Monotone Difference Equations, *Adv. Difference Equ.* (2016) 25 pp.

[2] A.M. Amleh, E. Camouzis, and G. Ladas, On the Dynamics of a Rational Difference Equation, Part 1, *Int. J. Difference Equ.*Vol. 3 (1) (2008), 1-35.

[3] A. Brett and M. R. S. Kulenović, Basins of attraction of equlilibrium points of monotone difference equations, *Sarajevo J. Math.*, Vol. 5 (18) (2009), 211-233.

[4] D. Burgić, S. Kalabušić, and M. R. S. Kulenović, Nonhyperbolic Dynamics for Competitive Systems in the Plane and Global Period-doubling Bifurcations, *Adv. Dyn. Syst. Appl.* 3 (2008), 229-249.

[5] D. Burton and S. Henson, A note on the onset of synchrony in avian ovulation cycles, *J. Difference Equ. Appl.*, Vol. 20 (4) (2014), 664–668.

[6] E. Camouzis and G. Ladas, When does local asymptotic stability imply global attractivity in rational equations? *J. Difference Equ. Appl.*, 12(2006), 863–885.

[7] S. Elaydi, *An introduction to difference equations.* Third edition. Undergraduate Texts in Mathematics. Springer, New York, 2005.

[8] H. El-Metwally, E.A. Grove, G. Ladas, R. Levins, and M. Radin, On the Difference Equation $x_{n+1} = \alpha + \beta x_{n-1}e^{-x_n}$, *Nonlinear Analysis TMA* Vol. 47 (2001), 4623-4634.

[9] N. Fotiades and G. Papaschinopoulos, Existence uniqueness and attractivity of prime period two solution for a difference equation of exponential form, *Appl. Math. Comp.*, 218 (2012), 11648-11653

[10] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures,* Chapman and Hall/CRC, Boca Raton, London, 2001.

[11] M. R. S. Kulenović and O. Merino, Global bifurcations for competitive system in the plane, *Discrete Contin. Dyn. Syst. Ser. B* 12 (2009), 133-149.

[12] M. R. S. Kulenović and O. Merino, Invariant manifolds for competitive discrete systems in the plane, *Int. J. Bifur. Chaos* 20 (2010),2471-2486.

[13] H. L. Smith, Periodic competitive differential equations and the discrete dynamics of competitive maps, *J. Differential Equations* 64 (1986), 163-194.

[14] H. L. Smith, Planar competitive and cooperative difference equations. *J. Difference Equ. Appl.* 3 (1998), 335–357.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Bounds for the Real Parts and Arguments of Normalized Analytic Functions Defined by the Srivastava-Attiya Operator

Young Jae Sim[1], Oh Sang Kwon[1], Nak Eun Cho[2,*] and H. M. Srivastava[3,4]

[1]Department of Mathematics, Kyungsung University,
Busan 48434, Republic of Korea

E-Mail: yjsim@ks.ac.kr oskwon@ks.ac.kr

[2]Department of Applied Mathematics, Pukyong National University,
Busan 48513, Republic of Korea

E-Mail: necho@pknu.ac.kr

[*]Corresponding Author

[3]Department of Mathematics and Statistics, University of Victoria,
Victoria, British Columbia V8W 3R4, Canada

[4]Department of Medical Research, China Medical University Hospital,
China Medical University, Taichung 40402, Taiwan, Republic of China

E-Mail: harimsri@math.uvic.ca

Abstract

In this paper, we derive some bounds for the real parts and arguments of the functionals given by

$$\frac{zJ'_{s,b}(f)(z)}{J_{s,b}(f)(z)}, \quad \frac{J_{s,b}(f)(z)}{z} \quad \text{and} \quad \frac{J_{s,b}(f)(z)}{J_{s+1,b}(f)(z)} \quad (z \in \mathbb{D}),$$

where $J_{s,b}$ is the widely-investigated Srivastava-Attiya operator defined on the class of normalized analytic functions f in the open unit disk

$$\mathbb{D} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

with suitable real parameters s and b. These results reduce upon specialization to some well-known inclusion relationships for several classes of functions with given geometric properties. We also make a comparison between one of the results obtained here and an already known result for some specific cases.

2010 Mathematics Subject Classification. Primary 30C45; Secondary 30C55.

Key Words and Phrases. Analytic functions; Univalent functions; Starlike functions; Convex functions; Srivastava-Attiya operator; Strongly starlike functions; Strongly convex functions; Principle of differential subordination; Inclusion relationships.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{D} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

The general Hurwitz-Lerch Zeta function $\Phi(z, s, b)$ is defined by

$$\Phi(z, s, b) = \sum_{n=0}^{\infty} \frac{z^n}{(b+n)^s}$$

$$\big(b \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \quad \text{when} \quad |z| < 1;\ \Re\{s\} > 1 \quad \text{when} \quad |z| = 1\big).$$

It is known that the function $\Phi(z, s, b)$ reduces to such more familiar functions of Analytic Number Theory as the Riemann and the Hurwitz Zeta functions, Lerch's Zeta function, the Polylogarithmic function and the Lipschitz-Lerch Zeta function (see, for details, [12]).

Srivastava and Attiya [11] introduced the linear operator $J_{s,b} : \mathcal{A} \to \mathcal{A}$ defined by

$$J_{s,b}(f)(z) = G_{s,b}(z) * f(z) \qquad (b \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}),$$

where the symbol $*$ denotes the Hadamard product (or convolution) of analytic functions and the function $G_{s,b}$ is defined by

$$G_{s,b}(z) = (b+1)^s[\Phi(z, s, b) - b^{-s}].$$

For a $f \in \mathcal{A}$ of the form given by (1.1), we get

$$J_{s,b}(f)(z) = z + \sum_{n=2}^{\infty} \left(\frac{b+1}{b+n}\right)^s a_n z^n \qquad (z \in \mathbb{D}). \tag{1.2}$$

Srivastava and Attiya [11] showed that (see also the recent work by Srivastava *et al.* [13])

$$J_{0,b}(f)(z) = f(z),$$

$$J_{1,0}(f)(z) = \int_0^z \frac{f(t)}{t} \mathrm{d}t =: A(f)(z),$$

$$J_{1,\gamma}(f)(z) = \frac{1+\gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t) \mathrm{d}t =: \mathcal{J}_\gamma(f)(z) \qquad (\gamma > -1)$$

and

$$J_{\sigma,1}(f)(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\sigma a_n z^n =: I^\sigma(f)(z) \qquad (\sigma > 0),$$

where A, $\mathcal{J}_\gamma$ and I^σ are the familiar Alexander [1], Bernardi [2] and Jung-Kim-Srivastava [4] integral operators, respectively.

From the equation (1.2), we can obtain the following recurrence relation:

$$zJ'_{s+1,b}(f)(z) = (1+b)J_{s,b}(f)(z) - bJ_{s+1,b}(f)(z). \tag{1.3}$$

For $\alpha \in [0, 1)$ and $\beta \in (0, 1]$, let $\Omega_{\alpha,\beta}$ denote a subset of $\mathbb{C}$ defined by

$$\Omega_{\alpha,\beta} = \left\{w : w \in \mathbb{C} \quad \text{and} \quad |\arg(w - \alpha)| < \frac{\pi}{2}\,\beta\right\}.$$

We denote by $\mathcal{S}^*(\alpha, \beta)$ and $\mathcal{C}(\alpha, \beta)$ the classes of functions $f \in \mathcal{A}$ satisfying the following conditions:

$$\frac{zf'(z)}{f(z)} \in \Omega_{\alpha,\beta} \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \in \Omega_{\alpha,\beta} \qquad (\forall\, z \in \mathbb{D}),$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

respectively. The function f in the classes $\mathcal{S}^*(\alpha,\beta)$ and $\mathcal{C}(\alpha,\beta)$ is called starlike of order β and type α in $\mathbb{D}$ and strongly convex of order β and type α in $\mathbb{D}$, respectively. We note that

$$\mathcal{S}^*(\alpha,1)\equiv\mathcal{S}^*(\alpha) \quad \text{and} \quad \mathcal{C}(\alpha,1)\equiv\mathcal{C}(\alpha),$$

which are the well-known classes of starlike functions of order α in $\mathbb{D}$ and convex functions of order α in $\mathbb{D}$.

Wilken and Feng [15] showed that $f\in\mathcal{C}(\alpha,1)$ implies that $f\in\mathcal{S}^*(\beta,1)$, where

$$\beta:=\beta(\alpha)=\begin{cases}\dfrac{1-2\alpha}{2^{2-2\alpha}[1-2^{2\alpha-1}]} & \left(\alpha\neq\dfrac{1}{2}\right)\\[2ex] \dfrac{1}{2\log 2} & \left(\alpha=\dfrac{1}{2}\right).\end{cases} \tag{1.4}$$

Nunokawa *et al.* [8] investigated relations between $\gamma\in(0,1)$ and $\delta\in(0,1)$ so that $\mathcal{S}^*(\alpha,\gamma)$ implies that $\mathcal{C}(\beta,\delta)$, where β is given by (1.4). We will discuss this relation in Section 4.

The relation given above can be represented by using the operator $J_{s,b}$ as follows:

$$\frac{zJ'_{s,b}(f)(z)}{J_{s,b}(f)(z)}\in\Omega_{\alpha,\gamma}\Longrightarrow\frac{zJ'_{s+1,b}(f)(z)}{J_{s+1,b}(f)(z)}\in\Omega_{\beta,\delta} \qquad (z\in\mathbb{D}), \tag{1.5}$$

for $s=-1$ and $b=0$.

In the present paper, we will consider the implication given in (1.5) for suitable values of s and b in $\mathbb{R}$. We also consider other similar problems associated with (1.5), which are related to the forms given by

$$\frac{J_{s,b}(f)(z)}{z} \quad \text{and} \quad \frac{J_{s,b}(f)(z)}{J_{s+1,b}(f)(z)}.$$

We say that f is subordinate to F in $\mathbb{D}$, written as $f\prec F$ or as $f(z)\prec F(z)$ in $\mathbb{D}$, if and only if $f(z)=F\big(\omega(z)\big)$ for some Schwarz function $\omega(z)$ with $\omega(0)=0$ and $|\omega(z)|<1$ for $z\in\mathbb{D}$. It is well known that, if F is univalent in $\mathbb{D}$, then $f\prec F$ is equivalent to $f(0)=F(0)$ and $f(\mathbb{D})\subset F(\mathbb{D})$ (see, for details, [10, p. 36]).

Let $\psi:\mathbb{C}^2\to\mathbb{C}$ and let h be univalent in $\mathbb{D}$. If p is analytic in $\mathbb{D}$ and satisfies the following differential subordination:

$$\psi\big(p(z),zp'(z)\big)\prec h(z) \qquad (z\in\mathbb{D}),$$

then p is called a solution of the differential subordination. A univalent function q is called a dominant of the solutions of the differential subordination (or, simply, a dominant) if $p\prec q$ in $\mathbb{D}$ for all solutions p. A function $\widetilde{q}$ is called best dominant if $\widetilde{q}\prec q$ in $\mathbb{D}$ for all dominants q.

We recall the following lemmas which are required in our present investigation.

Lemma 1. (see Hallenbeck and Ruscheweyh [3]; see also [6, p. 71]) *Let h be convex in $\mathbb{D}$ with $h(0)=a$, $\gamma\neq0$ and $\Re\{\gamma\}\geqq0$. If p is analytic in $\mathbb{D}$ with the form given by*

$$p(z)=a+c_nz^n+c_{n+1}z^{n+1}\cdots \qquad (n\in\mathbb{N}:=\{1,2,3,\cdots\})$$

and

$$p(z)+\frac{zp'(z)}{\gamma}\prec h(z) \quad (z\in\mathbb{D}), \tag{1.6}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

then

$$p(z) \prec q(z) \prec h(z) \quad (z \in \mathbb{D}),$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{(\gamma/n)-1}dt.$$

The function q is convex and is the best dominant of (1.6).

Lemma 2. (see Miller and Mocanu [5]) *If $-1 \leqq B < A \leqq 1$, $\beta > 0$ and the complex number γ satisfies the inequality*:

$$\Re\{\gamma\} \geqq -\frac{(1-A)\beta}{1-B},$$

then the following differential equation:

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{D})$$

has a univalent solution in $\mathbb{D}$ given by

$$q(z) = \begin{cases} \dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta \int_0^z t^{\beta+\gamma-1}(1+Bt)^{(A-B)\beta/B}\mathrm{d}t} - \dfrac{\gamma}{\beta} & (B \neq 0) \\[2ex] \dfrac{z^{\beta+\gamma}\exp(\beta Az)}{\beta \int_0^z t^{\beta+\gamma-1}\exp(\beta At)\mathrm{d}t} - \dfrac{\gamma}{\beta} & (B = 0). \end{cases}$$

If the function $p(z)$ given by

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

is analytic in $\mathbb{D}$ and satisfies the following subordination condition:

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{D}), \tag{1.7}$$

then

$$p(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{D}),$$

and q is the best dominant of (1.7).

The generalized hypergeometric function ${}_{\mathfrak{q}}F_{\mathfrak{s}}$ is defined by

$${}_{\mathfrak{q}}F_{\mathfrak{s}}(z) = {}_{\mathfrak{q}}F_{\mathfrak{s}}(\alpha_1, \cdots, \alpha_{\mathfrak{q}}; \beta_1, \cdots, \beta_{\mathfrak{s}}; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_{\mathfrak{q}})_n}{(\beta_1)_n \cdots (\beta_{\mathfrak{s}})_n} \frac{z^n}{n!} \qquad (z \in \mathbb{D}), \tag{1.8}$$

where $\alpha_j \in \mathbb{C}$ $(j = 1, \cdots, \mathfrak{q})$, $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $\mathbb{Z}_0^- := \{0, -1, -2, \cdots\}$ $(j = 1, \cdots, \mathfrak{s})$, $\mathfrak{q} \leqq \mathfrak{s} + 1$, $\mathfrak{q}, \mathfrak{s} \in \mathbb{N}_0$, and $(\alpha)_n$ is the Pochhammer symbol defined by

$$(\alpha)_0 = 1 \quad \text{and} \quad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+n-1) \qquad (n \in \mathbb{N}),$$

$\Gamma(z)$ being the Gamma function of the argument z.

We recall following well-known identities for the Gaussian hypergeometric function ${}_2F_1$, that is, the special case of (1.8) when $\mathfrak{q} - 1 = \mathfrak{s} = 1$:

Lemma 3. (see [14, pp. 285 and 293]) *For real or complex numbers a, b and c $(c \notin \mathbb{Z}_0^-)$, the following identities hold true*:

(i) $\displaystyle\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1\left(a,b;c;z\right)$
when $\Re\{c\} > \Re\{b\} > 0$*;*

(ii) ${}_2F_1\left(a,b;c;z\right) = {}_2F_1\left(b,a;c;z\right)$;

(iii) ${}_2F_1\left(a,b;c;z\right) = (1-z)^{-a}\;{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right)$.

The following lemmas will also be required in our present investigation.

Lemma 4. (see Wilken and Feng [15]) *Let* ν *be a positive measure on* $[0,1]$ *and let* h *be a complex-valued function defined on* $\mathbb{D}\times[0,1]$ *such that* $h(\cdot,t)$ *is analytic in* $\mathbb{D}$ *for each* $t\in[0,1]$ *and that* $h(z,\cdot)$ *is* ν*-integrable on* $[0,1]$ *for all* $z\in\mathbb{D}$. *In addition, suppose that* $\Re\{h(z,t)\} > 0$, $h(-r,t)$ *is real and*

$$\Re\left\{\frac{1}{h(z,t)}\right\} \geqq \frac{1}{h(-r,t)} \qquad (|z| \leqq r < 1;\ t\in[0,1]).$$

If the function H *is defined by*

$$H(z) = \int_0^1 h(z,t)\mathrm{d}\nu(t),$$

then

$$\Re\left\{\frac{1}{H(z)}\right\} \geqq \frac{1}{H(-r)} \qquad (|z| \leqq r < 1).$$

Lemma 5. (see Nunokawa [7]) *Let the function* P *be analytic in* $\mathbb{D}$, $P(0)=1$, $P(z)\neq 0$ *in* $\mathbb{D}$ *and suppose that there exists a point* $z_0\in\mathbb{D}$ *such that*

$$\left|\arg\left(P(z)\right)\right| < \frac{\pi}{2}\,\delta \qquad (|z| < |z_0|)$$

and

$$\left|\arg\left(P(z_0)\right)\right| = \frac{\pi}{2}\,\delta \qquad (\delta > 0).$$

Then

$$\frac{z_0P'(z_0)}{P(z_0)} = \mathrm{i}k\delta, \tag{1.9}$$

where

$$k \geqq \frac{1}{2}\left(a+\frac{1}{a}\right) \quad when \quad \arg\left(P(z_0)\right) = \frac{\pi}{2}\,\delta$$

and

$$k \leqq -\frac{1}{2}\left(a+\frac{1}{a}\right) \quad when \quad \arg\left(P(z_0)\right) = -\frac{\pi}{2}\,\delta,$$

and where

$$P(z_0)^{\frac{1}{\delta}} = \pm\mathrm{i}a$$

with $a > 0$.

2. Bounds for the Real Parts

In this section, we investigate the bounds for the real parts of normalized analytic functions defined by the Srivastava-Attiya operator $J_{s,b}$.

Theorem 1. *Let $f \in \mathcal{A}$ and*

$$\Re\left\{\frac{zJ'_{s,b}(f)(z)}{J_{s,b}(f)(z)}\right\} > \alpha \qquad (z \in \mathbb{D}), \tag{2.1}$$

where $s \in \mathbb{R}$, $0 \leqq \alpha < 1$ and $b \geqq -\alpha$. Then

$$\Re\left\{\frac{zJ'_{s+1,b}(f)(z)}{J_{s+1,b}(f)(z)}\right\} > -b + (b+1)\left[{}_2F_1\left(1, 2-2\alpha; b+2; \frac{1}{2}\right)\right]^{-1} \qquad (z \in \mathbb{D}). \tag{2.2}$$

This result is sharp.

Proof. Let us define a function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{zJ'_{s+1,b}(f)(z)}{J_{s+1,b}(f)(z)}.$$

Then p is analytic in $\mathbb{D}$ with $p(0) = 1$. Thus, from the recurrence relation (1.3), we have

$$\frac{zJ'_{s,b}(f)(z)}{J_{s,b}(f)(z)} = p(z) + \frac{zp'(z)}{p(z)+b}. \tag{2.3}$$

From (2.1), the above relation shows that

$$p(z) + \frac{zp'(z)}{p(z)+b} \prec \frac{1+(1-2\alpha)z}{1-z}. \tag{2.4}$$

Also, from Lemma 2 with $A = 1-2\alpha$, $B = -1$, $\beta = 1$ and $\gamma = b$, we find that

$$p(z) \prec \frac{1}{Q(z)} - b \qquad (z \in \mathbb{D}), \tag{2.5}$$

where Q is defined by

$$Q(z) = \int_0^1 t^b \left(\frac{1-zt}{1-z}\right)^{-2(1-\alpha)} \mathrm{d}t.$$

By applying Lemma 3, we have

$$Q(z) = \frac{\Gamma(b+1)}{\Gamma(b+2)}\, {}_2F_1\left(2-2\alpha, 1; b+2; \frac{z}{z-1}\right).$$

Moreover, the function Q is represented as follows:

$$Q(z) = \int_0^1 g(t,z)\mathrm{d}\mu(t),$$

where

$$g(t,z) = \frac{1-z}{1-(1-t)z}$$

and

$$\mathrm{d}\mu(t) = \frac{\Gamma(b+1)}{\Gamma(2-2\alpha)\Gamma(b+2\alpha)} t^{1-2\alpha}(1-t)^{b+2\alpha-1}\mathrm{d}t$$

with $t \in [0,1]$ and $z \in \mathbb{D}$. We note that $\mathrm{d}\mu(t)$ is a positive measure on $[0,1]$. We can easily verify that the assertions hold true:

(i) $g(\cdot,t)$ is analytic in $\mathbb{D}$ for each $t \in [0,1]$;
(ii) $g(z,\cdot)$ is integrable with respect to μ on $[0,1]$;
(iii) $\mathfrak{R}\{g(z,t)\} > 0$ for all $z \in \mathbb{D}$ and $t \in [0,1]$;
(iv) $g(-r,t)$ is real for all r and for $t \in [0,1]$.

Indeed, we have

$$\mathfrak{R}\left\{\frac{1}{g(z,t)}\right\} = \mathfrak{R}\left\{1+\frac{zt}{1-z}\right\} \geqq 1-\frac{tr}{1+r} = \frac{1}{g(-r,t)},$$

for $|z| \leqq r < 1$ and $t \in [0,1]$. Therefore, by applying Lemma 4, we obtain

$$\mathfrak{R}\left\{\frac{1}{Q(z)}\right\} \geqq \frac{\Gamma(b+2)}{\Gamma(b+1)}\left[{}_2F_1\left(2-2\alpha,1;b+2;\frac{r}{1+r}\right)\right]^{-1} \qquad (|z| \leqq r < 1). \tag{2.6}$$

Letting $r \to 1-$ in (2.6) we conclude that the inequality (2.2) holds true from the relation (2.5). The sharpness of this result follows from the fact that the function Q is the best dominant of (2.4). □

Theorem 2. *Let $f \in \mathcal{A}$ and suppose that*

$$\mathfrak{R}\left\{\frac{J_{s,b}(f)(z)}{z}\right\} > \alpha \qquad (z \in \mathbb{D}), \tag{2.7}$$

where $s \in \mathbb{R}$, $0 \leqq \alpha < 1$ and $b > -1$. Then

$$\mathfrak{R}\left\{\frac{J_{s+1,b}(f)(z)}{z}\right\} > 1-\frac{(1-\alpha)(b+1)}{b+2}\,{}_2F_1\left(1,1;b+3;\frac{1}{2}\right) \qquad (z \in \mathbb{D}). \tag{2.8}$$

This result is sharp.

Proof. Let us define a function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{J_{s+1,b}(f)(z)}{z} \qquad (z \in \mathbb{D}).$$

Then we have

$$\frac{J_{s,b}(f)(z)}{z} = p(z) + \frac{1}{b+1}zp'(z).$$

From (2.7), we see that

$$p(z) + \frac{zp'(z)}{b+1} \prec h(z) \qquad (z \in \mathbb{D}),$$

where

$$h(z) = \frac{1+(1-2\alpha)z}{1-z} \qquad (z \in \mathbb{D}).$$

Thus, by applying Lemma 1 with $\gamma = b+1$ and h given above, we have $p(z) \prec q(z)$ in $\mathbb{D}$, where q is a convex function in $\mathbb{D}$ defined by

$$q(z) = \frac{b+1}{z^{b+1}}\int_0^z \frac{1+(1-2\alpha)t}{1-t}t^b\mathrm{d}t,$$

which, in view of Lemma 3, yields

$$q(z) = 1 + \frac{2(1-\alpha)(b+1)z}{(b+2)(1-z)} \; {}_2F_1\left(1, 1; b+3; \frac{z}{z-1}\right).$$

Since the function q is convex with real coefficients, by the subordination relation:

$$p(z) \prec q(z) \qquad (z \in \mathbb{D}),$$

we obtain the inequality (2.8) by letting $z \to -1+$. The sharpness of this result follows from the fact that the function q is the best dominant of the differential subordination given by

$$p(z) + \frac{zp'(z)}{b+1} \prec h(z) \qquad (z \in \mathbb{D}).$$

□

We recall the following special case due to Prajapat and Bulboacă [9, Corollary 2.10].

Theorem 3. *Let $f \in \mathcal{A}$ and suppose that*

$$\Re\left\{\frac{J_{s,b}(f)(z)}{J_{s+1,b}(f)(z)}\right\} > \alpha \qquad (z \in \mathbb{D}),$$

where $s \in \mathbb{R}$, $0 \leqq \alpha < 1$ and $b \geqq -\alpha$. Then

$$\Re\left\{\frac{J_{s+1,b}(f)(z)}{J_{s+2,b}(f)(z)}\right\} > \left[{}_2F_1\left(1, 2-2\alpha; b+2; \frac{1}{2}\right)\right]^{-1} \qquad (z \in \mathbb{D}). \tag{2.9}$$

This result is sharp.

3. Bounds for the Arguments

For given $\alpha \in [0, 1)$, let the parameters β_1, β_2 and β_3 be real numbers defined by

$$\beta_1 = \beta_1(\alpha, b) := -b + (b+1)\left[{}_2F_1\left(1, 2-2\alpha; b+2; \frac{1}{2}\right)\right]^{-1} \qquad (b \geqq -\alpha), \tag{3.1}$$

$$\beta_2 = \beta_2(\alpha, b) := 1 - \frac{(1-\alpha)(b+1)}{b+2} \; {}_2F_1\left(1, 1; b+3; \frac{1}{2}\right) \qquad (b > -1) \tag{3.2}$$

and

$$\beta_3 = \beta_3(\alpha, b) := \left[{}_2F_1\left(1, 2-2\alpha; b+2; \frac{1}{2}\right)\right]^{-1} \qquad (b \geqq -\alpha). \tag{3.3}$$

We note that $\beta_j < 1$ $(j = 1, 2, 3)$. We also note that

$$\beta_j \geqq \alpha \qquad (j = 1, 2, 3).$$

These inequalities are immediate consequences of Lemma 1 or 2 with

$$h(z) = \frac{1 + Az}{1 + Bz} = \frac{1 + (1-2\alpha)z}{1-z} \qquad (A = 1 - 2\alpha; \; B = -1)$$

such that

$$\Re\{h(z)\} > \alpha \qquad (z \in \mathbb{D}).$$

In this section, we investigate the bounds for the arguments of normalized analytic functions defined by the Srivastava-Attiya operator $J_{s,b}$. In order to get our results, we need the following propositions.

Proposition 1. *Let $w_1, w_2, w_3 \in \mathbb{C}$ satisfy the following conditions*:

(i) $\mathfrak{R}\{w_1\} > 0$ *and* $\mathfrak{I}\{w_1\} < 0$;
(ii) $0 < \arg(w_3) \leqq \arg(w_2) < \frac{\pi}{2}$;
(iii) $|w_3| \leqq |w_2|$.

Then the inequality:

$$\arg(w_1 + w_3) \leqq \arg(w_1 + w_2) \tag{3.4}$$

holds true.

Proof. First of all, we consider a case for which $\arg(w_3) = \arg(w_2)$. In this case, we let

$$w_1 = x + \mathrm{i}y, \quad w_2 = R_2 \mathrm{e}^{\mathrm{i}\theta} \quad \text{and} \quad w_3 = R_3 \mathrm{e}^{\mathrm{i}\theta},$$

where $x > 0$, $y < 0$ and $R_2 \geqq R_3$. Then the inequality (3.4) is equivalent to

$$\frac{y + R_2 \sin\theta}{x + R_2 \cos\theta} \geqq \frac{y + R_3 \sin\theta}{x + R_3 \cos\theta}.$$

Furthermore, since $x > 0$ and $\theta \in (0, \pi/2)$, the above inequality is equivalent to

$$(R_2 - R_3)(x \sin\theta - y \cos\theta) \geqq 0.$$

Therefore, it follows from $x > 0$ and $y < 0$ that the above inequality holds true.

To complete the proof of Proposition 1, let $\Omega \subset \mathbb{C}$ be defined by

$$\Omega = \left\{ R\mathrm{e}^{\mathrm{i}\psi} \in \mathbb{C} : 0 < R \leqq R_2 \quad \text{and} \quad 0 < \psi \leqq \arg(w_2) \right\}.$$

Letting $w_3 \in \Omega$, we suppose that ℓ_1 be a straight line through the points $-w_1$ and w_2 and ℓ_2 be a straight line through the points $-w_1$ and w_3. From Condition (ii) of Proposition 1, we can take the unique intersection point denoted by $\widetilde{w}_3 \in \Omega$ of ℓ_1 and ℓ_2. For this point, we have

$$\arg\big(w_3 - (-w_1)\big) = \arg\big(\widetilde{w}_3 - (-w_1)\big) \geqq \arg\big(w_2 - (-w_1)\big),$$

which completes the proof of Proposition 1. □

The demonstration of Proposition 2 below is fairly straightforward.

Proposition 2. *Let w_1 and w_2 be in $\mathbb{C} \setminus \{0\}$. Then*

$$\arg(w_1 + w_2) \geqq \min\{\arg(w_1), \arg(w_2)\}.$$

Theorem 4. *Let $\beta \in \mathbb{R}$ be the parameter β_1 given by* (3.1). *Suppose also that $f \in \mathcal{A}$ and*

$$\left| \arg\left(\frac{zJ'_{s,b}(f)(z)}{J_{s,b}(f)(z)} - \alpha \right) \right| < \frac{\pi}{2}\,\gamma \qquad (z \in \mathbb{D}), \tag{3.5}$$

where $s \in \mathbb{R}$, $b \geqq -\beta$, $0 \leqq \alpha < 1$ and $0 < \gamma < 1$. Then

$$\left| \arg\left(\frac{zJ'_{s+1,b}(f)(z)}{J_{s+1,b}(f)(z)} - \beta \right) \right| < \frac{\pi}{2}\,\delta \qquad (z \in \mathbb{D};\ 0 < \delta < 1),$$

where $0 < \delta < 1$ and

$$\gamma = \min\left\{ \delta, \frac{2}{\pi}\ \arctan\left(\frac{\delta(1-\beta)(x_0^{1+\delta} + x_0^{\delta-1})}{2(\beta-\alpha)[(1-\beta)x_0^{\delta} + \beta + b]} \right) \right\}$$

and $x_0 \in (0,1)$ *is the root of the following equation*:

$$(1-\beta)(x^2-1)x^\delta = (\beta+b)[1-\delta-(1+\delta)x^\delta]. \tag{3.6}$$

Proof. Let us define the functions p and $P : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{zJ'_{s+1,b}(f)(z)}{J_{s+1,b}(f)(z)} \quad \text{and} \quad P(z) = \frac{p(z)-\beta}{1-\beta}.$$

Then the functions p and P are analytic in $\mathbb{D}$ with $p(0) = P(0) = 1$. From the recurrence relation (1.3), we have

$$\frac{zJ'_{s,b}(f)(z)}{J_{s,b}(f)(z)} = p(z) + \frac{zp'(z)}{p(z)+b}. \tag{3.7}$$

We now assume that there exists a point $z_0 \in \mathbb{R}$ such that

$$\left|\arg\left(P(z)\right)\right| = \left|\arg\left(p(z)-\beta\right)\right| < \frac{\pi}{2}\,\delta$$

for $|z| < |z_0|$ and

$$\left|\arg\left(P(z_0)\right)\right| = \left|\arg\left(p(z_0)-\beta\right)\right| = \frac{\pi}{2}\,\delta.$$

Consider the case when

$$\arg\left(P(z_0)\right) = \arg\left(p(z_0)-\beta\right) = \frac{\pi}{2}\,\delta.$$

Then, by Lemma 5, we have

$$\frac{z_0P'(z_0)}{P(z_0)} = \frac{z_0p'(z_0)}{p(z_0)-\beta} = \mathrm{i}\delta k, \tag{3.8}$$

where

$$k \geqq \frac{1}{2}\left(a + \frac{1}{a}\right) \tag{3.9}$$

with $a > 0$. Also, from (3.7) and (3.8), we have

$$\begin{aligned}
&\arg\left(\frac{z_0J'_{s,b}(f)(z_0)}{J_{s,b}(f)(z_0)} - \alpha\right) \\
&\quad = \arg\left(p(z_0) + \frac{z_0p'(z_0)}{p(z_0)+b} - \alpha\right) \\
&\quad = \arg\left(p(z_0)-\beta\right) + \arg\left(\frac{p(z_0)-\alpha}{p(z_0)-\beta} + \frac{z_0p'(z_0)}{p(z_0)-\beta}\cdot\frac{1}{p(z_0)+b}\right) \\
&\quad = \frac{\pi}{2}\,\delta + \arg\left(1-\beta+(\beta-\alpha)(\mathrm{i}a)^{-\delta} + \frac{\mathrm{i}\delta k(1-\beta)}{(1-\beta)(\mathrm{i}a)^\delta+\beta+b}\right).
\end{aligned} \tag{3.10}$$

Let us define w_1, w_2 and w_3 by

$$w_1 = 1-\beta+(\beta-\alpha)(\mathrm{i}a)^{-\delta},$$

$$w_2 = \frac{\mathrm{i}\delta k(1-\beta)}{(1-\beta)(\mathrm{i}a)^\delta+\beta+b}$$

and

$$w_3 = \frac{\mathrm{i}\delta k(1-\beta)}{[(1-\beta)a^\delta+\beta+b]\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\delta}}.$$

We note that

$$\mathfrak{R}\{w_1\} = 1-\beta+\frac{\beta-\alpha}{(\beta-\alpha)a^{\delta}}\cos\left(\frac{\pi}{2}\,\delta\right) > 0$$

and

$$\mathfrak{I}\{w_1\} = -\frac{\beta-\alpha}{(\beta-\alpha)a^{\delta}}\sin\left(\frac{\pi}{2}\,\delta\right) < 0.$$

Also, from the inequality $\beta + b \geqq 0$, we can easily verify that the inequality $\arg(w_2) \geqq \arg(w_3)$ holds true. Furthermore, the inequality $|w_2| \geqq |w_3|$ is true, since

$$\begin{aligned}
|w_2|^2 &= \frac{\delta^2k^2(1-\beta)^2}{(1-\beta)^2a^{2\delta}+2(1-\beta)(\beta+b)a^{\delta}\cos\left(\frac{\pi}{2}\,\delta\right)+(\beta+b)^2} \\
&\geqq \frac{\delta^2k^2(1-\beta)^2}{[(1-\beta)a^{\delta}+\beta+b]^2} \\
&= |w_3|^2.
\end{aligned}$$

Therefore, by applying Proposition 1 with (3.10), we have

$$\begin{aligned}
&\arg\left(\frac{z_0J'_{s,b}(f)(z_0)}{J_{s,b}(f)(z_0)}-\alpha\right) \\
&\geqq \frac{\pi}{2}\,\delta+\arg\left(1-\beta+(\beta-\alpha)(\mathrm{i}a)^{-\delta}+\frac{\mathrm{i}\delta k(1-\beta)}{[(1-\beta)a^{\delta}+\beta+b]\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\,\delta}}\right) \\
&= \frac{\pi}{2}\,\delta+\arg\left(\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}\,\delta}\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\,\delta}+\frac{\beta-\alpha}{(1-\beta)a^{\delta}}+\frac{\mathrm{i}\delta k}{(1-\beta)a^{\delta}+\beta+b}\right)\right) \\
&\geqq \arg\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\,\delta}+\frac{\beta-\alpha}{(1-\beta)a^{\delta}}+\frac{\mathrm{i}\delta(a+a^{-1})}{2[(1-\beta)a^{\delta}+\beta+b]}\right).
\end{aligned} \tag{3.11}$$

Let us now put

$$w_4 = \frac{\beta-\alpha}{(1-\beta)a^{\delta}}+\mathrm{i}\frac{\delta(a+a^{-1})}{2[(1-\beta)a^{\delta}+\beta+b]}.$$

Then

$$\arg(w_4) = \arctan\left(\frac{(1-\beta)\delta}{2(\beta-\alpha)}g(a)\right),$$

where $g : (0,\infty) \to \mathbb{R}$ is a function defined by

$$g(x) = \frac{x+x^{-1}}{1-\beta+(\beta+b)x^{-\delta}}.$$

Differentiating the function g with respect to x, we have

$$x^{\delta+2}[1-\beta+(\beta+b)x^{-\delta}]^2g'(x) = h(x),$$

where the function $h : (0,\infty) \to \mathbb{R}$ is defined by

$$h(x) = (x^2-1)[(1-\beta)x^{\delta}+\beta+b]+\delta(\beta+b)(x^2+1).$$

Since the function h is continuous on $(0,\infty)$ with

$$h(0) = -(\beta+b)(1-\delta) < 0 \quad \text{and} \quad h(1) = 2\delta(\beta+b) > 0,$$

there exists an $x_0 \in (0,1)$ such that $h(x_0) = 0$, which is equivalent to the equation given by (3.6). Differentiating the function h twice with respect to x, we find that

$$h''(x) = (2+\delta)(1-\beta)(1+\delta)x^{\delta} + \delta(1-\delta)(1-\beta)x^{\delta-2} + 2(1+\delta)(\beta+b) > 0$$

for all $x \in (0,1)$. Since $h(x) > 0$ for $x \in (1,\infty)$, it follows from the convexity of $h(x)$ on $(0,1)$ that the function $g'(x)$ vanishes only at $x_0 \in (0,1)$. Furthermore, we can easily verify that $g(x_0)$ is the minimum value of $g(x)$ on $(0,\infty)$. Therefore, we have

$$\begin{aligned} \arg(w_4) &\geqq \arctan\left(\frac{(1-\beta)\delta}{2(\beta-\alpha)}\, g(x_0)\right) \\ &= \arctan\left(\frac{\delta(1-\beta)(x_0^{1+\delta}+x_0^{\delta-1})}{2(\beta-\alpha)\big[(1-\beta)x_0^{\delta}+\beta+b\big]}\right). \end{aligned} \tag{3.12}$$

Finally, from (3.11), (3.12) and Proposition 2, we have

$$\begin{aligned} &\arg\left(\frac{z_0 J'_{s,b}(f)(z_0)}{J_{s,b}(f)(z_0)} - \alpha\right) \\ &\geqq \arg\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\,\delta} + w_4\right) \\ &\geqq \min\left\{\frac{\pi}{2}\,\delta, \arg(w_4)\right\} \\ &\geqq \min\left\{\frac{\pi}{2}\,\delta, \arctan\left(\frac{\delta(1-\beta)(x_0^{1+\delta}+x_0^{\delta-1})}{2(\beta-\alpha)[(1-\beta)x_0^{\delta}+\beta+b]}\right)\right\} \\ &= \frac{\pi}{2}\,\gamma, \end{aligned}$$

which leads to a contradiction to the hypothesis (3.5).

For the case when

$$\arg\big(P(z_0)\big) = \arg\big(p(z_0)-\beta\big) = -\frac{\pi}{2}\,\delta,$$

Lemma 5 yields

$$\frac{z_0 P'(z_0)}{P(z_0)} = \frac{z_0 p'(z_0)}{p(z_0)-\beta} = \mathrm{i}\delta k, \tag{3.13}$$

where

$$k \leqq -\frac{1}{2}\left(a + \frac{1}{a}\right) \tag{3.14}$$

with $a > 0$. We also have

$$\begin{aligned}
&\arg\left(\frac{z_0 J'_{s,b}(f)(z_0)}{J_{s,b}(f)(z_0)} - \alpha\right) \\
&= \arg\big(p(z_0) - \beta\big) + \arg\left(\frac{p(z_0) - \alpha}{p(z_0) - \beta} + \frac{z_0 p'(z_0)}{p(z_0) - \beta} \cdot \frac{1}{p(z_0) + b}\right) \\
&= -\frac{\pi}{2}\,\delta + \arg\left(\frac{(1-\beta)(-\mathrm{i}a)^\delta + \beta - \alpha}{(1-\beta)(-\mathrm{i}a)^\delta} + \frac{\mathrm{i}\delta k}{(1-\beta)(-\mathrm{i}a)^\delta + \beta + b}\right) \\
&= -\left[\frac{\pi}{2}\,\delta + \arg\left(1 - \beta + (\beta - \alpha)(\mathrm{i}a)^{-\delta} + \frac{\mathrm{i}\delta\widetilde{k}(1-\beta)}{(1-\beta)(\mathrm{i}a)^\delta + \beta + b}\right)\right],
\end{aligned}$$

where

$$\widetilde{k} := -k > \frac{a + a^{-1}}{2}.$$

Therefore, from the proof of the first case, we have

$$\arg\left(\frac{z_0 J'_{s,b}(f)(z_0)}{J_{s,b}(f)(z_0)} - \alpha\right) \leqq -\frac{\pi}{2}\,\gamma,$$

which also leads to a contradiction to the hypothesis (3.5). This completes the proof of Theorem 4. □

Theorem 5. *Let $\beta \in \mathbb{R}$ be the parameter β_2 given by (3.2). Let $f \in \mathcal{A}$ and suppose that*

$$\left|\arg\left(\frac{J_{s,b}(f)(z)}{z} - \alpha\right)\right| < \frac{\pi}{2}\,\gamma \qquad (z \in \mathbb{D}), \tag{3.15}$$

where $s \in \mathbb{R}$, $b > -1$, $0 \leqq \alpha < 1$ and $0 < \gamma < 1$. Then

$$\left|\arg\left(\frac{J_{s+1,b}(f)(z)}{z} - \beta\right)\right| < \frac{\pi}{2}\,\delta \qquad (z \in \mathbb{D};\ 0 < \delta < 1),$$

where

$$\gamma = \delta + \frac{2}{\pi}\,\arctan\left\{\frac{-2(b+1)(\beta-\alpha)\sin\left(\frac{\pi}{2}\,\delta\right) + \delta(1-\beta)(x_0^{\delta+1} + x_0^{\delta-1})}{2(b+1)\left[(1-\beta)x_0^\delta + (\beta-\alpha)\cos\left(\frac{\pi}{2}\,\delta\right)\right]}\right\},$$

and $x_0 \in (0, 1)$ is the unique zero of the function h defined by

$$h(x) = Cx^\delta(x^2 - 1) + AC(\delta + 1)x^2 + \delta Bx + AC(\delta - 1) \tag{3.16}$$

with

$$A = \frac{\beta - \alpha}{1 - \beta}\cos\left(\frac{\pi}{2}\,\delta\right), \quad B = \frac{\beta - \alpha}{1 - \beta}\sin\left(\frac{\pi}{2}\,\delta\right) \quad \text{and} \quad C = \frac{\delta}{2(b+1)}. \tag{3.17}$$

Proof. Let us define the functions p and $P : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{J_{s+1,b}(f)(z)}{z} \quad \text{and} \quad P(z) = \frac{p(z) - \beta}{1 - \beta}. \tag{3.18}$$

Then the functions p and P are analytic in $\mathbb{D}$ with $p(0) = P(0) = 1$. We also have

$$\frac{J_{s,b}(f)(z)}{z} = p(z) + \frac{1}{b+1} z p'(z). \tag{3.19}$$

We now assume that there exists a point $z_0 \in \mathbb{R}$ such that

$$\left|\arg\big(P(z)\big)\right| = \left|\arg\big(p(z)-\beta\big)\right| < \frac{\pi}{2}\,\delta,$$

for $|z| < |z_0|$ and

$$\left|\arg\big(P(z_0)\big)\right| = \left|\arg\big(p(z_0)-\beta\big)\right| = \frac{\pi}{2}\,\delta.$$

We consider the case when

$$\arg\big(P(z_0)\big) = \arg\big(p(z_0)-\beta\big) = \frac{\pi}{2}\,\delta.$$

Then, by Lemma 5, we have the relations given by (3.8) and (3.9) with $a > 0$. From (3.18) and (3.19), we have

$$\begin{aligned}
&\arg\left(\frac{J_{s,b}(f)(z_0)}{z_0} - \alpha\right)\\
&\quad= \arg\big(p(z_0)-\beta\big) + \arg\left(\frac{p(z_0)-\alpha}{p(z_0)-\beta} + \frac{z_0 p'(z_0)}{(b+1)\big(p(z_0)-\beta\big)}\right)\\
&\quad= \frac{\pi}{2}\,\delta + \arg\left(1 + \frac{\beta-\alpha}{(1-\beta)(\mathrm{i}a)^{\delta}} + \frac{\mathrm{i}\delta k}{b+1}\right)\\
&\quad= \frac{\pi}{2}\,\delta + \arctan\left(-\frac{\frac{\beta-\alpha}{(1-\beta)a^{\delta}}\sin\left(\frac{\pi}{2}\,\delta\right) + \frac{\delta k}{b+1}}{1 + \frac{\beta-\alpha}{(1-\beta)a^{\delta}}\cos\left(\frac{\pi}{2}\,\delta\right)}\right)\\
&\quad\geqq \frac{\pi}{2}\,\delta + \arctan\big(g(a)\big),
\end{aligned} \tag{3.20}$$

where

$$g(x) = \frac{-B + C(x^{\delta+1} + x^{\delta-1})}{x^{\delta} + A},$$

and A, B and C are positive constants given by (3.17). Differentiating the function $g(x)$ with respect to x, we have

$$a^{2-\delta}(a^{\delta}+A)^2 g'(x) = h(x),$$

where h is given by (3.16). Simple calculations show that

$$h(0) = -AC(1-\delta) < 0 \quad \text{and} \quad h(x) \geqq h(1) = \delta(2AC+B) > 0 \qquad (x \geqq 1)$$

and

$$h''(x) = C[(\delta+2)(\delta+1)x^{\delta} + \delta(1-\delta)x^{\delta-2}] + 2AC(\delta+1) > 0 \qquad (0 < x < 1).$$

Similar methods as in the proof of Theorem 4 would yield

$$g(x) \geqq g(x_0) \qquad (0 < x < \infty), \tag{3.21}$$

where x_0 is the unique zero of $h(x)$ on $(0,\infty)$. Therefore, by (3.20) and (3.21), we obtain

$$\begin{aligned}
&\arg\left(\frac{J_{s,b}(f)(z_0)}{z_0} - \alpha\right)\\
&\quad\geqq \frac{\pi}{2}\,\delta + \arctan\left(\frac{-2(b+1)(\beta-\alpha)\sin(\delta\pi/2) + \delta(1-\beta)(x_0^{\delta+1} + x_0^{\delta-1})}{2(b+1)[(1-\beta)x_0^{\delta} + (\beta-\alpha)\cos(\delta\pi/2)]}\right)\\
&\quad= \frac{\pi}{2}\,\gamma,
\end{aligned}$$

which provides a contradiction to the hypothesis (3.15).

For the case when

$$\arg\big(P(z_0)\big) = \arg\big(p(z_0) - \beta\big) = -\frac{\pi}{2}\,\delta,$$

we have the relations given by (3.13) and (3.14) with $a > 0$. Therefore, we have

$$\begin{aligned}
&\arg\left(\frac{J_{s,b}(f)(z_0)}{z_0} - \alpha\right) \\
&= -\left[\frac{\pi}{2}\,\delta + \arctan\left(-\frac{\frac{\beta-\alpha}{(1-\beta)a^\delta}\sin\left(\frac{\pi}{2}\,\delta\right) + \frac{\delta\widetilde{k}}{b+1}}{1 + \frac{\beta-\alpha}{(1-\beta)a^\delta}\cos\left(\frac{\pi}{2}\,\delta\right)}\right)\right] \\
&\leqq -\left(\frac{\pi}{2}\,\delta + \arctan\big(g(a)\big)\right),
\end{aligned} \tag{3.22}$$

where

$$\widetilde{k} := -k > \frac{a + a^{-1}}{2}.$$

Therefore, from (3.22) and (3.21), we have

$$\arg\left(\frac{J_{s,b}(f)(z_0)}{z_0} - \alpha\right) \leqq -\frac{\pi}{2}\,\gamma,$$

which also provides a contradiction to the hypothesis (3.15). This evidently completes the proof of Theorem 5. □

Next, for given suitable real of the parameters s and b and for $f \in \mathcal{A}$, we define a function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = \frac{J_{s+1,b}(f)(z)}{J_{s+2,b}(f)(z)}.$$

Then, by using the recurrence relation (1.3), we obtain

$$\frac{J_{s,b}(f)(z)}{J_{s+1,b}(f)(z)} = p(z) + \frac{zp'(z)}{(b+1)p(z)} \qquad (z \in \mathbb{D}). \tag{3.23}$$

By applying the same methods as in the proof of Theorem 4 to the differential equation (3.23) instead of (3.7), we can establish the following argument property associated with the Srivastava-Attiya operator.

Theorem 6. *Let $\beta \in \mathbb{R}$ be the parameter β_3 given by (3.3). Also let $f \in \mathcal{A}$ and*

$$\left|\arg\left(\frac{J_{s,b}(f)(z)}{J_{s+1,b}(f)(z)} - \alpha\right)\right| < \frac{\pi}{2}\,\gamma \qquad (z \in \mathbb{D}),$$

where $s \in \mathbb{R}$, $0 \leqq \alpha < 1$, $b \geqq -\alpha$ and $0 < \gamma < 1$. Then

$$\left|\arg\left(\frac{J_{s+1,b}(f)(z)}{J_{s+2,b}(f)(z)} - \beta\right)\right| < \frac{\pi}{2}\,\delta \qquad (z \in \mathbb{D}),$$

where $0 < \delta < 1$ and

$$\gamma = \min\left\{\delta, \frac{2}{\pi}\,\arctan\left(\frac{\delta(1-\beta)(x_0^{1+\delta} + x_0^{\delta-1})}{2(\beta-\alpha)(b+1)[(1-\beta)x_0^\delta + \beta]}\right)\right\}$$

and x_0 is the root in the interval $(0,1)$ of the following equation:

$$[(1-\beta)x^{\delta}+\beta](1-x^2)=\beta\delta(x^2+1). \tag{3.24}$$

4. Numerical and Computational Analysis

Let $s=-1$ and $b=0$. Since the following equalities:

$$J_{s,b}(f)(z)=zf'(z) \quad \text{and} \quad J_{s+1,b}(f)(z)=f(z)$$

hold true for $f\in\mathcal{A}$, it follows from Theorem 4 that $f\in\mathcal{C}(\alpha,\gamma_1)$ implies that $f\in\mathcal{S}^*(\beta,\delta)$, where

$$\gamma_1=\min\left\{\delta, \frac{2}{\pi}\ \arctan\left(\frac{\delta(1-\beta)(x_0^{1+\delta}+x_0^{\delta-1})}{2(\beta-\alpha)[(1-\beta)x_0^{\delta}+\beta]}\right)\right\} \tag{4.1}$$

and $x_0\in(0,1)$ is the root of the following equation:

$$(1-\beta)(x^2-1)x^{\delta}=\beta[1-\delta-(1+\delta)x^{\delta}]).$$

On the other hand, Nunokawa *et al.* [8] showed that $f\in\mathcal{C}(\alpha,\gamma_2)$ implies that $f\in\mathcal{S}^*(\beta,\delta)$, where

$$\gamma_2=\frac{2}{\pi}\ \arctan\left(\frac{\delta(1-\beta)(x_0^{1+\delta}+x_0^{\delta-1})}{(1-\beta)x_0^{\delta}+\beta}\right), \tag{4.2}$$

and $x_0\in(0,1)$ is the root of the following equation:

$$(1-\beta)(x^2-1)x^{\delta}=\beta(1-\delta-(1+\delta)x^2). \tag{4.3}$$

As it does not seem to be so easy to compare the values γ_1 and γ_2 for the whole ranges of the parameters $\alpha\in(0,1)$ and $\delta\in(0,1)$, we will compare them here in several particular cases of α and δ. Thus, if we fix $\alpha=\frac{1}{2}$, then we have

$$\beta=\frac{1}{2\log 2}.$$

With the aid of *Mathematica*, we can thus obtain Table 1 (see below) which gives the approximate values of $\gamma_1\in(0,1)$ and $\gamma_2\in(0,1)$ defined by (4.1) and (4.2), respectively, when δ is given by

$$\delta=\frac{j}{10} \qquad (j=1,2,\cdots,9).$$

As we see from Table 1, we can verify that the results in this paper would significantly improve the results in the earlier work [8] for the special cases considered above.

Finally, we give another table (Table 2 below) which gives the approximate values of γ defined in Theorem 5 and Theorem 6, respectively, when δ is given by

$$\delta=\frac{j}{10} \qquad (j=1,2,\cdots,9).$$

δ	γ_1	γ_2
0.9	0.44897	0.27427
0.8	0.43647	0.28576
0.7	0.41317	0.28270
0.6	0.38021	0.26598
0.5	0.33781	0.23485
0.4	0.28596	0.18916
0.3	0.22487	0.13310
0.2	0.15544	0.07889
0.1	0.07952	0.03626

TABLE 1. **The Approximate Values of γ_1 and γ_2**

δ	Theorem 5 (γ)	Theorem 6 (γ)
0.9	0.75302	0.28582
0.8	0.75151	0.27787
0.7	0.67933	0.26303
0.6	0.59106	0.24205
0.5	0.49662	0.21506
0.4	0.39926	0.18205
0.3	0.30036	0.14316
0.2	0.20061	0.09896
0.1	0.10041	0.05062

TABLE 2. **The Approximate Values of γ in Theorem 5 and Theorem 6**

Acknowledgements

The first-named author was supported by a research grant of the National Research Foundation of the Republic of Korea (NRF) funded by the Government of the Republic of Korea (MSIP; Ministry of Science, ICT & Future Planning) (Grant No. NRF-2017R1C1B5076778). The third-named author was supported by the Basic Science Research Program through the National Research Foundation of the Republic of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant No. 2016R1D1A1A09916450).

References

[1] J. W. Alexander, Functions which map the interior of the unit circle upon simple regions, *Ann. Math.* **17** (1915), 12–22.

[2] S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.* **135** (1969), 429–446.

[3] D. J. Hallenbeck and St. Ruscheweyh, Subordination by convex functions, *Proc. Amer. Math. Soc.* **52** (1975), 191–195.

[4] I. B. Jung, Y. C. Kim and H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operator, *J. Math. Anal. Appl.* **176** (1993), 138–147.

[5] S. S. Miller and P. T. Mocanu, Univalent solutions of Briot-Bouquet differential subordinations, *J. Differ. Equ.* **58** (1985). 297–309.

[6] S. S. Miller and P. T. Mocanu, *Differential Subordination: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, No. **225**, Marcel Dekker Incorporated, New York and Basel, 2000.

[7] M. Nunokawa, On the order of strongly starlikeness of strongly convex functions, *Proc. Japan. Acad. Ser. A Math. Sci.* **69** (1993), 234–237.

[8] M. Nunokawa, T. Hayami and S. Owa, On the order of strongly starlikeness and order of starlikeness of a certain convex functions, *Surikaisekikenkyusho Kokyuroku* **1772** (2011), 85–90.

[9] J. K. Prajapat and T. Bulboacă, Differential subordination for subclasses of analytic functions involving Srivastava-Attiya operator, *Thai J. Math.* **15** (2017), 655–672.

[10] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck und Ruprecht, Göttingen, 1975.

[11] H. M. Srivastava and A. A. Attiya, An integral operator associated with the Hurwitz-Lerch zeta function and differential subordination, *Integral Transforms Spec. Funct.* **18** (2007), 207–216.

[12] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

[13] H. M. Srivastava, A. Prajapati and P. Gochhayat, Third-order differential subordination and differential superordination results for analytic functions involving the Srivastava-Attiya operator, *Appl. Math. Inform. Sci.* **12** (2018), 469–481.

[14] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; with an Account of the Principal Transcendental Functions*, Fourth Edition, Cambridge University Press, Cambridge, London and New York, 1927.

[15] D. R. Wilken and J. Feng, A remark on convex and starlike functions, *J. London Math. Soc.* (*Ser.* 2) **21** (1980), 287–290.

SHARP BOUNDS FOR THE COMPLETE ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KINDS*

XIAO-HUI ZHANG[1,2], YU-MING CHU[3,**], AND WEN ZHANG[4]

ABSTRACT. In the article, we prove that $\alpha = 3$, $\beta = \log 4/(\pi/2 - \log 4) = 7.51371\cdots$, $\gamma = 1/4$ and $\delta = 1 + \log 2 - \pi/2 = 0.122351\cdots$ are the best possible constants such that the double inequalities

$$\frac{\beta+1}{\beta+r^2}\log\frac{4}{r'} < \mathcal{K}(r) < \frac{\alpha+1}{\alpha+r^2}\log\frac{4}{r'},$$

$$1+\left(\frac{1}{2}\log\frac{4}{r'}-\gamma\right)r'^2 < \mathcal{E}(r) < 1+\left(\frac{1}{2}\log\frac{4}{r'}-\delta\right)r'^2$$

hold for all $r \in (0,1)$, where $r' = \sqrt{1-r^2}$, and $\mathcal{K}(r) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-r^2\sin^2\theta}}$ and $\mathcal{E}(r) = \int_0^{\pi/2}\sqrt{1-r^2\sin^2\theta}d\theta$ are the complete elliptic integrals of the first and second kinds.

1. INTRODUCTION

The complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ [1-5] of the first and the second kinds are respectively defined by

$$\begin{cases} \mathcal{K}(r) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-r^2\sin^2\theta}}, \\ \mathcal{K}(0) = \frac{\pi}{2}, \qquad \mathcal{K}(1) = \infty \end{cases}$$

and

$$\begin{cases} \mathcal{E}(r) = \int_0^{\pi/2}\sqrt{1-r^2\sin^2\theta}d\theta, \\ \mathcal{E}(0) = \frac{\pi}{2}, \qquad \mathcal{E}(1) = 1. \end{cases}$$

It is well known that the function $r \to \mathcal{K}(r)$ is strictly increasing from $(0,1)$ onto $(\pi/2,\infty)$ and the function $r \to \mathcal{E}(r)$ is strictly decreasing from $(0,1)$ onto $(1,\pi/2)$. The complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ are the particular cases of the Gaussian hypergeometric function [6-15]

$$F(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} \quad (-1<x<1),$$

where $(a)_0 = 1$ for $a \neq 0$, $(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is the shifted factorial function and $\Gamma(x) = \int_0^{\infty}t^{x-1}e^{-t}dt$ $(x>0)$ is the gamma

2010 *Mathematics Subject Classification.* Primary: 33E15; Secondary: 33C05.

Key words and phrases. complete elliptic integrals, monotonicity, bound.

*The research was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11771400, 11301127, 11701176, 11626101, 11601485), the Science and Technology Research Program of Zhejiang Educational Committee (Grant no. Y201635325) and the Natural Science Foundation of Zhejiang Sci-Tech University (Grant No. 16062023Y).

**Corresponding author: Yu-Ming Chu, Email: chuyuming2005@126.com.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

function [16-21]. Indeed,

$$\mathcal{K}(r)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}r^{2n},$$

$$\mathcal{E}(r)=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(n!)^2}r^{2n}.$$

The complete elliptic integrals play a very important role in the study of geometric function theory and they have numerous applications in various problems of physics and engineering. In particular, Many remarkable inequalities and elementary approximations for the complete elliptic integrals can be found in the literature [22-34].

In the sequel, we will use the symbols $\mathcal{K}$ and $\mathcal{E}$ for $\mathcal{K}(r)$ and $\mathcal{E}(r)$, respectively. Throughout this paper we let $r'=\sqrt{1-r^2}$ for $0<r<1$. Then we use the symbols $\mathcal{K}'$ and $\mathcal{E}'$ for $\mathcal{K}(r')$ and $\mathcal{E}(r')$, respectively.

Carlson and Gustafson [35] proved that the double inequality

$$1<\frac{\mathcal{K}(r)}{\log(4/r')}<\frac{4}{3+r^2}$$

holds for all $0<r<1$.

Kühnau [36] proved the inequality

$$\frac{9}{8+r^2}<\frac{\mathcal{K}(r)}{\log(4/r')} \tag{1.1}$$

for all $0<r<1$.

It is well known that the double inequality

$$\frac{\pi}{2}M_{3/2}(1,r')<\mathcal{E}(r)<\frac{\pi}{2}M_2(1,r')$$

holds for all $0<r<1$ (see [37, 19.9.4]), where

$$M_p(a,b)=\left(\frac{a^p+b^p}{2}\right)^{1/p}\quad(p\neq 0),\qquad M_0(a,b)=\sqrt{ab}$$

is the pth power mean [38-51]

It is the aim of this paper to refine the inequality (1.1) for the complete elliptic integral of the first kind, and to obtain sharp upper and lower bounds for the complete elliptic integral of the second kind. Our main results are the following Theorems 1.1 and 1.2.

Theorem 1.1. *The double inequality*

$$\frac{\beta+1}{\beta+r^2}<\frac{\mathcal{K}(r)}{\log(4/r')}<\frac{\alpha+1}{\alpha+r^2} \tag{1.2}$$

holds for all $r\in(0,1)$ *with the best possible constants* $\alpha=3$ *and* $\beta=(\log 4)/(\pi/2-\log 4)=7.51371\cdots$.

Theorem 1.2. *The double inequality*

$$1+r'^2\left(\frac{1}{2}\log\frac{4}{r'}-\gamma\right)<\mathcal{E}(r)<1+r'^2\left(\frac{1}{2}\log\frac{4}{r'}-\delta\right) \tag{1.3}$$

holds for all $r\in(0,1)$ *with the best possible constants* $\gamma=1/4$ *and* $\delta=1+\log 2-\pi/2=0.122351\cdots$.

2. Proof of Theorems 1.1 and 1.2

In order to prove our main results we need to establish some monotonicity properties for the functions defined by the complete elliptic integrals. The following derivative formula can be found in the literature [52]:

$$\frac{d\mathcal{K}}{dr}=\frac{\mathcal{E}-r'^2\mathcal{K}}{rr'^2},\quad \frac{d\mathcal{E}}{dr}=\frac{\mathcal{E}-\mathcal{K}}{r},$$

$$\frac{d}{dr}(\mathcal{E}-r'^2\mathcal{K})=r\mathcal{K},\quad \frac{d}{dr}(\mathcal{K}-\mathcal{E})=\frac{r\mathcal{E}}{r'^2}.$$

Theorem 2.1. *The function*

$$F(r)=(3+r^2)\mathcal{K}-4\log(4/r')$$

is strictly increasing from $(0,1)$ *onto* $(-a,0)$ *with* $a=4\log 4-3\pi/2=0.832788\cdots$. *In particular, the double inequality*

$$\frac{4\log(4/r')-a}{3+r^2}<\mathcal{K}<\frac{4\log(4/r')}{3+r^2} \tag{2.1}$$

holds for all $r\in(0,1)$.

Proof. Let the functions g and h be defined by

$$g(r)=(3+r^2)\mathcal{E}-(r^4-4r^2+3)\mathcal{K}-4r^2,$$

$$h(r)=3r'^2\mathcal{K}+4\mathcal{E}-8.$$

Then differentiation gives

$$rr'^2\frac{d}{dr}F(r)=g(r),$$

$$\frac{1}{r}\frac{dg(r)}{dr}=h(r).$$

It follows from [52, Theorem 3.21(7)] that the function h is strictly decreasing from $(0,1)$ onto $(-4,7\pi/2-8)$ and there exists $r_0\in(0,1)$ such that h is positive on $(0,r_0)$ and negative on $(r_0,1)$. We conclude that g is strictly increasing on $(0,r_0)$ and strictly decreasing on $(r_0,1)$. From $g(0)=0=g(1)$ we clearly see that $g(r)>0$ for $r\in(0,1)$ and F is strictly increasing on $(0,1)$. It is easy to see that the limiting value $F(0)=-a$, and by [53, 112.01] $F(1^-)=0$. □

Theorem 2.2. *Let* $\beta=\log 4/(\pi/2-\log 4)=7.51371\cdots$. *Then there exists* $s_0\in(0,1)$ *such that the function*

$$G(r)=(\beta+r^2)\mathcal{K}-(\beta+1)\log(4/r')$$

is strictly increasing on $(0,s_0)$ *and strictly decreasing on* $(s_0,1)$ *with the limiting values* $G(0^+)=0=G(1^-)$. *In particular, the inequality*

$$\frac{(\beta+1)\log(4/r')}{\beta+r^2}<\mathcal{K} \tag{2.2}$$

holds for all $r\in(0,1)$.

Proof. Let the functions g, h, l and p be defined by

$$g(r) = (\beta + r^2)\mathcal{E} - (\beta - r^2)r'^2\mathcal{K} - (\beta + 1)r^2,$$
$$h(r) = 4\mathcal{E} + (\beta - 3r^2)\mathcal{K} - 2(\beta + 1),$$
$$l(r) = (\beta + 4 - 7r^2)\mathcal{E} - (\beta + 4 + 3r^2)r'^2\mathcal{K},$$
$$p(r) = (21r^2 + 8 + \beta)\mathcal{K} - 24\mathcal{E}.$$

Then differentiation leads to

$$rr'^2\frac{d}{dr}G(r) = g(r),$$
$$\frac{1}{r}\frac{d}{dr}g(r) = h(r),$$
$$rr'^2\frac{d}{dr}h(r) = l(r),$$
$$\frac{1}{r}\frac{d}{dr}l(r) = p(r).$$

It is easy to see that the function p is strictly increasing from $(0, 1)$ onto $((8 + \beta - 24)\pi/2, \infty)$. It follows from $(8 + \beta - 24)\pi/2 = -13.3302\cdots < 0$ that there exists $r_1 \in (0, 1)$ such that p is negative on $(0, r_1)$ and positive on $(r_1, 1)$. Hence the function l is strictly decreasing on $(0, r_1)$ and strictly increasing on $(r_1, 1)$. From the limiting values $l(0^+) = 0$ and $l(1^-) = \beta - 3 = 4.51371\cdots > 0$ we clearly see that there exists $r_2 \in (0, 1)$ such that l is negative on $(0, r_2)$ and positive on $(r_2, 1)$. We conclude that h is strictly decreasing on $(0, r_2)$ and strictly increasing on $(r_2, 1)$. This together with the values $h(0^+) = 2\pi - 2 + (\pi/2 - 1)\beta = 1.05827\cdots$, $h(0.8) = -0.760875\cdots$ and $h(1^-) = \infty$ implies that there exists $0 < r_3 < r_4 < 1$ such that h is positive on $(0, r_3) \cup (r_4, 1)$ and negative on (r_3, r_4). Hence g is strictly increasing on $(0, r_3)$ and $(r_4, 1)$, and strictly decreasing on (r_3, r_4). Since $g(0^+) = 0 = g(1^-)$, we conclude that there exists $s_0 \in (0, 1)$ such that g is positive on $(0, s_0)$ and negative on $(s_0, 1)$. Therefore, the function G is strictly increasing on $(0, s_0)$ and strictly decreasing on $(s_0, 1)$. It is easy to see that $G(0^+) = 0$ and

$$\lim_{r\to 1^-} G(r) = \lim_{r\to 1^-} (a + 1)(\mathcal{K} - \log(4/r')) - r'^2\mathcal{K} = 0.$$

□

Proof of Theorem 1.1. Inequality (1.2) follows from inequality (2.2) and the right-hand side inequality of (2.1) immediately.

Lemma 2.3. *The function*

$$u(r) = (1 + r^2)\mathcal{E} - r'^2\mathcal{K} - \frac{5}{2}r^2 + \frac{1}{2}r^4$$

is negative on $(0, 1)$.

Proof. Let the functions f and g be defined by

$$f(r) = 3\mathcal{E} + 2r^2 - 5,$$
$$g(r) = \frac{3(\mathcal{E} - \mathcal{K})}{r^2} + 4.$$

Then Applying the derivative formulas we get

$$\frac{d}{dr}u(r) = rf(r),$$

$$\frac{d}{dr}f(r) = rg(r).$$

Since the function $r \mapsto (\mathcal{E}-\mathcal{K})/r^2$ is strictly decreasing from $(0,1)$ onto $(-\infty, -\pi/4)$ (see [52, 3.43(11)]), the function g is strictly decreasing from $(0,1)$ onto $(-\infty, (16-3\pi)/4)$. Then from $(16-3\pi)/4) > 0$ we know that there exists $r_0 \in (0,1)$ such that $rg(r)$ is positive on $(0, r_0)$ and negative on $(r_0, 1)$. Hence f is strictly increasing on $(0, r_0)$ and strictly decreasing on $(r_0, 1)$. It is easy to see that $f(0^+) = 3\pi/2 - 5 < 0$ and $f(1^-) = 0$. We conclude that there exists $r_1 \in (0,1)$ such that $rf(r)$ is negative on $(0, r_1)$ and positive on $(r_1, 1)$. Therefore, the function u is strictly decreasing on $(0, r_1)$ and strictly increasing on $(r_1, 1)$. Then from the facts that $u(0^+) = 0 = u(1^-)$ we get $u(x) < 0$ for all $r \in (0,1)$. □

Theorem 2.4. *The function*

$$H(r) = \frac{\mathcal{E}-1}{r'^2} - \frac{1}{2}\log\frac{4}{r'}$$

is strictly decreasing from $(0,1)$ *onto* $(-1/4, -\delta)$ *with* $\delta = 1 + \log 2 - \pi/2 = 0.122351\cdots$.

Proof. Differentiation yields

$$rr'^4\frac{d}{dr}H(r) = (1+r^2)\mathcal{E} - r'^2\mathcal{K} - \frac{5}{2}r^2 + \frac{1}{2}r^4 = u(r) < 0$$

by Lemma 2.3. Hence, the function H is strictly decreasing on $(0,1)$.

We clearly see that

$$H(0^+) = \pi/2 - 1 - \log 2 = -\delta.$$

Let

$$h_1(r) = \mathcal{E} - 1 - \frac{1}{2}r'^2\log\frac{4}{r'}, \qquad h_2(r) = r'^2.$$

Then $h_1(1^-) = 0 = h_2(1^-)$, and by l'Hospital's rule one has

$$H(1^-) = \lim_{r\to 1^-}\frac{h_1'(r)}{h_2'(r)} = \lim_{r\to 1^-}\frac{1}{2}\left(\frac{r'^2\mathcal{K}}{r^2} + \mathcal{K} - \log\frac{4}{r'}\right) - \frac{\mathcal{E}}{2r^2} + \frac{1}{4} = -\frac{1}{4},$$

where the last equality follows from the facts (see [52, 3.21(7) and (3.25)] or [53, 112.01] that

$$\lim_{r\to 1^-} {r'}^2\mathcal{K} = 0, \qquad \lim_{r\to 1^-}\left(\mathcal{K} - \log\frac{4}{r'}\right) = 0.$$

□

Proof of Theorem 1.2. Inequality (1.3) follows easily from Theorem 2.4 immediately.

References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, U. S. Government Printing Office, Washington, 1964.

[2] M.-K. Wang, Y.-M. Chu, S.-L. Qiu and Y.-P. Jiang, Convexity of the complete elliptic integrals of the first kind with respect to Hölder means, J. Math. Anal. Appl., 2012, **388**(2), 1141–1146.

[3] Y.-M. Chu, M.-K. Wang, Y.-P. Jiang and S.-L. Qiu, Concavity of the complete elliptic integrals of the second kind with respect to Hölder means, J. Math. Anal. Appl., 2012, **395**(2), 637–642.

[4] Y.-M. Chu, M.-K. Wang, S.-L. Qiu and Y.-P. Jiang, Bounds for complete elliptic integrals of the second kind with applications, Comput. Math. Appl., 2012, **63**(7), 1177–1184.
[5] M.-K. Wang, S.-L. Qiu, Y.-M. Chu and Y.-P. Jiang, Generalized Hersch-Pfluger distortion function and complete elliptic integrals, J. Math. Anal. Appl., 2012, **385**(1), 221–229.
[6] Zh.-H. Yang, Y.-M. Chu and M.-K. Wang, Monotonicity criterion for the quotient of power series with applications, J. Math. Anal. Appl., 2015, **428**(1), 587–604.
[7] M.-K. Wang, Y.-M. Chu and Y.-P. Jiang, Ramanujan's cubic transformation inequalities for zero-balanced hypergeometric functions, Rocky Mountain J. Math., 2016, **46**(2), 679–691.
[8] M.-K. Wang, Y.-M. Chu and Y.-Q. Song, Asymptotical formulas for Gaussian and generalized hypergeometric functions, Appl. Math. Comput., 2016, **276**, 44–60.
[9] M.-K. Wang and Y.-M. Chu, Refinenemts of transformation inequalities for zero-balanced hypergeometric functions, Acta Math. Sci., 2017, **37B**(3), 607–622.
[10] M.-K. Wang, Y.-M. Li and Y.-M. Chu, Inequalities and infinite product formula for Ramanujan generalized modular equation function, Ramanujan J., 2018, **46**(1), 189–200.
[11] M.-K. Wang and Y.-M. Chu, Landen inequalities for a class of hypergeometric functions with applications, Math. Inequal. Appl., 2018, **21**(2), 521–537.
[12] M.-K. Wang, S.-L. Qiu and Y.-M. Chu, Infinite series formula for Hübner upper bound function with applications to Hersch-Pfluger distortion function, Math. Inequal. Appl., 2018, **21**(3), 629–648.
[13] T.-H. Zhao, M.-K. Wang, W. Zhang and Y.-M. Chu, Quadratic transformation inequalities for Gaussian hypergeometric function, J. Inequal. Appl., 2018, **2018**, Article 251, 15 pages.
[14] Y.-F. Yang, Special values of hypergeometric functions and periods of CM elliptic curves, Trans. Amer. Math. Soc., 2018, **370**(9), 6433–6467.
[15] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind, J. Math. Anal. Appl., 2018, **462**(2), 1714–1726
[16] T.-H. Zhao and Y.-M. Chu, A class of logarithmically completely monotonic functions associated with a gamma function, J. Inequal. Appl., 2010, **2010**, Article ID 392431, 11 pages.
[17] T.-H. Zhao, Y.-M. Chu and H. Wang, Logarithmically complete monotonicity properties relating to the gamma function, Abstr. Appl. Anal., 2011, **2011**, Article ID 896483, 13 pages.
[18] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On rational bounds for the gamma function, J. Inequal. Appl., 2017, **2017**, Article 210, 17 pages.
[19] Zh.-H. Yang, W. Zhang and Y.-M. Chu, Sharp Gautschi inequality for parameter $0 < p < 1$ with applications, Math. Inequal. Appl., 2017, **20**(4), 1107–1120.
[20] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the error function, Math. Inequal. Appl., 2018, **21**(2), 469–479.
[21] T.-R. Huang, B.-W. Han, X.-Y. Ma and Y.-M. Chu, Optimal bounds for the generalized Euler-Mascheronic constant, J. Inequal. Appl., 2018, **2018**, Article 118, 9 pages.
[22] Y.-M. Chu, M.-K. Wang and Y.-F. Qiu, On Alzer and Qiu's conjecture for complete elliptic integral and inverse hyperbolic tangent function, Abstr. Appl. Anal., 2011, **2011**, Article ID 697547, 7 pages.
[23] M.-K. Wang, Y.-M. Chu, Y.-F. Qiu and S.-L. Qiu, An optimal power mean inequality for the complete elliptic integrals, Appl. Math. Lett., 2011, **24**(6), 887–890.
[24] Y.-M. Chu, M.-K. Wang, Y.-P. Jiang and S.-L. Qiu, Monotonicity, convexity, and inequalities invoving the Agard distortion function, Abstr. Appl. Anal., 2011, **2011**, Article ID 671765, 8 pages.
[25] Y.-M. Chu and M.-K. Wang, Inequalities between arithmetic-geometric, Gini, and Toader means, Abstr. Appl. Anal., 2012, **2012**, Article ID 830585, 11 pages.
[26] Y.-M. Chu, M.-K. Wang and S.-L. Qiu, Optimal combinations bounds of root-square and arithmetic means for Toader mean, Proc. Indian Acad. Sci. Math. Sci., 2012, **122**(1), 41–51.
[27] Y.-M. Chu and M.-K. Wang, Optimal Lehmer mean bounds for the Toader mean, Results Math., 2012, **61**(3-4), 223–229.
[28] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, Hölder mean inequalities for the complete elliptic integrals, Integral Transforms Spec. Funct., 2012, **23**(7), 521–527.
[29] M.-K. Wang and Y.-M. Chu, Asymptotical bounds for complete elliptic integrals of the second kind, J. Math. Anal. Appl., 2013, **402**(1), 119–126.

[30] Y.-M. Chu and T.-H. Zhao, Concavity of the error functions with respect to Hölder means, Math. Inequal. Appl., 2016, **19**(2), 589–595.
[31] W.-M. Qian and Y.-M. Chu, Sharp bounds for a special quasi-arithmetic mean in terms of arithmetic and geometric means with two parameters, J. Inequal. Appl., 2017, **2017**, Article 274, 10 pages.
[32] Zh.-H. Yang, W.-M. Qian and Y.-M. Chu, Monotonicity properties and bounds involving the complete elliptic integrals of the first kind, Math. Inequal. Appl., 2018, **21**(4), 1185–1199.
[33] T.-R. Huang, S.-Y. Tan, X.-Y. Ma and Y.-M. Chu, Monotonicity properties and bounds for the complete p-elliptic integrals, J. Inequal. Appl., 2018, **2018**, Article 239, 11 pages.
[34] Zh.-H. Yang, Y.-M. Chu and W. Zhang, Accurate approximations for the complete elliptic of the second kind, J. Math. Anal. Appl., 2016, **438**(2), 875–888.
[35] B. C. Carlson and J. L. Gustafson, Asymptotic expansion of the first elliptic integral, SIAM J. Math. Anal., 1985, **16**(5), 1072–1092.
[36] R. Kühnau, Eine Methode, die Positivität einer Funktion zu prüfen, Z. Angew. Math. Mech., 1994, **74**(2), 140–143.
[37] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark (Ed.), NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge, 2010.
[38] B.-Y. Long and Y.-M. Chu, Optimal power mean bounds for the weighted geometric mean of calssical means, J. Inequal. Appl., 2010, **2010**, Article ID 905679, 6 pages.
[39] W.-F. Xia, Y.-M. Chu and G.-D. Wang, The optimal upper and lower power mean bounds for a convex combination of the arithmetic and logarithmic means, Abstr. Appl. Anal., 2010, **2010**, Article 604804, 9 pages.
[40] Y.-M. Chu and W.-F. Xia, Two optimal double inequalities between power mean and logarithmic mean, Comput. Math. Appl., 2010, **60**(1), 83–89.
[41] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, Sharp power mean bounds for the combination of Seiffert and geometric means, Abstr. Appl. Anal., 2010, **2010**, Article ID 108920, 12 pages.
[42] Y.-M. Chu, M.-K. Wang and Y.-F. Qiu, An optimal double inequality between power-type Heron and Seiffert means, J. Inequal. Appl., 2010, Article ID 146945, 11 pages.
[43] Y.-M. Chu, S.-S. Wang and C. Zong, Optimal lower power mean bound for the convex combination of harmonic and logarithmic means, Abstr. Appl. Anal., 2011, **2011**, Article ID 520648, 9 pages.
[44] G.-D. Wang, X.-H. Zhang and Y.-M. Chu, A power mean inequality for the Grötzsch ring function, Math. Inequal. Appl., 2011, **14**(4), 833–837.
[45] Y.-M. Li, B.-Y. Long, Y.-M. Chu and W.-M. Gong, Optimal inequalities for power mean, J. Appl. Math., 2012, **2012**, Article ID 182905, 8 pages.
[46] Y.-M. Li, B.-Y. Long and Y.-M. Chu, Sharp bounds by the power mean for the generalized Heronian mean, J. Inequal. Appl., 2012, **2012**, Article 129, 9 pages.
[47] W.-F. Xia, W. Janous and Y.-M. Chu, The optimal convex combination bounds of arithmetic and harmonic means in terms of power mean, J. Math. Inequal., 2012, **6**(2), 241–248.
[48] Y.-M. Li, B.-Y. Long and Y.-M. Chu, A best possible double inequality for power mean, J. Appl. Math., 2012, **2012**, Article ID 379785, 12 pages.
[49] Y.-M. Chu and B.-Y. Long, Bounds of the Neuman-Sándor mean using power and identric means, Abstr. Appl. Anal., 2013, **2013**, Article ID 832591, 6 pages.
[50] G.-D. Wang, X.-H. Zhang and Y.-M. Chu, A power mean inequality involving the complete elliptic integrals, Rocky Mountain J. Math., 2014, **44**(5), 1661–1667.
[51] Y.-M. Chu, Zh.-H. Yang and L.-M. Wu, Sharp power mean bounds for Sándor mean, Abstr. Appl. Anal., 2015, **2015**, Article ID 172867, 5 pages.
[52] G. D. Anderson, M. K. Vamanamurthy and M. K. Vuorinen, Conformal Invariants, Inequalities, and Quasiconformal Maps, John Wiley & Sons, New York, 1997.
[53] P. F. Byrd and M. D. Friedman, Handbook of elliptic integrals for engineers and Scientists, Springer-Verlag, New York, 1971.

Xiao-Hui Zhang, [1]College of Science, Hunan City University, Yiyang 413000, Hunan, China; [2]Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, Zhejiang, China
E-mail address: xhzhang31415926@126.com

Yu-Ming Chu (Corresponding author), [3]Department of Mathematics, Huzhou University, Huzhou 313000, Zhejiang, China
E-mail address: chuyuming2005@126.com

Wen Zhang, [4]Friedman Brain Institute, Icahn School of Medicine at Mount Sinai, New York, NY 10029, USA
E-mail address: zhang.wen81@gmail.com

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Symmetric identities for the second kind q-Bernoulli polynomials

C. S. Ryoo

Department of Mathematics, Hannam University, Daejeon 34430, Korea

Abstract : In [9], we studied the second kind q-Bernoulli numbers and polynomials. By using these numbers and polynomials, we investigated the zeros of the second kind q-Bernoulli polynomials. In this paper, by applying the symmetry of the fermionic p-adic q-integral on $\mathbb{Z}_p$, we give recurrence identities the second kind q-Bernoulli polynomials and the sums of powers of consecutive q-odd integers.

Key words : Symmetric properties, the sums of powers of consecutive q-odd integers, the second kind Bernoulli numbers and polynomials, the second kind q-Bernoulli numbers and polynomials.

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Bernoulli numbers, Bernoulli polynomials, q-Bernoulli numbers, q-Bernoulli polynomials, the second kind Bernoulli number, the second kind Bernoulli polynomials, Euler numbers, Euler polynomials, Genocchi numbers, Genocchi polynomials, tangent numbers, tangent polynomials, and Bell polynomials were studied by many authors (see for details [1-11]). Bernoulli numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. In [8], by using the second kind Euler numbers E_j and polynomials $E_j(x)$, we investigated the alternating sums of powers of consecutive odd integers. Let k be a positive integer. Then we obtain

$$T_j(k-1) = \sum_{n=0}^{k-1}(-1)^n(2n+1)^j = \frac{(-1)^{k+1}E_j(2k)+E_j}{2}.$$

In [9], we introduced the second kind q-Bernoulli numbers $B_{n,q}$ and polynomials $B_{n,q}(x)$. By using computer, we observed an interesting phenomenon of 'scattering' of the zeros of the second kind q-Bernoulli polynomials $B_{n,q}(x)$ in complex plane. Also we carried out computer experiments for doing demonstrate a remarkably regular structure of the complex roots of the second kind q-Bernoulli polynomials $B_{n,q}(x)$. The outline of this paper is as follows. We introduce the second kind q-Bernoulli numbers $B_{n,q}$ and polynomials $B_{n,q}(x)$. In Section 2, we obtain the sums of powers of consecutive q-odd integers. Finally, we give recurrence identities the second kind q-Bernoulli polynomials and the sums of powers of consecutive q-odd integers.

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the p-adic q-integral was defined by [1, 2, 3, 4, 6]

$$I_q(g) = \int_{\mathbb{Z}_p} g(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]} \sum_{x=0}^{p^N-1} g(x)q^x.$$

The bosonic integral was considered from a physical point of view to the bosonic limit $q \to 1$, as follows:

$$I_1(g) = \lim_{q\to 1} I_q(g) = \int_{\mathbb{Z}_p} g(x)d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} g(x) \text{ (see [1, 2, 3, 5])}. \tag{1.1}$$

By (1.1), we easily see that

$$I_1(g_1) = I_1(g) + g'(0), \text{ cf. [1, 2, 3, 4, 6, 7]}, \tag{1.2}$$

where $g_1(x) = g(x+1)$ and $g'(0) = \frac{dg(x)}{dx}|_{x=0}$.

First, we introduce the second kind Bernoulli numbers B_n and polynomials $B_n(x)$. The second kind Bernoulli numbers B_n and polynomials $B_n(x)$ are defined by means of the following generating functions (see [7]):

$$F(t) = \frac{2te^t}{e^{2t}-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

and

$$F(x,t) = \frac{2te^t}{e^{2t}-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!},$$

respectively.

The second kind q-Bernoulli polynomials, $B_{n,q}(x)$ are defined by means of the generating function:

$$F_q(x,t) = \frac{(\log q + 2t)e^t}{qe^{2t}-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!}. \tag{1.3}$$

The second kind q-Bernoulli numbers $B_{n,q}$ are defined by means of the generating function:

$$F_q(t) = \frac{(\log q + 2t)e^t}{qe^{2t}-1} = \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{n!}. \tag{1.4}$$

In (1.2), if we take $g(x) = q^x e^{(2x+1)t}$, then we have

$$\int_{\mathbb{Z}_p} q^x e^{(2x+1)t} d\mu_1(x) = \frac{(\log q + 2t)e^t}{q^h e^{2t}-1}. \tag{1.5}$$

for $|t| \le p^{-\frac{1}{p-1}}$. In (1.2), if we take $g(x) = e^{2nxt}$, then we also have

$$\int_{\mathbb{Z}_p} e^{2nxt} d\mu_1(x) = \frac{2nt}{e^{2nt}-1}. \tag{1.6}$$

It will be more convenient to write (1.2) as the equivalent bosonic integral form

$$\int_{\mathbb{Z}_p} g(x+1)d\mu_1(x) = \int_{\mathbb{Z}_p} g(x)d\mu_1(x) + g'(0), \quad \text{(see [1,2,3,4,6])}. \tag{1.7}$$

For $n \in \mathbb{N}$, we also derive the following bosonic integral form by (1.7),

$$\int_{\mathbb{Z}_p} g(x+n)d\mu_1(x) = \int_{\mathbb{Z}_p} g(x)d\mu_1(x) + \sum_{k=0}^{n-1} g'(k), \text{ where } g'(k) = \frac{dg(x)}{dx}|_{x=k}. \tag{1.8}$$

In [9], we introduced the second kind q-Bernoulli numbers $B_{n,q}$ and polynomials $B_{n,q}(x)$ and investigate their properties. The following elementary properties of the second kind q-Bernoulli numbers $B_{n,q}$ and polynomials $B_{n,q}(x)$ are readily derived form (1.1), (1.2), (1.3) and (1.4). We, therefore, choose to omit details involved.

Theorem 1(Witt formula). For $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$, we have

$$\int_{\mathbb{Z}_p} q^x (2x+1)^n d\mu_1(x) = B_{n,q},$$

$$\int_{\mathbb{Z}_p} q^y (x+2y+1)^n d\mu_1(y) = B_{n,q}(x).$$

Theorem 2. For any positive integer n, we have

$$B_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} B_{k,q} x^{n-k}.$$

Theorem 3(Distribution Relation). For any positive integer m, we obtain

$$B_{n,q}(x) = m^{n-1} \sum_{i=0}^{m-1} q^i B_{n,q^m}\left(\frac{2i+x+1-m}{m}\right) \text{ for } n \geq 0.$$

2. Symmetry identities for the second kind q-Bernoulli polynomials

In this section, we assume that $q \in \mathbb{C}_p$. In [2], Kim investigated interesting properties of symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli polynomials and Bernoulli polynomials. By using same method of [3], expect for obvious modifications, we obtain recurrence identities the second kind q-Bernoulli polynomials. By (1.7), we obtain

$$\begin{aligned} \frac{1}{h\log q + 2t} &\left(\int_{\mathbb{Z}_p} q^x q^n e^{(2x+2n+1)t} d\mu_1(x) - \int_{\mathbb{Z}_p} q^x e^{(2x+1)t} d\mu_1(x) \right) \\ &= \frac{n \int_{\mathbb{Z}_p} q^x e^{(2x+1)t} d\mu_1(x)}{\int_{\mathbb{Z}_p} q^{nx} e^{2ntx} d\mu_1(x)}. \end{aligned} \tag{2.1}$$

By (1.8), we obtain

$$\begin{aligned} \frac{1}{h\log q + 2t} &\left(\int_{\mathbb{Z}_p} q^x q^n e^{(2x+2n+1)t} d\mu_1(x) - \int_{\mathbb{Z}_p} q^x e^{(2x+1)t} d\mu_1(x) \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n-1} q^i (2i+1)^k \right) \frac{t^k}{k!}. \end{aligned} \tag{2.2}$$

For each integer $k \geq 0$, let

$$O_{k,q}(n) = 1^k + q3^k + q^2 5^k + q^3 7^k + \cdots + q^n (2n+1)^k.$$

The above sum $O_{k,q}(n)$ is called the sums of powers of consecutive q-odd integers.

From the above and (2.2), we obtain

$$\frac{1}{\log q + 2t} \left(\int_{\mathbb{Z}_p} q^x q^n e^{(2x+2n+1)t} d\mu_1(x) - \int_{\mathbb{Z}_p} q^x e^{(2x+1)t} d\mu_1(x) \right) \frac{t^k}{k!} = \sum_{k=0}^{\infty} O_{k,q}(n-1) \frac{t^k}{k!}. \tag{2.3}$$

Thus, we have

$$\sum_{k=0}^{\infty}\left(q^n\int_{\mathbb{Z}_p}q^x(2x+2n+1)^k d\mu_1(x)-\int_{\mathbb{Z}_p}q^x(2x+1)^k d\mu_1(x)\right)\frac{t^k}{k!}=\sum_{k=0}^{\infty}(\log q+2t)O_{k,q}(n-1)\frac{t^k}{k!}.$$

By comparing coefficients $\frac{t^k}{k!}$ in the above equation, we arrive at the following theorem:

Theorem 4. Let k be a positive integer. Then we obtain

$$q^n B_{n,q}(2n)-B_{n,q}=\log q O_{k,q}(n-1)+2kO_{k-1,q}(n-1). \tag{2.4}$$

Remark 5. For the sums of powers of consecutive odd integers, we have

$$\lim_{q\to 1}(\log q O_{k,q}(n-1)+2kO_{k-1,q}(n-1))=2k\sum_{i=0}^{n-1}(2i+1)^{k-1}=B_k(2n)-B_k \text{ for } k\in\mathbb{N}.$$

By using (2.1) and (2.3), we arrive at the following theorem:

Theorem 6. Let n be positive integer. Then we have

$$\frac{n\int_{\mathbb{Z}_p}q^x e^{(2x+1)t}d\mu_1(x)}{\int_{\mathbb{Z}_p}q^{nx}e^{2ntx}d\mu_1(x)}=\sum_{m=0}^{\infty}(O_{m,q}(n-1))\frac{t^m}{m!}. \tag{2.5}$$

Let w_1 and w_2 be positive integers. By using (1.5) and (1.6), we have

$$\begin{aligned}&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}q^{(w_1x_1+w_2x_2)}e^{(w_1(2x_1+1)+w_2(2x_2+1)+w_1w_2x)t}d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p}q^{w_1w_2x}e^{2w_1w_2xt}d\mu_1(x)}\\&=\frac{(\log q+2t)e^{w_1t}e^{w_2t}e^{w_1w_2xt}(q^{w_1w_2}e^{2w_1w_2t}-1)}{(q^{w_1}e^{2w_1t}-1)(q^{w_2}e^{2w_2t}-1)}.\end{aligned} \tag{2.6}$$

By using (2.4) and (2.6), after elementary calculations, we obtain

$$\begin{aligned}a&=\left(\frac{1}{w_1}\int_{\mathbb{Z}_p}q^{w_1x_1}e^{(w_1(2x_1+1)+w_1w_2x)t}d\mu_1(x_1)\right)\left(\frac{w_1\int_{\mathbb{Z}_p}q^{w_2x_2}e^{(2x_2+1)(w_2t)}d\mu_1(x_2)}{\int_{\mathbb{Z}_p}q^{w_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_1}\sum_{m=0}^{\infty}B_{m,q^{w_1}}(w_2x)w_1^m\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}O_{m,q^{w_2}}(w_1-1)w_2^m\frac{t^m}{m!}\right).\end{aligned} \tag{2.7}$$

By using Cauchy product in the above, we have

$$a=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}B_{j,q^{w_1}}(w_2x)w_1^{j-1}O_{m-j,q^{w_2}}(w_1-1)w_2^{m-j}\right)\frac{t^m}{m!}. \tag{2.8}$$

Again, by using the symmetry in (2.7), we have

$$\begin{aligned}a&=\left(\frac{1}{w_2}\int_{\mathbb{Z}_p}q^{w_2x_2}e^{(w_2(2x_2+1)+w_1w_2x)t}d\mu_1(x_2)\right)\left(\frac{w_2\int_{\mathbb{Z}_p}q^{w_1x_1}e^{(2x_1+1)(w_1t)}d\mu_1(x_1)}{\int_{\mathbb{Z}_p}q^{w_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_2}\sum_{m=0}^{\infty}B_{m,q^{w_2}}(w_1x)w_2^m\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}O_{m,q^{w_1}}(w_2-1)w_1^m\frac{t^m}{m!}\right).\end{aligned}$$

Thus we have

$$a=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}B_{j,q^{w_2}}(w_1x)w_2^{j-1}O_{m-j,q^{w_1}}(w_2-1)w_1^{m-j}\right)\frac{t^m}{m!}. \tag{2.9}$$

By comparing coefficients $\frac{t^m}{m!}$ on the both sides of (2.8) and (2.9), we arrive at the following theorem:

Theorem 7. Let w_1 and w_2 be positive integers. Then we obtain

$$\begin{aligned}
&\sum_{j=0}^{m}\binom{m}{j}B_{j,q^{w_1}}(w_2x)w_1^{j-1}O_{m-j,q^{w_2}}(w_1-1)w_2^{m-j}\\
&=\sum_{j=0}^{m}\binom{m}{j}B_{j,q^{w_2}}(w_1x)w_2^{j-1}O_{m-j,q^{w_1}}(w_2-1)w_1^{m-j},
\end{aligned}$$

where $B_{k,q}(x)$ and $O_{m,q}(k)$ denote the second kind q-Bernoulli polynomials and the sums of powers of consecutive q-odd integers, respectively.

By using Theorem 2, we have the following corollary:

Corollary 8. Let w_1 and w_2 be positive integers. Then we have

$$\begin{aligned}
&\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{m-k}w_2^{j-1}x^{j-k}B_{k,q^{w_2}}O_{m-j,q^{w_1}}(w_2-1)\\
&=\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{j-1}w_2^{m-k}x^{j-k}B_{k,q^{w_1}}O_{m-j,q^{w_2}}(w_1-1).
\end{aligned}$$

By using (2.6), we have

$$\begin{aligned}
a&=\left(\frac{1}{w_1}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{w_1x_1}e^{(2x_1+1)w_1t}d\mu_1(x_1)\right)\left(\frac{w_1\int_{\mathbb{Z}_p}q^{w_2x_2}e^{(2x_2+1)(w_2t)}d\mu_1(x_2)}{\int_{\mathbb{Z}_p}q^{w_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\
&=\left(\frac{1}{w_1}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{w_1x_1}e^{(2x_1+1)w_1t}d\mu_1(x_1)\right)\left(\sum_{j=0}^{w_1-1}q^{w_2j}e^{(2j+1)(w_2t)}\right)\\
&=\sum_{j=0}^{w_1-1}q^{w_2j}\int_{\mathbb{Z}_p}q^{w_1x_1}e^{\left(2x_1+1+w_2x+(2j+1)\frac{w_2}{w_1}\right)(w_1t)}d\mu_1(x_1)\\
&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_1-1}q^{w_2j}B_{n,q^{w_1}}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^{n-1}\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.10}$$

Again, by using the symmetry property in (2.10), we also have

$$\begin{aligned}
a&=\left(\frac{1}{w_2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{w_2x_2}e^{(2x_2+1)w_2t}d\mu_1(x_2)\right)\left(\frac{w_2\int_{\mathbb{Z}_p}q^{w_1x_1}e^{(2x_1+1)(w_1t)}d\mu_1(x_1)}{\int_{\mathbb{Z}_p}q^{w_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\
&=\left(\frac{1}{w_2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{w_2x_2}e^{(2x_2+1)w_2t}d\mu_1(x_2)\right)\left(\sum_{j=0}^{w_2-1}q^{w_1j}e^{(2j+1)(w_1t)}\right)\\
&=\sum_{j=0}^{w_2-1}q^{w_1j}\int_{\mathbb{Z}_p}q^{w_2x_2}e^{\left(2x_2+1+w_1x+(2j+1)\frac{w_1}{w_2}\right)(w_2t)}d\mu_1(x_2)\\
&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_2-1}q^{w_1j}B_{n,q^{w_2}}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right)w_2^{n-1}\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.11}$$

By comparing coefficients $\frac{t^n}{n!}$ on the both sides of (2.10) and (2.11), we have the following theorem.

Theorem 9. Let w_1 and w_2 be positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{w_1-1} q^{w_2 j} B_{n,q^{w_1}}\left(w_2 x+(2j+1)\frac{w_2}{w_1}\right) w_1^{n-1}\\ =&\sum_{j=0}^{w_2-1} q^{w_1 j} B_{n,q^{w_2}}\left(w_1 x+(2j+1)\frac{w_1}{w_2}\right) w_2^{n-1}.\end{aligned} \tag{2.12}$$

Substituting $w_1 = 1$ into (2.12), we arrive at the following corollary.

Corollary 10. Let w_2 be positive integer. Then we obtain

$$B_{n,q}(x) = w_2^{n-1} \sum_{j=0}^{w_2-1} q^j B_{n,q^{w_2}}\left(\frac{x-w_2+(2j+1)}{w_2}\right).$$

This last result(Corollary 10) is shown to yield the known Distribution Relation of the second kind q-Bernoulli polynomials(Theorem 3).

Acknowledgment

This work was supported by 2019 Hannam University Research Fund.

REFERENCES

1. Kim, T.(2002). *q-Volkenborn integration*, *Russ. J. Math. Phys.*, v.9, pp. 288-299.
2. Kim, T.(2008). *Euler numbers and polynomials associated with zeta function, Abstract and Applied Analysis*, Art. ID 581582.
3. Kim, T.(2008) *Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials, Journal of Difference Equations and Applications*, v.12, pp. 1267-1277.
4. Kim, T., Jang, L. C., Pak, H. K.(2001). *A note on q-Euler and Genocchi numbers*, *Proc. Japan Acad.*, v.77 A, pp. 139-141.
5. Liu, G.(2006). *Congruences for higher-order Euler numbers*, *Proc. Japan Acad.*, v.82 A, pp. 30-33.
6. Ryoo, C.S., Kim, T., Jang, L. C.(2007). *Some relationships between the analogs of Euler numbers and polynomials*, *Journal of Inequalities and Applications*, v.2007, ID 86052, pp. 1-22.
7. Ryoo, C.S.(2011). *Distribution of the roots of the second kind Bernoulli polynomials*, *Journal of Computational Analysis and Applications*, v.13, pp. 971-976.
8. Ryoo, C.S.(2011). *On the alternating sums of powers of consecutive odd integers*, *Journal of Computational Analysis and Applications*, v.13, pp. 1019-1024.
9. Ryoo, C.S.(2013). *Calculating zeros of q-extension of the second kind Bernoulli polynomials*, *Journal of Computational Analysis and Applications*, v.15, pp. 248-254.
10. Ryoo, C.S.(2016). *Differential equations associated with generalized Bell polynomials and their zeros*, *Open Mathematics*, v.14, pp. 807-815.
11. Ryoo, C.S.(2016) *Differential equations associated with tangent numbers, J. Appl. Math. & Informatics*, v.34, pp. 487-494.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

On some finite difference methods on the Shishkin mesh for the singularly perturbed problem *

Quan Zheng[†] Ying Liu, Jie He

College of Sciences, North China University of Technology, Beijing 100144, China

Abstract: This paper studies the convergence behavior of three finite difference schemes on the Shishkin mesh to solve the singularly perturbed two-point boundary value problem. Three new error estimates are proved for the hybrid finite difference scheme that combines the midpoint upwind scheme on the coarse part with the central difference scheme on the fine part, the midpoint upwind scheme and the simple upwind scheme, respectively. Finally, numerical experiments illustrate that these error estimates are sharp and the convergence is uniform with respect to the perturbation parameter.

Keywords: Singularly perturbed boundary value problem; Finite difference scheme; Piecewise equidistant mesh; Error estimate; Uniform convergence

1 Introduction

Consider the singularly perturbed two-point boundary value problem:

$$\begin{cases} Lu(x) := -\varepsilon u''(x) + b(x)u'(x) + c(x)u(x) = f(x),\ x \in (0,1), \\ u(0) = A,\ u(1) = B, \end{cases} \tag{1}$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter, A and B are given constants, and the functions $b(x)$, $c(x)$ and $f(x)$ are sufficiently smooth satisfying $0 < \beta < b(x) < \beta^*$ and $0 \leq c(x) < \gamma^*$, where β, β^* and γ^* are constants. Under these conditions, the singularly perturbed problem (1) has a unique solution that possesses a boundary layer at $x = 1$ (see [1–4]).

Among various numerical methods to solve singularly perturbed problems, finite difference schemes on layer-adapted meshes for the singularly perturbed two-point boundary value problem have been widely studied in the literature, see [1–10]. The simple upwind scheme was proved to have the error estimate $O(N^{-1})$ on the coarse part and $O(N^{-1}\ln N)$ on the fine part on the Shishkin mesh and the error estimate $O(N^{-1})$ on the whole interval on the Bakhvalov-Shishkin mesh, see, e.g., [5, 6]. The central difference scheme on the Shishkin mesh was proved

*This paper is supported by Natural Science Foundation of China (No. 11471019).

†E-mail: zhengq@ncut.edu.cn (Q. Zheng).

to have the convergence $O(N^{-2}\ln^2 N)$ on the whole nodes by discrete Green's functions, although the computed solution had small oscillations on the coarse part, see [3, 7]. In order to avoid oscillation, the midpoint upwind scheme on the Shishkin mesh was constructed and the convergence $O(\max\{N^{-2}, N^{-5+4i/N}\ln N\})$ were proved for $i = 1, \cdots, N$ in [8]. The midpoint upwind scheme on the Bakhvalov-Shishkin mesh was shown to be of order $O(N^{-2})$ on the coarse part and $O(N^{-1})$ on the fine part in [9]. To improve the convergence behaviour of boundary layer, the hybrid finite difference scheme was proposed and the convergence $O(N^{-2})$ on the coarse part and $O(N^{-2}\ln^2 N)$ on the fine part were proved for (1.1) with $c(x) \equiv 0$ in [8] and with geeneral $c(x) \geq 0$ in [4].

In this paper, we construct the hybrid finite difference scheme on the Shishkin mesh to slove (1) with $c(x) \geq 0$, not only give the suitable conditions especially for the $c(x)$ to guarantee an associated M-matrix and the discrete maximum principle, but also obtain a better error estimate. Furthermore, new error estimates for the midpoint upwind scheme and the simple upwind scheme are also obtained. Finally, the convergence behaviours according to these new error estimates for these schemes are confirmed by numerical experiments.

Note: Throughout the paper, the nontrivial case $\varepsilon \leq CN^{-1}$ is considered, C denotes a generic positive constant that is independent of both perturbation parameter ε and mesh parameter N, and C can take different values at each occurrence, even in the same argument.

2 Error estimates on the Shishkin mesh

Lemma 1 (see [1–3] The solution $u(x)$ of (1) can be decomposed as $u(x) = S(x) + E(x)$ on $[0, 1]$, where the smooth part S satisfies

$$LS(x) = f(x) \text{ and } \mid S^{(i)}(x) \mid\leq C,\ 0 \leq i \leq q,$$

while the layer part E satisfies

$$LE(x) = 0 \text{ and } \mid E^{(i)}(x) \mid\leq C\varepsilon^{-i}\exp\left(-\frac{\beta(1-x)}{\varepsilon}\right),\ 0 \leq i \leq q,$$

where the maximal order q depends on the smoothness of the data. □

Let N be a positive even integer and $\tau = \min\left\{\frac{1}{2}, \frac{4\varepsilon}{\beta}\ln N\right\}$. Since the singularly perturbed problem is considered, we generally take $\varepsilon \leq CN^{-1}$ and $\tau = \frac{4\varepsilon}{\beta}\ln N$. Choose $1-\tau$ be the transition point. Divide $[0, 1-\tau]$ and $[1-\tau, 1]$ uniformly into $N/2$ subintervals, respectively. Then the Shishkin mesh is:

$$x_i = \begin{cases} \frac{2(1-\tau)}{N}i,\ 0 \leq i \leq \frac{N}{2}, \\ 1 - 2\tau\left(1 - \frac{i}{N}\right),\ \frac{N}{2} \leq i \leq N. \end{cases} \tag{2}$$

Lemma 2 Denote $h_i = x_i - x_{i-1}$ for (2), then $N^{-1} \le h_i < 2N^{-1}$ and $h_{N/2+i} = \dfrac{8\varepsilon}{\beta} N^{-1} \ln N$ for $i = 1, 2, \cdots, N/2$. □

We construct the following hybrid finite difference scheme:

$$L^N u_i^N := \begin{cases} -\varepsilon D^+ D^- u_i^N + b_{i-1/2} D^- u_i^N + c_{i-1/2} u_{i-1/2}^N = f_{i-1/2},\ 0 < i \le N/2, \\ -\varepsilon D^+ D^- u_i^N + b_i D^0 u_i^N + c_i u_i^N = f_i,\ N/2 < i < N, \end{cases} \tag{3}$$
$$u_0^N = A,\ u_N^N = B,$$

where define L^N as a discrete operator, $D^+ u_i^N = \dfrac{u_{i+1}^N - u_i^N}{h_{i+1}}$, $D^- u_i^N = \dfrac{u_i^N - u_{i-1}^N}{h_i}$, $D^+ D^- u_i^N = \dfrac{2\left(D^+ u_i^N - D^- u_i^N\right)}{h_{i+1} + h_i}$, $D^0 u_i^N = \dfrac{u_{i+1}^N - u_{i-1}^N}{h_{i+1} + h_i}$, $u_{i-1/2}^N = \dfrac{u_{i-1}^N + u_i^N}{2}$, $b_{i-1/2} = b\left(x_{i-1/2}\right)$, $b_i = b\left(x_i\right)$, $f_{i-1/2} = f\left(x_{i-1/2}\right)$ and so on.

The scheme (3) is slightly different from the scheme (2.86) in [4] in the discretization of cu at $x_{i-1/2}$. The scheme (3) gives the following expression:

$$L^N u_i^N = \begin{cases} -\frac{2\varepsilon}{h_{i+1}(h_i+h_{i+1})} u_{i+1}^N + (\frac{2\varepsilon}{h_i h_{i+1}} + \frac{b_{i-1/2}}{h_i} + \frac{c_{i-1/2}}{2}) u_i^N - (\frac{2\varepsilon}{h_i(h_i+h_{i+1})} + \frac{b_{i-1/2}}{h_i} - \frac{c_{i-1/2}}{2}) u_{i-1}^N, \\ -(\frac{2\varepsilon}{h_{i+1}(h_i+h_{i+1})} - \frac{b_i}{h_i+h_{i+1}}) u_{i+1}^N + (\frac{2\varepsilon}{h_i h_{i+1}} + c_i) u_i^N - (\frac{2\varepsilon}{h_i(h_i+h_{i+1})} + \frac{b_i}{h_i+h_{i+1}}) u_{i-1}^N. \end{cases}$$

Lemma 3 (Discrete comparison principle) If $N > \dfrac{\gamma^*}{\beta}$ and $\dfrac{N}{\ln N} > \dfrac{4\beta^*}{\beta}$, then the operator L^N defined by (3) on (2) satisfies the discrete comparison principle, i.e., let $\{v_i\}$ and $\{w_i\}$ are mesh functions, if $v_0 \le w_0$, $v_N \le w_N$ and $L^N v_i \le L^N w_i$ for $i = 1, 2, \cdots, N-1$, then $v_i \le w_i$ for all i.
Proof. Under the conditions of Lemma 3, the coefficient matrix associated with L^N by the above expression is clearly an $(N-1) \times (N-1)$ strictly diagonally dominant matrix, and has positive diagonal entries and non-positive off diagonal entries. So it is an irreducible M-matrix. Hence, the operator satisfies the discrete comparison principle. □

So, the scheme (3) on (2) has a unique solution and the function w_i is defined as a barrier function for v_i by Lemma 3.

Lemma 4 Set $Z_0 = 1$, define the mesh function $Z_i = \displaystyle\prod_{j=1}^{i} \left(1 + \frac{\beta h_j}{2\varepsilon}\right)$ for $i = 1, 2, \cdots, N$. Then the operator L^N of (3) satisfies

$$L^N Z_i \ge \frac{C}{\max\{\varepsilon, h_i\}} Z_i \text{ for } i = 1, 2, \cdots, N-1.$$

Proof. Clearly

$$D^+ Z_i = \frac{\beta}{2\varepsilon} Z_i \text{ and } D^- Z_i = \frac{\beta}{2\varepsilon + \beta h_i} Z_i.$$

Hence

$$-\varepsilon D^+ D^- Z_i = -\frac{2\varepsilon}{h_{i+1} + h_i} \left(D^+ Z_i - D^- Z_i\right) = -\frac{\beta^2 h_i}{\left(h_{i+1} + h_i\right)\left(2\varepsilon + \beta h_i\right)} Z_i,$$

and
$$D^0 Z_i = \frac{h_{i+1} D^+ Z_i + h_i D^- Z_i}{h_{i+1}+h_i} = \left(\frac{\beta}{2\varepsilon + \beta h_i} + \frac{\beta^2 h_{i+1} h_i}{2\varepsilon\left(h_{i+1}+h_i\right)\left(2\varepsilon+\beta h_i\right)} \right) Z_i.$$
Thus, from (3), by using $c(x) \geq 0$ and $b(x) > \beta > 0$, we have
$$\begin{aligned}
L^N Z_i &\geq -\frac{\beta^2 h_i}{\left(h_{i+1}+h_i\right)\left(2\varepsilon+\beta h_i\right)} Z_i + b_{i-1/2} \cdot \frac{\beta}{2\varepsilon+\beta h_i} Z_i \\
&= \frac{\beta}{2\varepsilon+\beta h_i}\left(b_{i-1/2} - \frac{\beta h_i}{h_{i+1}+h_i} \right) Z_i, \quad i = 1,2,\cdots,N/2, \\
L^N Z_i &\geq -\frac{\beta^2 h_i}{\left(h_{i+1}+h_i\right)\left(2\varepsilon+\beta h_i\right)} Z_i + b_i \left(\frac{\beta}{2\varepsilon+\beta h_i} + \frac{\beta^2 h_{i+1} h_i}{2\varepsilon\left(h_{i+1}+h_i\right)\left(2\varepsilon+\beta h_i\right)} \right) Z_i \\
&\geq \frac{\beta}{2\varepsilon+\beta h_i}\left(b_i - \frac{\beta h_i}{h_{i+1}+h_i} \right) Z_i, \quad i = N/2+1,\cdots,N-1,
\end{aligned}$$
and obtain the result. □

Lemma 5 For the Shishkin mesh (2), there exists a constant C such that
$$\prod_{j=i+1}^{N} \left(1+\frac{\beta h_j}{2\varepsilon}\right)^{-1} \leq C N^{-4(1-i/N)} \text{ for } N/2 \leq i < N.$$

Proof. By Lemma 4.1(b) in [1] and noting $h_j = h$ for $j = N/2+1,\cdots,N$, we have
$$\prod_{j=i+1}^{N} \left(1+\frac{\beta h_j}{2\varepsilon}\right)^{-1} = \prod_{j=i+1}^{N} \left(1+\frac{\beta h}{2\varepsilon}\right)^{-1} \leq \mathrm{e}^{-\beta(1-x_i)/(2\varepsilon+\beta h)} = \mathrm{e}^{-4(1-i/N)\ln N/\left(1+4N^{-1}\ln N\right)}.$$
Then as the proof of Lemma 3.2 in [8], the result is proved. □

Lemma 6 Assuming that $u(x)$ be sufficiently smooth function defined on $[0,1]$, for the truncation error of (3) on the Shishkin mesh to solve (1), there exists a constant C such that
$$\left|L^N(u_i) - (Lu)(x_{i-1/2})\right| \leq C \left[\int_{x_{i-1}}^{x_{i+1}} \varepsilon |u'''(t)| \mathrm{d}t + h_i \int_{x_{i-1}}^{x_i} (|u'''(t)| + |u''(t)|) \mathrm{d}t \right], \ i = 1,\cdots,N/2,$$
$$\left|L^N(u_i) - (Lu)\left(x_i\right)\right| \leq C h_i \int_{x_{i-1}}^{x_{i+1}} \left(\varepsilon \left|u^{(4)}(t)\right| + \left|u'''(t)\right| \right) \mathrm{d}t, \ i = N/2+1,\cdots,N-1.$$
Proof. The results follow by noting that $c(x)u(x)$ contributes
$$\left|c_{i-1/2}\right| \left| \left(u(x_{i-1}) + u(x_i)\right)/2 - u\left(x_{i-1/2}\right) \right| \leq C h_i \int_{x_{i-1}}^{x_i} \left|u''(t)\right| \mathrm{d}t$$
for $i = 1,2,\cdots,N/2$ to the truncation error in the Lemma 2.4 in [8] and zero for $i = N/2+1,\cdots,N-1$ to the truncation error in Theorem 3.2 in [8], respectively. □

Similarly, the numerical solution can also be split into the smooth part and the layer part by $u_i^N = S_i^N + E_i^N$, where S_i^N satisfies $L^N S_i^N = f_{i-1/2}$, $i = 1,2,\cdots,N/2$, $L^N S_i^N = f_i$, $i = N/2+1,\cdots,N-1$, $S_0^N = S_0$ and $S_N^N = S_N$, and E_i^N satisfies $L^N E_i^N = 0$, $i = 1,2,\cdots,N-1$, $E_0^N = E_0$ and $E_N^N = E_N$, therefore
$$\left|u_i - u_i^N\right| \leq \left|S_i - S_i^N\right| + \left|E_i - E_i^N\right|. \tag{4}$$

Lemma 7 If $N > \dfrac{\gamma^*}{\beta}$ and $\dfrac{N}{\ln N} > \dfrac{4\beta^*}{\beta}$, then for the smooth part of the solutions of (1) and (3) on the Shishkin mesh, there exists a constant C such that

$$\left|S_i - S_i^N\right| \leq CN^{-2} \text{ for all } i.$$

Proof. By Lemma 1 and Lemma 6, we have

$$\left|L^N(S_i - S_i^N)\right| = \begin{cases} \left|L^N(S_i) - (LS)(x_{i-1/2})\right| \leq C(h_i + h_{i+1})(\varepsilon + h_i),\ i = 1, \cdots, N/2, \\ \left|L^N(S_i) - (LS)\left(x_i\right)\right| \leq Ch_i(h_i + h_{i+1})(\varepsilon + 1),\ i = N/2+1, \cdots, N-1. \end{cases}$$

Set $w_i = C_0 N^{-1}\left(\varepsilon + N^{-1}\right)x_i$ for all i, where constant C_0 is chosen sufficiently large. Then

$$L^N w_i = \begin{cases} b_{i-1/2}C_0N^{-1}(\varepsilon + N^{-1}) + c_{i-1/2}\left(w_{i-1} + w_i\right)/2 \geq CN^{-1}(\varepsilon + N^{-1}),\ i = 1, \cdots, N/2, \\ b_iC_0N^{-1}(\varepsilon + N^{-1}) + c_i w_i \geq CN^{-1}(\varepsilon + N^{-1}),\ i = N/2+1, \cdots, N-1. \end{cases}$$

Therefore, $L^N w_i \geq \left|L^N\left(S_i - S_i^N\right)\right|$ for $i = 1, \cdots, N-1$. Clearly, $w_0 = 0 = \left|S_0 - S_0^N\right|$ and $w_N = C_0N^{-1}\left(\varepsilon + N^{-1}\right) \geq 0 = \left|S_N - S_N^N\right|$. By Lemma 3, w_i is a barrier function for $\left|S_i - S_i^N\right|$ and then the proof is completed. $\square$

Lemma 8 If $N > \dfrac{\gamma^*}{\beta}$ and $\dfrac{N}{\ln N} > \dfrac{4\beta^*}{\beta}$, then for the layer part of the solutions of (1) and (3) on the Shishkin mesh, there exists a constant C such that

$$\left|E_i - E_i^N\right| \leq CN^{-2} \text{ for } i = 0, 1, \cdots, N/2.$$

Proof. For $i = 0, 1, \cdots, N/2$, from Lemma 1, we have

$$|E_i| \leq C\mathrm{e}^{-\dfrac{\beta(1-x_i)}{\varepsilon}} \leq C\mathrm{e}^{-\dfrac{\beta(1-x_i)}{2\varepsilon}} \leq C\mathrm{e}^{-\dfrac{\beta(1-x_{N/2})}{2\varepsilon}} = CN^{-2}. \tag{5}$$

Recall the function Z_i in Lemma 4. Now $\mathrm{e}^t \geq 1 + t$ for all $t \geq 0$. So,

$$\frac{Z_i}{Z_N} = \prod_{j=i+1}^{N}\left(1 + \frac{\beta h_j}{2\varepsilon}\right)^{-1} \geq \prod_{j=i+1}^{N} \mathrm{e}^{-\beta h_j/(2\varepsilon)} = \mathrm{e}^{-\beta(1-x_i)/(2\varepsilon)}. \tag{6}$$

Let $Y_i = C_0\dfrac{Z_i}{Z_N}$ for all i, where constant C_0 is chosen sufficiently large. From Lemma 4, we have $L^N Y_i = C_0/Z_N \cdot L^N Z_i \geq 0 = \left|L^N E_i^N\right|$ for $i = 1, \cdots, N-1$. By (6) and Lemma 1, $Y_0 = C_0Z_0/Z_N \geq C_0\mathrm{e}^{-\frac{\beta}{2\varepsilon}} \geq C_0\mathrm{e}^{-\frac{\beta}{\varepsilon}} \geq |E\left(0\right)| = \left|E_0^N\right|$ and $Y_N = C_0 \geq |E\left(1\right)| = \left|E_N^N\right|$. Thus, by Lemma 3, we have

$$\left|E_i^N\right| \leq Y_i = C\prod_{j=i+1}^{N}\left(1 + \frac{\beta h_j}{2\varepsilon}\right)^{-1} \text{ for all } i. \tag{7}$$

By Lemma 5, we have

$$\left|E_i^N\right| \leq C\prod_{j=N/2+1}^{N}\left(1 + \frac{\beta h_j}{2\varepsilon}\right)^{-1} \leq CN^{-2} \text{ for } i = 0, 1, \cdots, N/2.$$

Consequently, combining this inequality with (5), the proof is completed. □

Lemma 9 If $N > \dfrac{\gamma^*}{\beta}$ and $\dfrac{N}{\ln N} > \dfrac{4\beta^*}{\beta}$, then for the layer part of the solutions of (1) and (3) on the Shishkin mesh, there exists a constant C such that

$$\left|E_i - E_i^N\right| \le C \max\left\{N^{-2}, N^{-6+4i/N}\ln^2 N\right\} \text{ for } i = N/2+1, \cdots, N.$$

Proof. By Lemmas 6, 1 and 2, we have

$$\begin{aligned}
\left|L^N\left(E_i - E_i^N\right)\right| &= \left|L^N\left(E_i\right) - (LE)\left(x_i\right)\right| \\
&\le C h_i \int_{x_{i-1}}^{x_{i+1}} \left(\varepsilon\left|E^{(4)}(t)\right| + \left|E'''(t)\right|\right) \mathrm{d}t \\
&\le C h_i \int_{x_{i-1}}^{x_{i+1}} \varepsilon^{-3} \exp\left(-\frac{\beta(1-t)}{\varepsilon}\right) \mathrm{d}t \\
&= C\varepsilon^{-3} h_i \cdot \frac{\varepsilon}{\beta} \sinh\frac{\beta h_i}{\varepsilon} \cdot \mathrm{e}^{-\beta(1-x_i)/\varepsilon} \\
&\le C\varepsilon^{-3} h_i^2 \mathrm{e}^{-\beta(1-x_i)/(2\varepsilon)} \\
&\le C\varepsilon^{-1} N^{-2} \ln^2 N \prod_{j=i+1}^{N}\left(1+\frac{\beta h_j}{2\varepsilon}\right)^{-1}, \text{ since } \mathrm{e}^{-\beta(1-x_i)/(2\varepsilon)} \le \prod_{j=i+1}^{N}\left(1+\frac{\beta h_j}{2\varepsilon}\right)^{-1} \\
&= C\varepsilon^{-1} N^{-2} \ln^2 N \cdot Z_i/Z_N.
\end{aligned}$$

Set $\phi_i = C_0\left\{N^{-2} + N^{-2}\ln^2 N \cdot Z_i/Z_N\right\}$ for $i = N/2, \cdots, N$, where constant C_0 is chosen sufficiently large. By Lemma 4, we have $L^N\phi_i \ge CN^{-2}\ln^2 N/Z_N \cdot L^N Z_i \ge \left|L^N\left(E_i - E_i^N\right)\right|$ for $i = N/2+1, \cdots, N-1$. Clearly, $\phi_{N/2} \ge C_0 N^{-2} \ge \left|E_{N/2} - E_{N/2}^N\right|$ by Lemma 8 and $\phi_N \ge 0 = \left|E_N - E_N^N\right|$. Thus, ϕ_i is a barrier function for $\left|E_i - E_i^N\right|$ by Lemma 3. And by Lemma 5, the result follows. □

Theorem 1 Assuming that $N > \dfrac{\gamma^*}{\beta}$ and $\dfrac{N}{\ln N} > \dfrac{4\beta^*}{\beta}$, the hybrid finite difference scheme (3) on the Shishkin mesh (2) for (1) satisfies:

$$\left|u_i - u_i^N\right| \le C \max\left\{N^{-2}, N^{-6+4i/N}\ln^2 N\right\} \text{ for } i = 1, \cdots, N. \tag{8}$$

Furthermore,

$$\left|u_i - u_i^N\right| \le \begin{cases} CN^{-2}, \ 0 \le i \le p_h N, \\ CN^{-2}\ln^2 N, \ p_h N < i \le N, \end{cases} \tag{9}$$

where $p_h = 1 - \dfrac{1}{2\mathrm{e}} \approx 0.8161$.

Proof. From (4) and Lemmas 7, 8 and 9, we have the error estimate (8).

Furthermore, since $N^{-6+4i/N}\ln^2 N = N^{-2}N^{-4+4i/N}\ln^2 N$, we consider the function

$$f(x) = x^{-4+4p_h}\ln^2 x, \ x > 1.$$

From $f'(x) = x^{-4+4p_h-1} \ln x \left[(-4+4p_h)\ln x + 2\right] = 0$, we have $x = \mathrm{e}^{1/(2-2p_h)}$. So,

$$\max_{x>1}\{f(x)\} = \frac{1}{4(1-p_h)^2\mathrm{e}^2},$$

then

$$N^{-4+4p_h} \ln^2 N \le \frac{1}{4(1-p_h)^2\mathrm{e}^2} = 1.$$

Therefore, (9) is proved. □

Theorem 2 Assuming that $N > \dfrac{\gamma^*}{\beta}$, the midpoint upwind scheme on the Shishkin mesh (2) for (1) satisfies:

$$\left|u_i - u_i^N\right| \le \begin{cases} CN^{-2}, \ 0 \le i \le p_m N, \\ CN^{-1}\ln N, \ p_m N < i \le N, \end{cases} \tag{10}$$

where $p_m = \dfrac{3}{4} - \dfrac{1}{4\mathrm{e}} \approx 0.6580$.

Proof. Under the hypothesis of Theorem 2, the matrix associated with the midpoint upwind scheme is an M-matrix.

In [8], it is shown that $\left|E_i - E_i^N\right| \le C\max\left\{N^{-2}, N^{-5+4i/N}\ln N\right\}$ for $i = N/2, \cdots, N$. Combining this with (3.1) and (3.2) in [8] yields the result:

$$\left|u_i - u_i^N\right| \le C\max\left\{N^{-2}, N^{-5+4i/N}\ln N\right\} \text{ for all } i. \tag{11}$$

Further, as the proof of Theorem 1, Theorem 2 follows. □

Theorem 3 The simple upwind scheme on the Shishkin mesh (2) for (1) satisfies:

$$\left|u_i - u_i^N\right| \le \begin{cases} CN^{-1}, \ 0 \le i \le p_s N, \\ CN^{-1}\ln N, \ p_s N < i \le N, \end{cases} \tag{12}$$

where $p_s = 1 - \dfrac{1}{2\mathrm{e}} \approx 0.8161$.

Proof. The matrix associated with the simple upwind scheme is an M-matrix. From Lemma 2.95 in [3], we know $\left|L^N(E_i - E_i^N)\right| \le C\varepsilon^{-1}N^{-1}\mathrm{e}^{-\beta(1-x_i)/\varepsilon}$.

Set a new $\phi_i = C_0\left\{N^{-1} + N^{-1}\ln N \cdot Z_i/Z_N\right\}$ for $i = N/2, \cdots, N$, where constant C_0 is chosen sufficiently large. It is easy to verify that $\left|E_i - E_i^N\right| \le \phi_i$ for $i = N/2, \cdots, N$, by the discrete comparison principle.

Note that $\tau = \frac{2\varepsilon}{\beta}\ln N$ and $h_i = \frac{4\varepsilon}{\beta}N^{-1}\ln N$ for $i = N/2+1, \cdots, N$. As the proof of Lemma 5, we have $\prod_{j=i+1}^{N}\left(1 + \frac{\beta h_j}{2\varepsilon}\right)^{-1} \le CN^{-2(1-i/N)}$ for $N/2 \le i < N$. Thus $\left|E_i - E_i^N\right| \le C\max\left\{N^{-1}, N^{-3+2i/N}\ln N\right\}$ for $i = N/2, \cdots, N$. Combining this inequality with Lemma 2.86 and Corollary 2.95 in [3], we have

$$\left|u_i - u_i^N\right| \le C\max\left\{N^{-1}, N^{-3+2i/N}\ln N\right\} \text{ for all } i. \tag{13}$$

Consequently, as the proof of Theorem 1, the proof is completed. □

Remark. In this paper, for the hybrid scheme and the midpoint upwind scheme, the condition $N > \frac{\gamma^*}{\beta}$, not in [4] and [8], is added in Theorems 1 and 2 for $c(x) \geq 0$ in (1). Moreover, the constants p_h, p_m and p_s are much larger than $\frac{1}{2}$, and the factor C of new error estimates are uniform to ε and N.

3 Numerical results

Example 1 (see [10]). Consider the singularly perturbed problem

$$\begin{cases} -\varepsilon y'' + \dfrac{1}{x+1} y' + \dfrac{1}{x+2} y = f(x), \ 0 < x < 1, \\ y(0) = 1 + 2^{-\frac{1}{\varepsilon}}, \ y(1) = \mathrm{e} + 2, \end{cases}$$

where $f(x)$ is chosen such that $y(x) = \mathrm{e}^x + 2^{-\frac{1}{\varepsilon}}(x+1)^{1+\frac{1}{\varepsilon}}$ is the exact solution.

The numerical results of Example 1 by the hybrid scheme (3) on (2) are shown in Table 1, where the numerical convergence order is computed by $\log_2 \frac{\max_{0<i\leq p_h N} \left|u_i - u_i^N\right|}{\max_{0<i\leq p_h N} \left|u_i - u_i^{2N}\right|}$, and the numerical convergence constant is computed by $\max_{0<i\leq p_h N} \left|u_i - u_i^N\right| / N^{-2}$, with the corresponding formulas for $p_h N < i < N$, and for the midpoint upwind scheme and the simple upwind scheme.

Table 1. The numerical results of the hybrid scheme (3) on the Shishkin mesh (2)

	$\varepsilon = 10^{-6}$						$\varepsilon = 10^{-10}$					
N	$i \leq p_h N$	order	const	$i > p_h N$	order	const	$i \leq p_h N$	order	const	$i > p_h N$	order	const
16	0.0178	——	4.553	0.1711	——	5.698	0.0178	——	4.553	0.1711	——	5.698
32	0.0031	2.522	3.212	0.0639	1.421	5.446	0.0031	2.522	3.212	0.0639	1.421	5.446
64	5.1389e-4	2.593	2.105	0.0211	1.599	4.988	5.1395e-4	2.593	2.105	0.0211	1.599	4.988
128	8.4857e-5	2.599	1.390	0.0071	1.571	4.922	8.4882e-5	2.599	1.391	0.0071	1.572	4.922
256	1.5179e-5	2.483	0.995	0.0023	1.626	4.874	1.5190e-5	2.482	0.996	0.0023	1.626	4.874
512	3.1035e-6	2.290	0.814	7.2139e-4	1.673	4.859	3.1083e-6	2.289	0.815	7.2139e-4	1.673	4.859
1024	6.8326e-7	2.183	0.717	2.2239e-4	1.698	4.854	6.8554e-7	2.181	0.719	2.2279e-4	1.695	4.862
2048	1.5924e-7	2.101	0.668	6.7248e-5	1.726	4.852	1.6034e-7	2.096	0.673	6.7628e-5	1.720	4.879

The third and ninth columns in Table 1 show second-order convergence and agree with $(9)_1$ on $[0, x_{[p_h N]}]$. The sixth and twelfth columns show almost second-order convergence and agree with $(9)_2$ on $(x_{[p_h N]}, 1]$. Moreover, the columns of orders and constants in Table 1 show that the convergence is uniform to the different perturbation parameters.

Table 2. The numerical results of the midpoint upwind scheme on the Shishkin mesh

	$\varepsilon = 10^{-6}$						$\varepsilon = 10^{-10}$					
N	$i \leq p_m N$	order	const	$i > p_m N$	order	const	$i \leq p_m N$	order	const	$i > p_m N$	order	const
16	0.0058	——	1.476	0.3578	——	2.065	0.0058	——	1.476	0.3578	——	2.065
32	5.6856e-4	3.351	0.582	0.2550	0.489	2.355	5.6849e-4	3.351	0.582	0.2550	0.489	2.355
64	1.5345e-4	1.890	0.629	0.1735	0.556	2.670	1.5350e-4	1.889	0.629	0.1735	0.556	2.670
128	3.8769e-5	1.985	0.635	0.1091	0.670	2.877	3.8792e-5	1.984	0.636	0.1091	0.670	2.878
256	9.6954e-6	2.000	0.635	0.0655	0.736	3.023	9.7056e-6	1.999	0.636	0.0655	0.736	3.023
512	2.4218e-6	2.001	0.635	0.0381	0.782	3.127	2.4265e-6	2.000	0.636	0.0381	0.782	3.127
1024	6.0438e-7	2.003	0.634	0.0216	0.819	3.191	6.0664e-7	2.000	0.636	0.0216	0.819	3.191
2048	1.5056e-7	2.005	0.632	0.0120	0.848	3.225	1.5166e-7	2.000	0.636	0.0120	0.848	3.225

Table 2 shows the uniform convergence of second-order on $[0, x_{[p_m N]}]$ and almost first-order on $(x_{[p_m N]}, 1]$, and agrees with Theorem 2.

Table 3. The numerical results of the simple upwind scheme on the Shishkin mesh

	$\varepsilon = 10^{-6}$						$\varepsilon = 10^{-10}$					
N	$i \le p_s N$	order	const	$i > p_s N$	order	const	$i \le p_s N$	order	const	$i > p_s N$	order	const
16	0.2228	——	3.565	0.2409	——	1.390	0.2228	——	3.565	0.2409	——	1.390
32	0.1123	0.988	3.594	0.1540	0.646	1.422	0.1123	0.988	3.594	0.1540	0.646	1.422
64	0.0516	1.122	3.303	0.0968	0.670	1.490	0.0516	1.122	3.303	0.0968	0.670	1.490
128	0.0226	1.191	2.888	0.0599	0.693	1.581	0.0226	1.191	2.888	0.0599	0.693	1.581
256	0.0097	1.220	2.484	0.0356	0.751	1.644	0.0097	1.220	2.484	0.0356	0.751	1.644
512	0.0043	1.174	2.187	0.0205	0.796	1.686	0.0043	1.174	2.187	0.0205	0.796	1.686
1024	0.0019	1.178	1.942	0.0116	0.822	1.709	0.0019	1.178	1.942	0.0116	0.822	1.709
2048	8.5786e-4	1.147	1.757	0.0064	0.858	1.719	8.5793e-4	1.147	1.757	0.0064	0.858	1.719

Table 3 shows the uniform convergence of first-order on $[0, x_{[p_s N]}]$ and almost first-order on $(x_{[p_s N]}, 1]$, and verifies Theorem 3.

The log2-log2 graphs of errors to illustrate the convergence orders for the hybrid scheme on $[0, 1-\tau]$, $(1-\tau, x_{[p_h N]}]$ and $(x_{[p_h N]}, 1]$ on the Shishkin mesh are shown in Fig. 1 (a), those for the midpoint upwind scheme and the simple upwind scheme are in Figs. 1 (b) and (c).

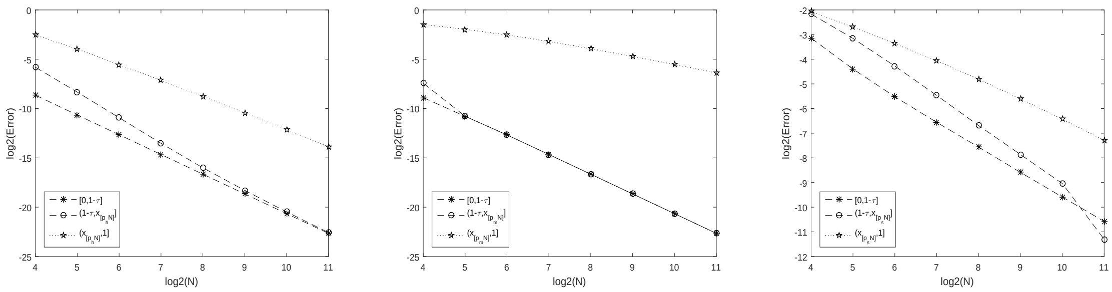


Fig. 1 The log2-log2 graphs of errors on the Shishkin mesh for: (a) the hybrid scheme, (b) the midpoint upwind scheme, (c) the simple upwind scheme.

4 Conclusions

In this paper, the hybrid finite difference scheme is constructed, which is slightly different from the schemes in [4] and [8]. The new estimates on the Shishkin mesh, which are $O(N^{-2})$ for $1 \le i \le p_h N$ and $O(N^{-2} \ln^2 N)$ for $p_h N < i < N$ with $p_h = 1 - \frac{1}{2\mathrm{e}}$ for the hybrid finite difference scheme, $O(N^{-2})$ for $1 \le i \le p_m N$ and $O(N^{-1} \ln N)$ for $p_m N < i < N$ with $p_m = \frac{3}{4} - \frac{1}{4\mathrm{e}}$ for the midpoint upwind scheme, and $O(N^{-1})$ for $1 \le i \le p_s N$ and $O(N^{-1} \ln N)$ for $p_s N < i < N$ with $p_s = 1 - \frac{1}{2\mathrm{e}}$ for the simple upwind scheme, are better than those in [4–6, 8]. The numerical example strongly support our results.

References

[1] R. B. Kellogg, A. Tsan, Analysis of some difference approximations for a singular perturbation problem without turning points, Math. Comput. 32 (1978) 1025-1039.

[2] J. J. H. Miller, E. O'Riordan, G. I. Shishkin, Fitted Numerical Methods for Singular Perturbation Problems, World Scientific, Singapore, 1996.

[3] H.-G. Roos, M. Stynes, L. Tobiska, Robust Numerical Methods for Singularly Perturbed Differential Equations, Springer-Verlag, Berlin, Heidelberg, 2008.

[4] T. Linß. Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems, Lecture Notes in Mathematics, Springer-Verlag, 2010.

[5] J.J.H. Miller, E. O'Riordan, G.I. Shishkin. On piecewise-uniform meshes for upwind-and central-difference operators for solving singularly perturbed problems, IMA J. Numer. Anal. 15 (1995) 89-99.

[6] H.-G. Roos, T. Linß, Sufficient conditions for uniform convergence on layer-adapted grids, Computing. 63 (1999) 27-45.

[7] V. B. Andreev and N. V. Kopteva, Investigation of difference schemes with an approximation of the first derivative by a central difference relation, Comput. Math. Math. Phys., 36 (1996) 1065-1078.

[8] M. Stynes, H.-G. Roos, The midpoint upwind scheme, Appl. Numer. Math. 23 (1997) 361-374.

[9] Q. Zheng, X.-L. Feng, X.-Z. Li, ε-uniform convergence of the midpoint upwind scheme on the Bakhvalov-Shishkin mesh for singularly perturbed problems, J. Comput. Anal. Appl. 17 (2014) 40-47.

[10] M. K. Kadalbajoo, K. C. Patidar, ε-Uniformly convergent fitted mesh finite difference methods for general singular perturbation problems, Appl. Math. Comput. 179 (2006) 248-266.

FEKETE SZEGÖ PROBLEM RELATED TO SIMPLE LOGISTIC ACTIVATION FUNCTION

C. RAMACHANDRAN AND D. KAVITHA

Abstract. In this present paper, we introduce the new subclass of analytic univalent functions associated with quasi-subordination in the field of sigmoid functions. We obtained the coefficient bounds and Fekete-Szego inequality belongs to the defined class. Also, we extracted the new subclasses from the dened class of analytic functions.

Mathematics Subject Classification: Primary:30C45; Secondary:30C50,33E99
Keywords: Univalent functions, Sigmoid function, Subordination, Quasi-subordination, Fekete-Szegö Inequality.

1. Introduction and preliminaries

Sigmoid function playing an important role in the branch of special functions which is the part of logistic activation function developed in eighteenth century. The theory of special functions has been developed by C. F.Gauss, C. G. J. Jacobi, F. Klein and many others in nineteenth century. However, in the twentieth century , from the perspective of fundamental science sigmoid functions are of special interest in abstract areas such as approximation theory, functional analysis, topology, differential equations and probability theory and so on.

A typical applications of the sigmoid function includes neural networks, image processing, artificial networks, biomathematics, chemistry, geoscience, probability theory, economics etc., We can find the similar kind of functions called gompertz function and ogee function which are used in modelling systems to saturate at more values of time period. The evaluation process of sigmoid function in many ways especially by truncated series expansion method was seen in [4, 10].

Recently Ramachandran et al. [13] discssed the problem of Hankel determinant for the subclass of analytic and univalent functions. The sigmoid function is of the form

$$h(z) = \frac{1}{1+e^{-z}} \tag{1.1}$$

is differentiable and has the following properties:

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

- Bound output real numbers between 0 and 1, it leads to the probability theory.
- It maps a very large input domain to a small range of outputs.
- It never loses information because it is an injective function.
- It increases monotonically.

The above properties permit us to use sigmoid function in the univalent function theory.

In computational networks, this sigmoid function leads the output as digital numbers 1 for ON and 0 for OFF. Kannan et al.[6] brought out contrast enhancement using modified sigmoid function provides the highest measure of contrast and can be effectively used for further analysis of sports color images.

Let $\mathcal{A}$ be the class of functions $f(z)$ which are analytic in the open disk $\mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\}$ is of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1.2}$$

and normalized by $f(0) = f'(0) - 1 = 0$ and let $\mathcal{S}$ be a class of all functions in $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$.

If $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write $f(z) \prec g(z)$, $z \in \mathbb{U}$ if there exits a Schwarz function $\omega(z)$, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad and \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that $f(z) = g(\omega(z)), z \in \mathbb{U}$. In particular, if the function g is univalent in $\mathbb{U}$, then We have that

$$f \prec g \quad or \quad f(z) \prec g(z), z \in \mathbb{U}$$

if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$ defined by[11].

In the year 1970, Robertson [15] introduced the concept of quasi-subordination. For, two analytic functions $f(z)$ and $g(z)$, the function $f(z)$ is quasi-subordinate to $g(z)$ in the open unit disc $\mathbb{U}$, written by

$$f(z) \prec_q g(z).$$

If there exist an analytic function φ and ω, with $|\varphi(z)| \leq 1, \omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$f(z) = \varphi(z) g(\omega(z)), \quad (z \in \mathbb{U}).$$

Observe that if $\varphi(z) \equiv 1$, then $f(z) = g(\omega(z))$, so that $f(z) \prec g(z)$ in $\mathbb{U}$. Furthermore, if $\omega(z) = z$, then $f(z) = \varphi(z)g(z)$, said to be that $f(z)$ is majorized by $g(z)$ and symbolically written as $f(z) \ll g(z)$ in $\mathbb{U}$. Hence it is obvious that the quasi-subordination is a generalization of subordination as well as majorization [2, 8, 16].

Haji Mohd and Darus [5] introduced the concepts of q-starlike and q-convex functions as follows:

Definition 1. *Let the class $\mathcal{S}_q^*(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfies the quasi-subordination*

$$\left(\frac{zf'(z)}{f(z)} - 1\right) \prec_q \varphi(z) - 1; \quad (z \in \mathbb{U}). \tag{1.3}$$

Example 1. *A function $f \in \mathbb{U} \to \mathbb{C}$ defined by*

$$\left(\frac{zf'(z)}{f(z)} - 1\right) = z(\varphi(z) - 1) \prec_q \varphi(z) - 1; \quad (z \in \mathbb{U}).$$

belongs to the class $\mathcal{S}_q^*(\varphi)$.

Definition 2. *Let the class $\mathcal{C}_q(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfies the quasi-subordination*

$$\left(\frac{zf''(z)}{f'(z)}\right) \prec_q \varphi(z) - 1; \quad (z \in \mathbb{U}). \tag{1.4}$$

Example 2. *A function $f \in \mathbb{U} \to \mathbb{C}$ defined by*

$$\left(\frac{zf''(z)}{f'(z)}\right) = z(\varphi(z) - 1) \prec_q \varphi(z) - 1; \quad (z \in \mathbb{U}).$$

belongs to the class $\mathcal{C}_q(\varphi)$.

To prove our main results, we need the following lemmas:

Lemma 1. [7] *Let ω be the analytic function in $\mathbb{D}$, with $\omega(0) = 0, |\omega(z)| < 1$ and $\omega(z) = \omega_1 z + \omega_2 z^2 + \dots$, then $|\omega_2 - \nu\omega_1^2| \le max[1; |\nu|]$, where $\nu \in \mathbb{C}$. The result is sharp for the functions $\omega(z) = z^2$ or $\omega(z) = z$.*

Lemma 2. [3] *Let ω be the analytic function in $\mathbb{D}$, with $\omega(0) = 0, |\omega(z)| < 1$ and $\omega(z) = \omega_1 z + \omega_2 z^2 + \dots$, then*

$$|\omega_n| \le \begin{cases} 1, & n = 1 \\ 1 - |\omega_1|^2, & n \ge 2. \end{cases}$$

The result is sharp for the functions $\omega(z) = z^2$ or $\omega(z) = z$.

Lemma 3. [11] *Let φ be an analytic function with positive real part in $\mathbb{D}$, with $|\varphi(z)| < 1$ and let $\varphi(z) = c_0 + c_1 z + c_2 z^2 + \dots$ Then $|c_0| \leq 1$ and $|c_n| \leq 1 - |c_0|^2 \leq 1$, for $n > 0$.*

Lemma 4. [4] *Let h be the sigmoid function defined in (1.1) and*

$$\Phi(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m, \tag{1.5}$$

then $\Phi(z) \in P, |z| < 1$ where $\Phi(z)$ is a modified sigmoid function.

Lemma 5. [4] *Let*

$$\Phi_{n,m}(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m$$

then $|\Phi_{n,m}(z)| < 2$.

Lemma 6. [4] *If $\Phi(z) \in P$ and it is starlike, then f is a normalized univalent function of the form (1.2). Taking m = 1, Joseph et al [4] remarked the following:*

Remark 1. *Let*

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

where $c_n = \dfrac{(-1)^{n+1}}{2n!}$ then $|c_n| \leq 2, n = 1, 2, 3 \dots$ this result is sharp for each n see [4].

Motivated by the earlier works of Ramachandran et al. [14], we define the class of function involving quasi-subordination in terms of sigmoid functions.

Definition 3. *A function $f \in \mathcal{A}$ is in the class $M_q^{\alpha,\lambda,\beta}(\Phi_{n,m})$ if*

$$\left[\frac{zf'(z)}{f(z)} \right]^{\alpha} \left[(1-\lambda) \frac{zf'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) \right]^{\beta} - 1 \prec_q \Phi(z) - 1 \tag{1.6}$$

here $0 < \beta \leq 1, \quad 0 \leq \alpha \leq 1, \quad 0 \leq \lambda \leq 1.$

With various choices of the parameters, the class $M_q^{\alpha,\lambda,\beta}(\Phi_{n,m})$ reduces to the following new classes,

(1) $M_q^{0,0,1}(\Phi_{n,m}) \equiv \mathcal{S}_q^*(\Phi_{n,m})$,
(2) $M_q^{0,1,1}(\Phi_{n,m}) \equiv \mathcal{C}_q(\Phi_{n,m})$,
(3) $M_q^{0,\lambda,1}(\Phi_{n,m}) \equiv M_q^{\lambda}(\Phi_{n,m})$.

In this present paper, we determine the coefficient estimates including a Fekete-Szegö inequality of functions belonging to the above defined class and the class

involving majorization. This result can assist us to represent various geometric interpretation as well as behaviours of the functions in complex domain.

Let $f(z)$ be of the form (1.2), $\varphi(z) = c_0 + c_1 z + c_2 z^2 + \ldots.$ and $\omega(z) = \omega_1 z + \omega_2 z^2 + \ldots$, throughout this paper unless otherwise mentioned.

2. Fekete-Szegö Inequality

In this section we obtain the first two coefficient estimates and the Fekete-Szegö Inequality for the class $M_q^{\alpha,\lambda,\beta}(\Phi_{n,m})$.

Theorem 1. *If $f(z) \in M_q^{\alpha,\lambda,\beta}(\Phi_{n,m})$, then*

$$|a_2| \leq \frac{1}{2[\alpha+\beta(1+\lambda)]},$$

$$|a_3| \leq \frac{1}{4[\alpha+\beta(1+2\lambda)]} max \left\{1, \left|\frac{\alpha(\alpha-3)+\beta(\beta-1)(1+\lambda)^2+2\alpha\beta(1+\lambda)-2\beta(1+3\lambda)}{4[\alpha+\beta(1+\lambda)]^2}\right|\right\},$$

and for any complex number μ, we have

$$|a_3 - \mu a_2^2| \leq \frac{1}{4[\alpha+\beta(1+2\lambda)]} max \left\{1, \left|\Lambda + \frac{\mu[\alpha+\beta(1+2\lambda)]}{[\alpha+\beta(1+\lambda)]^2}\right|\right\},$$

where

$$\Lambda = \frac{\alpha(\alpha-3)+\beta(\beta-1)(1+\lambda)^2+2\alpha\beta(1+\lambda)-2\beta(1+3\lambda)}{4[\alpha+\beta(1+\lambda)]^2}. \tag{2.1}$$

Proof. Since $f \in \mathcal{A}$ belongs to the class $M_q^{\alpha,\lambda,\beta}(\Phi_{n,m})$, then from (1.6) we have

$$\left[\frac{zf'(z)}{f(z)}\right]^\alpha \left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^\beta - 1 = \varphi(z)(\Phi(z)-1), z \in \mathbb{U}. \tag{2.2}$$

The modified sigmoid function $\Phi(z)$ can be expressed as

$$\Phi(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \frac{1}{64}z^6 + \frac{779}{20160}z^7 + \ldots$$

since $\varphi(z)$ as defined earlier, now we obtain

$$\begin{aligned} \varphi(z)(\Phi(\omega(z))-1) &= (c_0 + c_1 z + c_2 z^2 + \ldots.)\left[\frac{\omega_1}{2}z + \frac{\omega_2}{2}z^2 + \left(\frac{\omega_3}{2} - \frac{\omega_1^3}{24}\right)z^3 + \ldots\right] \\ &= \frac{c_0\omega_1}{2}z + \left(\frac{c_0\omega_2}{2} + \frac{c_1\omega_1}{2}\right)z^2 + \ldots \end{aligned} \tag{2.3}$$

by replacing the equivalent expressions of $f(z)$, $\frac{f'(z)}{f(z)}$ and $\frac{zf''(z)}{f'(z)}$ in (2.2) and the simple calculation yields the following,

$$\left[\frac{zf'(z)}{f(z)}\right]^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\beta}-1=[\alpha+\beta(1+\lambda)]a_2z$$
$$+\left\{2[\alpha+\beta(1+2\lambda)]a_3+\left[\frac{\alpha(\alpha-3)}{2}+\frac{\beta(\beta-1)}{2}(1+\lambda)^2+\alpha\beta(1+\lambda)-\beta(1+3\lambda)\right]a_2^2\right\}z^2+\dots \tag{2.4}$$

Equating right hand side part of (2.3) and (2.4), we get,

$$a_2=\frac{c_0\omega_1}{2[\alpha+\beta(1+\lambda)]}, \tag{2.5}$$

and

$$\begin{aligned} a_3=&\frac{1}{4[\alpha+\beta(1+2\lambda)]}\left\{c_1\omega_1+c_0\left[\omega_2\right.\right.\\ &\left.\left.-\left(\frac{\alpha(\alpha-3)+\beta(\beta-1)(1+\lambda)^2+2\alpha\beta(1+\lambda)-2\beta(1+3\lambda)}{4[\alpha+\beta(1+\lambda)]^2}\right)\omega_1^2c_0\right]\right\}. \end{aligned} \tag{2.6}$$

Using the hypothesis of Lemma 3 and the well-known inequality of Lemma 2, for $n>0$

$$|c_n|\le 1-|c_0|^2\le 1.$$

and

$$|\omega_1|\le 1$$

we have,

$$|a_2|\le\frac{1}{2[\alpha+\beta(1+\lambda)]},$$

and for any $\mu\in\mathbb{C}$, we obtain from (2.5) and (2.6)

$$a_3-\mu a_2^2=\frac{1}{4[\alpha+\beta(1+2\lambda)]}\left\{c_1\omega_1+c_0\left[\omega_2-\left(\Lambda+\frac{\mu[\alpha+\beta(1+2\lambda)]}{[\alpha+\beta(1+\lambda)]^2}\right)\omega_1^2c_0\right]\right\}.$$

Since $\varphi(z)$ is analytic and bounded in $\mathbb{U}$, using [11], for some $y, |y|\le 1$:

$$|c_0|\le 1 \qquad \textit{and} \qquad c_1=(1-c_0^2)y.$$

Now, replacing the value of c_1 as defined above, we get

$$a_3-\mu a_2^2=\frac{1}{4[\alpha+\beta(1+2\lambda)]}\left\{y\omega_1+c_0\omega_2-\left[\left(\Lambda+\frac{\mu[\alpha+\beta(1+2\lambda)]}{[\alpha+\beta(1+\lambda)]^2}\right)\omega_1^2+y\omega_1\right]c_0^2\right\}. \tag{2.7}$$

If $c_0 = 0$ then
$$|a_3 - \mu a_2^2| \leq \frac{|\omega_1||y|}{4[\alpha + \beta(1+2\lambda)]} = \frac{1}{4[\alpha + \beta(1+2\lambda)]}.$$
If $c_0 \neq 0$ then
$$|a_3 - \mu a_2^2| \leq \frac{1}{4[\alpha + \beta(1+2\lambda)]} max\left\{1, \left|\Lambda + \frac{\mu[\alpha + \beta(1+2\lambda)]}{[\alpha + \beta(1+\lambda)]^2}\right|\right\} \tag{2.8}$$
and the result is sharp. □

Further setting $\mu = 0$ in (2.8) we get the bound on $|a_3|$. This completes the proof of the Theorem 1.

Corollary 1. *Let $\alpha = 0$, $\lambda = 0$ and $\beta = 1$ the class $M_q^{\alpha,\lambda,\beta}(\Phi_{n,m})$ reduced to $\mathcal{S}_q^*(\Phi_{n,m})$ then we have,*
$$a_2 = \frac{c_0\omega_1}{2},$$
and
$$|a_3 - \mu a_2^2| \leq \frac{1}{4} max\left\{1, \left|\frac{2\mu - 1}{2}\right|\right\}.$$

Corollary 2. *Let $\alpha = 0$, $\lambda = 1$ and $\beta = 1$ the class $M_q^{\alpha,\lambda,\beta}(\Phi_{n,m})$ reduced to $\mathcal{C}_q(\Phi_{n,m})$ then we have,*
$$a_2 = \frac{c_0\omega_1}{4},$$
and
$$|a_3 - \mu a_2^2| \leq \frac{1}{12} max\left\{1, \left|\frac{3\mu - 2}{4}\right|\right\}.$$

Corollary 3. *Let $\alpha = 0$ and $\beta = 1$ the class $M_q^{\alpha,\lambda,\beta}(\Phi_{n,m})$ reduced to $M_q^{\lambda}(\Phi_{n,m})$ then we have,*
$$a_2 = \frac{c_0\omega_1}{2(1+\lambda)},$$
and
$$|a_3 - \mu a_2^2| \leq \frac{1}{4(1+2\lambda)} max\left\{1, \left|\frac{2\mu(1+2\lambda) - (1+3\lambda)}{2(1+\lambda)^2}\right|\right\}.$$

Theorem 2. *If $f \in \mathcal{A}$, such that the function*
$$\left[\frac{zf'(z)}{f(z)}\right]^{\alpha}\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\beta} - 1 \ll \Phi(z) - 1, \quad z \in \mathbb{U}$$
then,
$$|a_2| \leq \frac{1}{2|\alpha + \beta(1+\lambda)|},$$

and for any complex number μ

$$|a_3 - \mu a_2^2| \leq \frac{1}{4[\alpha+\beta(1+2\lambda)]} max \left\{1, \left|\Lambda + \frac{\mu[\alpha+\beta(1+2\lambda)]}{[\alpha+\beta(1+\lambda)]^2}\right|\right\}.$$

Proof. Taking $\omega(z) = z$ in the proof of Theorem 1, we get the desired result. □

3. Conclusion

Finding the estimates for various subclasses of analytic functions with normalization is the most important role of geometric function theory. These estimates characterise the behaviours of functions in complex domain. This characterisation provides a tool using the sigmoid function in wide range of fields like image processing, digital communications, neural sciences etc.,

References

[1] R. M. Ali, V. Ravichandran & N. Seenivasagan, Coefficient bounds for p-valent functions, Appl. Math. Comput. vol. 187, 2007, no. 1, pp. 35–46.

[2] O. Altintas & S. Owa, Majorizations and Quasi-subordinations for certain analytic functions, Proceedings of the Japan AcademyA, vol.68, 1992, no.7, pp. 181–185.

[3] P.L. Duren, Univalent Functions, Grundlehren der Mathematischen Wissenschaften, 259, Spinger, New York, 1983.

[4] O. A. Fadipe-Joseph, A. T. Oladipo & U. A. Ezeafulukwe, Modified sigmoid function in univalent function theory, International Journal of Mathematical Sciences and Engineering Application, vol.7, 2013, No. 7 ,pp. 313–317.

[5] M. Haji Mohd & M. Darus, Fekete-Szego problems for quasi-subordination classes, Abstr. Appl. Anal., 2012, Art. ID 192956, pp. 14

[6] P. Kannan, S. Deepa, & R. Ramakrishnan, Contrast Enhancement of Sports Images Using Two Comparative Approaches, Amer. Jour. Int. Sys., vol.2, 2012, no. 6, pp. 141–147.

[7] F. R. Keogh & E. P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc., vol.20, 1969, pp. 8–12.

[8] S. Y. Lee, Quasi-subordinate functions and coefficient conjectures, Journal of Korean Mathematical Socity, vol.12, no. 1, Art. ID 19750, pp. 43-50.

[9] W. C. Ma & D. Minda, A Unified treatment of some special classes of univalent functions, in: *Proceedings of the Conference on Complex Analysis*, Z. Li, F. Ren, L. Yang, & S. Zhang(Eds.), Int. Press, 1994, pp. 157-169.

[10] G. Murugusundaramoorthy, T. Janani, Sigmoid function in the space of univalent λ-pseudo starlike functions, IJPAM, Vol.101, 2015, no.1, pp. 33–41.

[11] Z. Nehari, Conformal mappings, McGraw-Hill, New York, 1952.

[12] C. Ramachandran & R. Ambrose Prabhu, Applications of geometric function theory related to mechanical systems, Rev. Tec. Ing. Univ. Zulia, Vol.39,2016, no.1, pp. 177–184.

[13] C. Ramachandran & K. Dhanalakshmi, The Fekete Szegö poblem for a subclass of analytic functions related to sigmoid function,IJPAM, vol.113, 2017, no.3, pp. 389–398.

[14] C. Ramachandran, S. Sivasubramanyan, H.M. Srivastava & A. Swaminathan, Coefficient inequalities for certain subclasses of ananlytic functions and their applications involving the Owa-srivastsva operator of fractional calculus, Math.Ineq. and Appl.

[15] M. S. Robertson, Quasi-subordination and Coefficient conjectures, Bull. Amer. S-Bolyai Math. Soc. vol.76, 1970, pp. 1–9.
[16] F. Y. Ren, S. Owa & S. Fukui, Some inequalities on quasi-subordinate functions, Bulletin of Australian Mathematical Socity, vol. 43, 1991, no.2, pp. 317–324.
[17] S. Owa, Properties of certain integral operators, Southeast Asian Bulletin of Mathematics, Vol. 24, 2000, no.3, pp. 411-419.

Department of Mathematics, University College of Engineering Villupuram,Kakuppam, Villupuram 605103, Anna University, Tamilnadu, India
E-mail address: crjsp2004@yahoo.com

Department of Mathematics, IFET College of Engineering, Villupuram 605108, Tamilnadu, India
E-mail address: soundarkavitha@gmail.com

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

On the second kind twisted q-Euler numbers and polynomials of higher order

Cheon Seoung Ryoo

Department of Mathematics, Hannam University, Daejeon 306-791, Korea

Abstract : In this paper, we introduce the second kind twisted q-Euler polynomials $E^{(k)}_{n,\omega,q}(x)$ of order k. We also get interesting properties related to the second kind twisted q-Euler numbers and polynomials. Finally, we construct twisted q-zeta function of order which interpolates the second kind twisted q-Euler numbers of higher order at negative integer.

Key words : Euler numbers, Euler polynomials, the second kind Euler numbers and polynomials, q-zeta function, twisted q-Euler numbers and polynomials, twisted q-Euler numbers and polynomials of higher order, twisted q-zeta function.

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Recently, mathematicians have studied Euler numbers, Euler polynomials, the second kind Euler numbers and the second kind Euler polynomials (see [1-9]). These numbers and polynomials possess many interesting properties and arising in many areas of mathematics, applied mathematics, and physics. In this paper, we introduce the second kind twisted q-Euler numbers $E^{(k)}_{n,\omega,q}$ and polynomials $E^{(k)}_{n,\omega,q}(x)$ of higher order. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of rational numbers, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic invariant integral on $\mathbb{Z}_p$ of the function $g \in UD(\mathbb{Z}_p)$ is defined by

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x, \text{ see [1, 3]}. \tag{1.1}$$

From (1.1), we note that

$$\int_{\mathbb{Z}_p} g(x+1)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = 2g(0). \tag{1.2}$$

First, we introduce the second kind q-Euler numbers $E^{(k)}_{n,q}$ of higher order k. The second kind q-Euler numbers $E^{(k)}_{n,q}$ of higher order k are defined by the generating function:

$$\left(\frac{2e^t}{qe^{2t}+1}\right)^k = \sum_{n=0}^{\infty} E^{(k)}_{n,q}\frac{t^n}{n!}, \quad (|\log q + 2t| < \pi). \tag{1.3}$$

Let $T_p = \cup_{N\geq 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \{\omega | \omega^{p^N} = 1\}$ is the cyclic group of order p^N. For $\omega \in T_p$, we denote by $\phi_\omega : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \omega^x$. We introduce the second kind twisted q-Euler polynomials $E_{n,\omega,q}(x)$ as follows:

$$\frac{2e^t}{\omega qe^{2t}+1}e^{xt} = \sum_{n=0}^{\infty} E_{n,w,q}(x)\frac{t^n}{n!}. \tag{1.4}$$

In [5], we obtain the second kind twisted q-Euler numbers $E_{n,\omega,q}$ polynomials $E_{n,\omega,q}(x)$ and investigate their properties.

Theorem 1. For positive integers $n, \omega \in T_p$, we have

$$\int_{\mathbb{Z}_p} \phi_\omega(x) q^x (2x+1)^n d\mu_{-1}(x) = E_{n,\omega,q},$$

$$\int_{\mathbb{Z}_p} \phi_\omega(y) q^y (x+2y+1)^n d\mu_{-1}(y) = E_{n,\omega,q}(x).$$

2. The second kind twisted q-Euler polynomials of higher order

The main purpose of this section is to study a systemic properties of the second kind twisted q-Euler numbers and polynomials of higher order. In this section, we assume that $q \in \mathbb{C}_p$. We construct the second kind twisted q-Euler numbers $E^{(k)}_{n,\omega,q}$ and polynomials $E^{(k)}_{n,\omega,q}(x)$ of higher order k. We use the notation

$$\sum_{k_1=0}^{m} \cdots \sum_{k_n=0}^{m} = \sum_{k_1 \cdots k_n=0}^{m}.$$

The binomial formulae are known as

$$(1-a)^n = \sum_{i=0}^{n} \binom{n}{i} (-a)^i, \text{ where } \binom{n}{i} = \frac{n(n-1)\dots(n-i+1)}{i!},$$

and

$$\frac{1}{(1-a)^n} = (1-a)^{-n} \sum_{i=0}^{n} \binom{-n}{i} (-a)^i = \sum_{i=0}^{n} \binom{n+i-1}{i} a^i$$

Now, using multiple of p-adic q-integral, we introduce the second kind twisted q-Euler polynomials $E^{(k)}_{n,w,q}(x)$ of higher order : For $k \in \mathbb{N}$, we define

$$\begin{aligned} &\sum_{n=0}^{\infty} E^{(k)}_{n,\omega,q}(x) \frac{t^n}{n!} \\ &\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k \text{ times}} \omega^{x_1+\cdots+x_k} q^{x_1+x_2+\cdots+x_k} e^{(x+2x_1+2x_2+\cdots+2x_k+k)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k). \end{aligned} \tag{2.1}$$

By using Taylor series of $e^{(x+2x_1+2x_2+\cdots+2x_k+k)t}$ in the above equation, we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \omega^{x_1+\cdots+x_k} q^{x_1+\cdots+x_k} (x+2x_1+\cdots+2x_k+k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} E^{(k)}_{n,w,q}(x) \frac{t^n}{n!}. \end{aligned}$$

By comparing coefficients $\frac{t^n}{n!}$ on the above equation, we arrive at the following theorem.

Theorem 2. For positive integers n and k, we have

$$E^{(k)}_{n,\omega,q}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \omega^{x_1+\cdots+x_k} q^{x_1+\cdots+x_k} (x+2x_1+\cdots+2x_k+k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k). \tag{2.2}$$

By (2.1), the second kind twisted q-Euler polynomials of higher order, $E^{(k)}_{n,\omega,q}(x)$ are defined by means of the following generating function

$$F^{(k)}_{\omega,q}(x,t) = \left(\frac{2e^t}{\omega q e^{2t}+1} \right)^k e^{xt} = \sum_{n=0}^{\infty} E^{(k)}_{n,\omega,q}(x) \frac{t^n}{n!}. \tag{2.3}$$

Again, by using (2,1), the second kind twisted q-Euler numbers of higher order, $E^{(k)}_{n,\omega,q}$ are defined by the following generating function

$$\left(\frac{2e^t}{\omega q e^{2t}+1}\right)^k = \sum_{n=0}^{\infty} E^{(k)}_{n,\omega,q}\frac{t^n}{n!}, \quad |t+\log q| < \frac{\pi}{2}. \tag{2.4}$$

When $k=1$, above (2.3) and (2.4) will become the corresponding definitions of the second kind twisted q-Euler polynomials $E_{n,\omega,q}(x)$ and the second kind twisted q-Euler numbers $E_{n,\omega,q}$. Observe that for $x=0$, the equation (2.4) reduces to (2.3). Note that when $k=1$, then we have (1.4), when $q\to 1$, then we have

$$\left(\frac{2e^t}{\omega e^{2t}+1}\right)^k e^{xt} = \sum_{n=0}^{\infty} E^{(k)}_{n,\omega}(x)\frac{t^n}{n!},$$

where $E^{(k)}_{n,\omega}(x)$ denote the second kind twisted Euler polynomials of higher order k. In the case when $x=0$ in (2.1), we have the following corollary.

Corollary 3. For positive integers n,k, we have

$$E^{(k)}_{n,\omega,q} = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\omega^{x_1+\cdots+x_k}q^{x_1+\cdots+x_k}(2x_1+\cdots+2x_k+k)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k).$$

By using binomial expansion in (2.2), we obtain

$$E^{(k)}_{n,\omega,q}(x) = \sum_{l=0}^{n}\binom{n}{l}x^{n-l}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\omega^{x_1+\cdots+x_k}q^{x_1+\cdots+x_k}(2x_1+\cdots+2x_k+k)^l d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k).$$

Again, by Corollary 3, we arrive at the following theorem.

Theorem 4. For positive integers n,k, we have

$$E^{(k)}_{n,\omega,q}(x) = \sum_{l=0}^{n}\binom{n}{l}E^{(k)}_{l,\omega,q}x^{n-l}.$$

We define distribution relation of the second kind twisted q-Euler polynomials of higher order as follows: For $m\in\mathbb{N}$ with $m\equiv 1(\bmod 2)$, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}E^{(k)}_{n,\omega,q}(x)\frac{t^n}{n!}\\
&=\left(\frac{2e^t}{\omega q e^{2t}+1}\right)\left(\frac{2e^t}{\omega q e^{2t}+1}\right)\cdots\left(\frac{2e^t}{\omega q e^{2t}+1}\right)e^{xt}\\
&=\left(\frac{2e^{mt}}{\omega^m q^m e^{2mt}+1}\right)^k\sum_{a_1,\cdots,a_k=0}^{m-1}\omega^{a_1+\cdots+a_k}(-q)^{a_1+\cdots+a_k}e^{\left(\frac{2a_1+\cdots+2a_k+k+x-mk}{m}\right)(mt)}.
\end{aligned}$$

From the above, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}E^{(k)}_{n,\omega,q}(x)\frac{t^n}{n!}\\
&=\sum_{a_1,\cdots,a_k=0}^{m-1}\omega^{a_1+\cdots+a_k}(-q)^{a_1+\cdots+a_k}\sum_{n=0}^{\infty}E^{(k)}_{n,\omega^m,q^m}\left(\frac{2a_1+\cdots+2a_k+k+x-mk}{m}\right)\frac{(mt)^n}{n!}.
\end{aligned}$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.

Theorem 5 (Distribution relation of the second kind twisted q-Euler polynomials of higher order). For $m \in \mathbb{N}$ with $m \equiv 1(\mod 2)$, we have

$$E_{n,\omega,q}^{(k)}(x) = m^n \sum_{a_1,\cdots,a_k=0}^{m-1} \omega^{a_1+\cdots+a_k}(-q)^{a_1+\cdots+a_k} E_{n,\omega^m,q^m}^{(k)}\left(\frac{2a_1+\cdots+2a_k+k+x-mk}{m}\right).$$

By (2.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,\omega,q}^{(k)}(x)\frac{t^n}{n!} &= 2^k \sum_{a_1,\cdots,a_k=0}^{\infty} \omega^{a_1+\cdots+a_k}(-q)^{a_1+\cdots+a_k} e^{(2a_1+\cdots+2a_k+k+x)t} \\ &= 2^k \sum_{m=0}^{\infty}\binom{m+k-1}{m}(-1)^m\omega^m q^m e^{(2m+k+x)t}.\end{aligned} \tag{2.5}$$

From the above, we obtain

$$\sum_{n=0}^{\infty} E_{n,\omega,q}^{(k)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(2^k \sum_{a_1,\cdots,a_k=0}^{\infty} \omega^{a_1+\cdots+a_k}(-q)^{a_1+\cdots+a_k}(x+2a_1+\cdots+2a_k+k)^n\right)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(2^k\sum_{m=0}^{\infty}\binom{m+k-1}{m}(-1)^m\omega^m q^m(2m+k+x)^n\right)\frac{t^n}{n!}.$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.

Theorem 6. For positive integers n and k, we have

$$\begin{aligned}E_{n,\omega,q}^{(k)}(x) &= 2^k \sum_{a_1,\cdots,a_k=0}^{\infty} \omega^{a_1+\cdots+a_k}(-q)^{a_1+\cdots+a_k}(2a_1+\cdots+2a_k+k+x)^n \\ &= 2^k\sum_{m=0}^{\infty}\binom{m+k-1}{m}(-1)^m\omega^m q^m(2m+k+x)^n.\end{aligned} \tag{2.6}$$

Since

$$\begin{aligned}\sum_{l=0}^{\infty} E_{l,\omega,q}^{(k)}(x+y)\frac{t^l}{l!} &= \left(\frac{2e^t}{\omega q e^{2t}+1}\right)^k e^{(x+y)t} \\ &= \sum_{n=0}^{\infty} E_{n,\omega,q}^{(k)}(x)\frac{t^n}{n!}\sum_{m=0}^{\infty} y^m\frac{t^m}{m!} \\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} E_{n,\omega,q}^{(k)}(x)\frac{t^n}{n!}y^{l-n}\frac{t^{l-n}}{(l-n)!}\right) \\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n}E_{n,\omega,q}^{(k)}(x)y^{l-n}\right)\frac{t^l}{l!},\end{aligned}$$

we have the following addition theorem.

Theorem 7. The second kind twisted q-Euler polynomials $E_{n,\omega,q}^{(k)}(x)$ of higher order satisfies the following relation:

$$E_{n,\omega,q}^{(k)}(x+y) = \sum_{l=0}^{n}\binom{n}{l}E_{l,\omega,q}^{(k)}(x)y^{n-l}.$$

3. Multiple twisted q-Euler zeta function

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. Let ω be the p^N-th root of unity. We define multiple twisted q-Euler zeta function. This function interpolates the second kind twisted q-Euler polynomials of higher order at negative integers.

By using (2.5), we have

$$F^{(k)}_{\omega,q}(x,t) = 2^k \sum_{a_1,\cdots,a_k=0}^{\infty} \omega^{a_1+\cdots+a_k}(-q)^{a_1+\cdots+a_k} e^{(2a_1+\cdots+2a_k+k+x)t} = \sum_{n=0}^{\infty} E^{(k)}_{n,\omega,q}(x)\frac{t^n}{n!}.$$

For $s, x \in \mathbb{C}$ with $\mathcal{R}(x) > 0$, we can derive the following Eq. (3.1) form the Mellin transformation of $F^{(k)}_{\omega,q}(x,t)$.

$$\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}F^{(k)}_{\omega,q}(x,-t)dt = 2^k \sum_{a_1,\cdots,a_k=0}^{\infty} \frac{(-1)^{a_1+\cdots+a_k}\omega^{a_1+\cdots+a_k}q^{a_1+\cdots+a_k}}{(2a_1+\cdots+2a_k+k+x)^s} \tag{3.1}$$

For $s, x \in \mathbb{C}$ with $\mathcal{R}(x) > 0$, we define the multiple twisted q-Euler zeta function as follows:

Definition 8. For $s, x \in \mathbb{C}$ with $\mathcal{R}(x) > 0$, we define

$$\zeta^{(k)}_{\omega,q}(s,x) = 2^k \sum_{a_1,\cdots,a_k=0}^{\infty} \frac{(-1)^{a_1+\cdots+a_k}\omega^{a_1+\cdots+a_k}q^{a_1+\cdots+a_k}}{(2a_1+\cdots+2a_k+k+x)^s}. \tag{3.2}$$

For $s = -l$ in (3.2) and using (2.6), we arrive at the following theorem.

Theorem 9. For positive integer l, we have

$$\zeta^{(k)}_{\omega,q}(-l,x) = E^{(k)}_{l,\omega,q}(x).$$

By (2.4), we have

$$\sum_{n=0}^{\infty} E^{(k)}_{n,\omega,q}\frac{t^n}{n!} = \left(\frac{2e^t}{\omega q e^{2t}+1}\right)^k = 2^k \sum_{m=0}^{\infty}\binom{m+k-1}{m}(-1)^m\omega^m q^m e^{(2m+k)t}.$$

By using Taylor series of $e^{(2m+k)t}$ in the above, we have

$$\sum_{n=0}^{\infty} E^{(k)}_{n,\omega,q}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(2^k\sum_{m=0}^{\infty}\binom{m+k-1}{m}(-1)^m\omega^m q^m(2m+k)^n\right)\frac{t^n}{n!}.$$

By comparing coefficients $\frac{t^n}{n!}$ in the above equation, we have

$$E^{(k)}_{n,\omega,q} = 2^k\sum_{m=0}^{\infty}\binom{m+k-1}{m}(-1)^m\omega^m q^m(2m+k)^n. \tag{3.3}$$

By using (3.3), we define twisted q-Euler zeta function as follows:

Definition 10. For $s \in \mathbb{C}$, we define

$$\zeta^{(k)}_{\omega,q}(s) = 2^k\sum_{m=0}^{\infty}\binom{m+k-1}{m}\frac{(-1)^m\omega^m q^m}{(2m+k)^s}. \tag{3.4}$$

The function $\zeta^{(k)}_{\omega,q}(s)$ interpolates the number $E^{(k)}_{n,\omega,q}$ at negative integers. Substituting $s = -n$ with $n \in \mathbb{Z}_+$ into (3.4), and using (3.3), we obtain the following theorem:

Theorem 11. Let $n \in \mathbb{Z}_+$, We have

$$\zeta^{(k)}_{\omega,q}(-n) = E^{(k)}_{n,\omega,q}.$$

Further, by (3.2) and (3.4), we have

$$\sum_{a_1,\cdots,a_k=0}^{\infty} \frac{(-1)^{a_1+\cdots+a_k}\omega^{a_1+\cdots+a_k}q^{a_1+\cdots+a_k}}{(2a_1+\cdots+2a_k+k)^s} = \sum_{m=0}^{\infty}\binom{m+k-1}{m}\frac{(-1)^m\omega^m q^m}{(2m+k)^s}.$$

Acknowledgment

This paper has been supported by the 2019 Hannam University Research Fund.

REFERENCES

1. Kim, T.(2008). *Euler numbers and polynomials associated with zeta function, Abstract and Applied Analysis*, Art. ID 581582.

2. Liu, G.(2006). *Congruences for higher-order Euler numbers*, *Proc. Japan Acad.* , v. 82 A, pp. 30-33.

3. Ryoo, C.S., Kim, T., Jang, L.C.(2007). *Some relationships between the analogs of Euler numbers and polynomials*, *Journal of Inequalities and Applications*, v.2007, ID 86052, pp. 1-22.

4. Ryoo, C.S.(2010). *Calculating zeros of the second kind Euler polynomials*, *Journal of Computational Analysis and Applications*, v.12, pp. 828-833.

5. Ryoo, C.S.(2013). *Exploring the zeros of the second kind twisted q-Euler polynomials*, *Far East Journal of Mathematical Sciences*, v.77, pp. 195-203.

6. Ryoo, C.S.(2014). *Note on the second kind Barnes' type multiple q-Euler polynomials*, *Journal of Computational Analysis and Applications*, v.16, pp. 246-250.

7. Ryoo, C.S.(2015). *On the second kind Barnes-type multiple twisted zeta function and twisted Euler polynomials*, *Journal of Computational Analysis and Applications*, v.18, pp. 423-429.

8. Ryoo, C.S.(2016). *Differential equations associated with generalized Bell polynomials and their zeros*, *Open Mathematics*, v.14, pp. 807-815.

9. Ryoo, C.S.(2016) *Differential equations associated with tangent numbers, J. Appl. Math. & Informatics*, v.34, pp. 487-494.

HERMITE-HADAMARD INEQUALITY AND GREEN'S FUNCTION WITH APPLICATIONS*

YING-QING SONG[1], YU-MING CHU[2,**], MUHAMMAD ADIL KHAN[3], AND ARSHAD IQBAL[3]

ABSTRACT. In the article, we derive the Hermite-Hadamard inequality by using Green's function, establish some Hermite-Hadamrd type inequalities for the class of monotonic as well as convex functions, and give applications for means, mid-point and trapezoid formulae.

1. INTRODUCTION

Convexity plays an important role in different fields of pure and applied sciences such as statistics, optimization theory, economics and finance etc. The fundamental justification for the significance of convexity is its meaningful relationship with the theory of inequalities. Many useful inequalities have been obtained by using convexity. Among those inequalities, the most extensively and intensively attractive inequality in the last decades is the well known Hermite-Hadamard inequality [1-9], which can be stated as follows: the double inequality

$$\psi\left(\frac{\alpha_1+\alpha_2}{2}\right) \leq \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx \leq \frac{\psi(\alpha_1)+\psi(\alpha_2)}{2} \tag{1.1}$$

holds if the function $\psi : [\alpha_1, \alpha_2] \to \mathbb{R}$ is a convex function. If ψ is a concave function then (1.1) holds in the reverse direction.

The Hermite-Hadamard inequality gives an upper as well as lower estimations for the integral mean of any convex function defined on closed and bounded interval which involves the the endpoints and midpoint of the domain of the function. Also inequality (1.1) provides the necessary and sufficient condition for the function to be convex. There are several applications of the Hermite-Hadamard inequality in the geometry of Banach spaces [10] and nonlinear analysis [11]. Some peculiar convex functions can be used in (1.1) to obtain classical inequalities for means. For some comprehensive surveys on various generalizations and developments of inequality (1.1) we recommend [12]. Due to the great importance of the convexity and the Hermite-Hadamard inequlity, in the recent years many generalizations, refinements and extensions can be found in the literature [13-37]

In the article, we give a new proof for the Hermite-Hadamard inequality by using Green's function, obtain some refinements of the Hermite-Hadamard inequality

2010 *Mathematics Subject Classification.* Primary: 26D15; Secondary: 26A51, 26E60.

Key words and phrases. Hermite-Hadamard inequality, Green's function, convexity.

*The research was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11301127, 11701176, 11626101, 11601485) and the Science and Technology Research Program of Zhejiang Educational Committee (Grant No. Y201635325).

**Corresponding author: Yu-Ming Chu, Email: chuyuming2005@126.com.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

for monotonic functions as well as convex functions. At the end, we give some applications for means, mid-point and trapezoid formulae.

2. Main Results

In order to obtain our main results we need to establish a lemma, which we present in this section.

Lemma 2.1. *Let $\mathcal{G}$ be the Green's function defined on $[\alpha_1, \alpha_2] \times [\alpha_1, \alpha_2]$ by*

$$\mathcal{G}(\lambda, \mu) = \begin{cases} \alpha_1 - \mu \ , & \alpha_1 \le \mu \le \lambda; \\ \alpha_1 - \lambda, & \lambda \le \mu \le \alpha_2. \end{cases} \tag{2.1}$$

Then any $\psi \in C^2([\alpha_1, \alpha_2])$ can be expressed as

$$\psi(x) = \psi(\alpha_1) + (x - \alpha_1)\psi'(\alpha_2) + \int_{\alpha_1}^{\alpha_2} \mathcal{G}(x, \mu)\psi''(\mu)d\mu. \tag{2.2}$$

Proof. By using the techniques of integration by parts in $\int_a^b \mathcal{G}(t, \mu)\psi''(\mu)d\mu$, we can easily obtain (2.2). □

The following Theorem 2.2 give a new proof for the Hermite-Hadamard inequality.

Theorem 2.2. *Let $\psi \in C^2([\alpha_1, \alpha_2])$. Then the double inequality*

$$\psi\left(\frac{\alpha_1 + \alpha_2}{2}\right) \le \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \psi(x)dx \le \frac{\psi(\alpha_1) + \psi(\alpha_2)}{2} \tag{2.3}$$

holds if ψ is convex on $[\alpha_1, \alpha_2]$.

Proof. Let $x = (\alpha_1 + \alpha_2)/2$. Then (2.2) leads to

$$\psi\left(\frac{\alpha_1 + \alpha_2}{2}\right) = \psi(\alpha_1) + \left(\frac{\alpha_1 + \alpha_2}{2} - \alpha_1\right)\psi'(\alpha_2) + \int_{\alpha_1}^{\alpha_2} \mathcal{G}\left(\frac{\alpha_1 + \alpha_2}{2}, \mu\right)\psi''(\mu)d\mu,$$

$$\psi\left(\frac{\alpha_1 + \alpha_2}{2}\right) = \psi(\alpha_1) + \left(\frac{\alpha_2 - \alpha_1}{2}\right)\psi'(\alpha_2) + \int_{\alpha_1}^{\alpha_2} \mathcal{G}\left(\frac{\alpha_1 + \alpha_2}{2}, \mu\right)\psi''(\mu)d\mu. \tag{2.4}$$

Taking integral of (2.2) with respect to x and dividing by $\alpha_2 - \alpha_1$, we get

$$\frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2} \psi(x)dx = \psi(\alpha_1) + \frac{1}{\alpha_2 - \alpha_1}\left(\frac{\alpha_2^2 - \alpha_1^2}{2} - \alpha_1(\alpha_2 - \alpha_1)\right)\psi'(\alpha_2)$$
$$+ \frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2}\int_{\alpha_1}^{\alpha_2} \mathcal{G}(x, \mu)\psi''(\mu)d\mu dx,$$

$$\frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2} \psi(x)dx = \psi(\alpha_1) + \left(\frac{\alpha_2 - \alpha_1}{2}\right)\psi'(\alpha_2)$$
$$+ \frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2}\int_{\alpha_1}^{\alpha_2} \mathcal{G}(x, \mu)\psi''(\mu)d\mu dx. \tag{2.5}$$

Subtracting (2.5) from (2.4) we obtain

$$\psi\left(\frac{\alpha_1 + \alpha_2}{2}\right) - \frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2} \psi(x)dx$$

$$= \int_{\alpha_1}^{\alpha_2} \mathcal{G}\left(\frac{\alpha_1+\alpha_2}{2}, \mu\right) \psi''(\mu)d\mu - \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} \int_{\alpha_1}^{\alpha_2} \mathcal{G}(x,\mu)\psi''(\mu)d\mu dx$$

$$= \int_{\alpha_1}^{\alpha_2} \left[\mathcal{G}\left(\frac{\alpha_1+\alpha_2}{2}, \mu\right) - \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} \mathcal{G}(x,\mu)dx\right] \psi''(\mu)d\mu. \tag{2.6}$$

Note that

$$\int_{\alpha_1}^{\alpha_2} \mathcal{G}(x,\mu)dx = \frac{\mu^2}{2} - \frac{\alpha_1^2}{2} + \alpha_1\alpha_2 - \alpha_2\mu, \tag{2.7}$$

$$\mathcal{G}\left(\frac{\alpha_1+\alpha_2}{2}, \mu\right) = \begin{cases} \alpha_1 - \mu\ , & \alpha_1 \le \mu \le \frac{\alpha_1+\alpha_2}{2}; \\ \frac{\alpha_1-\alpha_2}{2}, & \frac{\alpha_1+\alpha_2}{2} \le \mu \le \alpha_2. \end{cases}$$

If $\alpha_1 \le \mu \le \frac{\alpha_1+\alpha_2}{2}$, then

$$\mathcal{G}\left(\frac{\alpha_1+\alpha_2}{2}, \mu\right) - \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} \mathcal{G}(x,\mu)dx$$

$$= \alpha_1 - \mu - \frac{1}{\alpha_2-\alpha_1}\left(\frac{\mu^2}{2} - \frac{\alpha_1^2}{2} + \alpha_1\alpha_2 - \alpha_2\mu\right)$$

$$= \frac{-\left(\mu-\alpha_1\right)^2}{2(\alpha_2-\alpha_1)} \le 0. \tag{2.8}$$

If $\frac{\alpha_1+\alpha_2}{2} \le \mu \le \alpha_2$, then

$$\mathcal{G}\left(\frac{\alpha_1+\alpha_2}{2}, \mu\right) - \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} \mathcal{G}(x,\mu)dx$$

$$= \frac{\alpha_1-\alpha_2}{2} - \frac{1}{\alpha_2-\alpha_1}\left(\frac{\mu^2}{2} - \frac{\alpha_1^2}{2} + \alpha_1\alpha_2 - \alpha_2\mu\right)$$

$$= \frac{-\left(\alpha_2-\mu\right)^2}{2(\alpha_2-\alpha_1)} \le 0. \tag{2.9}$$

From the convexity of ψ we know that $\psi''(\mu) \ge 0$. Therefore, the first inequality of (2.3) follows easily from (2.6), (2.8) and (2.9).

Next, we prove second inequality of (2.3).

Let $x = \alpha_2$. Then (2.2) gives

$$\psi(\alpha_2) = \psi(\alpha_1) + (\alpha_2-\alpha_1)\psi'(\alpha_2) + \int_{\alpha_1}^{\alpha_2} \mathcal{G}(\alpha_2,\mu)\psi''(\mu)d\mu,$$

$$\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2} = \psi(\alpha_1) + \frac{1}{2}(\alpha_2-\alpha_1)\psi'(\alpha_2) + \frac{1}{2}\int_{\alpha_1}^{\alpha_2} \mathcal{G}(\alpha_2,\mu)\psi''(\mu)d\mu. \tag{2.10}$$

It follows from (2.5) and (2.10) that

$$\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2} - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} \psi(x)dx$$

$$= \int_{\alpha_1}^{\alpha_2} \left(\frac{1}{2}\mathcal{G}(\alpha_2,\mu) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} \mathcal{G}(x,\mu)dx\right)\psi''(\mu)d\mu. \tag{2.11}$$

From (2.1) one has

$$\mathcal{G}(\alpha_2,\mu) = \alpha_1 - \mu \tag{2.12}$$

if $\alpha_1 \leq \mu \leq \alpha_2$.

It follows from (2.7) and (2.12) that

$$\begin{aligned}
&\frac{1}{2}\mathcal{G}(\alpha_2,\mu) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathcal{G}(x,\mu)dx \\
&= \frac{\alpha_1-\mu}{2} - \frac{1}{\alpha_2-\alpha_1}\left(\frac{\mu^2}{2} - \frac{\alpha_1^2}{2} + \alpha_1\alpha_2 - \alpha_2\mu\right) \\
&= \frac{1}{2(\alpha_2-\alpha_1)}\left((\alpha_1-\mu)(\alpha_2-\alpha_1) - \mu^2 + \alpha_1^2 - 2\alpha_1\alpha_2 + 2\alpha_2\mu\right) \\
&= \frac{1}{2(\alpha_2-\alpha_1)}\left((\mu-\alpha_1)(\alpha_2-\mu)\right) \geq 0. \qquad (2.13)
\end{aligned}$$

Therefore,

$$\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2} - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx \geq 0$$

follows from (2.11) and (2.13) together with $\psi''(\mu) \geq 0$. □

Next, we give some refinements of the Hermite-Hadamard inequality for the class of monotonic and convex functions.

Theorem 2.3. *Let $\psi \in C^2([\alpha_1,\alpha_2])$. Then the following statements are true:*

(1) If $|\psi''|$ is increasing, then

$$\left|\psi\left(\frac{\alpha_1+\alpha_2}{2}\right) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right| \leq \frac{(\alpha_2-\alpha_1)^2}{48}\left[|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)| + |\psi''(\alpha_2)|\right];$$

(2) If $|\psi''|$ is decreasing, then

$$\left|\psi\left(\frac{\alpha_1+\alpha_2}{2}\right) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right| \leq \frac{(\alpha_2-\alpha_1)^2}{48}\left[|\psi''(\alpha_1)| + |\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)|\right];$$

(3) If $|\psi''|$ is convex, then

$$\begin{aligned}
&\left|\psi\left(\frac{\alpha_1+\alpha_2}{2}\right) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right| \\
&\leq \frac{(\alpha_2-\alpha_1)^2}{48}\left[\max\left\{\left|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|, |\psi''(\alpha_1)|\right\} + \max\left\{\left|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|, |\psi''(\alpha_2)|\right\}\right].
\end{aligned}$$

Proof. (1) By using (2.6) we have

$$\begin{aligned}
&\psi\left(\frac{\alpha_1+\alpha_2}{2}\right) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx \\
&= \int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}\left[\mathcal{G}\left(\frac{\alpha_1+\alpha_2}{2},\mu\right) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathcal{G}(x,\mu)dx\right]\psi''(\mu)d\mu \\
&\quad + \int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}\left[\mathcal{G}\left(\frac{\alpha_1+\alpha_2}{2},\mu\right) - \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\mathcal{G}(x,\mu)dx\right]\psi''(\mu)d\mu \\
&= \frac{-1}{2(\alpha_2-\alpha_1)}\left[\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}(\mu-\alpha_1)^2\psi''(\mu)d\mu + \int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}(\alpha_2-\mu)^2\psi''(\mu)d\mu\right].
\end{aligned}$$

Taking absolute and using triangular inequality we obtain

$$\left|\psi\left(\frac{\alpha_1+\alpha_2}{2}\right)-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right|$$

$$\leq \frac{1}{2(\alpha_2-\alpha_1)}\left[\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}(\mu-\alpha_1)^2\left|\psi''(\mu)\right|d\mu+\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}(\alpha_2-\mu)^2\left|\psi''(\mu)\right|d\mu\right]$$

$$\leq \frac{1}{2(\alpha_2-\alpha_1)}\left[\left|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}(\mu-\alpha_1)^2d\mu+\left|\psi''(\alpha_2)\right|\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}(\alpha_2-\mu)^2d\mu\right]$$

$$=\frac{1}{2(\alpha_2-\alpha_1)}\left[|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)|\times\frac{1}{24}(\alpha_2-\alpha_1)^3+|\psi''(\alpha_2)|\times\frac{1}{24}(\alpha_2-\alpha_1)^3\right]$$

$$=\frac{(\alpha_2-\alpha_1)^2}{48}\left[|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)|+|\psi''(\alpha_2)|\right]. \tag{2.14}$$

Similarly we can prove part (2).

For part (3), using (2.14) and the fact that every convex function ψ defined on the interval $[\alpha_1,\alpha_2]$ is bounded above by $\max\{\psi(\alpha_1),\psi(\alpha_2)\}$, we obtain

$$\left|\psi\left(\frac{\alpha_1+\alpha_2}{2}\right)-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right|$$

$$\leq \frac{1}{2(\alpha_2-\alpha_1)}\left[\max\left\{\left|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|,|\psi''(\alpha_1)|\right\}\int_{\alpha_1}^{\frac{\alpha_1+\alpha_2}{2}}(\mu-\alpha_1)^2d\mu\right.$$
$$\left.+\max\left\{\left|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|,|\psi''(\alpha_2)|\right\}\int_{\frac{\alpha_1+\alpha_2}{2}}^{\alpha_2}(\alpha_2-\mu)^2d\mu\right]$$

$$=\frac{(\alpha_2-\alpha_1)^2}{48}\left[\max\left\{\left|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|,|\psi''(\alpha_1)|\right\}+\max\left\{\left|\psi''\left(\frac{\alpha_1+\alpha_2}{2}\right)\right|,|\psi''(\alpha_2)|\right\}\right].$$

□

Theorem 2.4. *Let $\psi\in C^2([\alpha_1,\alpha_2])$. Then the following statements are true:*

(1) If $|\psi''|$ is increasing, then

$$\left|\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right|\leq\frac{|\psi''(\alpha_2)|(\alpha_2-\alpha_1)^2}{12};$$

(2) If $|\psi''|$ is decreasing, then

$$\left|\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right|\leq\frac{|\psi''(\alpha_1)|(\alpha_2-\alpha_1)^2}{12};$$

(3) If $|\psi''|$ is a convex function, then

$$\left|\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right|\leq\frac{|\max\{|\psi''(\alpha_1)|,|\psi''(\alpha_2)|\}|(\alpha_2-\alpha_1)^2}{12}.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Proof. It follows from (2.11) that

$$\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx=\frac{1}{2(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}(\mu-\alpha_1)(\alpha_2-\mu)\psi''(\mu)d\mu.$$

Taking absolute and using triangular inequality one has

$$\left|\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right|$$

$$\leq\frac{1}{2(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}\left((\mu-\alpha_1)(\alpha_2-\mu)\right)|\psi''(\mu)|d\mu. \tag{2.15}$$

Since $(\mu-\alpha_1)(\alpha_2-\mu)\geq 0$ and $|\psi''|$ is increasing, therefore

$$\begin{aligned}&\left|\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right|\\ &\leq\frac{|\psi''(\alpha_2)|}{2(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}(\mu-\alpha_1)(\alpha_2-\mu)d\mu\\ &=\frac{|\psi''(\alpha_2)|(\alpha_2-\alpha_1)^2}{12}.\end{aligned}$$

Similarly we can prove part (2)

For part (3), using (2.15) and the fact that every convex function f defined on the interval $[\alpha_1,\alpha_2]$ is bounded above by $\max\{f(\alpha_1),f(\alpha_2)\}$, we have

$$\begin{aligned}&\left|\frac{\psi(\alpha_1)+\psi(\alpha_2)}{2}-\frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2}\psi(x)dx\right|\\ &\leq\frac{\max\{|\psi''(\alpha_1)|,|\psi''(\alpha_2)|\}}{2(\alpha_2-\alpha_1)}\int_{\alpha_1}^{\alpha_2}\left((\mu-\alpha_1)(\alpha_2-\mu)\right)d\mu.\end{aligned}$$

□

3. Applications to Means

A bivariate function $M:(0,\infty)\times(0,\infty)\mapsto(0,\infty)$ is said to be a mean if $\min\{a,b\}\leq M(a,b)\leq\max\{a,b\}$, $M(a,b)=M(b,a)$ and $M(\lambda a,\lambda b)=\lambda M(a,b)$ for all $a,b,\lambda\in(0,\infty)$.

Let $a,b>0$ with $a\neq b$. Then the arithmetic mean $A(a,b)$ [38-43], logarithmic mean $L(a,b)$ [44-48] and (α,r)-th generalized logarithmic mean $L_{(\alpha,r)}(a,b)$ [49-52] are defined by

$$A(a,b)=\frac{a+b}{2},\quad L(a,b)=\frac{b-a}{\log b-\log a},\quad L_{(\alpha,r)}(a,b)=\left[\frac{\alpha(b^{r+\alpha}-a^{r+\alpha})}{(r+\alpha)(b^\alpha-a^\alpha)}\right]^{1/r},$$

respectively. Recently, the bivariate means have been the subject of intensive research [53-67] and many remarkable inequalities for the bivariate means and related special functions can be found in the literature [68-90].

In this section we present several new inequalities the arithmetic, logarithmic and generalized logarithmic means by using our results.

Theorem 3.1. *Let $0 < \alpha_1 < \alpha_2$. Then the following statements are true:*

(1) if $r \geq 2$, then

$$\left| A^r(\alpha_1, \alpha_2) - L_r^r(\alpha_1, \alpha_2) \right| \leq \frac{r(r-1)(\alpha_2 - \alpha_1)^2}{48} \left[\left(\frac{\alpha_1 + \alpha_2}{2} \right)^{r-2} + \alpha_2^{r-2} \right]; \tag{3.1}$$

(2) if $r < 2$ and $r \neq 0, -1$, then

$$\left| A^r(\alpha_1, \alpha_2) - L_r^r(\alpha_1, \alpha_2) \right| \leq \frac{r(r-1)(\alpha_2 - \alpha_1)^2}{48} \left[\left(\frac{\alpha_1 + \alpha_2}{2} \right)^{r-2} + \alpha_1^{r-2} \right]. \tag{3.2}$$

Proof. Let $\psi(x) = x^r$ $(x > 0)$ and $r \geq 2$. Then we clearly see that $|\psi''|$ is increasing and inequality (3.1) follows easily from Theorem 2.3(1). Similarly, we can prove inequality (3.2). ☐

Theorem 3.2. *Let $0 < \alpha_1 < \alpha_2$. Then the following statements are true:*

(1) if $r \geq 2$, then

$$|A(\alpha_1^r, \alpha_2^r) - L_r^r(\alpha_1, \alpha_2)| \leq \frac{r(r-1)(\alpha_2 - \alpha_1)^2 \alpha_2^{r-2}}{48};$$

(2) if $r < 2$ and $r \neq 0, -1$, then

$$|A(\alpha_1^r, \alpha_2^r) - L_r^r(\alpha_1, \alpha_2)| \leq \frac{r(r-1)(\alpha_2 - \alpha_1)^2 \alpha_1^{r-2}}{48}.$$

Proof. By using Theorem 2.4 and the same arguments as given in the proof of Theorem 3.1, we can obtain the desired results. ☐

Theorem 3.3. *The inequalities*

$$\left| A^{-1}(\alpha_1, \alpha_2) - L^{-1}(\alpha_1, \alpha_2) \right| \leq \frac{(\alpha_2 - \alpha_1)^2}{24} \left[\frac{8}{(\alpha_1 + \alpha_2)^3} + \frac{1}{\alpha_1^3} \right],$$

$$\left| A^{-1}(\alpha_1, \alpha_2) - L^{-1}(\alpha_1, \alpha_2) \right|$$
$$\leq \frac{(\alpha_2 - \alpha_1)^2}{24} \left[\max\left\{ \frac{8}{(\alpha_1 + \alpha_2)^3}, \frac{1}{\alpha_2^3} \right\} + \max\left\{ \frac{8}{(\alpha_1 + \alpha_2)^3}, \frac{1}{\alpha_1^3} \right\} \right]$$

hold for all $\alpha_1, \alpha_2 \in \mathbb{R}^+$ with $\alpha_1 < \alpha_2$.

Proof. Let $x > 0$ and $\psi(x) = 1/x$. Then we clearly see that $|\psi''|$ is decreasing and convex and Theorem 3.3 follows easily from Theorem 2.3(2) and (3). ☐

Theorem 3.4. *The inequality*

$$\left| A\left(\alpha_1^{-1}, \alpha_2^{-1}\right) - L^{-1}(\alpha_1, \alpha_2) \right| \geq \frac{(\alpha_2 - \alpha_1)^2}{6\alpha_1^3}$$

holds for all $\alpha_1, \alpha_2 \in \mathbb{R}^+$ with $\alpha_1 < \alpha_2$.

Proof. Similar proof as in Theorem 3.3 but use Theorem 2.4 instead of Theorem 2.3 ☐

4. Applications to Trapezoidal and Mid-Point Formulae

In this section we provide some new error estimations for the trapezoidal and mid-point formulae.

Let d be a division $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of the interval $[a, b]$ and consider the quadrature formula

$$\int_a^b \psi(x)dx = \mathtt{T}(\psi, d) + \mathtt{E}(\psi, d),$$

where

$$\mathtt{T}(\psi, d) = \sum_{i=0}^{n-1} \frac{\psi(x_i) + \psi(x_{i+1})}{2} (x_{i+1} - x_i)$$

for the trapezoidal version and

$$\mathtt{T}(\psi, d) = \sum_{i=0}^{n-1} \psi\Big(\frac{x_i + x_{i+1}}{2}\Big) (x_{i+1} - x_i)$$

for the midpoint version and $\mathtt{E}(\psi, d)$ denotes the associated approximation error.

Theorem 4.1. *Let d be a division $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of the interval $[a, b]$, $\psi \in C^2([a, b])$ and $E(\psi, d)$ be the trapezoidal error. Then one has*

$$|E(\psi, d)| \leq \sum_{i=0}^{n-1} \frac{|\psi''(x_{i+1})| (x_{i+1} - x_i)^3}{12} \tag{4.1}$$

if $|\psi''|$ is an increasing function;

$$|E(\psi, d)| \leq \sum_{i=0}^{n-1} \frac{\max\{|\psi''(x_i)|, |\psi''(x_{i+1})|\} (x_{i+1} - x_i)^3}{12}$$

if $|\psi''|$ is a decreasing function;

$$|E(\psi, d)| \leq \sum_{i=0}^{n-1} \frac{|\psi''(x_{i+1})| (x_{i+1} - x_i)^3}{12}$$

if $|\psi''|$ is a convex function.

Proof. Applying Theorem 2.4 on each subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, 2, \cdots, n-1)$ of the division d, we have

$$\left|\frac{\psi(x_i) + \psi(x_{i+1})}{2} - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} \psi(x)dx\right| \leq \frac{|\psi''(x_{i+1})|(x_{i+1} - x_i)^2}{12}.$$

Multiplying both sides by $x_{i+1} - x_i$ and taking summation we obtain

$$\left|\int_a^b \psi(x)dx - \mathtt{T}(\psi, d)\right| \leq \sum_{i=0}^{n-1} \left\{\frac{|\psi''(x_{i+1})|}{12} (x_{i+1} - x_i)^3\right\} \leq \sum_{i=0}^{n-1} \left|\left\{\frac{|\psi''(x_{i+1})|}{12} (x_{i+1} - x_i)^3\right\}\right|,$$

which is equivalent to (4.1). Similarly we can prove other parts. □

Theorem 4.2. *Let d be a division $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of the interval $[a, b]$, $\psi \in C^2([a, b])$ and $E(\psi, d)$ be the mid-point error. Then one has*

$$|E(\psi, d)| \leq \frac{1}{48} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3 \left[\left|\psi''\Big(\frac{x_{i+1} + x_i}{2}\Big)\right| + |\psi''(x_{i+1})|\right]$$

if $|\psi''|$ is an increasing function;

$$|E(\psi,d)| \le \frac{1}{48}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\left[\left|\psi''\Big(\frac{x_{i+1}+x_i}{2}\Big)\right| + \left|\psi''(x_i)\right|\right]$$

if $|\psi''|$ is a decreasing function;

$$|E(\psi,d)| \le \frac{1}{48}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\left[\max\{\left|\psi''\Big(\frac{x_{i+1}+x_i}{2}\Big)\right|, \left|\psi''(x_i)\right|\}\right.$$
$$\left. + \max\{\left|\psi''\Big(\frac{x_{i+1}+x_i}{2}\Big)\right|, \left|\psi''(x_{i+1})\right|\}\right]$$

if $|\psi''|$ is convex function.

Proof. The proof is analogous to the proof of Theorem 4.1. □

References

[1] J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier dune fonction considérée par Riemann, J. Math. Pures Appl., **58**, (1893) 171–215.

[2] X.-M. Zhang, Y.-M. Chu and X.-H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its applications, J. Inequal. Appl., 2010, **2010**, Article ID 507560, 11 pages.

[3] Y.-M. Chu, G.-D. Wang and X.-H. Zhang, Schur convexity and Hadamard's inequality, Math. Inequal. Appl., 2010, **13**(4), 725–731.

[4] M. Adil Khan, A. Iqbal, M. Suleman and Y.-M. Chu, Hermite-Hadamard typ inequalities for fractional integrals via Green's function, J. Inequal. Appl., 2018, **2018**, Article 161, 15 pages.

[5] M. Adil Khan, Y.-M. Chu, A. Kashuri, R. Liko and G. Ali, Conformable fractional integrals version of Hermite-Hadamard inequalities and their applications, J. Funct. Spaces, 2018, **2018**, Article ID 6928130, 9 pages.

[6] M. Adil Khan, Y. Khurshid, T.-S. Du and Y.-M. Chu, Generalization of Hermite-Hadamard type inequalities via conformable fractional integrals, J. Funct. Spaces, 2018, **2018**, Article ID 5357463, 12 pages.

[7] A. Iqbal, M. Adil Khan, S. Ullah, Y.-M. Chu and A. Kashuri, Hermite-Hadamard type inequalities pertaining conformable fractional integrals and their applications, AIP Advances, 2018, **8**, Article ID 075101, 18 pages, DOI: 10.1063/1.5031954.

[8] M. Adil Khan, Z. M. Al-sahwi and Y.-M. Chu, New estimations for Shannon and Zipf-Mandelbrot entropies, Entropy, 2018, **20**, Article 608, 10 pages, DOI: 10.3390/e2008608.

[9] M. Adil Khan, Y.-M. Chu, A. Kashuri and R. Liko, Hermite-Hadamard type fractional integral inequalities for $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions, J. Comput. Anal. Appl., 2019, **26**(8), 1487–1503.

[10] E. Kikianty, Hermite-Hadamard inequality in the geometry of Banach spaces, Ph.D. thesis, Victoria University, 2010.

[11] M. A. Noor, Hermite-Hadamard integral inequalities for log-ϕ-convex functions, Nonlinear Anal. Forum, 2008, **13**(2), 119–124.

[12] C. P. Niculescu and L.-E. Persson, Old and new on the Hermite-Hadamard inequality, Real Anal. Exchange, 2003/04, **29**(2), 663–685.

[13] M. Alomari, M. Darus and S. S. Dragomir, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are quasi-convex, Tamkang J. Math., 2010, **41**(4), 353–359.

[14] S. S. Dragomir and K. Nikodem, Jensen's and Hermite-Hadamard's type inequalities for lower and strongly convex functions on normed spaces, Bull. Iranian Math. Soc., 2018, **44**(5), 1337–1349.

[15] J. Makó and Z. Páles, Approximate Hermite-Hadamard type inequalities for approximately convex functions, Math. Inequal. Appl., 2013, **16**(2), 507–526.

[16] J. Park, Some Hermite-Hadamard-like type inequalities for logarithmically convex functions, Int. J. Math. Anal., 2013, **7**(45-48), 2217–2233.

[17] P. Spurek and Ja. Tabor, Numerical verification of condition for approximately midconvex functions, Aequationes Math., 2012, **83**(3), 223–237.
[18] P. Mereau and J.-G. Paquet, Second order conditions for pseudo-convex functions, SIAM J. Anal. Math., 1974, **27**, 131–137.
[19] S. S. Dragomir, Inequalities of Hermite-Hadamard type for φ-convex functions, Konuralp J. Math., 2016, **4**(1), 54–67.
[20] M. Adamek, A characterization of λ-convex functions, JIPAM. J. Inequal. Pure Appl. Math., 2004, **5**(3), Article 71, 5 pages.
[21] M. Bombardelli and S. Varošanec, Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities, Comput. Math. Appl., 2009, **58**(9), 1869–1877.
[22] T. Rajba, On strong delta-convexity and Hermite-Hadamard type inequalities for delta-convex functions of higher order, Math. Inequal. Appl., 2015, **18**(1), 267–293.
[23] Y.-M. Chu and T.-C. Sun, The Schur harmonic convexity for a class of symmetric functions, Acta Math. Sci., 2010, **30B**(5), 1501–1506.
[24] Y.-M. Chu, W.-F. Xia and T.-H. Zhao, Schur convexity for a class of symmetric functions, Sci. China Math., 2010, **53**(2), 465–474.
[25] W.-F. Xia and Y.-M. Chu, On Schur convexity of some symmetric functions, J. Inequal. Appl., 2010, **2010**, Article ID 543250, 12 pages.
[26] W.-F. Xia and Y.-M. Chu, Necessary and sufficient conditions for the Schur harmonic convexity or concavity of the extended mean values, Rev. Un. Mat. Argentina, 2011, **52**(1), 121–132.
[27] W.-F. Xia and Y.-M. Chu, The Schur convexity of Gini mean values in the sense of harmonic mean, Acta Math. Sci., 2011, **31B**(3), 1103–1112.
[28] Y.-M. Chu, W.-F. Xia and X.-H. Zhang, The schur concavity, Schur multiplicative and harmonic convexities of the second dual form of the Hamy symmetric function with applications, J. Multivariate Anal., 2012, **105**, 412–421.
[29] Y.-M. Chu, G.-D. Wang and X.-H. Zhang, The Schur multiplicative and harmonic convexities of the complete symmetric function, Math. Nachr., 2011, **284**(5-6), 653–663.
[30] W.-D. Jiang, M.-K. Wang, Y.-M. Chu, Y.-P. Jiang and F. Qi, Convexity of the generalized sine function and the generalized hyperbolic sine function, J. Approx. Theory, 2013, **174**, 1–9.
[31] M.-K. Wang, Y.-M. Chu, S.-L. Qiu and Y.-P. Jiang, Convexity of the complete elliptic integrals of the first kind with respect to Hölder means, J. Math. Anal. Appl., 2012, **388**(2), 1141–1146.
[32] Y.-M. Chu, M.-K. Wang, Y.-P. Jiang and S.-L. Qiu, Monotonicity, convexity, and inequalities invoving the Agard distortion function, Abstr. Appl. Anal., 2011, **2011**, Article ID 671765, 8 pages.
[33] X.-M. Zhang and Y.-M. Chu, Convexity of the integral arithmetic mean of a convex function, Rocky Mountain J. Math., 2010, **40**(3), 1061–1068.
[34] Y.-M. Chu and T.-H. Zhao, Concavity of the error functions with respect to Hölder means, Math. Inequal. Appl., 2016, **19**(2), 589–595.
[35] Y.-M. Chu, M.-K. Wang, Y.-P. Jiang and S.-L. Qiu, Concavity of the complete elliptic integrals of the second kind with respect to Hölder means, J. Math. Anal. Appl., 2012, **395**(2), 637–642.
[36] Y.-M. Chu and X.-M. Zhang, Multiplicative concavity of the integral of multiplicatively concave functions, J. Inequal. Appl., 2010, **2010**, Article ID 845930, 8 pages.
[37] S. S. Dragomir and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett., 1998, **5**, 91–95.
[38] H.-Z. Xu, Y.-M. Chu and W.-M. Qian, Sharp bounds for the Sándor-Yang means in terms of arithmetic and contra-harmonic means, J. Inequal. Appl., 2018, **2018**, Article 127, 13 pages.
[39] W.-F. Xia, W. Janous and Y.-M. Chu, The optimal convex combination bounds of arithmetic and harmonic means in terms of power mean, J. Math. Inequal., 2012, **6**(2), 241–248.
[40] W.-M. Gong, Y.-Q. Song, M.-K. Wang and Y.-M. Chu, A sharp double inequality between Seiffert, arithmetic, and geometric means, Abstr. Appl. Anal., 2012, **2012**, Article ID 684834, 7 pages.
[41] Y.-M. Chu and M.-K. Wang, Inequalities between arithmetic-geometric, Gini, and Toader means, Abstr. Appl. Anal., 2012, **2012**, Article ID 830585, 11 pages.

[42] Y.-M. Chu, Y.-F. Qiu, M.-K. Wang and G.-D. Wang, The optimal convex combination bounds of arithmetic and harmonic means for the Seiffert's mean, J. Inequal. Appl., 2010, **2010**, Article ID 436457, 7 pages.
[43] W.-F. Xia, Y.-M. Chu and G.-D. Wang, The optimal upper and lower power mean bounds for a convex combiantion of the arithmetic and logarithmic means, Abstr. Appl. Anal., 2010, **2010**, Article ID 604804, 9 pages.
[44] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, Best possible inequalities among harmonic, geometric, logarithmic and Seiffert means, Math. Inequal. Appl., 2012, **15**(2), 415–422.
[45] Y.-M. Chu and M.-K. Wang, Optimal inequalities betwen harmonic, geometric, logarithmic, and arithmetic-geometric means, J. Appl. Math., 2011, **2011**, Article ID 618929, 9 pages.
[46] Y.-M. Chu, S.-S. Wang and C. Zong, Optimal lower power mean bound for the convex combination of harmonic and logarithmic means, Abstr. Appl. Anal., 2011, **2011**, Article ID 520648, 9 pages.
[47] Y.-F. Qiu, M.-K. Wang, Y.-M. Chu and G.-D. Wang, Two sharp inequalities for Lehmer mean, identric mean and logarithmic mean, J. Math. Inequal., 2011, **5**(3), 301–306.
[48] Y.-M. Chu and W.-F. Xia, Two double inequalities between power mean and logarithmic mean, Comput. Math. Appl., 2010, **60**(1), 83–89.
[49] Y.-M. Chu and B.-Y. Long, Best possible inequalities between generalized logarithmic mean and classical means, Abstr. Appl. Anal., 2010, **2010**, Article ID 303286, 13 pages.
[50] W.-M. Qian and Y.-M. Chu, Best possible bounds for Yang mean using generalized logarithmic mean, Math. Probl. Eng., 2016, **2016**, Article ID 8901258, 7 pages.
[51] Y.-M. Li, B.-Y. Long and Y.-M. Chu, Sharp bounds for the Neuman-Sándor mean in terms of generalized logarithmic mean, J. Math. Inequal., 2012, **6**(4), 567–577.
[52] Y.-M. Chu, M.-K. Wang and G.-D. Wang, The optimal generalized logarithmic mean bounds for Seiffert's mean, Acta Math. Sci., 2012, **32B**(4), 1619–1626.
[53] Y.-M. Chu, Y.-M. Li, W.-F. Xia and X.-H. Zhang, Best possible inequalities for the harmonic mean of error function, J. Inequal. Appl., 2014, **2014**, Article 525, 9 pages.
[54] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, A best-possible double inequality between Seiffert and harmonic means, J. Inequal. Appl., 2011, **2011**, Article 94, 7 pages.
[55] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, A sharp double inequality between harmonic and identric means, Abstr. Appl. Anal., 2011, **2011**, Article ID 657935, 7 pages.
[56] Y.-M. Chu, M.-K. Wang and S.-L. Qiu, Optimal combinations bounds of root-square and arithmetic means for Toader mean, Proc. Indian Acad. Sci. Math. Sci., 2012, **122**(1), 41–51.
[57] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind, J. Math. Anal. Appl., 2018, **462**(2), 1714–1726.
[58] W.-M. Qian and Y.-M. Chu, Sharp bounds for a special quasi-arithmetic mean in terms of arithmetic and geometric means with two parameters, J. Inequal. Appl., 2017, **2017**, Article 274, 10 pages.
[59] Y.-M. Chu and M.-K. Wang, Optimal Lehmer mean bounds for the Toader mean, Results Math., 2012, **61**(3-4), 223–229.
[60] M.-K. Wang, Y.-M. Chu, Y.-F. Qiu and S.-L. Qiu, An optimal power mean inequality for the complete elliptic integrals, Appl. Math. Lett., 2011, **24**(6), 887–890.
[61] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, An optimal double inequality between Seiffert and geoemtric means, J. Appl. Math., 2011, **2011**, Article ID 261237, 6 pages.
[62] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, Sharp power mean bounds for the combination of Seiffert and geoemtric means, Abstr. Appl. Anal., 2010, **2010**, Article ID 108920, 12 pages.
[63] B.-Y. Long and Y.-M. Chu, Optimal power mean bounds for the weighted geometric mean of classical means, J. Inequal. Appl., 2010, **2010**, Article ID 905697, 6 pages.
[64] Y.-M. Chu, B.-Y. Long and B.-Y. Liu, Bounds of the Neuman-Sándor mean using power and identric means, Abstr. Appl. Anal., 2013, **2013**, Article ID 832591, 6 pages.
[65] M.-K. Wang, Y.-M. Chu and Y.-F. Qiu, Some comparison inequalities for generalized Muirhead and identric means, J. Inequal. Appl., 2010, **2010**, Article ID 295620, 10 pages.
[66] M.-K. Wang, Z.-K. Wang and Y.-M. Chu, An double inequality between geometric and identric means, Appl. Math. Lett., 2012, **25**(3), 471–475.
[67] W.-M. Qian, X.-H. Zhang and Y.-M. Chu, Sharp bounds for the Toader-Qi mean in terms of harmonic and geometric means, J. Math. Inequal., 2017, **11**(1), 121–127.

[68] Zh.-H. Yang, W.-M. Qian and Y.-M. Chu, Monotonicity properties and bounds involving the complete elliptic integrals of the first kind, Math. Inequal. Appl., 2018, **21**(4), 1185–1199.
[69] T.-H. Zhao, M.-K. Wang, W. Zhang and Y.-M. Chu, Quadratic transformation inequalities for Gaussian hypergeometric function, J. Inequal. Appl., 2018, **2018**, Article 251, 15 pages.
[70] T.-R. Huang, S.-Y. Tan, X.-Y. Ma and Y.-M. Chu, Monotonicity properties and bounds for the complete p-elliptic integrals, J. Inequal. Appl., 2018, **2018**, Article 239, 11 pages.
[71] M.-K. Wang, S.-L. Qiu and Y.-M. Chu, Infinite series formula for Hübner upper bound function with applications to Hersch-Pfluger distortion function, Math. Inequal. Appl., 2018, **21**(3), 629–648.
[72] M.-K. Wang and Y.-M. Chu, Landen inequalities for a class of hypergeometric functions with applications, Math. Inequal. Appl., 2018, **21**(2), 521–537.
[73] Zh.-H. Yang and Y.-M. Chu, A monotonicity property involving the generalized elliptic integral of the first kind, Math. Inequal. Appl., 2017, **20**(3), 729–735.
[74] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, Monotonicity rule for the quotient of two function and its applications, J. Inequal. Appl., 2017, **2017**, Article 106, 13 pages.
[75] M.-K. Wang, Y.-M. Chu and Y.-Q. Song, Asymptotical formulas for Gaussian and generalized hypergeometric functions, Appl. Math. Comput., 2016, **276**, 44–60.
[76] M.-K. Wang, Y.-M. Chu and Y.-P. Jiang, Ramanujan's cubic transformaation inequalities for zero-balanced hypergeometric functions, Rocky Mountain J. Math., 2016, **46**(2), 679–691.
[77] Y.-M. Chu, M.-K. Wang and Y.-F. Qiu, On Alzer and Qiu's conjecture for complete elliptic integral and inverse hyperbolic tangent function, Abstr. Appl. Anal., 2011, **2011**, Article ID 697547, 7 pages.
[78] M.-K. Wang and Y.-M. Chu, Asymptotical bounds for complete elliptic integrals of the second kind, J. Math. Anal. Appl., 2013, **402**(1), 119–126.
[79] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, Hölder mean inequalities for the complete elliptic integrals, Integral Transforms Spec. Funct., 2012, **23**(7), 521–527.
[80] Y.-M. Chu, M.-K. Wang, S.-L. Qiu and Y.-P. Jiang, Bounds for complete elliptic integrals of the second kind with applications, Comput. Math. Appl., 2012, **63**(7), 1177–1184.
[81] M.-K. Wang, S.-L. Qiu, Y.-M. Chu and Y.-P. Jiang, Generalized Hersch-Pfluger distortion function and complete elliptic integrals, J. Math. Anal. Appl., 2012, **385**(1), 221–229.
[82] M.-K. Wang, Y.-M. Li and Y.-M. Chu, Inequalities and infinite product formula for Ramanujan generalized modular equation function, Ramanujan J., 2018, **46**(1), 189–200.
[83] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On rational bounds for the gamma function, J. Inequal. Appl., 2017, **2017**, Article 210, 17 pages.
[84] T.-H. Zhao and Y.-M. Chu, A class of logarithmically completely monotonic functions associated with gamma function, J. Inequal. Appl., 2010, **2010**, Article ID 392431, 11 pages.
[85] T.-H. Zhao, Y.-M. Chu and H. Wang, Logarithmically complete monotonicity properties relating to the gamma function, Abstr. Appl. Anal., 2011, **2011**, Article ID 896483, 13 pages.
[86] T.-R. Huang, B.-W. Han, X.-Y. Ma and Y.-M. Chu, Optimal bounds for the generalized Euler-Mascheronic constant, J. Inequal. Appl., 2018, **2018**, Article 118, 9 pages.
[87] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the error function, Math. Inequal. Appl., 2018, **21**(2), 469–479.
[88] Y.-Q. Song, P.-G. Zhou and Y.-M. Chu, Inequalities for the Gaussian hypergeometric function, Sci. China Math., 2014, **57**(11), 2369–2380.
[89] Zh.-H. Yang, W. Zhang and Y.-M. Chu, Sharp Gautschi inequality for parameter $0 < p < 1$ with applications, Math. Inequal. Appl., 2017, **20**(4), 1107–1120.
[90] Zh.-H. Yang, Y.-M. Chu and W. Zhang, Accurate approximations for the complete elliptic of the second kind, J. Math. Anal. Appl., 2016, **438**(2), 875–888.

Ying-Qing Song, College of Science, Hunan City University, Yiyang 413000, China
E-mail address: 1452225875@qq.com

Yu-Ming Chu (Corresponding author), Department of Mathematics, Huzhou University, Huzhou 313000, China
E-mail address: chuyuming2005@126.com

Muhammad Adil Khan, Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan
E-mail address: adilswati@gmail.com

Arshad Iqbal, Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan
E-mail address: arshadiqbal80@gmail.com

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

On the Applications of the Girard-Waring Identities

Tian-Xiao He[1] and Peter J.-S. Shiue[2*]
[1]Department of Mathematics
Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
[2] Department of Mathematical Sciences
University of Nevada, Las Vegas
Las Vegas, Nevada, 89154-4020, USA

Abstract

We present here some applications of Girard-Waring identities. Many various identities for things like elementary mathematics and other advanced mathematics come from those identities.

AMS Subject Classification: 05A15, 05A05, 15B36, 15A06, 05A19, 11B83.

Key Words and Phrases: Girard-Waring identities, Recursive sequences, Fibonacci numbers and polynomials, Lucas numbers, Chebysheve polynomials of the first kind, Chebysheve polynomials of the second kind.

1 Introduction

In this paper, we are concerned with the applications of the following Girard-Waring identities:

$$x^n + y^n = \sum_{0\leq k\leq [n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(x+y)^{n-2k}(xy)^k \tag{1}$$

and

$$\frac{x^{n+1}-y^{n+1}}{x-y} = \sum_{0\leq k\leq [n/2]} (-1)^k \binom{n-k}{k}(x+y)^{n-2k}(xy)^k. \tag{2}$$

*This work was completed while on sabbatical leave from University of Nevada, Las Vegas, and the author would like to thank UNLV for its support.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Albert Girard published these identities in Amsterdam in 1629 and Edward Waring published similar material in Cambridge in 1762-1782. These may be derived from the earlier work of Sir Isaac Newton. It worth noting that $(-1)^k \frac{n}{n-k}\binom{n-k}{k}$ is an integer because

$$\begin{aligned}&\frac{n}{n-k}\binom{n-k}{k} = \binom{n-k}{k} + \binom{n-k-1}{k-1}\\ &= 2\binom{n-k}{k} - \binom{n-k-1}{k}.\end{aligned}$$

The proofs of formulas (1) and (2) can be seen in Comtet [3] (P. 198) and the survey paper by Gould [6]. Recently, Shapiro and one the authors [8] gave a different proof of (2) by using Riordan arrays.

There are some alternative forms of formula (1). As an example, we give the following one. If $x+y+z=0$, then (1) gives

$$\begin{aligned}x^n + y^n &= \sum_{0\leq k\leq [n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(-z)^{n-2k}(xy)^k\\ &= (-1)^n z^n + \sum_{1\leq k\leq [n/2]} (-1)^{n-k}\frac{n}{n-k}\binom{n-k}{k} z^{n-2k}(xy)^k,\end{aligned}$$

which implies

$$x^n + y^n - (-1)^n z^n = \sum_{1\leq k\leq [n/2]} (-1)^{n-k}\frac{n}{n-k}\binom{n-k}{k} z^{n-2k}(xy)^k.$$

Thus, when n is even, we have formula

$$x^n + y^n - z^n = \sum_{1\leq k\leq [n/2]} (-1)^{n-k}\frac{n}{n-k}\binom{n-k}{k} z^{n-2k}(xy)^k, \quad (3)$$

while for odd n we have

$$x^n + y^n + z^n = \sum_{1\leq k\leq [n/2]} (-1)^{n-k}\frac{n}{n-k}\binom{n-k}{k} z^{n-2k}(xy)^k, \quad (4)$$

where $x+y+z=0$. Particularly, if $n=3$, then

$$x^3 + y^3 + z^3 = 3xyz, \quad (5)$$

which will be shown in Corollary **??** and applied in the following examples. The formulas (3) and (4) can be considered as analogies of the results for the case of $xy+yz+zx=0$ shown in Ma [11]. Draim and Bicknell [4] use sums

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

and products of two roots of a quadratic equation to derive a class of Girard-Waring identities. Using Girard-Waring formulas to derive combinatorial identities is also an attractive topic. For instance, Filipponi [5] uses Girard-Waring formula (1) to derive some unusual binomial Fibonacci identities. Furthermore, some well-known identities can be re-derived by using Girard-Waring formulas. As an example, we substitute $x = u + \sqrt{u^2-4}$, $y = u - \sqrt{u^2-4}$, and $z = -x - y$ into (1) and obtain the following identity shown on page 57 of Riordan [13]:

$$\sum_{k=0}^{m}(-1)^k \frac{n}{n-k}\binom{n-k}{k}u^{n-2k} = 2^{-n}\left[(u+\sqrt{u^2-4})^n + (u-\sqrt{u^2-4})^n\right] \tag{6}$$

for $n = 1, 2, \ldots$, where $m = [n/2]$. In particular, if $u = 2$, above identity (6) reduces to

$$\sum_{k=0}^{m}(-1)^k \frac{n}{n-k}\binom{n-k}{k}2^{n-2k} = 2$$

for $n = 1, 2, \ldots$. It worth mentioning that Vasil'ev and Zelevinskii [18] denoted the function shown on the right-hand side of (6) by Q_n and obtained (see (4') on Page 57 of [18])

$$Q_n(x) = \Pi_{1\le k\le n}\left(x - 2\cos\frac{(2k-1)\pi}{2n}\right),$$

which implies

$$\Pi_{1\le k\le m}\cos\frac{(2k-1)\pi}{4m} = \frac{\sqrt{2}}{2^m}$$

for $m \ge 1$ (see (d) on Page 58 of [18]).

In the next section, we present some applications of Girard-Waring identities to the trigonometric identities. In section 3, some applications of Girard-Waring identities to the linear recurrence relations of order 2 will be given.

2 Girard-Waring identities and trigonometric identities

Girard-Waring identities can be applied to construct many interesting trigonometric identities related to the roots of some quadratic equations.

Our idea of the first application of Girard-Waring identities can be presented as follows: In formulas (1) and (2), there are two terms $x+y$, and xy. If we consider x and y the two roots r_1 and r_2 of a given quadratic equation, $ax^2+bx+c=0$, then we have the sums of, and differences of, n-th powers of the roots of the quadratic equation. Therefore we have $r_1+r_2=-\frac{b}{a}=: p$ and $r_1 r_2=\frac{c}{a}=: q$. Thus formula (1) and (2) give:

$$r_1^n+r_2^n=\sum_{0\le k\le[n/2]}(-1)^k\frac{n}{n-k}\binom{n-k}{k}p^{n-2k}q^k \tag{7}$$

and

$$\frac{r_1^{n+1}-r_2^{n+1}}{r_1-r_2}=\sum_{0\le k\le[n/2]}(-1)^k\binom{n-k}{k}p^{n-2k}q^k. \tag{8}$$

We first consider a simple quadratic equation $x^2+c=0$. Then two roots, r_1 and r_2, of the equation satisfy

$$r_1+r_2=0 \quad \text{and} \quad r_1r_2=c.$$

From (7) we have the identity

$$r_1^n+r_2^n=\sum_{0\le k\le[n/2]}(-1)^k\frac{n}{n-k}\binom{n-k}{k}(r_1+r_2)^{n-2k}(r_1r_2)^k,$$

which implies

$$r_1^{2\ell}+r_2^{2\ell}=2(-c)^\ell \tag{9}$$

and $r_1^{2\ell+1}+r_2^{2\ell+1}=0$. For instance, if $c=-3$, then $r_1=2\cos(\pi/6)$ and $r_2=2\cos(5\pi/6)$. From (9) we obtain

$$\cos^{2\ell}\left(\frac{\pi}{6}\right)+\cos^{2\ell}\left(\frac{5\pi}{6}\right)=2\left(\frac{3}{4}\right)^\ell.$$

When $\ell=1$ and 2, when $\cos^2\left(\frac{\pi}{6}\right)+\cos^2\left(\frac{5\pi}{6}\right)=1.5$ and $\cos^4\left(\frac{\pi}{6}\right)+\cos^4\left(\frac{5\pi}{6}\right)=9/8$, respectively.

Consider a quadratic equation $ax^2+bx+c=0$, we have $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. If $b^2-4ac<0$, then

$$x=A\pm Bi=\rho(\cos\theta\pm i\sin\theta),$$

where $\theta=tan^{-1}\frac{B}{A}$.

Then two roots r_1 and r_2 are:

$$r_1 = \rho(\cos\theta + i\sin\theta) \quad \text{and} \quad r_2 = \rho(\cos\theta - i\sin\theta),$$

which implies $p = r_1 + r_2 = 2\rho\cos\theta$ and $q = r_1 r_2 = \rho$. Thus, equation (7) gives

$$r_1^n + r_2^n = \rho^n \sum_{0\le k\le[n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(2\cos\theta)^{n-2k},$$

which implies

$$2\cos n\theta = \sum_{0\le k\le[n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(2\cos\theta)^{n-2k}.$$

Note that

$$\frac{n}{n-k}\binom{n-k}{k} = \frac{n}{k}\binom{n-k-1}{k-1}, \quad k \ge 1.$$

Thus

$$\cos n\theta = \frac{1}{2}\Big\{(2\cos\theta)^n - \frac{n}{1}(2\cos\theta)^{n-2} + \frac{n}{2}\binom{n-3}{1}(2\cos\theta)^{n-4} - \frac{n}{3}\binom{n-4}{2}(2\cos\theta)^{n-6} + \cdots\Big\}.$$

Similarly, from (8) we have

$$\sin(n+1)\theta = \sin\theta \sum_{0\le k\le[n/2]} (-1)^k \binom{n-k}{k}(2\cos\theta)^{n-2k}.$$

Example 2.1 On Page 50 of Comtet [3], it can be seen that

$$\frac{\sin(n+1)\theta}{\sin\theta} = U_n(\cos\theta),$$

where $U_n(x)$ are the Chebyshev polynomials of the second kind. Thus,

$$U_n(\cos\theta) = \sum_{0\le k\le[n/2]} (-1)^k \binom{n-k}{k}(2\cos\theta)^{n-2k}.$$

On Page 88 of Comtet [3], we also find that

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} = \begin{vmatrix} 2\cos\theta & 1 & 0 & 0 & \dots \\ 1 & 2\cos\theta & 1 & 0 & \dots \\ 0 & 1 & 2\cos\theta & 1 & \dots \\ 0 & 0 & 1 & 2\cos\theta & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

Hence, the determinant of the tridiangonal matrix on the rightmost side of the above equation is equal to

$$\begin{vmatrix} 2\cos\theta & 1 & 0 & 0 & \dots \\ 1 & 2\cos\theta & 1 & 0 & \dots \\ 0 & 1 & 2\cos\theta & 1 & \dots \\ 0 & 0 & 1 & 2\cos\theta & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = \sum_{0\leq k\leq [n/2]} (-1)^k \binom{n-k}{k} (2\cos\theta)^{n-2k}.$$

Recall that the Chebyshev polynomials of the first kind $T_n(x)$ are defined by $T_n(x) = \cos(n \cos^{-1} x)$. Thus,

$$T_n(x) = \cos(n \cos^{-1} x)$$
$$= \frac{1}{2}\left\{(2x)^n - \frac{n}{1}(2x)^{n-2} + \frac{n}{2}\binom{n-3}{1}(2x)^{n-4} - \cdots\right\}$$

From Page 88 of [3],

$$T_n(\cos\theta) = \cos n\theta = \begin{vmatrix} \cos\theta & 1 & 0 & 0 & \dots \\ 1 & 2\cos\theta & 1 & 0 & \dots \\ 0 & 1 & 2\cos\theta & 1 & \dots \\ 0 & 0 & 1 & 2\cos\theta & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

Thus,

$$\begin{vmatrix} x & 1 & 0 & 0 & \dots \\ 1 & 2x & 1 & 0 & \dots \\ 0 & 1 & 2x & 1 & \dots \\ 0 & 0 & 1 & 2x & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = \frac{1}{2}\left\{(2x)^n - \frac{n}{1}(2x)^{n-2} + \frac{n}{2}\binom{n-3}{1}(2x)^{n-4} - \cdots\right\}.$$

From [19] (see Page 696),

$$T_n(x) = 2^{n-1}\Pi_{k=1}^n \left\{x - \cos\left(\frac{(2k-1)\pi}{2n}\right)\right\}.$$

Since $T_n(\cos\theta) = \cos n\theta$, the above formula implies that

$$\cos n\theta = 2^{n-1}\Pi_{k=1}^n \left\{\cos\theta - \cos\left(\frac{(2k-1)\pi}{2n}\right)\right\}. \tag{10}$$

Remark 2.1 It is well known (see, for example, [19]) that

$$U_n(x) = 2^n\Pi_{k=1}^n \left\{x - \cos\left(\frac{k\pi}{n+1}\right)\right\}.$$

Thus,

$$\frac{\sin(n+1)\theta}{\sin\theta} = U_n(\cos\theta) = 2^n \Pi_{k=1}^n \left\{\cos\theta - \cos\left(\frac{k\pi}{n+1}\right)\right\}. \tag{11}$$

Substituting different values of θ into (11), we may obtain a type of trigonometric identities. For instance, let $\theta = \pi/2$. Then (11) yields

$$\sin(n+1)\frac{\pi}{2} = -2^n \Pi_{k=1}^n \cos\left(\frac{k\pi}{n+1}\right).$$

If $n = 2m$, $m = 0, 1, , 2 \ldots$, because of $\sin(2m+1)\pi/2 = (-1)^m$, the last equation implies

$$(-1)^m = 2^{2m}\Pi_{k=1}^{2m} \cos\left(\frac{k\pi}{2m+1}\right) = 4^m (-1)^m \Pi_{k=1}^m \cos^2\left(\frac{k\pi}{2m+1}\right),$$

where in the last step we use the fact

$$\cos\left(\frac{k\pi}{2m+1}\right) = -\cos\left(\pi - \frac{k\pi}{2m+1}\right) = -\cos\left(\frac{(2m-k+1)\pi}{2m+1}\right)$$

for $k = m+1, m+2, \ldots, 2m$. Thus we obtain the identity

$$\Pi_{k=1}^m \cos^2\left(\frac{k\pi}{2m+1}\right) = \frac{1}{4^m}.$$

Other identities can be obtained by substituting $\theta = \pi/6$, $\pi/4$, $\pi/3$, etc.

Recall also that

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Let $r_1 = e^x$ and $r_2 = e^{-x}$. Then (7) gives

$$\cosh(nx) = \frac{1}{2} \sum_{0 \le k \le [n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (2\cosh x)^{n-2k}$$

and

$$\sinh(nx) = \sinh x \sum_{0 \le k \le [n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (2\cosh x)^{n-2k}.$$

Other applications of Girard-Waring identities to the product expansions of trigonometric functions similar to the results shown in [2] will be presented in the author's further work.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

3 Girard-Waring Identities and linear recurrence relations of order 2

Girard-Waring identities can be applied to construct the expressions of the linear recursive sequences of order 2.

Note that $x^n + y^n = (x+y)\left(x^{n-1} + y^{n-1}\right) - xy\left(x^{n-2} + y^{n-2}\right)$ $(n \geq 2)$. Let $w_n(x,y) = x^n + y^n$. Then

$$w_n(x,y) = (x+y)w_{n-1}(x,y) - xyw_{n-2}(x,y), n \geq 2, \tag{12}$$

with the initial conditions $w_0(x,y) = 2$ and $w_1(x,y) = x+y$. The characteristic equation of the above recurrence relation is $t^2 - (x+y)t + xy = 0$. Thus $t = x, y$

Proposition 3.1. *Let* $a_n(x,y) = p(x,y)a_{n-1}(x,y) + q(x,y)a_{n-2}(x,y)$, $n \geq 2$, *with given* $a_0(x,y)$ *and* $a_1(x,y)$. *Then*

$$\begin{aligned} & a_n(x,y) \\ = \;& \frac{a_1(x,y) - \beta(x,y)a_0(x,y)}{\alpha(x,y) - \beta(x,y)}\alpha^n(x,y) - \frac{a_1(xy) - \alpha(x,y)a_0(x,y)}{\alpha(x,y) - \beta(x,y0}\beta^n(x,y), \end{aligned}$$

where $\alpha(x,y) \neq \beta(x,y)$ *are the roots of the characteristic equation* $t^2 - p(x,y)t - q(x,y) = 0$.

By using this proposition 3.1, the solution of (12) is $w_n(x,y) = x^n + y^n$.
Example 3.1 The generalized Lucas polynomials (Lucas 1891, see Swamy [16]) $V_n(x,y)$ are defined by

$$V_n(x,y) = xV_{n-1}(x,y) + yV_{n-2}(x,y), \quad V_0(x,y) = 2,\ V_1(x,y) = x.$$

The characteristic equation is $t^2 - xt - y = 0$. Thus

$$t = \frac{x \pm \sqrt{x^2 + 4y}}{2}.$$

By Proposition 3.1 and the Girard-Waring identity (1)

$$V_n(x,y) = \alpha^n(x,y) + \beta^n(x,y) = \sum_{0 \leq k \leq [n/2]} \frac{n}{n-k}\binom{n-k}{k} x^{n-2k} y^k.$$

Example 3.2 Dickson polynomials of the first kind of degree n (Dickson 1897, see Lidl, Mullen, and Turnwald [10]) are defined by

$$D_n(x,a) = xD_{n-1}(x,a) - aD_{n-2}(x,a), \quad D_0(x,a) = 2,\ D_1(x,a) = x.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Thus from Proposition 3.1 and the Girard-Waring identity (1),

$$\begin{aligned} & V_n(x,-a) = D_n(x,a) \\ = & \sum_{0\leq k\leq [n/2]} \frac{n}{n-k}\binom{n-k}{k}(-a)^k x^{n-2k}. \end{aligned}$$

Example 3.3 For the Lucas polynomials (see Bicknell [1]) $\{\mathcal{L}_n(x) = V_n(x,1)\}$, i.e., let $y=1$ in $\{V_n(x,y)\}$, we have

$$\mathcal{L}_n(x) = \sum_{0\leq k\leq [n/2]} \frac{n}{n-k}\binom{n-k}{k} x^{n-2k}.$$

Note that $D_n(x,-1) = \mathcal{L}_n(x)$. For the Lucas numbers $\mathcal{L}_n = \mathcal{L}_n(1)$, we have

$$\mathcal{L}_n = \sum_{0\leq k\leq [n/2]} \frac{n}{n-k}\binom{n-k}{k}.$$

Example 3.4 The Chebysheve polynomials of the first kind (Chehysher 1821-1894, see Rivlin [14] and Zwillinger [19]) are defined by

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \qquad n \geq 2,$$

with the initial conditions $T_0(x) = 1$ and $T_1(x) = x$. Thus, from Proposition 3.1 and the Girard-Waring identity (1), we have

$$T_n(x) = \frac{1}{2} \sum_{0\leq k\leq [n/2]} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(2x)^{n-2k}.$$

Note that from (2), we have

$$\frac{x^{n+1}-y^{n+1}}{x-y} = (x+y)\frac{x^n-y^n}{x-y} - xy\frac{x^{n-1}-y^{n-1}}{x-y}, \quad n\geq 2.$$

Let $W_{n+1}(x,y) = (x^{n+1}-y^{n+1})/(x-y)$. Then

$$W_n(x,y) = (x+y)W_{n-1}(x,y) - xyW_{n-2}(x,y), \quad n \geq 2, \tag{13}$$

with the initial conditions $W_0(x,y) = 0$ and $W_1(x,y) = 1$. Thus

$$W_n(x,y) = \frac{x^n-y^n}{x-y} = \sum_{0\leq k\leq [(n-1)/2]} (-1)^k \binom{n-1-k}{k}(x+y)^{n-1-k}(xy)^k.$$

Remark 3.1 From the expression of $W_n(x,y)$ and noting the initial condition $W_0(x,y) = 0$, we know $\{W_n(x,y)\}$ is a linear divisibility sequence.

More precisely, from authors' recent work [9], if x and y be distinct real (or complex) numbers, then sequence $(W_n(x,y))$ is a second order linear homogenous recursive sequence with $W_0 = 0$ and $W_1 = 1$ and a linear divisibility sequence of order 2. For instance, when $r \neq 1$ and $a_1 = 1$, the geometric sequence $\{s_n = a_1(1-r^n)/(1-r) = (1-r^n)/(1-r)\}_{n\geq 1}$ is a linear divisibility sequence because $s_n = W_n(1,r)$.

Example 3.5 The Generalized Fibonacci polynomials $F_n(x,y)$ (see Swammy [17]) are defined by $F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y)$ $(n \geq 2)$ with the initial conditions $F_0(x,y) = 0$ and $F_1(x,y) = 1$. From Proposition 3.1 and the Girard-Waring identity (2), we have

$$F_n(x,y) = \sum_{0\leq k\leq [(n-1)/2]} \binom{n-1-k}{k} x^{n-1-k} y^k.$$

Thus, for the Fibonacci polynomials

$$F_n(x) = F_n(x,1) = \sum_{0\leq k\leq [(n-1)/2]} \binom{n-1-k}{k} x^{n-1-k}.$$

For Fibonacci sequence $\{F_n\}$

$$F_n = F_n(1) = \sum_{0\leq k\leq [(n-1)/2]} \binom{n-1-k}{k}.$$

For the Pell sequence $\{P_n\}$

$$P_n = F_n(2) = \sum_{0\leq k\leq [(n-1)/2]} \binom{n-1-k}{k} 2^{n-1-k}.$$

References

[1] M. Bicknell, A primer for the Fibonacci numbers VII, *Fibonacci Quart.* 8 (1970) pp. 407–420.

[2] G. Brouwer, A generalization of the angle doubling formulas for trigonometric functions, *Math. Magazine*, 90 (2017), No.1, 12–18.

[3] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.

[4] N. A. Draim and M. Bicknell, Sums of n-th powers of roots of a given quadratic equation, *Fibonacci Quart.* 4 (1966), no. 3, 170–178.

[5] P. Filipponi, Some binomial Fibonacci identities. *Fibonacci Quart.* 33 (1995), no. 3, 251–257.

[6] H. W. Gould, The Girard-Waring power sum formulas for symmetric functions and Fibonacci sequences, *Fibonacci Quart.* 37 (1999), no. 2, 135–140.

[7] R. P. Grimaldi, *Fibonacci and Catalan numbers, An introduction*, John Wiley & Sons, Inc., Hoboken, NJ, 2012.

[8] T.-X. He and L. W. Shapiro, Row sums and alternating sums of Riordan arrays, *Linear Algebra Appl.* 507 (2016), 77–95.

[9] T.-X. He and P. J.-S. Shiue, An approach to the construction of linear divisibility sequences of high orders, *J. Integer Sequences*, 20 (2017), Artical 17. 9. 3.

[10] R. Lidl, G. L. Mullen, and G. Turnwald, *Dickson Polynomials*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 65. Longman Scientific & Technical and John Wiley & Sons, Inc., 1993.

[11] X. Ma, A generalization of the Kummer identity and its application to Fibonacci-Lucas sequences, *Fibonacci Quart.* 36 (1998), no. 4, 339–347.

[12] R. W. D. Nickalls, Viéte, Descartes and the cubic equation, *Math. Gazette*, 90 (2006), July, 203–208.

[13] J. Riordan, *Combinatorial identities*, John Wiley & Sons, Inc. New York-London-Sydney, 1968.

[14] T. J. Rivlin, *Chebyshev Polynomials*, New York: Wiley, 1990.

[15] M. Saul and T. Andreescu, Symmelltry in algebra, part lll, Quantum, 8 (1998), no. 6, July/August, 41–42.

[16] M. N. S. Swamy, On a class of generalized polynomials, *Fibonacci Quart.* 35 (1997), no. 4, 329–334.

[17] M. N. S. Swamy, Generalized Fibonacci and Lucas polynomials and their associated diagonal polynomials, *Fibonacci Quart.* 37 (1999), no. 3, 213–222.

[18] N. Vasil'ev and A. Zelevinskii, Chebyshev polynomials and recurrence relations [Kvant 1982, no. 1, 12–19], Kvant selecta: algebra and analysis, II, 51–61, Math. World, 15, Amer. Math. Soc., Providence, RI, 1999.

[19] D. Zwillinger (Ed.), *CRC Standard Mathematical Tables and Formulae*, Boca Raton, FL: CRC Press, 1995.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Local Fractional Taylor Formula

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we derive an appropiate local fractional Taylor formula. We provide a complete description of the formula.

2010 AMS Mathematics Subject Classification: 26A33.
Keywords and Phrases: Local fractional derivative, Riemann-Liouville fractional derivative, Fractional Taylor formula.

1 Introduction

In [3], [4] was first introduced the local fractional derivative and presented an incomplete local fractional Taylor formula, all done by the use of Riemann-Liouville fractional derivative. Similar work was done in [1], but again with some gaps. The author is greatly motivated by the pioneering work of [1]-[4] and presents a local fractional Taylor formula in a complete suitable form and without any gaps.

2 Main Results

We mention

Definition 1 *([5], pp. 68, 89) Let $x, x' \in [a,b]$, $f \in C([a,b])$. The Riemann-Lioville fractional derivative of a function f of order q $(0<q<1)$ is defined as*

$$D_x^q f(x') = \left\{ \begin{array}{l} D_{x+}^q f(x'), \quad x' > x, \\ D_{x-}^q f(x'), \quad x' < x \end{array} \right\} =$$

$$\frac{1}{\Gamma(1-q)} \left\{ \begin{array}{l} \frac{d}{dx'} \int_x^{x'} (x'-t)^{-q} f(t)\, dt, \quad x' > x, \\ -\frac{d}{dx'} \int_{x'}^{x} (t-x')^{-q} f(t)\, dt, \quad x' < x. \end{array} \right. \tag{1}$$

1

We need

Definition 2 *([3]) The local fractional derivative of order q $(0 < q < 1)$ of a function $f \in C([a,b])$ is defined as*

$$\mathcal{D}^q f(x) = \lim_{x' \to x} D_x^q \left(f(x') - f(x)\right). \tag{2}$$

More generally we define

Definition 3 *(see also [1]) Let $N \in \mathbb{Z}_+$, $0 < q < 1$, the local fractional derivative of order $(N+q)$ of a function $f \in C^N([a,b])$ is defined by*

$$\mathcal{D}^{N+q} f(x) = \lim_{x' \to x} D_x^q \left(f(x') - \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} (x'-x)^n\right). \tag{3}$$

If $N = 0$, then Definition 3 collapses to Definition 2.

We need

Definition 4 *(related to Definition 3) Let $f \in C^N([a,b])$, $N \in \mathbb{Z}_+$. Set*

$$F(x, x'-x; q, N) := D_x^q \left(f(x') - \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} (x'-x)^n\right). \tag{4}$$

Let $x' - x := t$, then $x' = x + t$, and

$$F(x, t; q, N) = D_x^q \left(f(x+t) - \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} t^n\right). \tag{5}$$

We make

Remark 5 *Here $x', x \in [a,b]$, and $a \leq x+t \leq b$, equivalently $a - x \leq t \leq b - x$. From $a \leq x \leq b$, we get $a - x \leq 0 \leq b - x$.*

We assume here that $F(x, \cdot; q, N) \in C^1([a-x, b-x])$. Clearly, then it holds

$$\mathcal{D}^{N+q} f(x) = F(x, 0; q, N), \tag{6}$$

and $\mathcal{D}^{N+q} f(x)$ exists in $\mathbb{R}$.

We make

Remark 6 *We observe that:*

I) Let $x' > x$ $(x' - x > 0)$ then

$$f(x') - \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} (x'-x)^n \overset{([2])}{=}$$

2

$$D_x^{-q}\left[D_x^q\left(f(x') - \sum_{n=0}^{N}\frac{f^{(n)}(x)}{n!}(x'-x)^n\right)\right] =$$

$$D_x^{-q}(F(x,x'-x;q,N)) = \frac{1}{\Gamma(q)}\int_x^{x'}(x'-z)^{q-1}F(x,z-x;q,N)\,dz =$$

$$\frac{1}{\Gamma(q)}\int_0^{x'-x}\frac{F(x,t;q,N)}{(x'-x-t)^{-q+1}}dt =$$

(integration by parts)

$$\frac{1}{\Gamma(q)}\left[F(x,t;q,N)\int(x'-x-t)^{q-1}\,dt\right]_0^{x'-x} + \tag{7}$$

$$\frac{1}{\Gamma(q)}\int_0^{x'-x}\frac{dF(x,t;q,N)}{dt}\frac{(x'-x-t)^q}{q}dt.$$

Thus,

$$f(x') - \sum_{n=0}^{N}\frac{f^{(n)}(x)}{n!}(x'-x)^n = \frac{\mathcal{D}^{N+q}f(x)}{\Gamma(q+1)}(x'-x)^q +$$

$$\frac{1}{\Gamma(q+1)}\int_0^{x'-x}\frac{dF(x,t;q,N)}{dt}(x'-x-t)^q\,dt, \quad \text{for } x' > x, \tag{8}$$

$N \in \mathbb{Z}_+$.

II) Let $x' < x$ $(x'-x<0)$: We have similarly,

$$f(x') - \sum_{n=0}^{N}\frac{f^{(n)}(x)}{n!}(x'-x)^n \overset{([2])}{=}$$

$$D_x^{-q}\left[D_x^q\left(f(x') - \sum_{n=0}^{N}\frac{f^{(n)}(x)}{n!}(x'-x)^n\right)\right] =$$

$$D_x^{-q}(F(x,x'-x;q,N)) = \frac{1}{\Gamma(q)}\int_{x'}^{x}(z-x')^{q-1}F(x,z-x;q,N)\,dz = \tag{9}$$

$$\frac{1}{\Gamma(q)}\int_{x'-x}^{0}(x-x'+t)^{q-1}F(x,t;q,N)\,dt =$$

(integration by parts)

$$\frac{1}{\Gamma(q)}\left[F(x,t;q,N)\int(t+x-x')^{q-1}\,dt\right]_{x'-x}^{0} -$$

$$\frac{1}{\Gamma(q)}\int_{x'-x}^{0}\frac{dF(x,t;q,N)}{dt}\frac{(t+x-x')^q}{q}dt = \tag{10}$$

3

$$\frac{1}{\Gamma(q)}\left[F(x,0;q,N)\frac{(x-x')^q}{q}\right]+\frac{1}{\Gamma(q)}\int_0^{x'-x}\frac{dF(x,t;q,N)}{dt}\frac{(t+x-x')^q}{q}dt=$$

$$\frac{1}{\Gamma(q+1)}\mathcal{D}^{N+q}f(x)(x-x')^q+\frac{1}{\Gamma(q+1)}\int_0^{x'-x}\frac{dF(x,t;q,N)}{dt}(t-x'+x)^q\,dt. \tag{11}$$

Conclusion:

We have proved that ($N\in\mathbb{Z}_+$)

I)

$$f(x')=\sum_{n=0}^{N}\frac{f^{(n)}(x)}{n!}(x'-x)^n+\frac{\mathcal{D}^{N+q}f(x)}{\Gamma(q+1)}(x'-x)^q+$$

$$\frac{1}{\Gamma(q+1)}\int_0^{x'-x}\frac{dF(x,t;q,N)}{dt}(x'-x-t)^q\,dt,\ \ when\ x'>x, \tag{12}$$

and

II)

$$f(x')=\sum_{n=0}^{N}\frac{f^{(n)}(x)}{n!}(x'-x)^n+\frac{\mathcal{D}^{N+q}f(x)(x-x')^q}{\Gamma(q+1)}+$$

$$\frac{1}{\Gamma(q+1)}\int_0^{x'-x}\frac{dF(x,t;q,N)}{dt}(t-x'+x)^q\,dt,\ \ when\ x'<x. \tag{13}$$

We have derived

Theorem 7 *Let $f\in C^N([a,b])$, $N\in\mathbb{Z}_+$. Here $x,x'\in[a,b]$, and $F(x,\cdot;q,N)\in C^1([a-x,b-x])$. Then*

$$f(x')=\sum_{n=0}^{N}\frac{f^{(n)}(x)}{n!}(x'-x)^n+\frac{\mathcal{D}^{N+q}f(x)}{\Gamma(q+1)}|x'-x|^q+ \tag{14}$$

$$\frac{1}{\Gamma(q+1)}\int_0^{x'-x}\frac{dF(x,t;q,N)}{dt}|(x'-x)-t|^q\,dt.$$

In particular we get

Corollary 8 *(to Theorem 7, $N=0$) Let $f\in C([a,b])$; $x,x'\in[a,b]$, and $F(x,\cdot;q,0)\in C^1([a-x,b-x])$. Then*

$$f(x')=f(x)+\frac{\mathcal{D}^qf(x)}{\Gamma(q+1)}|x'-x|^q+ \tag{15}$$

$$\frac{1}{\Gamma(q+1)}\int_0^{x'-x}\frac{dF(x,t;q,0)}{dt}|(x'-x)-t|^q\,dt.$$

References

[1] F.B. Adda, J.Cresson, *About Non-differential Functions*, J. Math. Anal. & Appl., 263 (2001), 721-737.

[2] F.B. Adda, J. Cresso, *Fractional differential equations and the schrödinger equation*, Applied Math. & Computation, 161 (2005), 323-345.

[3] K.M. Kolwankar, *Local fractional calculus: a Review*, arXiv: 1307:0739v1 [nlin.CD] 2 Jul. 2013.

[4] K.M. Kolwankar and A.D. Gangal, *Local fractional calculus: a calculus for fractal space-time*, *Fractals: theory and applications in engineering*, 171-181, London, New York, Springer, 1999.

[5] I. Podlubny, *Fractional Differential equations*, Academic Press, San Diego, New York, 1999.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

ON VORONOVSKAJA TYPE ESTIMATES OF BERNSTEIN-STANCU OPERATORS

RONGRONG XIA AND DANSHENG YU*

ABSTRACT. In the present paper, we obtain the Voronovaskaja-type results of approximation by a type of Bernstein-Stancu operators with shifted knots.

1. Introduction and The Main Results

For any $f(x) \in C_{[0,1]}$, the corresponding Bernstein operators $B_n(f, x)$ are defined as follows:

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x),$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, k = 0, 1, \cdots, n$. The approximation properties of Bernstein operators for continuous functions or functions of smoothness have been investigated extensively. Among them, many authors have studied the Voronovaskaja-type asymptotical estimates (see [5]-[7], [13]).

Stancu ([11]) generalized the Bernstein operators to the following so called Bernstein-Stancu operators:

$$B_{n,\alpha,\beta}(f; x) = \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right) p_{n,k}(x). \tag{1.1}$$

It was showed that $B_{n,\alpha,\beta}(f; x)$ converges to continuous function $f(x)$ uniformly in $[0, 1]$ for α,β satisfying $0 \le \alpha \le \beta$.

Recently, Gadjiev and Ghorbanalizadeh ([4]) further generalized Bernstein-Stancu operators by using shifted knots as follows:

$$S_{n,\alpha,\beta}(f; x) = \left(\frac{n+\beta_2}{n}\right)^n \sum_{k=0}^{n} f\left(\frac{k+\alpha_1}{n+\beta_1}\right) q_{n,k}(x), \tag{1.2}$$

where $x \in \left[\frac{\alpha_2}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}\right]$, $q_{n,k}(x) = \binom{n}{k}\left(x - \frac{\alpha_2}{n+\beta_2}\right)^k \left(\frac{n+\alpha_2}{n+\beta_2} - x\right)^{n-k}$, $k = 0, 1, \cdots, n$, and $\alpha_k, \beta_k, k = 1, 2$ are positive real numbers satisfying $0 \le \alpha_1 \le \beta_1$, $0 \le \alpha_2 \le \beta_2$. They estimated the approximation rate of approxiamtion by $S_{n,\alpha,\beta}(f, x)$ for continuous functions in A_n. In fact, they established the following:

2010 *Mathematics Subject Classification.* 41A25, 41A35.

Key words and phrases. Bernstein operators with shifted knots, Voronovaskaja-type results.

* Corresponding author.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Theorem 1.1. *([4]) Let f be continuous function on $[0,1]$. Then the following inequalities hold:*

$$|S_{n,\alpha,\beta}(f,x)-f(x)|\le\begin{cases}\frac{3}{2}\omega\left(f,\sqrt{\frac{4(\beta_2-\beta_1)^2\left(\frac{n+\alpha_2}{n+\beta_2}\right)^2+n}{(n+\beta_1)^2}}\right), & \text{if } (\beta_2-\beta_1)\ge(\alpha_2-\alpha_1),\\ \frac{3}{2}\omega\left(f,\sqrt{\frac{4(\alpha_2-\alpha_1)^2+n}{(n+\beta_1)^2}}\right), & \text{if } (\beta_2-\beta_1)\le(\alpha_2-\alpha_1).\end{cases}$$

In Theorem 1.1, the approximation properties of $S_{n,\alpha,\beta}(f,x)$ in $A_n:=\left[\frac{\alpha_2}{n+\beta_2},\frac{n+\alpha_2}{n+\beta_2}\right]$ are considered. As we know, $S_{n,\alpha,\beta}$ is positive and linear in the set A_n. Although, $S_{n,\alpha,\beta}$ is still definable on $[0,1]\backslash A_n$, but it is not positive in this case. Then, a natural problem is whether $S_{n,\alpha,\beta}(f,x)$ can be used to approximate the continuous functions on the whole interval $[0,1]$. Wang, Yu and Zhou ([14]) give a positive answer by establishing the following:

Theorem 1.2. *Let f be a continuous function on $[0,1]$, $\lambda\in[0,1]$ be a fixed positive number. Then there exists a positive constant C only depending on $\lambda,\alpha_1,\alpha_2,\beta_1$ and β_2 such that*

$$|S_{n,\alpha,\beta}(f,x)-f(x)|\le C\omega_{\varphi^\lambda}\left(f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right), \tag{1.3}$$

where $\varphi(x)=\sqrt{x(1-x)}$, $\delta_n(x):=\varphi(x)+\frac{1}{\sqrt{n}}$, and

$$\omega_{\varphi^\lambda}(f,t):=\sup_{0<h\le t}\sup_{x\pm\frac{h\phi^\lambda(x)}{2}\in[0,1]}\left|f\left(x+\frac{h\varphi^\lambda(x)}{2}\right)-f\left(x-\frac{h\varphi^\lambda(x)}{2}\right)\right|.$$

Many authors have generalized $S_{n,\alpha,\beta}(f,x)$ in many ways (see [1], [3], [8]-[10], [12]).

Our purpose of the paper is to give the Voronovskaja type estimates of approximation by $S_{n,\alpha,\beta}(f,x)$ on A_n.

Theorem 1.3. *Let $f\in C^2(A_n)$, $\lambda\in[0,1]$ be a fixed positive number. Then there exists a positive constant C only depending on $\lambda,\alpha_1,\alpha_2,\beta_1$ and β_2 such that*

$$\left|S_{n,\alpha,\beta}(f,x)-f(\theta_n(x))-\frac{1}{2}f''(x)M_n(x)\right|\le C\frac{\delta_n^2(x)}{n}\omega_{\phi^\lambda}\left(f'',\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)_{A_n}, \tag{1.4}$$

where $\delta_n(x)=\phi(x)+\frac{1}{\sqrt{n}}$, $\phi(x)=\sqrt{\left(x-\frac{\alpha_2}{n+\beta_2}\right)\left(\frac{n+\alpha_2}{n+\beta_2}-x\right)}$,

$$\begin{aligned}M_n(x) &:= -\frac{1}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)^2\left(x-\frac{\alpha_2}{n+\beta_2}\right)^2+\left(\frac{(n+\beta_2)(1+2\alpha_2)}{(n+\beta_1)^2}-\frac{2\alpha_1}{n+\beta_2}\right)\left(x-\frac{\alpha_2}{n+\beta_2}\right)\\ &\quad+\frac{\alpha_1^2}{(n+\beta_1)^2}-\frac{\alpha_1^2}{(n+\beta_2)^2},\end{aligned}$$

$$\theta_n(x):=S_{n,\alpha,\beta}(t,x)=\left(\frac{n+\beta_2}{n+\beta_1}\right)x-\left(\frac{\alpha_2-\alpha_1}{n+\beta_1}\right).$$

When $\alpha_1=\alpha_2=\beta_1=\beta_2=0$, we get the results of [7] for Bernstein operators. Noting that $|\theta(x)-x|\le\left|\frac{(\beta_2-\beta_1)x-\alpha_2+\alpha_1}{n+\beta_1}\right|$, we have

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Corollary 1. *Let $f \in C^2(A_n)$, $\lambda \in [0,1]$ be a fixed positive number. Then there exists a positive constants C_1 and C_2 only depending on $\lambda, \alpha_1, \alpha_2, \beta_1$ and β_2 such that*

$$\begin{aligned}\left|S_{n,\alpha,\beta}(f,x) - f(x) - \frac{1}{2}f''(x)M_n(x)\right| \leq\ & C_1\left(\frac{\delta_n^2(x)}{n}\omega_{\phi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)_{A_n}\right.\\ & \left.+\omega\left(f, \frac{(\beta_2-\beta_1)x-\alpha_2+\alpha_1}{n+\beta_1}\right)_{A_n}\right), \quad (1.5)\end{aligned}$$

where $\omega(f,t)_{A_n}$ is the usual modulus of continuity of f on A_n.

Throughout the paper, C denotes either a positive absolute constant or a positive constant that may depend on some parameters but not on f, x and n, their values may be different at different occurrences. The symbol $x \sim y$ means that there exists a positive constant C such that $C^{-1} \leq x \leq Cy$.

2. Auxiliary Lemmas

Lemma 2.1. *([3], Lemma 3) For any given $\gamma \geq 0$, we have*

$$\Delta_{n,\gamma}(x) := \sum_{k=0}^{n}\left|\frac{k+\alpha_1}{n+\beta_1} - x\right|^\gamma q_{n,k}(x) \leq C\frac{\delta_n^\gamma(x)}{n^{\frac{\gamma}{2}}}, \quad x \in A_n. \quad (2.1)$$

Lemma 2.2. *If $g \in D_\lambda := \{g : g' \in AC_{loc}, \|\phi^\lambda g'\| < \infty, \|g'\| < \infty\}$, then for any $x \in A_n = [\frac{\alpha_2}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}]$, we have*

$$\left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)(g(u)-g(x))du, x\right)\right| \leq C\frac{\delta_n^2(x)}{n}\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\phi^\lambda g'\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{1}{1-\frac{\lambda}{2}}}\|g'\|\right). \quad (2.2)$$

Proof. We need the following inequality:

$$\int_x^t \frac{1}{\phi^\lambda(u)}du \leq C\frac{|t-x|}{\phi^\lambda(x)}, \text{ for any } x,t \in A_n. \quad (2.3)$$

In fact, when $\lambda = 1$, (2.3) is obvious. When $0 < \lambda \leq 1$, we have (by using Hölder's inequality for $0 < \lambda < 1$)

$$\begin{aligned}
\int_x^t \frac{1}{\phi^\lambda(u)}du &\leq \left|\int_x^t \frac{1}{\phi(u)}du\right|^\lambda \left|\int_x^t du\right|^{1-\lambda} \\
&\leq C|t-x|^{1-\lambda}\left|\int_x^t \left(\frac{1}{\sqrt{u-\frac{\alpha_2}{n+\beta_2}}}+\frac{1}{\sqrt{\frac{n+\alpha_2}{n+\beta_2}-u}}\right)du\right|^\lambda \\
&\leq C|t-x|^{1-\lambda}\left(\left|\sqrt{t-\frac{\alpha_2}{n+\beta_2}}-\sqrt{x-\frac{\alpha_2}{n+\beta_2}}\right|+\left|\sqrt{\frac{n+\alpha_2}{n+\beta_2}-t}-\sqrt{\frac{n+\alpha_2}{n+\beta_2}-x}\right|\right)^\lambda \\
&\leq C|t-x|\left(\frac{1}{\sqrt{t-\frac{\alpha_2}{n+\beta_2}}+\sqrt{x-\frac{\alpha_2}{n+\beta_2}}}+\frac{1}{\sqrt{\frac{n+\alpha_2}{n+\beta_2}-t}+\sqrt{\frac{n+\alpha_2}{n+\beta_2}-x}}\right)^\lambda \\
&\leq C|t-x|\left(\frac{1}{\sqrt{x-\frac{\alpha_2}{n+\beta_2}}}+\frac{1}{\sqrt{\frac{n+\alpha_2}{n+\beta_2}-x}}\right)^\lambda \\
&\leq C\frac{|t-x|}{\phi^\lambda(x)}, \qquad (2.4)
\end{aligned}$$

which proves (2.3).

Now, we prove (2.2) by considering the following two difference cases: $x \in B_n = \left[\frac{\alpha_2+1}{n+\beta_2}, \frac{n+\alpha_2-1}{n+\beta_2}\right]$ and $x \in B_n^c = [\frac{\alpha_2}{n+\beta_2}, \frac{\alpha_2+1}{n+\beta_2}] \cup [\frac{n+\alpha_2-1}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}]$, respectively.

When $x \in B_n = \left[\frac{\alpha_2+1}{n+\beta_2}, \frac{n+\alpha_2-1}{n+\beta_2}\right]$, we have

$$\phi(x) \geq \min\left(\phi\left(\frac{\alpha_2+1}{n+\beta_2}\right), \phi\left(\frac{n+\alpha_2-1}{n+\beta_2}\right)\right) \geq \frac{C}{\sqrt{n}},$$

which means that

$$\delta_n(x) \sim \phi(x), \quad x \in B_n. \qquad (2.5)$$

Then, by Lemma 2.1, we have

$$\begin{aligned}
&\left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)(g(u)-g(x))du, x\right)\right| \\
\leq\ &\left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)\left(\int_x^u \frac{\phi^\lambda(s)g'(s)}{\phi^\lambda(s)}ds\right)du, x\right)\right| \\
\leq\ &C\|\phi^\lambda g'\|\left|S_{n,\alpha,\beta}\left(\int_x^t |t-u|\frac{|x-u|}{\phi^\lambda(x)}du, x\right)\right| \\
\leq\ &C\frac{\|\phi^\lambda g'\|}{\phi^\lambda(x)}\left|S_{n,\alpha,\beta}\left(|t-x|^3, x\right)\right| \\
=\ &C\frac{\|\phi^\lambda g'\|}{\phi^\lambda(x)}\left(\frac{n+\beta_2}{n}\right)^n\sum_{k=0}^n\left|\frac{k+\alpha_1}{n+\beta_1}-x\right|^3|q_{n,k}(x)| \\
\leq\ &C\frac{\delta_n^2(x)}{n}\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\phi^\lambda g'\|. \qquad (2.6)
\end{aligned}$$

When $x \in B_n^c = [\frac{\alpha_2}{n+\beta_2}, \frac{\alpha_2+1}{n+\beta_2}] \cup [\frac{n+\alpha_2-1}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}]$, we have $\delta_n(x) \sim \frac{1}{\sqrt{n}}$. Then,

$$\begin{aligned}\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\delta_n^\lambda g'\| &\leq C\frac{\delta_n^2(x)}{n}\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\phi^\lambda g'\| + \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}\right)^\lambda \|g'\|\right) \\ &\leq C\frac{\delta_n^2(x)}{n}\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\phi^\lambda g'\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{1}{1-\frac{\lambda}{2}}} \|g'\|\right). \qquad (2.7)\end{aligned}$$

By Lemma 2.1 again, we get

$$\begin{aligned}&\left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)(g(u)-g(x))du, x\right)\right| \\ \leq\ & C\|\delta_n^\lambda g'\| \left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)\left(\int_x^u \frac{1}{\delta_n^\lambda(s)}ds\right)du, x\right)\right| \\ \leq\ & C\|\delta_n^\lambda g'\| \left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)^2\left(\frac{1}{\delta_n^\lambda(u)} + \frac{1}{\delta_n^\lambda(s)}\right)du, x\right)\right| \\ \leq\ & C\|\delta_n^\lambda g'\| \left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)^2\frac{1}{\delta_n^\lambda(u)}du, x\right)\right| \\ \leq\ & C\|\delta_n^\lambda g'\| \sum_{k=0}^n \left(\frac{1}{\delta_n^\lambda(x)} + \frac{1}{\delta_n^\lambda\left(\frac{k+\alpha_1}{n+\beta_1}\right)}\right)\left|x - \frac{k+\alpha_1}{n+\beta_1}\right|^3 q_{n,k}(x) \\ \leq\ & C\frac{\|\delta_n^\lambda g'\|}{\delta_n^\lambda(x)} \sum_{k=0}^n \left|x - \frac{k+\alpha_1}{n+\beta_1}\right|^3 q_{n,k}(x) \\ \leq\ & C\frac{\delta_n^2(x)}{n}\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\phi^\lambda g'\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{1}{1-\frac{\lambda}{2}}} \|g'\|\right). \qquad (2.8)\end{aligned}$$

We prove Lemma 2.2 by combining (2.6), (2.7) and (2.8). □

Lemma 2.3. *Under the conditions of Lemma 2.2, we have for $x \in A_n = [\frac{\alpha_2}{n+\beta_2}, \frac{n+\alpha_2}{n+\beta_2}]$ that*

$$\left|\int_x^{S_{n,\alpha,\beta}(t,x)} (S_{n,\alpha,\beta}(t,x) - u)(g(u)-g(x))du\right| \leq C\frac{\delta_n^2(x)}{n}\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\phi^\lambda g'\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{1}{1-\frac{\lambda}{2}}} \|g'\|\right). \qquad (2.9)$$

Proof. If $x \in B_n = [\frac{\alpha_2+1}{n+\beta_2}, \frac{n+\alpha_2-1}{n+\beta_2}]$, by (2.3) and the fact (see, [4])

$$S_{n,\alpha,\beta}(t,x) = \left(\frac{n+\beta_2}{n+\beta_1}\right)x - \left(\frac{\alpha_2-\alpha_1}{n+\beta_1}\right). \qquad (2.10)$$

we have for any $\gamma \geq 0$ that

$$|S_{n,\alpha,\beta}(t,x) - x|^\gamma \leq \frac{C}{n^\gamma}. \qquad (2.11)$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

By (2.3) and (2.11), we get

$$\begin{aligned}
&\left|\int_x^{S_{n,\alpha,\beta}(t,x)}(S_{n,\alpha,\beta}(t,x)-u)(g(u)-g(x))du\right| \\
\leq\ & \left|\int_x^{S_{n,\alpha,\beta}(t,x)}(S_{n,\alpha,\beta}(t,x)-u)\left(\int_x^u g'(s)ds\right)du\right| \\
\leq\ & C\|\phi^\lambda g'\|\left|\int_x^{S_{n,\alpha,\beta}(t,x)}(S_{n,\alpha,\beta}(t,x)-u)\left(\int_x^u \frac{1}{\phi^\lambda(s)}ds\right)du\right| \\
\leq\ & C\frac{\|\phi^\lambda g'\|}{\phi^\lambda(x)}|S_{n,\alpha,\beta}(t,x)-x|^3 \\
\leq\ & C\frac{1}{n^3}\frac{\|\phi^\lambda g'\|}{\phi^\lambda(x)} \\
\leq\ & C\frac{\delta_n^2(x)}{n}\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\phi^\lambda g'\|. \qquad (2.12)
\end{aligned}$$

If $x\in B_n^c=\left[\frac{\alpha_2}{n+\beta_2},\frac{\alpha_2+1}{n+\beta_2}\right]\cup\left[\frac{n+\alpha_2-1}{n+\beta_2},\frac{n+\alpha_2}{n+\beta_2}\right]$, we have

$$S_{n,\alpha,\beta}(t,x)=\left(\frac{n+\beta_2}{n+\beta_1}\right)x-\left(\frac{\alpha_2-\alpha_1}{n+\beta_1}\right)\in\left[\frac{\alpha_1}{n+\beta_2},\frac{\alpha_1+1}{n+\beta_2}\right]\cup\left[\frac{n+\alpha_1-1}{n+\beta_2},\frac{n+\alpha_1}{n+\beta_2}\right].$$

It is easy to observe that

$$\begin{aligned}
\delta_n(S_{n,\alpha,\beta}(t,x)) &= \delta_n\left(\frac{n+\beta_2}{n+\beta_1}x-\frac{\alpha_2-\alpha_1}{n+\beta_1}\right) \\
&= \sqrt{\left[\frac{n+\beta_2}{n+\beta_1}x-\frac{\alpha_2-\alpha_1}{n+\beta_1}-\frac{\alpha_2}{n+\beta_1}\right]\left[\frac{n+\alpha_2}{n+\beta_1}-\frac{n+\beta_2}{n+\beta_1}x+\frac{\alpha_2-\alpha_1}{n+\beta_1}\right]}+\frac{1}{\sqrt{n}} \\
&= \sqrt{\left[\frac{n+\beta_2}{n+\beta_1}x-\frac{2\alpha_2-\alpha_1}{n+\beta_1}\right]\left[\frac{n+2\alpha_2-\alpha_1}{n+\beta_1}-\frac{n+\beta_2}{n+\beta_1}x\right]}+\frac{1}{\sqrt{n}} \\
&\sim \delta_n(x)\sim\frac{1}{\sqrt{n}}.
\end{aligned}$$

Therefore, by (2.11) again, we get

$$\begin{aligned}
&\left|\int_x^{S_{n,\alpha,\beta}(t,x)}(S_{n,\alpha,\beta}(t,x)-u)(g(u)-g(x))du\right| \\
\leq\ & C\|\delta_n^\lambda g'\|\left|\int_x^{S_{n,\alpha,\beta}(t,x)}(S_{n,\alpha,\beta}(t,x)-u)^2\left(\frac{1}{\delta_n^\lambda(x)}+\frac{1}{\delta_n^\lambda(u)}\right)du\right| \\
\leq\ & C\|\delta_n^\lambda g'\|\left||S_{n,\alpha,\beta}(t,x)-x|^3\cdot\left(\frac{1}{\delta_n^\lambda(x)}+\frac{1}{\delta_n^\lambda(S_{n,\alpha,\beta}(t,x))}\right)\right| \\
\leq\ & C\frac{\delta_n^2(x)}{n}\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\delta_n^\lambda g'\|. \qquad (2.13)
\end{aligned}$$

We get Lemma 2.3 by combining (2.7), (2.12) and (2.13). □

3. Proof of Results

Define the auxiliary operators $\overline{S}_{n,\alpha,\beta}(f,x)$ as follows:

$$\overline{S}_{n,\alpha,\beta}(f,x) = S_{n,\alpha,\beta}(f,x) + L_{n,\alpha,\beta}(f,x), \tag{3.1}$$

where $L_{n,\alpha,\beta}(f,x) = f(x) - f(\theta_n(x))$, and $\theta_n(x) = S_{n,\alpha,\beta}(t,x)$.

It follows from the facts $S_{n,\alpha,\beta}(1,x) = 1$ and (2.10) that

$$\overline{S}_{n,\alpha,\beta}(1,x) = 1, \;\; \overline{S}_{n,\alpha,\beta}((t-x),x) = 0. \tag{3.2}$$

For any $f(x) \in C(A_n)$, $0 \le \lambda \le 1$, define the K-functional:

$$K_{\varphi^\lambda}(f,t) := \inf_{g \in A.C_{loc}} \left\{ \|f-g\| + t\left\|\phi^\lambda g'\right\| + t^{\frac{1}{1-\frac{\lambda}{2}}} \|g'\| \right\},$$

Then ([2])

$$K_{\phi^\lambda}(f,t) \sim \omega_{\phi^\lambda}(f,t).$$

Then, by taking $t = \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}$, there is a $g \in AC_{loc}$ such that

$$\left\|f''-g\right\| \le C\omega_{\phi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)_{A_n}. \tag{3.3}$$

$$\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\left\|\phi^\lambda g'\right\| \le C\omega_{\phi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)_{A_n}. \tag{3.4}$$

$$\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{1}{1-\frac{\lambda}{2}}} \|g'\| \le C\omega_{\phi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)_{A_n}. \tag{3.5}$$

By (3.1), we have

$$\left|S_{n,\alpha,\beta}(f,x) - f(\theta_n(x)) - \frac{1}{2}f''(x)M_n(x)\right| = \left|\overline{S}_{n,\alpha,\beta}(f,x) - f(x) - \frac{1}{2}f''(x)M_n(x)\right|.$$

Hence, we only need to prove the following inequality:

$$\left|\overline{S}_{n,\alpha,\beta}(f,x) - f(x) - \frac{1}{2}f''(x)M_n(x)\right| \le C\frac{\delta_n^2(x)}{n}\omega_{\phi^\lambda}\left(f'', \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)_{A_n}. \tag{3.6}$$

It follows from (3.1) that

$$\begin{aligned}
\overline{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)du, x\right) &= S_{n,\alpha,\beta}\left(\int_x^t (t-u)du, x\right) - \int_x^{S_{n,\alpha,\beta}(t,x)} \left(S_{n,\alpha,\beta}(t,x) - u\right) du \\
&= \frac{1}{2}\left[S_{n,\alpha,\beta}\left((t-x)^2, x\right) - \left(S_{n,\alpha,\beta}(t,x) - x\right)^2\right] \\
&= \frac{1}{2}\left[S_{n,\alpha,\beta}(t^2,x) - S^2_{n,\alpha,\beta}(t,x)\right]
\end{aligned}$$

It was proved in ([4]) that

$$\begin{aligned}
S_{n,\alpha,\beta}(t^2,x) &= \left(\frac{n+\beta_2}{n+\beta_1}\right)^2\left(x - \frac{\alpha_2}{n+\beta_2}\right)^2 - \frac{1}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)^2\left(x - \frac{\alpha_2}{n+\beta_2}\right)^2 \\
&+ \left(\frac{n+\beta_2}{n+\beta_1}\right)\frac{1}{n+\beta_1}\left(x - \frac{\alpha_2}{n+\beta_2}\right) + \left(\frac{n+\beta_2}{n+\beta_1}\right)\frac{2\alpha_2}{n+\beta_1}\left(x - \frac{\alpha_2}{n+\beta_2}\right) + \frac{\alpha_1^2}{(n+\beta_1)^2}.
\end{aligned}$$

By (2.10), we can rewrite $S_{n,\alpha,\beta}(t,x)$ as follows:

$$\begin{aligned} S_{n,\alpha,\beta}(t,x) &= \left(\frac{n+\beta_2}{n+\beta_1}\right)x-\left(\frac{\alpha_2-\alpha_1}{n+\beta_1}\right)=\frac{n+\beta_2}{n+\beta_1}\left(x-\frac{\alpha_2-\alpha_1}{n+\beta_2}\right)\\ &= \frac{n+\beta_2}{n+\beta_1}\left(x-\frac{\alpha_2}{n+\beta_2}+\frac{\alpha_1}{n+\beta_2}\right), \end{aligned}$$

which means that

$$\begin{aligned} (S_{n,\alpha,\beta}(t,x))^2 &= \left(\frac{n+\beta_2}{n+\beta_1}\right)^2\left(x-\frac{\alpha_2}{n+\beta_2}+\frac{\alpha_1}{n+\beta_2}\right)^2\\ &= \left(\frac{n+\beta_2}{n+\beta_1}\right)^2\left(x-\frac{\alpha_2}{n+\beta_2}\right)^2+\left(\frac{\alpha_1}{n+\beta_2}\right)^2+\frac{2\alpha_1}{n+\beta_2}\cdot\left(x-\frac{\alpha_2}{n+\beta_2}\right). \end{aligned}$$

Then

$$\begin{aligned} S_{n,\alpha,\beta}(t^2,x)-S^2_{n,\alpha,\beta}(t,x) &= \left(\frac{n+\beta_2}{n+\beta_1}\right)^2\left(x-\frac{\alpha_2}{n+\beta_2}\right)^2-\frac{1}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)^2\left(x-\frac{\alpha_2}{n+\beta_2}\right)^2\\ &\quad+\left(\frac{n+\beta_2}{n+\beta_1}\right)\frac{1}{n+\beta_1}\left(x-\frac{\alpha_2}{n+\beta_2}\right)+\left(\frac{n+\beta_2}{n+\beta_1}\right)\frac{2\alpha_1}{n+\beta_1}\left(x-\frac{\alpha_2}{n+\beta_2}\right)\\ &\quad+\frac{\alpha_1^2}{(n+\beta_1)^2}-\left(\frac{n+\beta_2}{n+\beta_1}\right)^2\left(x-\frac{\alpha_2}{n+\beta_2}\right)^2-\left(\frac{\alpha_1}{n+\beta_2}\right)^2\\ &\quad-\frac{2\alpha_1}{n+\beta_2}\cdot\left(\frac{\alpha_2}{n+\beta_2}\right)\\ &= -\frac{1}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)^2\left(x-\frac{\alpha_2}{n+\beta_2}\right)^2+\left(\frac{n+\beta_2}{n+\beta_1}\right)\frac{1}{n+\beta_1}\left(x-\frac{\alpha_2}{n+\beta_2}\right)\\ &\quad+\frac{n+\beta_2}{n+\beta_1}\cdot\frac{2\alpha_1}{n+\beta_1}\left(x-\frac{\alpha_2}{n+\beta_2}\right)+\frac{\alpha_1^2}{(n+\beta_1)^2}\\ &\quad-\frac{\alpha_1^2}{(n+\beta_2)}-\frac{2\alpha_1}{n+\beta_2}\left(x-\frac{\alpha_2}{n+\beta_2}\right)\\ &= -\frac{1}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)^2\left(x-\frac{\alpha_2}{n+\beta_2}\right)^2+\left[\frac{n+\beta_2}{(n+\beta_1)^2}+\frac{2\alpha_1(n+\beta_2)}{(n+\beta_1)^2}-\frac{2\alpha_1}{n+\beta_2}\right]\\ &\cdot\quad\left(x-\frac{\alpha_2}{n+\beta_2}\right)+\frac{\alpha_1^2}{(n+\beta_1)^2}-\frac{\alpha_1^2}{(n+\beta_2)^2}\\ &= -\frac{1}{n}\left(\frac{n+\beta_2}{n+\beta_1}\right)^2\left(x-\frac{\alpha_2}{n+\beta_2}\right)^2+\left(\frac{(n+\beta_2)(1+2\alpha_1)}{(n+\beta_1)^2}-\frac{2\alpha_1}{n+\beta_2}\right)\\ &\quad\times\left(x-\frac{\alpha_2}{n+\beta_2}\right)+\frac{\alpha_1^2}{(n+\beta_1)^2}-\frac{\alpha_1^2}{(n+\beta_2)^2}\\ &=: M_n(x) \end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

By Taylor's formula: $f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u)f''(u)du$, we have

$$\begin{aligned}
& \left|\overline{S}_{n,\alpha,\beta}(f,x) - f(x) - \frac{1}{2}f''(x)M_n(x)\right| \\
\leq\ & \left|\overline{S}_{n,\alpha,\beta}(f,x) - f(x) - f''(x)\overline{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)du, x\right)\right| \\
\leq\ & \left|\overline{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)f''(u)du, x\right) - \overline{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)f''(x)du, x\right)\right| \\
\leq\ & \left|\overline{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)(f''(u) - f''(x))du, x\right)\right| \\
\leq\ & \left|\overline{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)|f''(u) - g(u)|du, x\right)\right| + \left|\overline{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)|f''(x) - g(x)|du, x\right)\right| \\
+\ & \left|\overline{S}_{n,\alpha,\beta}\left(\int_x^t (t-u)|g(u) - g(x)|du, x\right)\right| \\
=:\ & I_1 + I_2 + I_2. \qquad (3.7)
\end{aligned}$$

For I_1, by (2.11), (3.1) and Lemma 2.1, we have

$$\begin{aligned}
I_1 \ \leq\ & \left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)(f''(u) - g(u))du, x\right)\right| \\
+\ & \left|\int_x^{S_{n,\alpha,\beta}(t,x)} (S_{n,\alpha,\beta}(t,x) - u)|f''(u) - g(u)|du\right| \\
\leq\ & C\|f'' - g\|\left(\frac{\delta_n^2(x)}{n} + \frac{1}{n^2}\right) \\
\leq\ & C\frac{\delta_n^2(x)}{n}\|f'' - g\|. \qquad (3.8)
\end{aligned}$$

Similarly, we also have

$$I_2 \leq C\frac{\delta_n^2(x)}{n}\|f'' - g\|. \qquad (3.9)$$

For I_3, by Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}
I_3 \ \leq\ & \left|S_{n,\alpha,\beta}\left(\int_x^t (t-u)(g(u) - g(x))du, x\right)\right| \\
+\ & \left|\int_x^{S_{n,\alpha,\beta}(t,x)} (S_{n,\alpha,\beta}(t,x) - u)(g(u) - g(x))du\right| \\
\leq\ & C\frac{\delta_n^2(x)}{n}\left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\|\phi^\lambda g'\| + \left(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right)^{\frac{1}{1-\frac{\lambda}{2}}}\|g'\|\right). \qquad (3.10)
\end{aligned}$$

We finish the proof of (3.6) by combining (3.3)-(3.5), (3.7)-(3.10) .

References

[1] T. Acar, A. Aral, V. Gupta, On approximation properties of a new type Bernstein-Durrmeyer operators, Math. Slovaca, 65(2015), 1107-1122.

[2] Z. Ditzian and V. Totik, Moduli of Smoothness, Springer-Verlag(Berlin, New York, 1987).

[3] L. X. Dong, D. S. Yu, Pointwise approximation by a Durrmeyer variant of Bernstein-Stancu operators, Journal of Inequalities and Applications, 2017, 28: 2-13.

[4] A. D. Gadjiev, A. M. Ghorbanalizaeh. Approximation properties of a new type Bernsten-Stancu polynomials of one and two variables, Appl Math Comput, 2010, 216(3):890-901.

[5] H. Gonska, On the degree of approximation in Voronovskaja's theorem, Studia Univ. "Babes-Bolyai", Mathematica, 52(2007), 103-115.

[6] H. Gonska, P. Pitul, I. Rasa, On Peano's form of Taylor remainer, Voronoskaja's theorem and the commutator of positive linear operators. In:"Numerical Analysis and Approximation Theory" (Proc. Int. Conf. Cluj-Napoca 2006; ed. by O.Agratini and P.Blaga), 55-80. Cluj-Napoca: Casa Cartii de Stiinta 2006.

[7] H. Gonska, G. Tachev, A Quantitative variant of voronovskaja's theorem, Result. Math., 2009, 53(3-4): 287-294.

[8] G. İçöz, A Kantorovich variant of a new type Bernstein-Stancu polynomials, Appl. Math. Comput., 218(2012), 8552-8560.

[9] H. S. Jung, N. Deo, M. Dhamija, Pointwise approximation by Bernstein type operators in mobile interval, Appl. Math. Comput., 244 (2014) 683-694.

[10] S. Sezgin, E. Ibikli, Approximation by means of Kantorovich-Stancu type operators, Numer. Func. Anal. Opt., 34(2013), 557-575.

[11] D. D. Stancu, Approimation of functions by a new class of linear polynomial operators, Rev. Roum. Math. Pure Appl., 1968, 13(8): 1173-1194.

[12] F. Taşdelen, G. Başcanbaz-Tunca, A. Erençin, On a new type Bernstein-Stancu operators, Fasci. Math., 48(2012), 119-128.

[13] V. S. Videnskij, Linear Positive Opeartors of Finite Rank (Russian), Leningrd:"A.I.Gerzen" State Pedagogical Institute 1998.

[14] M. L. Wang and D. S. Yu, P. Zhou. On the approximatin by operators of Bernstein-Stancu type, Appl Math Comput, 2014, 246(1):79-87.

Department of Mathematics, Hangzhou Normal University, Hangzhou, Zhejiang 310036, China.
E-mail address: `xuanxia2014@163.com`

Department of Mathematics, Hangzhou Normal University, Hangzhou, Zhejiang 310036, China.
E-mail address: `dsyu@hznu.edu.cn`

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Double-sided Inequalities of Ostrowski's Type and Some Applications

Waseem Ghazi Alshanti and Gradimir V. Milovanović

Abstract

We construct a new general Ostrowski type inequality for differentiable mappings whose first derivatives are bounded in terms of pre-assigned continuous functions. Applications to composite quadrature rules are also given.

1 Introduction

In 1938, A. Ostrowski [14] introduced the following interesting and useful integral inequality for differentiable mappings with bounded derivatives:

Theorem 1.1 *Let $f : [a,b] \to \mathbb{R}$ be continuous mapping on $[a,b]$ and differentiable on (a,b), whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in[a,b]} |f'(t)| < \infty$, then for all $x \in [a,b]$*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\,\|f'\|_\infty . \tag{1.1}$$

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

Ostrowski's inequality is one of the most famous inequalities in the integral calculus. It measures the deviation of a function from its integral mean. Also, an estimation of approximating area under the curve of a function by a rectangle can be obtained in this case.

In 1975, Milovanović [10] (see also [12, pp. 26–29]) proposed a generalization of (1.1) for a function f of several variables as follows:

Theorem 1.2 *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a differentiable function defined on $\overline{D}$ and let $\left|\frac{\partial f}{\partial x_i}\right| \le M_i$ in D, where $M_i > 0$ for each $i = 1,\dots,m$. Then, for every $X = (x_1,\dots,x_m) \in \overline{D}$, we have*

$$\begin{aligned} \Bigg| f(x_1,\dots,x_m) &- \frac{1}{\prod_{i=1}^{n}(b_i-a_i)} \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} f(y_1,\dots,y_m)\,\mathrm{d}y_1\cdots\mathrm{d}y_m \Bigg| \\ &\le \sum_{i=1}^{m} \left[\frac{1}{4} + \frac{\left(x_i - \frac{a_i+b_i}{2}\right)^2}{(b_i-a_i)^2} \right] (b_i-a_i)\,M_i. \end{aligned}$$

1

One year later, in 1976, Milovanović and Pečarić [11] presented the following generalization when $\left|f^{(n)}(x)\right| \leq M$ $(\forall x \in (a,b))$, and $n > 1$:

Theorem 1.3 *Let $f : \mathbb{R} \to \mathbb{R}$ be $n\,(>1)$ times differentiable function such that $\left|f^{(n)}(x)\right| \leq M$ $\big(\forall x \in (a,b)\big)$. Then, for every $x \in [a,b]$*

$$\left|\frac{1}{n}\left(f(x) + \sum_{k=1}^{n-1} F_k\right) - \frac{1}{b-a}\int_a^b f(y)\mathrm{d}y\right| \leq \frac{M}{n(n+1)!} \cdot \frac{(x-a)^{n+1} + (b-x)^{n+1}}{b-a},$$

where F_k is defined by

$$F_k \equiv F_k(f;n;x;a;b) \equiv \frac{n-k}{k!} \cdot \frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a}.$$

For $n = 2$, Theorem 1.3 gives

$$\begin{aligned}\left|\frac{1}{2}\left(f(x) \right.\right. & + \left.\left.\frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right) - \frac{1}{b-a}\int_a^b f(y)\mathrm{d}y\right| \\ & \leq \frac{M\,(b-a)^2}{4}\left[\frac{1}{12} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right].\end{aligned}$$

2 Preliminaries

Associated with differentiable mappings, there has been extensive research in the literature on related results. Over the past few decades, many studies on obtaining sharp bounds of Ostrowski's tpye inequalities have been conducted. Most of the calculations within these sharp bounds depend mainly on the magnitudes of Lebesgue norms of derivatives of given functions.

In [5]–[8], Dragomir and Wang obtained the following bounds on the deviation of an absolutely continuous mapping f, defined over the interval $[a,b]$, from its integral mean

$$\left|f(x) - \frac{1}{b-a}\int_a^x f(t)\mathrm{d}t\right| \leq \begin{cases} \left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right]\frac{\|f'\|_\infty}{b-a}, & f' \in L_\infty[a,b]; \\ \frac{1}{q+1}\left[(x-a)^{q+1} + (b-x)^{q+1}\right]^{1/q}\frac{\|f'\|_p}{b-a}, & \begin{array}{l} f' \in L_p[a,b], \\ \frac{1}{p}+\frac{1}{q}=1,\ p>1; \end{array} \\ \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]\frac{\|f'\|_1}{b-a}, & f' \in L_1[a,b]. \end{cases}$$

In [9], Masjed-Jamei and Dragomir provided the following analogues of the Ostrowski's inequality for a differentiable function f whose first derivative f' is bounded, bounded from below, and bounded from above in terms of two functions $\alpha, \beta \in C[a,b]$ as follows:

Theorem 2.1 *Let $f : I \to \mathbb{R}$, where I is an interval, be a function differentiable in the interior $\mathring{I}$ of I, and let $[a,b] \subset \mathring{I}$. For any $\alpha, \beta \in C[a,b]$ and $x \in [a,b]$, we have the following three cases:*

2

1° *If* $\alpha(x) \le f'(x) \le \beta(x)$, *then*

$$\frac{1}{b-a}\left(\int_a^x (t-a)\,\alpha(t)\mathrm{d}t + \int_x^b (t-b)\,\beta(t)\mathrm{d}t\right) \le f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$$
$$\le \frac{1}{b-a}\left(\int_a^x (t-a)\,\beta(t)\mathrm{d}t + \int_x^b (t-b)\,\alpha(t)\mathrm{d}t\right); \tag{2.1}$$

2° *If* $\alpha(x) \le f'(x)$, *then*

$$\frac{1}{b-a}\Bigg[\int_a^x (t-a)\alpha(t)\mathrm{d}t + \int_x^b (t-b)\alpha(t)\mathrm{d}t - \max\{x-a, b-x\}\left(f(b)-f(a)-\int_a^b \alpha(t)\mathrm{d}t\right)\Bigg]$$
$$\le f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$$
$$\le \frac{1}{b-a}\Bigg[\int_a^x (t-a)\alpha(t)\mathrm{d}t + \int_x^b (t-b)\alpha(t)\mathrm{d}t + \max\{x-a, b-x\}\left(f(b)-f(a)-\int_a^b \alpha(t)\mathrm{d}t\right)\Bigg], \tag{2.2}$$

3° *If* $f'(x) \le \beta(x)$, *then*

$$\frac{1}{b-a}\Bigg[\int_a^x (t-a)\beta(t)\mathrm{d}t + \int_x^b (t-b)\beta(t)\mathrm{d}t - \max\{x-a, b-x\}\left(\int_a^b \beta(t)\mathrm{d}t - f(b)+f(a)\right)\Bigg]$$
$$\le f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$$
$$\le \frac{1}{b-a}\Bigg[\int_a^x (t-a)\beta(t)\mathrm{d}t + \int_x^b (t-b)\beta(t)\mathrm{d}t + \max\{x-a, b-x\}\left(\int_a^b \beta(t)\mathrm{d}t - f(b)+f(a)\right)\Bigg], \tag{2.3}$$

The listed inequalities in Theorem 2.1 are significant as they improve all previous results in which the Lebesgue norms of f' come into play when handling the bounds calculations. In this case, the required computations in bounds are just in terms of pre-assigned functions. For other related general results, the reader may be refer to [3], [15], [16], [18], [1], and [2].

In this paper, motivated by [9], new integral inequalities of Ostrowski type are obtained. Namely, under certain conditions on f', we give the lower and upper bounds for the difference

$$E(f;h) = \frac{h}{2}[f(a) + f(b)] + (1-h)f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t, \tag{2.4}$$

where $h \in [0,1]$ and $x \in [a+h\frac{b-a}{2}, b-h\frac{b-a}{2}]$. Our results provides range of estimates including those given by [9] and [5]–[8]. Utilizing general Peano kernel, we recapture the three inequalities (2.1)–(2.3) obtained by [9]. Some special cases of our result and applications to numerical quadrature rules are also given.

3 Main Results

In order to formulate our main results, we need a kernel $K(t;\cdot) : [a,b] \to \mathbb{R}$ defined by

$$K(t;x) = \begin{cases} t - \left(a + h\frac{b-a}{2}\right), & t \in [a,x], \\ t - \left(b - h\frac{b-a}{2}\right), & t \in (x,b], \end{cases} \tag{3.1}$$

for all $h \in [0,1]$ and $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$. Also, for two functions $\alpha, \beta \in C[a,b]$, such that $\alpha(t) \le \beta(t)$ for each $t \in [a,b]$, we define the functions $A(t;\cdot) : [a,b] \to \mathbb{R}$ and $B(t;\cdot) : [a,b] \to \mathbb{R}$ by

$$A(t;x) = \frac{1}{2}\Big\{\big[1 - \operatorname{sgn} K(t;x)\big]\beta(t) + \big[1 + \operatorname{sgn} K(t;x)\big]\alpha(t)\Big\} \tag{3.2}$$

and

$$B(t;x) = \frac{1}{2}\Big\{\big[1 - \operatorname{sgn} K(t;x)\big]\alpha(t) + \big[1 + \operatorname{sgn} K(t;x)\big]\beta(t)\Big\}, \tag{3.3}$$

respectively. We note that

$$\operatorname{sgn} K(t;x) = \begin{cases} -1, & t \in \left[a, a + h\frac{b-a}{2}\right), \\ 1, & t \in \left(a + h\frac{b-a}{2}, x\right], \\ -1, & t \in \left(x, b - h\frac{b-a}{2}\right), \\ 1, & t \in \left(b - h\frac{b-a}{2}, b\right], \end{cases} \tag{3.4}$$

and equal to zero at $t = a + h\frac{b-a}{2}$ and $t = b - h\frac{b-a}{2}$.

Obviously, (2.4) provides range of estimates including those introduced by [9] and [5]–[8]. For

4

instance, when $h = 0$, $h = 1/2$, and $h = 1$, (2.4) can be, respectively, reduced to

$$E(f;0) = f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad (x \in [a,b]), \tag{3.5}$$

$$E(f;1/2) = \frac{1}{4}[f(a) + f(b) + 2f(x)] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x \in \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]\right), \tag{3.6}$$

$$E(f;1) = \frac{1}{2}[f(a) + f(b)] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t. \tag{3.7}$$

Theorem 3.1 *Let $f : I \to \mathbb{R}$, where I is an interval, be a function differentiable in the interior $\mathring{I}$ of I, and let $[a,b] \subset \mathring{I}$. Also, let $E(f;h)$, $K(t;x)$, $A(t;x)$, and $B(t;x)$ be given by (2.4), (3.1), (3.2), and (3.3), respectively.*

For any $\alpha, \beta \in C[a,b]$, $h \in [0,1]$, and $x \in [a + h\frac{b-a}{2}, b - h\frac{b-a}{2}]$, we have the following three cases:

1° If $\alpha(x) \le f'(x) \le \beta(x)$, then

$$\frac{1}{b-a}\int_a^b K(t;x)A(t;x)\mathrm{d}t \le E(f;h) \le \frac{1}{b-a}\int_a^b K(t;x)B(t;x)\mathrm{d}t; \tag{3.8}$$

2° If $\alpha(x) \le f'(x)$, then

$$\begin{aligned}\frac{1}{b-a}\left\{\int_a^b K(t;x)\alpha(t)\mathrm{d}t - L(x,h)\left(f(b) - f(a) - \int_a^b \alpha(t)\mathrm{d}t\right)\right\} &\le E(f;h) \\ \le \frac{1}{b-a}\left\{\int_a^b K(t;x)\alpha(t)\mathrm{d}t + L(x,h)\left(f(b) - f(a) - \int_a^b \alpha(t)\mathrm{d}t\right)\right\}&,\end{aligned} \tag{3.9}$$

where

$$L(x,h) = \max_{t\in[a,b]}\left|K(t;x)\right| = \max\left\{x - a - h\frac{b-a}{2}, b - x - h\frac{b-a}{2}, h\frac{b-a}{2}\right\}; \tag{3.10}$$

3° If $f'(x) \le \beta(x)$, then

$$\begin{aligned}\frac{1}{b-a}\left\{\int_a^b K(t;x)\beta(t)\mathrm{d}t - L(x,h)\left(\int_a^b \beta(t)\mathrm{d}t - f(b) + f(a)\right)\right\} &\le E(f;h) \\ \le \frac{1}{b-a}\left\{\int_a^b K(t;x)\beta(t)\mathrm{d}t + L(x,h)\left(\int_a^b \beta(t)\mathrm{d}t - f(b) + f(a)\right)\right\}&,\end{aligned} \tag{3.11}$$

where $L(x,h)$ is defined by (3.10).

5

Proof. By considering the kernel $K(t;x)$ in (3.1), we have

$$\begin{aligned}\int_a^b K(t;x)\left(f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right)\mathrm{d}t &= \int_a^b K(t;x)f'(t)\mathrm{d}t-\frac{1}{2}\int_a^b K(t;x)\big(\alpha(t)+\beta(t)\big)\mathrm{d}t\\ &= (b-a)E(f;h)-\frac{1}{2}\int_a^b K(t;x)\big(\alpha(t)+\beta(t)\big)\mathrm{d}t, \qquad (3.12)\end{aligned}$$

because of

$$\begin{aligned}\int_a^b K(t;x)f'(t)\mathrm{d}t &= \int_a^x \left[t-\left(a+h\frac{b-a}{2}\right)\right]f'(x)\mathrm{d}t+\int_x^b \left[t-\left(b-h\frac{b-a}{2}\right)\right]f'(x)\mathrm{d}t\\ &= \int_a^b tf'(t)\mathrm{d}t-\left(a+h\frac{b-a}{2}\right)(f(x)-f(a))-\left(b-h\frac{b-a}{2}\right)(f(b)-f(x))\\ &= (b-a)\left[\frac{h}{2}[f(a)+f(b)]+(1-h)f(x)\right]-\int_a^b f(t)\mathrm{d}t\\ &= (b-a)E(f;h).\end{aligned}$$

Now, for the first inequality (3.8), the given assumption $\alpha(x)\le f'(x)\le\beta(x)$ yields

$$\left|f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right|\le\frac{\beta(t)-\alpha(t)}{2}. \qquad (3.13)$$

Therefore, from (3.12) and (3.13), we get

$$\begin{aligned}\left|(b-a)E(f;h)-\frac{1}{2}\int_a^b K(t;x)\big(\alpha(t)+\beta(t)\big)\mathrm{d}t\right| &= \left|\int_a^b K(t;x)\left(f'(t)-\frac{\alpha(t)+\beta(t)}{2}\right)\mathrm{d}t\right|\\ &= \frac{1}{2}\int_a^b \big|K(t;x)\big|\big(\beta(t)-\alpha(t)\big)\mathrm{d}t,\end{aligned}$$

i.e.,

$$\begin{aligned}-\frac{1}{2}\int_a^b \big|K(t;x)\big|\big(\beta(t)-\alpha(t)\big)\mathrm{d}t+\frac{1}{2}\int_a^b K(t;x)\big(\alpha(t)+\beta(t)\big)\mathrm{d}t &\le (b-a)E(f;h)\\ \le\frac{1}{2}\int_a^b \big|K(t;x)\big|\big(\beta(t)-\alpha(t)\big)\mathrm{d}t+\frac{1}{2}\int_a^b K(t;x)\big(\alpha(t)+\beta(t)\big)\mathrm{d}t.\end{aligned}$$

Since $\big|K(t;x)\big|=K(t;x)\,\mathrm{sgn}\,K(t;x)$, and $A(t;x)$ and $B(t;x)$ are defined by (3.2) and (3.3), respectively, the previous inequalities reduce to (3.8).

For the second case, when $\alpha(x) \leq f'(x)$, we have

$$\begin{aligned}\int_a^b K(t;x)\left(f'(t)-\alpha(t)\right)\mathrm{d}t &= \int_a^b K(t;x)f'(t) - \int_a^b K(t;x)\alpha(t)\mathrm{d}t \\ &= (b-a)E(f;h) - \int_a^b K(t;x)\alpha(t)\mathrm{d}t.\end{aligned}$$

Hence,

$$\begin{aligned}\left|(b-a)E(f;h) - \int_a^b K(t;x)\alpha(t)\mathrm{d}t\right| &\leq \left|\int_a^b K(x,t)\left(f'(t)-\alpha(t)\right)\mathrm{d}t\right| \\ &\leq \int_a^x \big|K(t;x)\big|\big(f'(t)-\alpha(t)\big)\mathrm{d}t \\ &\leq \left(\max_{t\in[a,b]}|K(t;x)|\right)\int_a^b \big(f'(t)-\alpha(t)\big)\mathrm{d}t \\ &= L(x,h)\left(f(b)-f(a)-\int_a^b \alpha(t)\mathrm{d}t\right), \qquad (3.14)\end{aligned}$$

where $L(x,h)$ is defined by (3.10). Then, (3.14) gives (3.9).

Finally, for the third case, when $f'(x) \leq \beta(x)$, we have

$$\begin{aligned}\int_a^b K(t;x)\left(f'(t)-\beta(t)\right)\mathrm{d}t &= \int_a^b K(t;x)f'(t) - \int_a^b K(t;x)\beta(t)\mathrm{d}t \\ &= (b-a)E(f;h) - \int_a^b K(t;x)\beta(t)\mathrm{d}t,\end{aligned}$$

from which, as before, we obtain

$$\begin{aligned}\left|(b-a)E(f;h) - \int_a^b K(t;x)\beta(t)\mathrm{d}t\right| &\leq \left|\int_a^b K(x,t)\left(f'(t)-\beta(t)\right)\mathrm{d}t\right| \\ &\leq \int_a^x \big|K(t;x)\big|\big(\beta(t)-f'(t)\big)\mathrm{d}t \\ &\leq \left(\max_{t\in[a,b]}|K(t;x)|\right)\int_a^b \big(\beta(t)-f'(t)\big)\mathrm{d}t \\ &= L(x,h)\Big(\int_a^b \beta(t)\mathrm{d}t - f(b) + f(a)\Big),\end{aligned}$$

i.e., (3.11).

The proof of this theorem is completed. □

7

Remark 3.1 According to (3.10) and $\max\{u, v\} = \frac{1}{2}(u + v + |u - v|)$, we can see that

$$L(x, h) = \frac{b-a}{2}(1-h) + \left|x - \frac{a+b}{2}\right| \qquad \text{if} \quad h \leq \frac{1}{2}.$$

This expression holds also for $h > \frac{1}{2}$, but only when $|x| > 2h - 1$. However, for $|x| \leq 2h - 1$, the

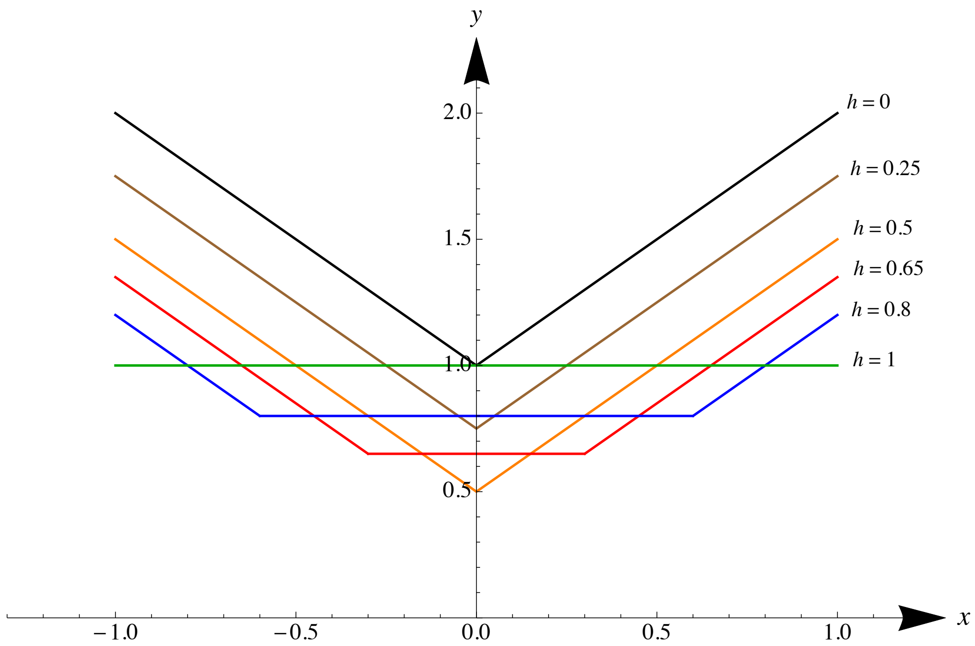


Figure 1: The function $x \mapsto L(x, h)$ for $h = 0$, 0.25, 0.5, 0.65, 0.8, and 1.

function $x \mapsto L(x, h)$ is a constant, i.e.,

$$L(x, h) = \frac{b-a}{2} h.$$

This function on $[a, b] = [-1, 1]$ for different value of h is presented in Figure 1.

Now, we consider cases with constant functions α and β, i.e., when $\alpha(x) = \alpha_0$ and $\beta(x) = \beta_0$ on $[a, b]$.

According to (3.2), (3.3), and (3.4), we get

$$\big(A(t; x), B(t; x)\big) = \begin{cases} (\beta_0, \alpha_0), & t \in \left[a, a + h\frac{b-a}{2}\right), \\ (\alpha_0, \beta_0), & t \in \left(a + h\frac{b-a}{2}, x\right], \\ (\beta_0, \alpha_0), & t \in \left(x, b - h\frac{b-a}{2}\right), \\ (\alpha_0, \beta_0), & t \in \left(b - h\frac{b-a}{2}, b\right], \end{cases}$$

8

so that the corresponding bounds in (3.8) become

$$\begin{aligned}\underline{B}^{(1)} &= \frac{1}{b-a}\int_a^b K(t;x)A(t;x)\mathrm{d}t \\ &= -\frac{1}{2(b-a)}\big[b^2\beta_0 - a^2\alpha_0 - 2(b\beta_0 - a\alpha_0)x + (\beta_0-\alpha_0)x^2\big] \\ &\quad +\frac{1}{2}\big[a\alpha_0 + b\beta_0 - (\alpha_0+\beta_0)x\big]h - \frac{1}{4}(b-a)(\beta_0-\alpha_0)h^2 \qquad (3.15)\end{aligned}$$

and

$$\begin{aligned}\overline{B}^{(1)} &= \frac{1}{b-a}\int_a^b K(t;x)B(t;x)\mathrm{d}t \\ &= \frac{1}{2(b-a)}\big[a^2\beta_0 - b^2\alpha_0 + 2(b\alpha_0 - a\beta_0)x + (\beta_0-\alpha_0)x^2\big] \\ &\quad +\frac{1}{2}\big[b\alpha_0 + a\beta_0 - (\alpha_0+\beta_0)x\big]h + \frac{1}{4}(b-a)(\beta_0-\alpha_0)h^2. \qquad (3.16)\end{aligned}$$

Also,

$$\begin{aligned}\frac{1}{b-a}\int_a^b K(t;x)\mathrm{d}t &= \frac{1}{b-a}\left\{\int_a^x \left[t-\left(a+h\frac{b-a}{2}\right)\right]\mathrm{d}t + \int_x^b \left[t-\left(b-h\frac{b-a}{2}\right)\right]\mathrm{d}t\right\} \\ &= \frac{1}{2}(1-h)(2x-a-b),\end{aligned}$$

so that we can find the corresponding lower and upper bounds in the inequalities (3.9) and (3.11):

$$\underline{B}^{(2)} = \frac{\alpha_0}{2}(1-h)(2x-a-b) - L(x,h)\left(\frac{f(b)-f(a)}{b-a} - \alpha_0\right), \qquad (3.17)$$

$$\overline{B}^{(2)} = \frac{\alpha_0}{2}(1-h)(2x-a-b) + L(x,h)\left(\frac{f(b)-f(a)}{b-a} - \alpha_0\right), \qquad (3.18)$$

$$\underline{B}^{(3)} = \frac{\beta_0}{2}(1-h)(2x-a-b) - L(x,h)\left(\beta_0 - \frac{f(b)-f(a)}{b-a}\right), \qquad (3.19)$$

$$\overline{B}^{(3)} = \frac{\beta_0}{2}(1-h)(2x-a-b) + L(x,h)\left(\beta_0 - \frac{f(b)-f(a)}{b-a}\right), \qquad (3.20)$$

where $L(x,h)$ is defined by (3.10).

Thus, for constant functions α and β on $[a,b]$, we get the following result:

Corollary 3.1 *Under the assumptions of Theorem* 3.1 *with* $\alpha(x)=\alpha_0$ *and* $\beta(x)=\beta_0$*, we have:*

1° *If* $\alpha_0 \le f'(x) \le \beta_0$*, then* $\underline{B}^{(1)} \le E(f;h) \le \overline{B}^{(1)}$*;*

2° *If* $\alpha_0 \le f'(x)$*, then* $\underline{B}^{(2)} \le E(f;h) \le \overline{B}^{(2)}$*;*

3° *If* $f'(x) \le \beta_0$*, then* $\underline{B}^{(3)} \le E(f;h) \le \overline{B}^{(3)}$*,*

where the bounds are given in (3.15)–(3.19).

4 Some Applications in Numerical Integration

Inequalities of Ostrowski's type have attracted considerable interest over the years. Many authors have worked on this subject and proved many extensions and generalizations, including applications in numerical integration (cf. [4]). These inequalities can be considered as error estimates of certain elementary quadrature rules in some classes of functions.

Beside the bounds of (3.5)–(3.7), in this section we consider also ones for $h = 1/3,\ 1/4,\ 2/3$, and $3/4$, i.e.,

$$\begin{aligned} E(f;1/3) &= \frac{1}{6}\left[f(a)+f(b)+4f(x)\right]-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x\in\left[\frac{5a+b}{6},\frac{a+5b}{6}\right]\right), \qquad (4.1)\\ E(f;1/4) &= \frac{1}{8}[f(a)+f(b)+6f(x)]-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x\in\left[\frac{7a+b}{8},\frac{a+7b}{8}\right]\right),\\ E(f;2/3) &= \frac{1}{3}[f(a)+f(b)+f(x)]-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x\in\left[\frac{2a+b}{3},\frac{a+2b}{3}\right]\right),\\ E(f;3/4) &= \frac{1}{8}[3f(a)+3f(b)+2f(x)]-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x\in\left[\frac{5a+3b}{8},\frac{3a+5b}{8}\right]\right), \end{aligned}$$

respectively.

For $x = (a + b)/2$, $E(f; 1/3)$, given before by (4.1), represents the error in the well-known Simpson formula (cf. [13, pp. 343–350]).

In order to get the corresponding estimates of (2.4), i.e.,

$$E(f;h)=\frac{h}{2}\big[f(a)+f(b)\big]+(1-h)f(x)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \quad \left(x\in\left[a+h\frac{b-a}{2},b-h\frac{b-a}{2}\right]\right),$$

for different values of h, we use here Corollary 3.1 (Case 1°).

Case $h = 0$. Here, the value of x can be arbitrary in $[a, b]$. Then, $\underline{B}^{(1)}$ and $\overline{B}^{(1)}$ reduce to

$$\underline{B}^{(1)} = -\frac{1}{2(b-a)}\big[b^2\beta_0-a^2\alpha_0-2(b\beta_0-a\alpha_0)x+(\beta_0-\alpha_0)x^2\big]$$

and

$$\overline{B}^{(1)} = \frac{1}{2(b-a)}\big[a^2\beta_0-b^2\alpha_0+2(b\alpha_0-a\beta_0)x+(\beta_0-\alpha_0)x^2\big],$$

so that, under the condition $\alpha_0 \le f'(x) \le \beta_0$, for each $x \in [a, b]$, we have

$$\underline{B}^{(1)} \le f(x)-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \le \overline{B}^{(1)}. \qquad (4.2)$$

For the symmetric bounds of the first derivative f' ($|f'(x)| \le \beta_0$), i.e., if $\alpha_0 = -\beta_0$, (4.2) reduces to

$$\left| f(x) - \frac{1}{b-a} \int\limits_a^b f(t)\mathrm{d}t \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\beta_0,$$

which is, in fact, the original Ostrowski inequality (1.1).

Otherwise, (4.2) for $x = (a+b)/2$ gives the error estimate for the midpoint rule,

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int\limits_a^b f(t)\mathrm{d}t \right| \le \frac{1}{8}(b-a)(\beta_0 - \alpha_0),$$

while for $x = b$ it gives the error estimate for the so-called endpoint rule

$$\frac{1}{2}(b-a)\alpha_0 \le f(b) - \frac{1}{b-a} \int\limits_a^b f(t)\mathrm{d}t \le \frac{1}{2}(b-a)\beta_0.$$

Case $h = 1$. Here x must be $(b-a)/2$! Taking $h = 1$ in (3.15) and (3.15), for the trapezoidal rule (3.7), we obtain the same bound as for the midpoint rule,

$$\left| \frac{1}{2}\big[f(a) + f(b)\big] - \frac{1}{b-a} \int\limits_a^b f(t)\mathrm{d}t \right| \le \frac{1}{8}(b-a)(\beta_0 - \alpha_0).$$

Case $0 < h < 1$. Now we take $x = (a+b)/2$ in (3.15) and (3.15). Since, in that case,

$$-\underline{B}^{(1)} = \overline{B}^{(1)} = \frac{1}{8}(b-a)\big(1 - 2h + 2h^2\big)(\beta_0 - \alpha_0),$$

we get

$$\left| \frac{h}{2}\big[f(a) + f(b)\big] + (1-h) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int\limits_a^b f(t)\mathrm{d}t \right| \le \frac{b-a}{4}\Big(\frac{1}{2} - h + h^2\Big)(\beta_0 - \alpha_0), \tag{4.3}$$

provided that $\alpha_0 \le f'(x) \le \beta_0$ for $x \in [a,b]$.

For $h = 1/2,\ 1/3,\ 1/4,\ 2/3$, and $3/4$, the inequality (4.3) reduces to

$$\left| \frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int\limits_a^b f(t)\mathrm{d}t \right| \le \frac{b-a}{16}(\beta_0 - \alpha_0),$$

$$\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int\limits_a^b f(t)\mathrm{d}t \right| \le \frac{5(b-a)}{72}(\beta_0 - \alpha_0),$$

$$\left| \frac{1}{8}\left[f(a) + 6f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int\limits_a^b f(t)\mathrm{d}t \right| \le \frac{5(b-a)}{64}(\beta_0 - \alpha_0),$$

$$\left| \frac{1}{3}\left[f(a) + f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int\limits_a^b f(t)\mathrm{d}t \right| \le \frac{13(b-a)}{144}(\beta_0 - \alpha_0),$$

11

and

$$\left|\frac{1}{8}\left[3f(a)+2f\left(\frac{a+b}{2}\right)+3f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right|\leq\frac{5(b-a)}{64}(\beta_0-\alpha_0),$$

respectively.

5 Conclusion

Inspired and motivated by the work of Masjed-Jamei and Dragomir [9], new integral inequalities of Ostrowski type are obtained with bounds are just in terms of pre-assigned functions. Our results provides a generalization of error bounds that is independent of Lebesgue norms including those given by [9] and [5]–[8]. We utilize general Peano kernel to recapture the three inequalities (3.8), (3.9), and (3.11), obtained in [9]. Some special cases and applications to numerical quadrature rules are also proposed.

References

[1] W.G. Alshanti and A. Qayyum, *A note on new Ostrowski type inequalities using a generalized kernel*, Bull. Math. Anal. Appl. **9**(1) (2017), 74–91.

[2] W.G. Alshanti, A. Qayyum and M.A. Majid, *Ostrowski type inequalities by using general quadratic kernel*, J. Ineq. Spec. Funct. **8**(4) (2017), 111–135.

[3] P. Cerone, *A new Ostrowski type inequality involving integral means over end intervals*, Tamkang J. Math. **33** (2002), 109–118.

[4] S. S. Dragomir and Th. M. Rassias (Eds.), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht, 2002.

[5] S. S. Dragomir and S. Wang, *A new inequality Ostrowski's type in L_1 norm and applications to some special means and some numerical quadrature rules*. Tamkang J. Math. **28** (1997), 239–244.

[6] S. S. Dragomir and S. Wang, *An inequality Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules*, Comput. Math. Appl. **33** (1997), 15–22.

[7] S. S. Dragomir and S. Wang, *A new inequality Ostrowski's type in L_p norm*, Indian J. Math. **40** (1998), 299–304.

[8] S. S. Dragomir and S. Wang, *Applications of Ostrowski's inequality to the estimation of error bounds for some special means and to some numerical quadrature rules*, Appl. Math. Lett. **11** (1998), 105–109.

[9] M. Masjed-Jamei and S. S. Dragomir, *An analogue of the Ostrowski inequality and applications*, Filomat **28** (2014), 373–381.

[10] G. V. Milovanović, *On some integral inequalities.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Math. Fiz. No. 498–541 (1975), 119–124.

[11] G. V. Milovanović and J. E. Pečarić, *On generalization of the inequality of A. Ostrowski and some related applications.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Math. Fiz. No. 544–576 (1976), 155–158.

[12] G. V. Milovanović, *On some functional inequalities.* Univ. Beograd. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Math. Fiz. No. 599 (1977), 1–59.

[13] G. V. Milovanović, *Numerical Analysis, Part I*, University of Niš, Niš, 1979 (Serbian).

[14] A. Ostrowski, *Über die Absolutabweichung einer differentienbaren Funktion von ihrem Integralimittelwert.* Comment. Math. Helv. **10** (1938), 226–227.

[15] N. Ujević, *New bounds for the first inequality of Ostrowski-Grüss type and applications*, Comput. Math. Appl. **46** (2003), 421–427.

[16] N. Ujević, *A generalization of Ostrowski's inequality and applications in numerical integration*, Appl. Math. Lett. **17** (2004), 133–137.

[17] A. Qayyum, M. Shoaib and I. Faye, *Some new generalized results on Ostrowski type integral inequalities with application*, J. Comput. Anal. Appl. **19** (2015), 693–712.

[18] L. Zeng, *More on Ostrowski type inequalities for some s-convex functions in the second sense*, Demonstratio Mathematica. **49**(4) (2016), 398–412.

(W. G. Alshanti) Department of General Studies, Jubail University College, Saudi Arabia

E-mail address: shantiw@ucj.edu.sa

(G. V. Milovanović) Serbian Academy of Sciences and Arts, 11000 Belgrade, Serbia & Faculty of Sciences and Mathematics, University of Niš, 18000 Niš, Serbia

E-mail address: gvm@mi.sanu.ac.rs

ITERATES OF CHENEY-SHARMA TYPE OPERATORS ON A TRIANGLE WITH CURVED SIDE

TEODORA CĂTINAŞ*, DIANA OTROCOL**

Abstract. We consider some Cheney–Sharma type operators as well as their product and Boolean sum for a function defined on a triangle with one curved side. Using the weakly Picard operators technique and the contraction principle, we study the convergence of the iterates of these operators.

Keywords: Triangle with curved side, Cheney-Sharma operators, contraction principle, weakly Picard operators.

MSC 2010 Subject Classification: 41A36, 41A25, 39B12, 47H10.

1. Cheney-Sharma type operators

We recall some results regarding Cheney-Sharma type operators on a triangle with one curved side, introduced in [6]. Similar operators were introduced and studied in [3], [4], [5] and [9].

We consider the standard triangle $\widetilde{T}_h$ with vertices $V_1 = (0, h)$, $V_2 = (h, 0)$ and $V_3 = (0, 0)$, with two straight sides Γ_1, Γ_2, along the coordinate axes, and with the third side Γ_3 (opposite to the vertex V_3) defined by the one-to-one functions f and g, where g is the inverse of the function f, i.e., $y = f(x)$ and $x = g(y)$, with $f(0) = g(0) = h$, for $h > 0$. Also, we have $f(x) \leq h$ and $g(y) \leq h$, for $x, y \in [0, h]$.

Let F be a real-valued function defined on $\widetilde{T}_h$ and $(0, y)$, $(g(y), y)$, respectively, $(x,\ 0)$, $(x, f(x))$ be the points in which the parallel lines to the coordinate axes, passing through the point $(x, y) \in \widetilde{T}_h$, intersect the sides Γ_i, $i = 1, 2, 3$. (See Figure 1.)

In [6], we have obtained the following extensions of Cheney-Sharma operator of second kind, to the case of functions defined on $\widetilde{T}_h$:

$$
\begin{aligned}
(1.1) \qquad (Q_m^x F)(x,y) &= \sum_{i=0}^{m} q_{m,i}(x,y) F\left(i\tfrac{g(y)}{m}, y\right),\\
(Q_n^y F)(x,y) &= \sum_{j=0}^{n} q_{n,j}(x,y) F\left(x, j\tfrac{f(x)}{n}\right),
\end{aligned}
$$

with

$$
q_{m,i}(x,y) = \binom{m}{i}\frac{1}{(1+m\beta)^{m-1}}\frac{x}{g(y)}\left(\frac{x}{g(y)}+i\beta\right)^{i-1}\left(1-\frac{x}{g(y)}\right)\left[1-\frac{x}{g(y)}+(m-i)\beta\right]^{m-i-1},
$$

$$
q_{n,j}(x,y) = \binom{n}{j}\frac{1}{(1+nb)^{n-1}}\frac{y}{f(x)}\left(\frac{y}{f(x)}+jb\right)^{j-1}\left(1-\frac{y}{f(x)}\right)\left[1-\frac{y}{f(x)}+(n-j)b\right]^{n-j-1},
$$

*Babeş-Bolyai University, Faculty of Mathematics and Computer Science, M. Kogălniceanu St. 1, RO-400084 Cluj Napoca, Romania, E-mail: tcatinas@math.ubbcluj.ro.

**T. Popoviciu Institute of Numerical Analysis, Romanian Academy, Cluj Napoca, Romania; Technical University of Cluj Napoca, Memorandumului St. 28, RO-400114, Cluj Napoca, Romania, E-mail: Diana.Otrocol@math.utcluj.ro.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

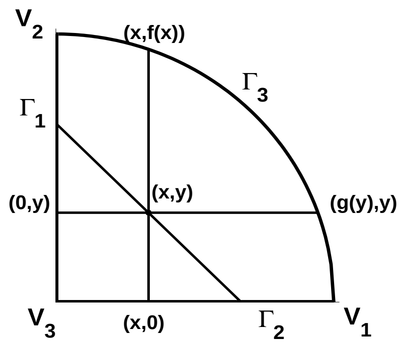


FIGURE 1. Triangle $\tilde{T}_h$.

where

$$\Delta_m^x = \left\{ i\tfrac{g(y)}{m} \middle| \ i = \overline{0,m} \right\} \text{ and } \Delta_n^y = \left\{ j\tfrac{f(x)}{n} \middle| \ j = \overline{0,n} \right\}$$

are uniform partitions of the intervals $[0, g(y)]$ and $[0, f(x)]$ and $m, n \in \mathbb{N}$, $\beta, b \in \mathbb{R}_+$.

Remark 1.1. *As the Cheney-Sharma operator of second kind interpolates a given function at the endpoints of the interval, we may use the operators Q_m^x and Q_n^y as interpolation operators on $\widetilde{T}_h$.*

Theorem 1.2. [6] *If F is a real-valued function defined on $\widetilde{T}_h$ then the following properties hold:*

(i) $Q_m^x F = F$ on $\Gamma_1 \cup \Gamma_3$;
(ii) $Q_n^y F = F$ on $\Gamma_2 \cup \Gamma_3$;
(iii) $(Q_m^x e_{ij})(x, y) = x^i y^j$, $\ i = 0, 1$; $j \in \mathbb{N}$;
(iv) $(Q_n^y e_{ij})(x, y) = x^i y^j$, $\ i \in \mathbb{N}$; $j = 0, 1$, where $e_{ij}(x, y) = x^i y^j$, $\ i, j \in \mathbb{N}$.

Let $P_{mn}^1 = Q_m^x Q_n^y$, respectively, $P_{nm}^2 = Q_n^y Q_m^x$ be the products of the operators Q_m^x and Q_n^y. We have

$$(P_{mn}^1 F)(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} q_{m,i}(x, y)\, q_{n,j}\left(i\tfrac{g(y)}{m}, y\right) F\left(i\tfrac{g(y)}{m}, j\frac{f(i\frac{g(y)}{m})}{n}\right), \tag{1.2}$$

respectively,

$$(P_{nm}^2 F)(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} q_{m,i}\left(x, j\tfrac{f(x)}{n}\right) q_{n,j}(x, y)\, F\left(i\frac{g(j\frac{f(x)}{n})}{m}, j\tfrac{f(x)}{n}\right).$$

Theorem 1.3. *If F is a real-valued function defined on $\widetilde{T}_h$ then*

(i) $(P_{mn}^1 F)(V_i) = F(V_i), \quad i = 1, ..., 3;$
$\quad (P_{mn}^1 F)(\Gamma_3) = F(\Gamma_3),$
(ii) $(P_{nm}^2 F)(V_i) = F(V_i), \quad i = 1, ..., 3;$
$\quad (P_{nm}^2 F)(\Gamma_3) = F(\Gamma_3).$

We consider the Boolean sums of the operators Q_m^x and Q_n^y,

$$\begin{aligned} S_{mn}^1 &:= Q_m^x \oplus Q_n^y = Q_m^x + Q_n^y - Q_m^x Q_n^y, \\ S_{nm}^2 &:= Q_n^y \oplus Q_m^x = Q_n^y + Q_m^x - Q_n^y Q_m^x. \end{aligned} \tag{1.3}$$

Theorem 1.4. *If F is a real-valued function defined on $\widetilde{T}_h$, then*

$$S^1_{mn}F\Big|_{\partial\widetilde{T}_h} = F\Big|_{\partial\widetilde{T}_h},$$
$$S^2_{mn}F\Big|_{\partial\widetilde{T}_h} = F\Big|_{\partial\widetilde{T}_h}.$$

2. Weakly Picard operators

We recall some results regarding weakly Picard operators that will be used in the sequel (see, e.g., [21]).

Let (X,d) be a metric space and $A : X \to X$ an operator. We denote by

$F_A := \{x \in X \mid A(x) = x\}$-the fixed points set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y,\ Y \neq \emptyset\}$-the family of the nonempty invariant subsets of A;

$A^0 := 1_X,\ A^1 := A,\ \ldots,\ A^{n+1} := A \circ A^n,\quad n \in \mathbb{N}.$

Definition 2.1. *The operator $A : X \to X$ is a Picard operator if there exists $x^* \in X$ such that:*
(i) $F_A = \{x^*\}$;
(ii) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Definition 2.2. *The operator A is a weakly Picard operator if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of A.*

Definition 2.3. *If A is a weakly Picard operator then we consider the operator A^∞, $A^\infty : X \to X$, defined by*

$$A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Theorem 2.4. *An operator A is a weakly Picard operator if and only if there exists a partition of X, $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$, such that*

(a) $X_\lambda \in I(A)$, $\forall \lambda \in \Lambda$;
(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator,* $\forall \lambda \in \Lambda$.

3. Iterates of Cheney-Sharma type operators

We study the convergence of the iterates of the Cheney-Sharma type operators (1.1) and of their product and Boolean sum operators, using the weakly Picard operators technique and the contraction principle. The same approach for some other linear and positive operators lead to similar results in [1], [2], [7], [8], [22]-[24].

The limit behavior for the iterates of some classes of positive linear operators were also studied, for example, in [10]-[20]. In the papers [10]-[12] were introduced new methods for the study of the asymptotic behavior of the iterates of positive linear operators. These techniques enlarge the class of operators for which the limit of the iterates can be calculated.

Let F be a real-valued function defined on $\widetilde{T}_h$, $h \in \mathbb{R}_+$. First we study the convergence of the iterates of the Cheney–Sharma type operators given in (1.1).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Theorem 3.1. *The operators Q_m^x and Q_n^y are weakly Picard operators and*

$$(Q_m^{x,\infty} F)(x,y) = \frac{F(g(y),y) - F(0,y)}{g(y)} x + F(0,y), \tag{3.1}$$

$$(Q_n^{y,\infty} F)(x,y) = \frac{F(x,f(x)) - F(x,0)}{f(x)} y + F(x,0). \tag{3.2}$$

Proof. Taking into account the interpolation properties of Q_m^x and Q_n^y (from Theorem 1.2), let us consider the following sets:

(3.3)

$$X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}} = \{F \in C(\widetilde{T}_h) \mid F(0,y) = \varphi|_{\Gamma_1},\ F(g(y),y) = \varphi|_{\Gamma_3}\}, \quad \text{for } y \in [0,h],$$
$$X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}} = \{F \in C(\widetilde{T}_h) \mid F(x,0) = \psi|_{\Gamma_2},\ F(x,f(x)) = \psi|_{\Gamma_3}\}, \quad \text{for } x \in [0,h],$$

and for $\varphi, \psi \in C(\widetilde{T}_h)$ we denote by

$$F^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}(x,y) = \frac{\varphi|_{\Gamma_3} - \varphi|_{\Gamma_1}}{g(y)} x + \varphi|_{\Gamma_1},$$
$$F^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}(x,y) = \frac{\psi|_{\Gamma_3} - \psi|_{\Gamma_2}}{f(x)} y + \psi|_{\Gamma_2}.$$

We have the following properties:

(i) $X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}$ and $X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}$ are closed subsets of $C(\widetilde{T}_h)$;

(ii) $X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}$ is an invariant subset of Q_m^x and $X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}$ is an invariant subset of Q_n^y, for $\varphi, \psi \in C(\widetilde{T}_h)$ and $n, m \in \mathbb{N}^*$;

(iii) $C(\widetilde{T}_h) = \underset{\varphi \in C(\widetilde{T}_h)}{\cup} X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}$ and $C(\widetilde{T}_h) = \underset{\psi \in C(\widetilde{T}_h)}{\cup} X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}$ are partitions of $C(\widetilde{T}_h)$;

(iv) $F^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}} \in X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}} \cap F_{Q_m^x}$ and $F^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}} \in X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}} \cap F_{Q_n^y}$, where $F_{Q_m^x}$ and $F_{Q_n^y}$ denote the fixed points sets of Q_m^x and Q_n^y.

The statements (i) and (iii) are obvious.

(ii) By linearity of Cheney-Sharma operators and Theorem 1.2, it follows that $\forall F^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}} \in X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}$ and $\forall F^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}} \in X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}$ we have

$$Q_m^x F^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}(x,y) = F^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}(x,y),$$
$$Q_n^y F^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}(x,y) = F^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}(x,y).$$

So, $X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}$ and $X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}$ are invariant subsets of Q_m^x and, respectively, of Q_n^y, for $\varphi, \psi \in C(\widetilde{T}_h)$ and $n, m \in \mathbb{N}^*$;

(iv) We prove that

$$Q_m^x|_{X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}} : X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}} \to X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}} \text{ and } Q_n^y|_{X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}} : X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}} \to X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}$$

are contractions for $\varphi, \psi \in C(\widetilde{T}_h)$ and $n, m \in \mathbb{N}^*$.

Let $F,G\in X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}$. From (1.1) and (3.3) we get

$$\begin{aligned}
&|Q^x_m(F)(x,y)-Q^x_m(G)(x,y)|=\\
&=|Q^x_m(F-G)(x,y)|\leq\\
&\leq|q_{m,0}\left(x;y\right)\left[F\left(0,0\right)-G(0,0)\right]|\\
&\quad+\left|\sum_{i=1}^{m}q_{m,i}\left(x;y\right)\left[F\left(\tfrac{ig(y)}{m},y\right)-G\left(x,\tfrac{jf(x)}{n}\right)\right]\right|\\
&=\left|\sum_{i=1}^{m}q_{m,i}\left(x;y\right)\left[F\left(\tfrac{ig(y)}{m},y\right)-G\left(x,\tfrac{jf(x)}{n}\right)\right]\right|\\
&\leq\sum_{i=1}^{m}q_{m,i}\left(x;y\right)\|F-G\|_\infty\\
&=\left[\sum_{i=0}^{m}q_{m,i}\left(x;y\right)-q_{m,0}\left(x;y\right)\right]\|F-G\|_\infty\\
&=\left\{1-\left(1-\tfrac{x}{g(y)}\right)\left[1-\tfrac{x}{g(y)(1+m\beta)}\right]^{m-1}\right\}\|F-G\|_\infty\\
&\leq\left[1-\left(1-\tfrac{1}{1+m\beta}\right)^{m-1}\right]\|F-G\|_\infty,
\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the Chebyshev norm.

Hence,

$$\begin{aligned}
&\|Q^x_m(F)(x,y)-Q^x_m(G)(x,y)\|_\infty\leq\\
&\leq\left[1-\left(1-\tfrac{1}{1+m\beta}\right)^{m-1}\right]\|F-G\|_\infty,\ \forall F,G\in X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}},
\end{aligned}\tag{3.4}$$

i.e., $Q^x_m|_{X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}}$ is a contraction for $\varphi\in C(\widetilde{T}_h)$.

Analogously, we prove that $Q^y_n|_{X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}}$ is a contraction for $\psi\in C(\widetilde{T}_h)$.

On the other hand, $\frac{\varphi|_{\Gamma_3}-\varphi|_{\Gamma_1}}{g(y)}(\cdot)+\varphi|_{\Gamma_1}\in X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}$ and $\frac{\psi|_{\Gamma_3}-\psi|_{\Gamma_2}}{f(x)}(\cdot)+\psi|_{\Gamma_2}\in X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}$ are fixed points of Q^x_m and Q^y_n, i.e.,

$$\begin{aligned}
Q^x_m\left(\tfrac{\varphi|_{\Gamma_3}-\varphi|_{\Gamma_1}}{g(y)}(\cdot)+\varphi|_{\Gamma_1}\right)&=\tfrac{\varphi|_{\Gamma_3}-\varphi|_{\Gamma_1}}{g(y)}(\cdot)+\varphi|_{\Gamma_1},\\
Q^y_n\left(\tfrac{\psi|_{\Gamma_3}-\psi|_{\Gamma_2}}{f(x)}(\cdot)+\psi|_{\Gamma_2}\right)&=\tfrac{\psi|_{\Gamma_3}-\psi|_{\Gamma_2}}{f(x)}(\cdot)+\psi|_{\Gamma_2}.
\end{aligned}$$

From the contraction principle, $F^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}(x,y):=\frac{\varphi|_{\Gamma_3}-\varphi|_{\Gamma_1}}{g(y)}x+\varphi|_{\Gamma_1}$ is the unique fixed point of Q^x_m in $X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}$ and $Q^x_m|_{X^{(1)}_{\varphi|_{\Gamma_1},\varphi|_{\Gamma_3}}}$ is a Picard operator, with

$$\left(Q^{x,\infty}_m F\right)(x,y)=\tfrac{F(g(y),y)-F(0,y)}{g(y)}x+F\left(0,y\right),$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

and, similarly, $F^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}(x,y) := \frac{\psi|_{\Gamma_3}-\psi|_{\Gamma_2}}{f(x)}y + \psi|_{\Gamma_2}$ is the unique fixed point of Q^y_n in $X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}$ and $Q^y_n|_{X^{(2)}_{\psi|_{\Gamma_2},\psi|_{\Gamma_3}}}$ is a Picard operator, with

$$(Q^{y,\infty}_n F)(x,y) = \tfrac{F(x,f(x))-F(x,0)}{f(x)}y + F(x,0).$$

Consequently, taking into account (ii), by Theorem 2.4, it follows that the operators Q^x_m and Q^y_n are weakly Picard operators. □

Further we study the convergence of the product and Boolean sum operators given in (1.2) and (1.3).

Theorem 3.2. *The operator P^1_{mn} is a weakly Picard operator and*

$$\left(P^{1,\infty}_{mn}F\right)(x,y) = \frac{F\left(g(y),y\right)}{g(y)}x. \tag{3.5}$$

Proof. Let

$$X_\alpha = \{F \in C(\widetilde{T}_h) \mid F(g(y),y) = \alpha\}, \qquad \alpha \in \mathbb{R}$$

and denote by

$$F_\alpha(x,y) := \frac{\alpha}{g(y)}x.$$

We remark that:

(i) X_α is a closed subset of $C(\widetilde{T}_h)$;
(ii) X_α is an invariant subset of P^1_{mn}, for $\alpha \in \mathbb{R}$ and $n,m \in \mathbb{N}^*$;
(iii) $C(\widetilde{T}_h) = \underset{\alpha}{\cup} X_\alpha$ is a partition of $C(\widetilde{T}_h)$;
(iv) $F_\alpha \in X_\alpha \cap F_{P^1_{mn}}$, where $F_{P^1_{mn}}$ denote the fixed points sets of P^1_{mn}.

The statements (i) and (iii) are obvious.

(ii) Similarly with the proof of Theorem 3.1, by linearity of Cheney-Sharma operators and Theorem 1.3, it follows that X_α is an invariant subset of P^1_{mn}, for $\alpha \in \mathbb{R}$ and $n,m \in \mathbb{N}^*$;

(iv) We prove that

$$P^1_{mn}\big|_{X_\alpha} : X_\alpha \to X_\alpha$$

is a contraction for $\alpha \in \mathbb{R}$ and $n,m \in \mathbb{N}^*$. Let $F,G \in X_\alpha$. From [2, Lemma 8] and (3.4), it follows that

$$\begin{aligned}&\left|P^1_{mn}(F)(x,y) - P^1_{mn}(G)(x,y)\right| = \left|P^1_{mn}(F-G)(x,y)\right|\\ &\leq \left[1 - \left(\tfrac{m\beta}{1+m\beta}\right)^{m-1}\left(\tfrac{nb}{1+nb}\right)^{n-1}\right]\|F-G\|_\infty,\end{aligned}$$

so, $P^1_{mn}\big|_{X_\alpha}$ is a contraction for $\alpha \in \mathbb{R}$.

From the contraction principle we have that F_α is the unique fixed point of P^1_{mn} in X_α and $P^1_{mn}\big|_{X_\alpha}$ is a Picard operator, so (3.5) holds. Consequently, taking into account (ii), by Theorem 2.4, it follows that the operators P^1_{mn} is a weakly Picard operator. □

Remark 3.3. *Similar results can be obtained for the operator P^2_{mn}.*

Theorem 3.4. *The operator S^1_{mn} is a weakly Picard operator and*

$$\left(S^{1,\infty}_{mn}F\right)(x,y) = \tfrac{-F(0,y)}{g(y)}x + \tfrac{F(x,f(x))-F(x,0)}{f(x)}y + F(x,0) + F(0,y).$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Proof. The proof follows the same steps as the proof of Theorem 3.2, using the following inequality

$$\begin{aligned}&\|S_{mn}(F)(x,y)-S_{mn}(G)(x,y)\|_\infty\\&\leq\left\{1-\left[\left(\tfrac{m\beta}{1+m\beta}\right)^{m-1}+\left(\tfrac{nb}{1+nb}\right)^{n-1}-\left(\tfrac{m\beta}{1+m\beta}\right)^{m-1}\left(\tfrac{nb}{1+nb}\right)^{n-1}\right]\right\}\|F-G\|_\infty,\end{aligned}$$

for proving that S^1_{mn} is a contraction. □

Remark 3.5. *We have a similar result for the operator S^2_{nm}.*

References

[1] Agratini, O., Rus, I.A., *Iterates of a class of discrete linear operators via contraction principle*, Comment. Math. Univ. Caroline **44** (2003), 555-563.
[2] Agratini, O., Rus, I.A., *Iterates of some bivariate approximation process via weakly Picard operators*, Nonlinear Anal. Forum **8**(2) (2003), 159-168.
[3] Blaga, P., Cătinaş, T., Coman, G., *Bernstein-type operators on triangle with one curved side*, Mediterr. J. Math. **9** (2012), No. 4, 833-845.
[4] Blaga, P., Cătinaş, T., Coman, G., *Bernstein-type operators on a square with one and two curved sides*, Studia Univ. Babeş–Bolyai Math. **55** (2010), No. 3, 51-67.
[5] Blaga, P., Cătinaş, T., Coman, G., *Bernstein-type operators on triangle with all curved sides*, Appl. Math. Comput. **218** (2011), 3072-3082.
[6] Cătinaş, T., *Extension of some Cheney-Sharma type operators to a triangle with one curved side,* 2017, submitted.
[7] Cătinaş, T., Otrocol, D., *Iterates of Bernstein type operators on a square with one curved side via contraction principle*, Fixed Point Theory **14** (2013), No. 1, 97-106.
[8] Cătinaş, T., Otrocol, D., *Iterates of multivariate Cheney-Sharma operators*, J. Comput. Anal. Appl. **15** (2013), No. 7, 1240-1246.
[9] Coman, G., Cătinaş, T., *Interpolation operators on a triangle with one curved side*, BIT Numerical Mathematics **50** (2010), No. 2, 243-267.
[10] Gavrea, I., Ivan, M., *The iterates of positive linear operators preserving the affine functions*, J. Math. Anal. Appl. **372** (2010), 366-368.
[11] Gavrea, I., Ivan, M., *The iterates of positive linear operators preserving the constants*, Appl. Math. Lett. **24** (2011), No. 12, 2068-2071.
[12] Gavrea, I., Ivan, M., *On the iterates of positive linear operators*, J. Approx. Theory **163** (2011), No. 9, 1076-1079.
[13] Gonska, H., Kacso, D., Piţul, P., *The degree of convergence of over-iterated positive linear operators*, J. Appl. Funct. Anal. **1** (2006), 403-423.
[14] Gonska, H., Piţul, P., Raşa, I., *Over-iterates of Bernstein-Stancu operators*, Calcolo **44** (2007), 117-125.
[15] Gonska, H., Raşa, I., *The limiting semigroup of the Bernstein iterates: degree of convergence*, Acta Math. Hungar. **111** (2006), No. 1-2, 119-130.
[16] Gonska, H., Raşa, I., *On infinite products of positive linear operators reproducing linear functions*, Positivity **17** (2013), No. 1, 67-79.
[17] Gwóźdź-Łukawska, G., Jachymski, J., *IFS on a metric space with a graph structure and extensions of the Kelisky-Rivlin theorem*, J. Math. Anal. Appl. **356(2)** (2009), 453-463.
[18] Karlin, S. , Ziegler, Z., *Iteration of positive approximation operators*, J. Approx. Theory **3** (1970), 310-339.
[19] Kelisky, R.P., Rivlin, T.J., *Iterates of Bernstein polynomials*, Pacific J. Math. **21** (1967), 511-520.
[20] Raşa, I., *C_0-Semigroups and iterates of positive linear operators: asymptotic behaviour*, Rend. Circ. Mat. Palermo, Ser. II, Suppl. **82** (2010), 123-142.
[21] Rus, I.A., *Picard operators and applications*, Sci. Math. Jpn. **58** (2003), 191-219.
[22] Rus, I.A., *Iterates of Stancu operators, via contraction principle*, Studia Univ. Babeş–Bolyai Math. **47** (2002), No. 4, 101-104.
[23] Rus, I.A., *Iterates of Bernstein operators, via contraction principle*, J. Math. Anal. Appl. **292** (2004), 259-261.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.4, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

[24] Rus, I.A., *Fixed point and interpolation point set of a positive linear operator on* $C(\overline{D})$, Studia Univ. Babeş–Bolyai Math. **55** (2010), No. 4, 243-248.

Contemporary Concepts of Neutrosophic Fuzzy Soft BCK-submodules

R. S. Alghamdi[1] and Noura Omair Alshehri[2]

[1]Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia; ralgamdi0068@stu.kau.edu.sa, rsaalghamdi@uj.edu.sa

[2]Department of Mathematics, Faculty of Science, University of Jeddah, P.O. Box 80327, Jeddah 21589, Saudi Arabia; noal-shehri@uj.edu.sa

Abstract

In this paper, we introduce the concept of neutrosophic fuzzy soft translations and neutrosophic fuzzy soft extensions of neutrosophic fuzzy soft BCK-submodules and discusse the relation between them. Also, we define the notion of neutrosophic fuzzy soft multiplications of neutrosophic fuzzy soft BCK-submodules. Finally, we investigate some resultes.

Keywords: BCK-algebras, BCK-modules, soft sets, fuzzy soft sets, neutrosophic sets, neutrosophic soft sets, neutrosophic fuzzy soft BCK-submodules, neutrosophic fuzzy soft translations, neutrosophic fuzzy soft multiplications and neutrosophic fuzzy soft extensions.

1 Introduction

Fuzzy set theory which was developed by Zadeh [23] is an appropriate theory for dealing with vagueness. It is consedered as the one of theories can be handled with uncertainties. Combining fuzzy set models with other mathematical models has attracted the attention of many researchers. Interval-valued fuzzy sets [24], hesitant fuzzy sets [21] , intuitionistic fuzzy sets [3, 4], Intutionistic Fuzzy BCK-submodules [5] and $(\epsilon, \epsilon \vee q)$-fuzzy BCK-submodules [2] are some of the researches that have dealt this subject.

Neutrosophic algebraic structure is a very recent study. It was applied in many fields in order to solve problems related to uncertainties and indeterminacy where they happens to be one of the major factors in almost all real-world problems. Neutrosophic set is a generalizations of the fuzzy set especially of intuitionistic fuzzy set. The intuitionistic fuzzy set has the degree of non-membership

as introduced by K. Atanassov [3]. Smarandache in 1998 [19] has introduced the degree of indeterminacy as an independent component and defined the neutrosophic set on three components: truth, indeterminacy and falsity.

The concept of BCK-algebra was first initiated by Imai and Iseki [8]. In 1994, the notion of BCK-modules was introduced by H. Abujable, M. Aslam and A. Thaheem as an action of BCK-algebras on abelian group [1]. BCK-modules theory then was developed by Z. perveen, M. Aslam and A. Thaheem [18]. Bakhshi [6] presented the concept of fuzzy BCK-submodules and investigated their properties. Recently, H. Bashir and Z. Zahid applied the theory of soft sets on BCK-modules in [12].

Translations, multiplications and extensions are very interested mathematical tools. They are types of operations that researchers like to apply with fuzzy set theory. In this paper, we introduce the concept of neutrosophic fuzzy soft translations and neutrosophic fuzzy soft extensions of neutrosophic fuzzy soft BCK-submodules and discusse the relation between them. Also, we define the notion of neutrosophic fuzzy soft multiplications of neutrosophic fuzzy soft BCK-submodules. Finally, we investigate some resultes.

2 Preliminaries

In this section, some preliminaries from the soft set theory, neutrosophic soft sets, BCK-algebras and BCK-modules are induced.

Definition 2.1.[17] Let U be an initial universe and E be a set of parameters. Let $P(U)$ denote the power set of U and let A be a nonempty subset of E. A pair $F_A = (F, A)$ is called a soft set over U, where $A \subseteq E$ and $F : A \to P(U)$ is a set-valued mapping, called the approximate function of the soft set (F, A). It is easy to represent a soft set (F, A) by a set of ordered pairs as follows:

$$(F, A) = \{(x, F(x)) : x \in A\}$$

Definition 2.2.[20] A neutrosophic set A on the universe of discourse U is defined as $A = \{(x, T_A(x), I_A(x), F_A(x)), x \in U\}$ where $T_A : X \to]^-0, 1^+[$ is a truth membership function, $I_A : U \to]^-0, 1^+[$ is an indeterminate membership function, and $F_A : X \to]^-0, 1^+[$ is a false membership function and $^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the subset of $[0, 1]$.

Definition 2.3.[13] Let U be an initial universe set and E be a set of parameters. Consider $A \subset E$. Let $P(U)$ denotes the set of all neutrosophic sets of U. The collection (F, A) is termed to be the neutrosophic soft set (NSS) over U, where F is a mapping given by $F : A \to P(U)$.

Definition 2.4.[8, 9] An algebra $(X, *, 0)$ of type $(2, 0)$ is called BCK-algebra if it satisfying the following axioms:

(BCK-1) $((x * y) * (x * z)) * (z * y) = 0$,

(BCK-2) $(x * (x * y)) * y = 0$,

(BCK-3) $x * x = 0$,

(BCK-4) $0 * x = 0$,

(BCK-5) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x, y, z \in X$.

A partial ordering "$\leq$ " is defined on X by $x \leq y \Leftrightarrow x * y = 0$. A BCK-algebra X is said to be bounded if there is an element $1 \in X$ such that $x \leq 1$, for all $x \in X$, commutative if it satisfies the identity $x \wedge y = y \wedge x$, where $x \wedge y = y * (y * x)$, for all $x, y \in X$ and implicative if $x * (y * x) = x$, for all $x, y \in X$.

Definition 2.5.[1] Let X be a BCK-algebra. Then by a left X-module (abbreviated X-module), we mean an abelian group M with an operation $X \times M \rightarrow M$ with $(x, m) \longmapsto xm$ satisfies the following axioms for all $x, y \in X$ and $m, n \in M$:

(i) $(x \wedge y)m = x(ym)$,

(ii) $x(m + n) = xm + xn$,

(iii) $0m = 0$.

If X is bounded and M satisfies $1m = m$, for all $m \in M$, then M is said to be unitary.

A mapping $\mu : X \rightarrow [0, 1]$ is called a fuzzy set in a BCK-algebra X. For any fuzzy set μ in X and any $t \in [0, 1]$, we define set $U(\mu; t) = \mu^t = \{x \in X | \mu(x) \geq t\}$, which is called upper t-level cut of μ.

Definition 2.6.[6] A fuzzy subset μ of M is said to be a fuzzy BCK-submodule if for all m, m_1, $m_2 \in M$ and $x \in X$, the following axioms hold:

(FBCKM1) $\mu(m_1 + m_2) \geq \min\{\mu(m_1), \mu(m_2)\}$,

(FBCKM2) $\mu(-m) = \mu(m)$,

(FBCKM3) $\mu(xm) \geq \mu(m)$.

Definition 2.7.[6] Let M, N be modules over a BCK-algebra X. A mapping $f : M \rightarrow N$ is called BCK-module homomorphism if

(1) $f(m_1 + m_2) = f(m_1) + f(m_2)$,

(2) $f(xm) = xf(m)$ for all $m, m_1, m_2 \in M$ and $x \in X$.

A BCK-module homomorphism is said to be monomorphism (epimorphism) if it is one to one (onto). If it is both one to one and onto, then we say that it is an isomorphism.

Definition 2.8.[12] Let (F, A) and (G, B) be two soft modules over M and N respectively, $f : M \rightarrow N$, $g : A \rightarrow B$ be two functions. Then we say that (f, g) is a soft BCK-homomorphism if the following conditions are satisfied:

(1) f is a homomorphism from M onto N,

(2) g is a mapping from A onto B, and

(3) $f(F(x)) = G(g(x))$ for all $x \in A$.

3 Neutrosophic fuzzy soft BCK-submodules

Definition 3.1. A neutrosophic fuzzy soft set (F, A) over a BCK-module M is said to be a neutrosophic fuzzy soft BCK-submodule over M if for all $m, m_1, m_2 \in M$, $x \in X$ and $\varepsilon \in A$ the following axioms hold :

(NFSS1) $T_{F(\varepsilon)}(m_1 + m_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$,

$I_{F(\varepsilon)}(m_1 + m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$,

$F_{F(\varepsilon)}(m_1 + m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}$,

(NFSS2) $T_{F(\varepsilon)}(-m) = T_{F(\varepsilon)}(m)$,

$I_{F(\varepsilon)}(-m) = I_{F(\varepsilon)}(m)$,

$F_{F(\varepsilon)}(-m) = F_{F(\varepsilon)}(m)$,

(NFSS3) $T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m)$,

$I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m)$,

$F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m)$.

Example 3.2. Let $X = \{0, a, b, c, d\}$ be a set along with a binary operation $*$ defined in Table 1, then $(X, *, 0)$ forms a commutative BCK-algebra which is not bounded (see [16]). Let $M = \{0, a, b, c\}$ be a subset of X along with an operation $+$ defined by Table 2. Then $(M, +)$ forms a commutative group. Table 3 explains the action of X on M under the operation $xm = x \wedge m$ for all $x \in X$ and $m \in M$. Consequently, M forms an X-module (see [11]).

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	d
d	d	d	d	d	0

Table 1

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Table 2

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c
d	0	0	0	0

Table 3

Let $A = \{0, a\}$. Define a neutrosophic fuzzy soft set (F, A) over M as shown in Table 4

Consequently, a routine exercise of calculations show that (F, A) forms a neutrosophic fuzzy soft BCK-submodule over M.

(F,A)	0	a	b	c
$T_{F(0)}$	0.9	0.7	0.8	0.7
$I_{F(0)}$	0.8	0.5	0.6	0.5
$F_{F(0)}$	0.1	0.1	0.1	0.1
$T_{F(a)}$	0.5	0.2	0.3	0.2
$I_{F(a)}$	0.3	0.1	0.3	0.1
$F_{F(a)}$	0.1	0.5	0.4	0.5

Table 4

For the sake of simplicity, we shall use the symbols $NFS(M)$ and $NFSS(M)$ for the set of all neutrosophic fuzzy soft sets over M and the set of all neutrosophic fuzzy soft BCK-submodules over M, respectively.

Theorem 3.3. A neutrosophic fuzzy soft set $(F,A) \in NFSS(M)$ if and only if

(i) $T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m), \quad I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m), \quad F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m)$,

(ii) $T_{F(\varepsilon)}(m_1 - m_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$,

$I_{F(\varepsilon)}(m_1 - m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$,

$F_{F(\varepsilon)}(m_1 - m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}$.

for all $m, m_1, m_2 \in M$, $x \in X$ and $\varepsilon \in A$.

Proof. Let (F,A) be a neutrosophic fuzzy soft BCK-submodule over M then by the definition(3.1) condition (i) is hold.

(ii) $T_{F(\varepsilon)}(m_1 - m_2) = T_{F(\varepsilon)}(m_1 + (-m_2)) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(-m_2)\} = \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$,

$I_{F(\varepsilon)}(m_1 - m_2) = I_{F(\varepsilon)}(m_1 + (-m_2)) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(-m_2)\} = \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$,

$F_{F(\varepsilon)}(m_1 - m_2) = F_{F(\varepsilon)}(m_1 + (-m_2)) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(-m_2)\} = \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}$.

Conversely suppose (F,A) satisfies the conditions (i),(ii). Then we have by (i)

$$T_{F(\varepsilon)}(-m) = T_{F(\varepsilon)}((-1)m) \geq T_{F(\varepsilon)}(m),$$

and

$$T_{F(\varepsilon)}(m) = T_{F(\varepsilon)}((-1)(-1)m) \geq T_{F(\varepsilon)}(-m).$$

Thus, $T_{F(\varepsilon)}(m) = T_{F(\varepsilon)}(-m)$. Similarly for $I_{F(\varepsilon)}(-m) = I_{F(\varepsilon)}(m)$ and $F_{F(\varepsilon)}(-m) = F_{F(\varepsilon)}(m)$.

$T_{F(\varepsilon)}(m_1 + m_2) = T_{F(\varepsilon)}(m_1 - (-m_2)) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(-m_2)\} = \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$,

$I_{F(\varepsilon)}(m_1 + m_2) = I_{F(\varepsilon)}(m_1 - (-m_2)) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(-m_2)\} = \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$,

$F_{F(\varepsilon)}(m_1 + m_2) = F_{F(\varepsilon)}(m_1 - (-m_2)) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(-m_2)\{= \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}$.

Hence (F, A) is a neutrosophic fuzzy soft BCK-submodule over M.

Theorem 3.4. A neutrosophic fuzzy soft set $(F, A) \in NFSS(M)$ if and only if for all $m, m_1, m_2 \in M$, $x, y \in X$ and $\varepsilon \in A$ the following statements hold:

(i) $T_{F(\varepsilon)}(0) \geq T_{F(\varepsilon)}(m), \quad I_{F(\varepsilon)}(0) \geq I_{F(\varepsilon)}(m), \quad F_{F(\varepsilon)}(0) \leq F_{F(\varepsilon)}(m)$,

(ii) $T_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}$,

$I_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\}$,

$F_{F(\varepsilon)}(xm_1 - ym_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}$.

Proof. Let $(F, A) \in NFSS(M)$ then by theorem (3.3) and since $0m = 0$ for all $m \in M$,we have

$$\begin{aligned}
\text{(i) } & T_{F(\varepsilon)}(0) = T_{F(\varepsilon)}(0m) \geq T_{F(\varepsilon)}(m), \\
& I_{F(\varepsilon)}(0) = I_{F(\varepsilon)}(0m) \geq I_{F(\varepsilon)}(m), \text{ and} \\
& F_{F(\varepsilon)}(0) = F_{F(\varepsilon)}(0m) \leq F_{F(\varepsilon)}(m). \\
\text{(ii) } & T_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{T_{F(\varepsilon)}(xm_1), T_{F(\varepsilon)}(ym_2)\} \\
& \qquad \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}.
\end{aligned}$$

Similarly for

$$I_{F(\varepsilon)}(xm_1 - ym_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\},$$

and

$$F_{F(\varepsilon)}(xm_1 - ym_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$$

Conversely suppose (F, A) satisfies (i),(ii), then we have

$$T_{F(\varepsilon)}(0) \geq T_{F(\varepsilon)}(m), \quad I_{F(\varepsilon)}(0) \geq I_{F(\varepsilon)}(m) and \;\; F_{F(\varepsilon)}(0) \leq F_{F(\varepsilon)}(m).$$

Then

$$T_{F(\varepsilon)}(xm) = T_{F(\varepsilon)}(x(m - 0)) \geq \min\{T_{F(\varepsilon)}(m), T_{F(\varepsilon)}(0)\} = T_{F(\varepsilon)}(m).$$

Similarly for

$$I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m) \text{ and } F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m).$$

Also,

$$T_{F(\varepsilon)}(m_1 - m_2) = T_{F(\varepsilon)}(1m_1 - 1m_2) \geq \min\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\}.$$

Similarly for

$$I_{F(\varepsilon)}(m_1 - m_2) \geq \min\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\} \text{ and } F_{F(\varepsilon)}(m_1 - m_2) \leq \max\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\}.$$

Hence (F, A) is a neutrosophic fuzzy soft BCK-submodule over M.

Definition 3.5. Let (F, A) be a neutrosophic fuzzy soft set over a BCK-module M and $\alpha \in [0, \perp]$ such that $\perp = 1 - \sup\left\{F_{F(\varepsilon)}(m) : m \in M, \varepsilon \in A\right\}$ .Then $\tilde{T}_\alpha[(F, A)] = (G, A_\alpha^T)$ is called a neutrosophic fuzzy soft α-translation of (F, A) if it satisfies:

$$G(\varepsilon) = \left(\left(T_{F(\varepsilon)}\right)_\alpha^T(m), \left(I_{F(\varepsilon)}\right)_\alpha^T(m), \left(F_{F(\varepsilon)}\right)_\alpha^T(m)\right),$$

for all $\varepsilon \in A, m \in M$ where:

$$\begin{aligned}
\left(T_{F(\varepsilon)}\right)_\alpha^T(m) &= T_{F(\varepsilon)}(m) + \alpha, \\
\left(I_{F(\varepsilon)}\right)_\alpha^T(m) &= I_{F(\varepsilon)}(m), \\
\left(F_{F(\varepsilon)}\right)_\alpha^T(m) &= F_{F(\varepsilon)}(m) - \alpha.
\end{aligned}$$

Theorem 3.6. A neutrosophic fuzzy soft set (F, A) is said to be a neutrosophic fuzzy soft BCK-submodule over M if and only if the α-translation neutrosophic fuzzy soft set $\tilde{T}_\alpha[(F, A)]$ is a neutrosophic fuzzy soft BCK-submodule over M for all $\alpha \in [0, \perp]$.

Proof. Let (F, A) be a neutrosophic fuzzy soft BCK-submodule over M and $\alpha \in [0, \perp]$, then by Theorem (3.3)

$$\begin{aligned}
\left(T_{F(\varepsilon)}\right)_\alpha^T(xm) &= T_{F(\varepsilon)}(xm) + \alpha \geq T_{F(\varepsilon)}(m) + \alpha = \left(T_{F(\varepsilon)}\right)_\alpha^T(m), \\
\left(F_{F(\varepsilon)}\right)_\alpha^T(xm) &= F_{F(\varepsilon)}(xm) - \alpha \leq F_{F(\varepsilon)}(m) - \alpha = \left(F_{F(\varepsilon)}\right)_\alpha^T(m),
\end{aligned}$$

for all $m \in M$, $x \in X$. Also, for all $m_1, m_2 \in M$ we have

$$\begin{aligned}
\left(T_{F(\varepsilon)}\right)_\alpha^T(m_1 - m_2) &= T_{F(\varepsilon)}(m_1 - m_2) + \alpha \\
&\geq \min\left\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\right\} + \alpha \\
&= \min\left\{T_{F(\varepsilon)}(m_1) + \alpha, T_{F(\varepsilon)}(m_2) + \alpha\right\} \\
&= \min\left\{\left(T_{F(\varepsilon)}\right)_\alpha^T(m_1), \left(T_{F(\varepsilon)}\right)_\alpha^T(m_2)\right\},
\end{aligned}$$

and

$$\begin{aligned}
\left(F_{F(\varepsilon)}\right)_\alpha^T(m_1 - m_2) &= F_{F(\varepsilon)}(m_1 - m_2) - \alpha \\
&\leq \max\left\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\right\} - \alpha \\
&= \max\left\{F_{F(\varepsilon)}(m_1) - \alpha, F_{F(\varepsilon)}(m_2) - \alpha\right\} \\
&= \max\left\{\left(F_{F(\varepsilon)}\right)_\alpha^T(m_1), \left(F_{F(\varepsilon)}\right)_\alpha^T(m_2)\right\}.
\end{aligned}$$

Hence $\tilde{T}_\alpha[(F, A)]$ is a neutrosophic fuzzy soft BCK-submodule over M.

Conversely, assume that $\tilde{T}_\alpha[(F, A)]$ is a neutrosophic fuzzy soft BCK-submodule over M for some $\alpha \in [0, \perp]$. Then for all $m \in M$, $x \in X$

$$\begin{aligned}
T_{F(\varepsilon)}(xm) + \alpha = \left(T_{F(\varepsilon)}\right)_\alpha^T(xm) &\geq \left(T_{F(\varepsilon)}\right)_\alpha^T(m) = T_{F(\varepsilon)}(m) + \alpha \\
\Longrightarrow T_{F(\varepsilon)}(xm) &\geq T_{F(\varepsilon)}(m).
\end{aligned}$$

Also,

$$F_{F(\varepsilon)}(xm) - \alpha = \left(F_{F(\varepsilon)}\right)_{\alpha}^{T}(xm) \leq \left(F_{F(\varepsilon)}\right)_{\alpha}^{T}(m) = F_{F(\varepsilon)}(m) - \alpha$$
$$\Longrightarrow F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m).$$

Now let $m_1, m_2 \in M$, then

$$\begin{aligned} T_{F(\varepsilon)}(m_1 - m_2) + \alpha &= \left(T_{F(\varepsilon)}\right)_{\alpha}^{T}(m_1 - m_2) \\ &\geq \min\left\{\left(T_{F(\varepsilon)}\right)_{\alpha}^{T}(m_1), \left(T_{F(\varepsilon)}\right)_{\alpha}^{T}(m_2)\right\} \\ &= \min\left\{T_{F(\varepsilon)}(m_1) + \alpha, T_{F(\varepsilon)}(m_2) + \alpha\right\} \\ &= \min\left\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\right\} + \alpha \end{aligned}$$

$$\Longrightarrow T_{F(\varepsilon)}(m_1 - m_2) \geq \min\left\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\right\},$$

and

$$\begin{aligned} F_{F(\varepsilon)}(m_1 - m_2) - \alpha &= \left(F_{F(\varepsilon)}\right)_{\alpha}^{T}(m_1 - m_2) \\ &\leq \max\left\{\left(F_{F(\varepsilon)}\right)_{\alpha}^{T}(m_1), \left(F_{F(\varepsilon)}\right)_{\alpha}^{T}(m_2)\right\} \\ &= \max\left\{F_{F(\varepsilon)}(m_1) - \alpha, F_{F(\varepsilon)}(m_2) - \alpha\right\} \\ &= \max\left\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\right\} - \alpha \end{aligned}$$

$$\Longrightarrow F_{F(\varepsilon)}(m_1 - m_2) \leq \max\left\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\right\}.$$

Hence by Theorem (3.3), (F, A) is a neutrosophic fuzzy soft BCK-submodule over M.

Definition 3.7. Let (F, A) and (G, B) be two neutrosophic fuzzy soft sets over a BCK-module M. If $A \subset B$ and $T_{F(\varepsilon)}(m) \leq T_{G(\varepsilon)}(m)$, $I_{F(\varepsilon)}(m) \leq I_{G(\varepsilon)}(m)$, $F_{F(\varepsilon)}(m) \geq F_{G(\varepsilon)}(m)$, $\forall \varepsilon \in A$ and $m \in M$. Then we say that (G, B) is a neutrosophic fuzzy soft extinsion of (F, A).

Definition 3.8. Let (F, A) and (G, B) be two neutrosophic fuzzy soft sets over a BCK-module M. Then (G, B) is a neutrosophic fuzzy soft s-extinsion of (F, A) if the following assertions hold:

(i) (G, B) is a neutrosophic fuzzy soft extinsion of (F, A).

(ii) If (F, A) is a neutrosophic fuzzy soft BCK-submodule over M, then so (G, B).

Theorem 3.9. Let (F, A) be a neutrosophic fuzzy soft BCK-submodule over M and $\alpha \in [0, \perp]$. Then the neutrosophic fuzzy soft α-translation $\tilde{T}_{\alpha}[(F, A)]$ is a neutrosophic fuzzy soft s-extinsion of (F, A).

Proof. Since $\tilde{T}_{\alpha}[(F, A)]$ is an α-translation, we know that $\left(T_{F(\varepsilon)}\right)_{\alpha}^{T}(m) \geq T_{F(\varepsilon)}(m)$, $\left(I_{F(\varepsilon)}\right)_{\alpha}^{T}(m) = I_{F(\varepsilon)}(m)$ and $\left(F_{F(\varepsilon)}\right)_{\alpha}^{T}(m) \leq F_{F(\varepsilon)}(m)$ for all $m \in M, \varepsilon \in A$. Hence $\tilde{T}_{\alpha}[(F, A)]$ is a neutrosophic fuzzy soft extinsion of (F, A). According to Theorem (3.6), $\tilde{T}_{\alpha}[(F, A)]$ is a neutrosophic fuzzy soft s-extinsion of (F, A).

The converse of Theorem (3.9) is not true in general as seen in the following example:

Example 3.10. Let $X = \{0, a, b, c\}$ along with a binary operation $*$ defined in Table 5, then $(X, *, 0)$ forms a bounded implicative BCK-algebra (see [16]). Let $M = \{0, a\}$ be a subset of X with a binary operation $+$ defined by $x + y = (x * y) \vee (y * x)$. Then M is a commutative group as shown in table 6. Define scalar multiplication $(X, M) \to M$ by $xm = x \wedge m$ for all $x \in X$ and $m \in M$ that is given in Table 7. Consequently, M forms an X-module (see [11]).

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Table 5

$+$	0	a
0	0	a
a	a	0

Table 6

$\wedge$	0	a
0	0	0
a	0	a
b	0	0
c	0	a

Table 7

Let $A = M$. Define a neutrosophic fuzzy soft set (F, A) over M as shown in Table 8.

(F, A)	0	a
$T_{F(0)}$	0.9	0.5
$I_{F(0)}$	0.8	0.6
$F_{F(0)}$	0.1	0.3
$T_{F(a)}$	0.3	0.3
$I_{F(a)}$	0.2	0.2
$F_{F(a)}$	0.3	0.5

Table 8

Then (F, A) is a neutrosophic fuzzy soft BCK-submodule over M. Let (G, B) be a neutrosophic fuzzy soft set over M given by Table 9.

Then (G, B) is also a neutrosophic fuzzy soft BCK-submodule over M. Since $T_{F(\varepsilon)}(m) \geq T_{G(\varepsilon)}(m)$, $I_{F(\varepsilon)}(m) \geq I_{G(\varepsilon)}(m)$ and $F_{F(\varepsilon)}(m) \leq F_{G(\varepsilon)}(m)$ for all $m \in M$ and $\varepsilon \in A \subset B$, hence (F, A) is a neutrosophic fuzzy soft s-extension of (G, B), but since $I_{F(0)}(0) = 0.8 \neq I_{G(0)}(0) = 0.7$ then (F, A) is not a neutrosophic fuzzy soft α-translation of (G, B) for all $\alpha \in [0, \perp]$.

(G,B)	0	a
$T_{G(0)}$	0.5	0.3
$I_{G(0)}$	0.7	0.6
$F_{G(0)}$	0.1	0.4
$T_{G(a)}$	0.2	0.2
$I_{G(a)}$	0.1	0.1
$F_{G(a)}$	0.4	0.5

Table 9

Definition 3.11. Let (F,A) be a neutrosophic fuzzy soft set over a BCK-module M and $\upsilon \in [0,1]$. A neutrosophic fuzzy soft υ-multiplication of (F,A) denoted by $\tilde{M}_\upsilon [(F,A)] = (G, m_\upsilon (A))$ is defined as:

$$G(\varepsilon) = \left(m_\upsilon \left(T_{F(\varepsilon)}\right)(m), m_\upsilon \left(I_{F(\varepsilon)}\right)(m), m_\upsilon \left(F_{F(\varepsilon)}\right)(m)\right),$$

where

$$\begin{aligned} m_\upsilon \left(T_{F(\varepsilon)}\right)(m) &= T_{F(\varepsilon)}(m).\upsilon, \\ m_\upsilon \left(I_{F(\varepsilon)}\right)(m) &= I_{F(\varepsilon)}(m), \\ m_\upsilon \left(F_{F(\varepsilon)}\right)(m) &= F_{F(\varepsilon)}(m).\upsilon, \end{aligned}$$

for all $\varepsilon \in A$ and $m \in M$.

Theorem.3.12. If $(F,A) \in NFSS(M)$, then the neutrosophic fuzzy soft υ-multiplication $\tilde{M}_\upsilon [(F,A)] \in NFSS(M)$ for all $\upsilon \in [0,1]$.

Proof. Assume that (F,A) is a neutrosophic fuzzy soft BCK-submodule over M and let $m, m_1, m_2 \in M$, $x \in X$ and $\varepsilon \in A$. Then

$$\begin{aligned} m_\upsilon \left(T_{F(\varepsilon)}\right)(xm) &= T_{F(\varepsilon)}(xm).\upsilon \geq T_{F(\varepsilon)}(m).\upsilon = m_\upsilon \left(T_{F(\varepsilon)}\right)(m), \\ m_\upsilon \left(I_{F(\varepsilon)}\right)(xm) &= I_{F(\varepsilon)}(xm) \geq I_{F(\varepsilon)}(m) = m_\upsilon \left(I_{F(\varepsilon)}\right)(m), \\ m_\upsilon \left(F_{F(\varepsilon)}\right)(xm) &= F_{F(\varepsilon)}(xm).\upsilon \leq F_{F(\varepsilon)}(m).\upsilon = m_\upsilon \left(F_{F(\varepsilon)}\right)(m). \end{aligned}$$

Moreover,

$$\begin{aligned} m_\upsilon \left(T_{F(\varepsilon)}\right)(m_1 - m_2) &= T_{F(\varepsilon)}(m_1 - m_2).\upsilon \\ &\geq \min\left\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\right\}.\upsilon \\ &= \min\left\{T_{F(\varepsilon)}(m_1).\upsilon, T_{F(\varepsilon)}(m_2).\upsilon\right\} \\ &= \min\left\{m_\upsilon \left(T_{F(\varepsilon)}\right)(m_1), m_\upsilon \left(T_{F(\varepsilon)}\right)(m_2)\right\}, \end{aligned}$$

10

$$\begin{aligned} m_{\upsilon}\left(I_{F(\varepsilon)}\right)(m_1-m_2) &= I_{F(\varepsilon)}(m_1-m_2) \\ &\geq \min\left\{I_{F(\varepsilon)}(m_1), I_{F(\varepsilon)}(m_2)\right\} \\ &= \min\left\{m_{\upsilon}\left(I_{F(\varepsilon)}\right)(m_1), m_{\upsilon}\left(I_{F(\varepsilon)}\right)(m_2)\right\}, \end{aligned}$$

$$\begin{aligned} m_{\upsilon}\left(F_{F(\varepsilon)}\right)(m_1-m_2) &= F_{F(\varepsilon)}(m_1-m_2).\upsilon \\ &\leq \max\left\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\right\}.\upsilon \\ &= \max\left\{F_{F(\varepsilon)}(m_1).\upsilon, F_{F(\varepsilon)}(m_2).\upsilon\right\} \\ &= \max\left\{m_{\upsilon}\left(F_{F(\varepsilon)}\right)(m_1), m_{\upsilon}\left(F_{F(\varepsilon)}\right)(m_2)\right\}. \end{aligned}$$

Therefore by Theorem (3.3), $\tilde{M}_{\upsilon}[(F,A)]$ is a neutrosophic fuzzy soft BCK-submodule over M.

The converse of Theorem (3.12) is not true in general as seen in the following example:

Example 3.13. Consider a BCK-algebra $X=\{0,a,b,c\}$ and X-module $M=\{0,a\}$ that are defined in Example 3.10. Table 10 defines a neutrosophic fuzzy soft set (F,A) over M

(F,A)	0	a
$T_{F(0)}$	0.3	0.4
$I_{F(0)}$	0.7	0.5
$F_{F(0)}$	0.1	0.5
$T_{F(a)}$	0.1	0.1
$I_{F(a)}$	0.1	0.1
$F_{F(a)}$	0.5	0.6

Table 10

If we take $\upsilon=0$, then the υ-multiplication is a neutrosophic fuzzy soft BCK-submodule over M since

$$\begin{aligned} m_0\left(T_{F(\varepsilon)}\right)(xm) &= 0 = m_0\left(T_{F(\varepsilon)}\right)(m), \\ m_0\left(I_{F(\varepsilon)}\right)(xm) &\geq m_0\left(I_{F(\varepsilon)}\right)(m), \\ m_0\left(F_{F(\varepsilon)}\right)(xm) &= 0 = m_0\left(F_{F(\varepsilon)}\right)(m), \end{aligned}$$

and

$$\begin{aligned} m_0\left(T_{F(\varepsilon)}\right)(m_1-m_2) &= 0 = \min\left\{m_0\left(T_{F(\varepsilon)}\right)(m_1), m_0\left(T_{F(\varepsilon)}\right)(m_2)\right\}, \\ m_0\left(I_{F(\varepsilon)}\right)(m_1-m_2) &\geq \min\left\{m_0\left(I_{F(\varepsilon)}\right)(m_1), m_0\left(I_{F(\varepsilon)}\right)(m_2)\right\}, \\ m_0\left(F_{F(\varepsilon)}\right)(m_1-m_2) &= 0 = \min\left\{m_0\left(F_{F(\varepsilon)}\right)(m_1), m_0\left(F_{F(\varepsilon)}\right)(m_2)\right\}, \end{aligned}$$

for all $m, m_1, m_2 \in M$ and $x \in X$. But if we take $m_1 = 0, m_2 = a$ and $\varepsilon = 0$ then

$$T_{F(0)}(0+a) = T_{F(0)}(a) = 0.4 \ngeq \min\left\{T_{F(0)}(0), T_{F(0)}(a)\right\} = 0.3.$$

Hence (F, A) is not a neutrosophic fuzzy soft BCK-submodule over M.

Theorem.3.14. A neutrosophic fuzzy soft set (F, A) is said to be a neutrosophic fuzzy soft BCK-submodule over M if and only if the υ-multiplication neutrosophic fuzzy set $\tilde{M}_\upsilon[(F, A)]$ is a neutrosophic fuzzy soft BCK-submodule over M for all $\upsilon \in (0, 1]$.

Proof. Let (F, A) be a neutrosophic fuzzy soft BCK-submodule over M then by Theorem (3.12) $\tilde{M}_\upsilon[(F, A)]$ is a neutrosophic fuzzy soft BCK-submodule over M for all $\upsilon \in (0, 1]$.

Now let $\upsilon \in (0, 1]$ be such that $\tilde{M}_\upsilon[(F, A)]$ is a neutrosophic fuzzy soft BCK-submodule over M and let $m, m_1, m_2 \in M$, $x \in X$ and $\varepsilon \in A$. Then

$$\begin{aligned} T_{F(\varepsilon)}(xm).\upsilon &= m_\upsilon\left(T_{F(\varepsilon)}\right)(xm) \geq m_\upsilon\left(T_{F(\varepsilon)}\right)(m) = T_{F(\varepsilon)}(m).\upsilon, \\ I_{F(\varepsilon)}(xm) &= m_\upsilon\left(I_{F(\varepsilon)}\right)(xm) \geq m_\upsilon\left(I_{F(\varepsilon)}\right)(m) = I_{F(\varepsilon)}(m), \\ F_{F(\varepsilon)}(xm).\upsilon &= m_\upsilon\left(F_{F(\varepsilon)}\right)(xm) \leq m_\upsilon\left(F_{F(\varepsilon)}\right)(m) = F_{F(\varepsilon)}(m).\upsilon, \end{aligned}$$

and since $\upsilon \neq 0$, then $T_{F(\varepsilon)}(xm) \geq T_{F(\varepsilon)}(m)$ and $F_{F(\varepsilon)}(xm) \leq F_{F(\varepsilon)}(m)$. Now

$$\begin{aligned} T_{F(\varepsilon)}(m_1 - m_2).\upsilon &= m_\upsilon\left(T_{F(\varepsilon)}\right)(m_1 - m_2) \\ &\geq \min\left\{m_\upsilon\left(T_{F(\varepsilon)}\right)(m_1), m_\upsilon\left(T_{F(\varepsilon)}\right)(m_2)\right\} \\ &= \min\left\{T_{F(\varepsilon)}(m_1).\upsilon, T_{F(\varepsilon)}(m_2).\upsilon\right\} \\ &= \min\left\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\right\}.\upsilon, \end{aligned}$$

which means that

$$T_{F(\varepsilon)}(m_1 - m_2) \geq \min\left\{T_{F(\varepsilon)}(m_1), T_{F(\varepsilon)}(m_2)\right\}.$$

Similarly,

$$F_{F(\varepsilon)}(m_1 - m_2) \leq \max\left\{F_{F(\varepsilon)}(m_1), F_{F(\varepsilon)}(m_2)\right\}.$$

Hence (F, A) is a neutrosophic fuzzy soft BCK-submodule over M.

4 Ismorphism Theorem Of Neutrosophic Fuzzy Soft BCK-submodules

Definition 4.1. Let M and N be two BCK-modules over a BCK-algebra X. Let $f : M \longrightarrow N$ be a BCK-submodule homomorphism and let (F, A), (G, B) be two neutrosophic fuzzy soft BCK-submodule over M and N respectively. Then the image of (F, A) is a neutrosophic fuzzy soft set over N defined as follows for all $x \in M$, $y \in N$ and $\varepsilon \in A$.

$$f(F(\varepsilon))(x) = \left(T_{f(F)}(y), I_{f(F)}(y), F_{f(F)}(y)\right) = \left(f(T_F)(y), f(I_F)(y), f(F_F)(y)\right),$$

where

$$f\left(T_F\right)(y)=\begin{cases}\sup T_F\left(x\right) & \text{if } x\in f^{-1}\left(y\right)\\ 0 & \text{otherwise}\end{cases},$$
$$f\left(I_F\right)(y)=\begin{cases}\sup I_F\left(x\right) & \text{if } x\in f^{-1}\left(y\right)\\ 0 & \text{otherwise}\end{cases},$$
$$f\left(F_F\right)(y)=\begin{cases}\inf F_F\left(x\right) & \text{if } x\in f^{-1}\left(y\right)\\ 0 & \text{otherwise}\end{cases},$$

and the preimage of (G,B) is a neutrosophic fuzzy soft set over M defined as

$$f^{-1}\left(G\left(\delta\right)\right)(y)=\left(T_{f^{-1}(G)}(x),I_{f^{-1}(G)}(x),F_{f^{-1}(G)}(x)\right)=\left(T_G\left(f\left(x\right)\right),I_G\left(f\left(x\right)\right),F_G\left(f\left(x\right)\right)\right),$$

where $\delta\in B$.

Theorem 4.2. Let $(X,*,0)$ be a BCK-algebra, M and N are modules of X. A mapping $f:M\longrightarrow N$ is a BCK-submodule homomorphism and $(F,A)\in NFSS(N)$, then the inverse image $\left(f^{-1}\left(F\right),A\right)\in NFSS(M)$.

Proof. Since (F,A) is a neutrosophic fuzzy soft BCK-submodule over N. Let $m\in M$, $\varepsilon\in A$ then by Theorem (3.4)

$$T_{f^{-1}(F)}(0)=T_{F(\varepsilon)}(f\left(0\right))=T_{F(\varepsilon)}(0)\geq T_{F(\varepsilon)}(f\left(m\right))=T_{f^{-1}(F)}(m),$$
$$I_{f^{-1}(F)}(0)=I_{F(\varepsilon)}(f\left(0\right))=I_{F(\varepsilon)}(0)\geq I_{F(\varepsilon)}(f\left(m\right))=I_{f^{-1}(F)}(m),$$
$$F_{f^{-1}(F)}(0)=F_{F(\varepsilon)}(f\left(0\right))=F_{F(\varepsilon)}(0)\leq F_{F(\varepsilon)}(f\left(m\right))=F_{f^{-1}(F)}(m).$$

Now let $m_1,m_2\in M$, $x,y\in X$, and $\varepsilon\in A$, then

$$\begin{aligned}T_{f^{-1}(F)}(xm_1-ym_2)&=T_{F(\varepsilon)}(f\left(xm-ym_2\right))\\&=T_{F(\varepsilon)}(xf\left(m_1\right)-yf\left(m_2\right))\\&\geq\min\left\{T_{F(\varepsilon)}(f\left(m_1\right)),T_{F(\varepsilon)}(f\left(m_2\right))\right\}\\&=\min\left\{T_{f^{-1}(F)}(m_1),T_{f^{-1}(F)}(m_2)\right\}.\end{aligned}$$

Similarly for

$$I_{f^{-1}(F)}(xm_1-ym_2)\geq\min\left\{I_{f^{-1}(F)}(m_1),I_{f^{-1}(F)}(m_2)\right\},$$

and

$$F_{f^{-1}(F)}(xm_1-ym_2)\leq\max\left\{F_{f^{-1}(F)}(m_1),F_{f^{-1}(F)}(m_2)\right\}.$$

Hence $\left(f^{-1}\left(F\right),A\right)$ is a neutrosophic fuzzy soft BCK-submodule over M.

Theorem.4.3. Let $(X,*,0)$ be a BCK-algebra, M and N are modules of X. A mapping $f:M\longrightarrow N$ is a BCK-submodule epimorphism. If (F,A) is a neutrosophic fuzzy soft set over N such that $\left(f^{-1}\left(F\right),A\right)\in NFSS(M)$, then $(F,A)\in NFSS(N)$.

Proof. Assume that $\left(f^{-1}(F), A\right)$ is a neutrosophic fuzzy soft BCK-submodule over M. Let $n \in N$ then there exist $m \in M$ such that $f(m) = n$. Then for all $\varepsilon \in A$

$$T_{F(\varepsilon)}(n) = T_{F(\varepsilon)}(f(m)) = T_{f^{-1}(F)}(m) \leq T_{f^{-1}(F)}(0) = T_{F(\varepsilon)}(f(0)) = T_{F(\varepsilon)}(0),$$
$$I_{F(\varepsilon)}(n) = I_{F(\varepsilon)}(f(m)) = I_{f^{-1}(F)}(m) \leq I_{f^{-1}(F)}(0) = I_{F(\varepsilon)}(f(0)) = I_{F(\varepsilon)}(0),$$
$$F_{F(\varepsilon)}(n) = F_{F(\varepsilon)}(f(m)) = F_{f^{-1}(F)}(m) \geq F_{f^{-1}(F)}(0) = F_{F(\varepsilon)}(f(0)) = F_{F(\varepsilon)}(0).$$

Let $m, \grave{m} \in M$, $n, \grave{n} \in N$ such that $f(m) = n$ and $f(\grave{m}) = \grave{n}$ and $x, y \in X$ then

$$\begin{aligned} T_{F(\varepsilon)}(xn - y\grave{n}) &= T_{F(\varepsilon)}(xf(m) - yf(\grave{m})) \\ &= T_{F(\varepsilon)}(f(xm - y\grave{m})) \\ &= T_{f^{-1}(F)}(xm - y\grave{m}) \\ &\geq \min\left\{T_{f^{-1}(F)}(m), T_{f^{-1}(F)}(\grave{m})\right\} \\ &= \min\left\{T_{F(\varepsilon)}(f(m)), T_{F(\varepsilon)}(f(\grave{m}))\right\} \\ &= \min\left\{T_{F(\varepsilon)}(n), T_{F(\varepsilon)}(\grave{n})\right\}. \end{aligned}$$

Similarly for

$$I_{F(\varepsilon)}(xn - y\grave{n}) \geq \min\left\{I_{F(\varepsilon)}(n), I_{F(\varepsilon)}(\grave{n})\right\},$$

and

$$F_{F(\varepsilon)}(xn - y\grave{n}) \leq \max\left\{F_{F(\varepsilon)}(n), F_{F(\varepsilon)}(\grave{n})\right\}.$$

Hence according to Theorem (3.4), (F, A) is a neutrosophic fuzzy soft BCK-submodule over N.

Theorem.4.4. Let $(X, *, 0)$ be a BCK-algebra, M and N are modules of X. A mapping $f : M \longrightarrow N$ is a BCK-submodule epimorphism and let (F, A) be a neutrosophic fuzzy soft BCK-submodule over M. Then the homomorphic image $(f(F), A)$ is a neutrosophic fuzzy soft BCK-submodule over N.

Proof. Assume that (F, A) is a neutrosophic fuzzy soft BCK-submodule over M. Let $n \in N$ then there exist $m \in M$ such that $f(m) = n$. Then

$$T_{f(F)}(n) = f(T_F)(n) = \sup T_F(m) \leq \sup T(0) = f(T_F)(0) = T_{f(F)}(0),$$
$$I_{f(F)}(n) = f(I_F)(n) = \sup I_F(m) \leq \sup I(0) = f(I_F)(0) = I_{f(F)}(0),$$
$$F_{f(F)}(n) = f(F_F)(n) = \inf F_F(m) \geq \inf F(0) = f(F_F)(0) = F_{f(F)}(0).$$

Let $m_1, m_2 \in M$, $n_1, n_2 \in N$ such that $f(m_1) = n_1$ and $f(m_2) = n_2$ and $x, y \in X$ then

$$\begin{aligned}
T_{f(F)}(xn_1 - yn_2) &= f(T_F)(xn_1 - yn_2) \\
&= \sup T_F(xm_1 - ym_2) \\
&\geq \sup\{\min\{T_F(m_1), T_F(m_2)\}\} \\
&= \min\{\sup T_F(m_1), \sup T_F(m_2)\} \\
&= \min\{f(T_F)(n_1), f(T_F)(n_2)\} \\
&= \min\{T_{f(F)}(n_1), T_{f(F)}(n_2)\}.
\end{aligned}$$

Similarly for

$$I_{f(F)}(xn_1 - yn_2) \geq \min\{I_{f(F)}(n_1), I_{f(F)}(n_2)\},$$

and

$$F_{f(F)}(xn_1 - yn_2) \leq \max\{F_{f(F)}(n_1), F_{f(F)}(n_2)\}.$$

Hence by Theorem (3.4), $(f(F), A)$ is a neutrosophic fuzzy soft BCK-submodule over N.

Corollary 4.5. Let $f : M \longrightarrow N$ be a homomorphism of BCK-submodules and (F, A) is a neutrosophic fuzzy soft set over N. If (F, A) is a neutrosophic fuzzy soft BCK-submodule, then so is $\left(f^{-1}(F), A_\alpha^T\right)$ for any α-translation $\tilde{T}_\alpha[(F, A)]$ of (F, A) with $\alpha \in [0, \perp]$.

Proof. Directly by Theorem(3.6) and Theorem(4.2).

Joining Theorems (3.6), (4.3) and (4.4) we have the following corollaries:

Corollary 4.6. Let $f : M \longrightarrow N$ be an epimorphism of BCK-submodules and (F, A) is a neutrosophic fuzzy soft set over N. If the inverse image of a neutrosophic fuzzy soft α-translation of (F, A) is a neutrosophic fuzzy soft BCK-submodule for some $\alpha \in [0, \perp]$, then so is (F, A).

Corollary 4.7. Let $f : M \longrightarrow N$ be an epimorphism of BCK-submodules and (F, A) is a neutrosophic fuzzy soft BCK-submodule over M, then the homomorphic image of a neutrosophic fuzzy soft α-translation of (F, A) is a neutrosophic fuzzy soft BCK-submodule over N for any $\alpha \in [0, \perp]$.

Using Theorems (3,14), (4.2), (4.3) and (4.4), we deduce the following results:

Corollary 4.8. Let $f : M \longrightarrow N$ be a homomorphism of BCK-submodules and (F, A) is a neutrosophic fuzzy soft BCK-submodule over N, then the inverse image of a neutrosophic fuzzy soft υ-multiplication of (F, A) is a neutrosophic fuzzy soft BCK-submodule over M for any υ-multiplication of (F, A) with $\upsilon \in [0, 1]$.

Corollary 4.9. Let $f : M \longrightarrow N$ be an epimorphism of BCK-submodules. If the inverse image of a neutrosophic fuzzy soft υ-multiplication of (F, A) is a neutrosophic fuzzy soft BCK-submodule over M for some $\upsilon \in (0, 1]$, then (F, A) is a neutrosophic fuzzy soft BCK-submodule over N.

Corollary 4.10. Let $f : M \longrightarrow N$ be an epimorphism of BCK-submodules and (F, A) is a neutrosophic fuzzy soft BCK-submodule over M, then the homomorphic image of a neutrosophic

fuzzy soft υ-multiplication of (F, A) is a neutrosophic fuzzy soft BCK-submodule over N for any $\upsilon \in (0,1]$.

5 Conclusion

Translations, multiplications and extensions are very interested mathematical tools. They are types of operations that researchers like to apply with fuzzy set theory. In this paper, the concept of neutrosophic fuzzy soft translations and neutrosophic fuzzy soft extensions of neutrosophic fuzzy soft BCK-submodules were introduced and the relation between them were discussed. Also, the notion of neutrosophic fuzzy soft multiplications of neutrosophic fuzzy soft BCK-submodules was defined. Finally, some results were investigated.

6 Compliance with Ethical Standards

Conflict of Interest: The authors declare that there is no conflict of interests regarding the publication of this paper.

Ethical Approval: This artical does not contain any studies with human participants or animals performed by any of the authors.

References

[1] H. A. S. Abujabal, M. Aslam and A. B. Thaheem, On actions of BCK-algebras on groups, Pan American Math. Journal 4, (1994) 727–735.

[2] M. M. Altaiary, S. A. Bashammakh, and N. O. Alshehri, "$(\epsilon, \epsilon \vee q)$-Fuzzy BCK-Submodules," Discrete Dynamics in Nature and Society, vol. 2016, Article ID 8746275, 7 pages, 2016.

[3] K .T. Atanassov," Intuitionistic fuzzy sets". Fuzzy Sets and Systems 20(1), (1986), pp.87-96.

[4] K .T. Atanassov," Intuitionistic fuzzy sets". Springer Physica-Verlag, Heidelberg, (1999).

[5] L.B.Badhurays , S.A.Bashammakh and N. O. Alshehri, Intuitionistic fuzzy BCK-submodules, The Journal of Fuzzy Mathematics, (2015).

[6] M. Bakhshi, Fuzzy set theory applied to BCK-modules, Advances in Fuzzy Sets and Systems, vol. 2, no. 8, pp. 61–87, 2011.

[7] D. Chen, E.C.C. Tsang, D. S. Yeung and X. Wang, The parameterization reduction of soft sets and its applications, Computers and Mathematics with Applications 49(5-6)(2005)757-763.

[8] Imai and K. Iseki, On axiom systems of propositional calculi, XIV, Proc. Japan Academy, 42 (1996), 19-22.

[9] K. Iseki, An algebraic related with a propositional calculus, Proc. Japan Academy, 42 (1996), 26-29.

[10] W. B. V. Kandasamy, Florentin Smarandache, Some neutrosophic algebraic structures and neutrosophic N-algebraic structures, Hexis, Phoenix, Arizona, 2006.

[11] A. Kashif and M. Aslam, "Homology theory of BCK-modules", Southeast Asian Bulletin ofMathematics, vol. 38,no. 1, pp. 61–72, 2014.

[12] A. Kashif, H. Bashir and Z. Zahid, On soft BCK- modules, Punjab University, Journal of Mathematics (ISSN 1016-2526), Vol. 50(1)(2018) pp. 67-78.

[13] P. K. Maji, Neutrosophic soft set, Annals of Fuzzy Mathematics and Informatics, Volume 5, No. 1, (January 2013), pp. 157-168.

[14] P.K. Maji, R. Biswas and R. Roy, Soft set theory, Comput. Math. Appl. 45(2003) 555-562.

[15] P.K. Maji, A.R. Roy and R. Biswas, An application of soft sets in a decision making problem, Comput. Math. Appl. 44 (2002) 1077-1083.

[16] J. Meng, Jie and Y. B. Jun, BCK-algebras, Kyung Moon Sa Co., Seoul, 1994.

[17] D. Molodtsov, Soft set theory first results, Comput. Math. Appl. 37(1999) 19-31.

[18] Z. Perveen, M. Aslam and A. B. Thaheem, On BCK- modules, Southeast Asian Bulletin of Mathematics, 30 (2006), 317-329.Bulletin of Mathematics, 30 (2006), 317-329.

[19] F. Smarandache, Neutrosophy, Neutrosophic Probability, Set, and Logic, Amer. Res. Press, Rehoboth, USA, 105 p., 1998.

[20] F. Smarandache, Neutrosophic set, a generalisation of the intuitionistic fuzzy sets. Int. J. Pure Appl. Math. 24, 287–297 (2005).

[21] V. Torra, Y.Narukawa,"On hesitant fuzzy sets and decision". The 18th IEEE international Conference on Fuzzy Systems, Jeju Island, Korea, (2009), pp.1378-1382.

[22] Y. Yin and J. Zhan, The characterizations of hemirings in terms of fuzzy soft h-ideals, Neural Computing and Applications 21(2012) S43-S57.

[23] L. A. Zadeh, Fuzzy sets, Inform. and Conntrol 8 (1965) 338-353.

[24] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning". Information Sciences 8(3), (1975), pp.199-249.

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO. 4, 2020

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO. 4, 2020

(continued)

Volume 28, Number 5 October 2020
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(six times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, St.Martin Univ.,Olympia,WA,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Editorial Board
Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics, Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

T. E. Simos
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

<u>Journal Article</u>

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

<u>Book</u>

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

<u>Contribution to a Book</u>

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus <u>homepage</u>.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

ON HARMONIC MULTIVALENT FUNCTIONS DEFINED BY A NEW DERIVATIVE OPERATOR

ADRIANA CĂTAŞ[1]*, ROXANA ŞENDRUŢIU[2]

ABSTRACT. In the present paper, we define and investigate a new class of multivalent harmonic functions in the open unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$, under certain conditions involving a new generalized differential operator. Coefficient inequalities, distortion bounds and a covering result are also obtained.

Keywords: differential operator, harmonic function, coefficient bounds.
2000 Mathematical Subject Classification: 30C45.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simple connected domain we can write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$. (See [4] for more details.)

Denote by $S_{\mathcal{H}}(p, n), (p, n \in \mathbb{N} = \{1, 2, \ldots\})$ the class of functions $f = h + \bar{g}$ that are harmonic multivalent and sense-preserving in the unit disc U for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_{\mathcal{H}}(p, n)$ we may express the analytic functions h and g as

$$h(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad g(z) = \sum_{k=p+n-1}^{\infty} b_k z^k, \quad |b_{p+n-1}| < 1. \tag{1.1}$$

Let $\tilde{S}_{\mathcal{H}}(p, n, m), (p, n \in \mathbb{N}, m \in \mathbb{N}_0 \cup \{0\})$ denote the family of functions $f_m = h + \bar{g}_m$ that are harmonic in D with the normalization

$$h(z) = z^p - \sum_{k=p+n}^{\infty} |a_k| z^k, \quad g_m(z) = (-1)^m \sum_{k=p+n-1}^{\infty} |b_k| z^k, \quad |b_{p+n-1}| < 1. \tag{1.2}$$

2. Coefficient bounds for the new classes $AL_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$ and $\widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$

We propose for the beginning a new generalized differential operator as follows.

Definition 2.1. Let $H(U)$ denote the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$ and let $\mathcal{A}(p)$ be the subclass of the functions belonging

to $H(U)$ of the form $h(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k$. For $m \in \mathbb{N}_0$, $\lambda \geq 0$, $\delta \in \mathbb{N}_0$, $l \geq 0$ we define the generalized differential operator $I^m_{\lambda,\delta}(p,l)$ on $\mathcal{A}(p)$

$$I^m_{\lambda,\delta}(p,l)h(z) = (p+l)^m z^p + \sum_{k=p+n}^{\infty} [p + \lambda(k-p) + l]^m C(\delta,k) a_k z^k, \tag{2.1}$$

$$C(\delta,k) = \left(\begin{array}{c} k+\delta-1 \\ \delta \end{array} \right) = \frac{\Gamma(k+\delta)}{\Gamma(k)\Gamma(\delta+1)}. \tag{2.2}$$

Remark 2.2. When $\lambda = 1$, $p = 1$, $l = 0$, $\delta = 0$ we get Sălăgean differential operator [10]; $p = 1$, $m = 0$ gives Ruscheweyh operator [9]; $p = 1$, $l = 0$, $\delta = 0$ implies Al-Oboudi differential operator of order m (see [1]); $\lambda = 1$, $p = 1$, $l = 0$ operator (2.1) reduces to Al-Shaqsi and Darus differential operator [2] and when $p = 1$, $l = 0$ we reobtain the operator introduced by Darus and Ibrahim in [5].

Definition 2.3. Let $f \in S_{\mathcal{H}}(p,n)$, $p \in \mathbb{N}$. Using the operator (2.1) for $f = h + \bar{g}$ given by (1.1) we define the differential operator of f as

$$I^m_{\lambda,\delta}(p,l)f(z) = I^m_{\lambda,\delta}(p,l)h(z) + (-1)^m \overline{I^m_{\lambda,\delta}(p,l)g(z)} \tag{2.3}$$

$$I^m_{\lambda,\delta}(p,l)h(z) = (p+l)^m z^p + \sum_{k=p+n}^{\infty} [p + \lambda(k-p) + l]^m C(\delta,k) a_k z^k \tag{2.4}$$

$$I^m_{\lambda,\delta}(p,l)g(z) = \sum_{k=p+n-1}^{\infty} [p + \lambda(k-p) + l]^m C(\delta,k) b_k z^k. \tag{2.5}$$

Remark 2.4. When $\lambda = 1$, $l = 0$, $\delta = 0$ the operator (2.3) reduces to the operator introduced earlier in [7] by Jahangiri et al.

Definition 2.5. A function $f \in S_{\mathcal{H}}(p,n)$ belongs to the class $AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ if

$$\frac{1}{p+l} \mathrm{Re} \left\{ \frac{I^{m+1}_{\lambda,\delta}(p,l)f(z)}{I^m_{\lambda,\delta}(p,l)f(z)} \right\} \geq \alpha, \quad 0 \leq \alpha < 1, \tag{2.6}$$

where $I^m_{\lambda,\delta} f$ is defined by (2.3), for $m \in \mathbb{N}_0$. Finally, we define the subclass

$$\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l) \equiv AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l) \cap \tilde{S}_{\mathcal{H}}(p,n,m). \tag{2.7}$$

Remark 2.6. The class $AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ includes a variety of well-known subclasses of $S_{\mathcal{H}}(p,n)$. For example, letting $n = 1$ we get $AL_{\mathcal{H}}(1,1,0,\alpha,1,0) \equiv HK(\alpha)$ in [6], for $n = 1$, $AL_{\mathcal{H}}(1,m-1,0,\alpha,1,0) \equiv S_H(t,u,\alpha)$ in [11], $AL_{\mathcal{H}}(p,n+p,0,\alpha,1,0) \equiv SH_p(n,\alpha)$ in [8] and $n = 1$, $AL_{\mathcal{H}}(1,m,\delta,\alpha,1,0) \equiv M_{\mathcal{H}}(m,\delta,\alpha)$ in [3].

Theorem 2.7. *Let $f = h + \bar{g}$ be given by (1.1). If*

$$\sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha) + \lambda(k-p)] d_{p,k}(m,\lambda,l) C(\delta,k)}{(p+l)^{m+1}(1-\alpha)} |a_k| + \tag{2.8}$$

$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|\leq 1,$$

with $\lambda n\geq\alpha(p+l)$*, where*

$$d_{p,k}(m,\lambda,l)=[p+\lambda(k-p)+l]^m \tag{2.9}$$

then $f\in AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$*.*

Proof. Using the fact that $\frac{1}{p+l}\mathrm{Re}\, w\geq\alpha$ if and only if $|(p+l)-(p+l)\alpha+w|\geq|(p+l)+(p+l)\alpha-w|$, it is sufficient to show that

$$\begin{aligned}&|(p+l)(1-\alpha)I^m_{\lambda,\delta}(p,l)f(z)+I^{m+1}_{\lambda,\delta}(p,l)f(z)|-\\ &-|(p+l)(1+\alpha)I^m_{\lambda,\delta}(p,l)f(z)-I^{m+1}_{\lambda,\delta}(p,l)f(z)|\geq 0.\end{aligned} \tag{2.10}$$

Substituting $I^m_{\lambda,\delta}(p,l)f(z)$ and $I^{m+1}_{\lambda,\delta}(p,l)f(z)$ in (2.10) yields by (2.8)

$$\begin{aligned}&|(p+l)(1-\alpha)I^m_{\lambda,\delta}(p,l)f(z)+I^{m+1}_{\lambda,\delta}(p,l)f(z)|-\\ &-|(p+l)(1+\alpha)I^m_{\lambda,\delta}(p,l)f(z)-I^{m+1}_{\lambda,\delta}(p,l)f(z)|>\\ &>2(p+l)^{m+1}(1-\alpha)\left\{1-\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|-\right.\\ &\left.-\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|\right\}.\end{aligned}$$

The last expression is nonnegative by (2.8) and therefore the proof is complete. □

Remark 2.8. The harmonic function

$$\begin{aligned}f(z)&=z^p+\sum_{k=p+n}^{\infty}\frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}x_kz^k+\\ &+\sum_{k=p+n-1}^{\infty}\frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}\overline{y_kz^k},\end{aligned} \tag{2.11}$$

where $\sum_{k=p+n}^{\infty}|x_k|+\sum_{k=p+n-1}^{\infty}|y_k|=1$, $0\leq\alpha<1$, $m\in\mathbb{N}_0$, $\lambda n\geq\alpha(p+l)$, $\lambda\geq 0$ and $d_{p,k}(m,\lambda,l)$ is given in (2.9), show that the coefficient bound expressed by (2.8) is sharp.

Theorem 2.9. *Let* $f_m=h+\bar{g_m}$ *be given by (1.2). Then* $f_m\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ *if and only if*

$$\begin{aligned}&\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|+\\ &+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|\leq 1,\end{aligned} \tag{2.12}$$

where $\lambda n\geq\alpha(p+l)$*,* $0\leq\alpha<1$*,* $m\in\mathbb{N}_0$*,* $\lambda\geq 0$ *and* $d_{p,k}(m,\lambda,l)$ *is given in (2.9).*

Proof. Since $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l) \subset AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$, we only need to prove the "only if" part of the theorem. For this part we consider that $f_m \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. Then

$$\mathrm{Re}\left\{\frac{I^{m+1}_{\lambda,\delta}(p,l)f(z)}{I^{m}_{\lambda,\delta}(p,l)f(z)} - \alpha(p+l)\right\} =$$

$$\mathrm{Re}\left\{\frac{(p+l)^{m+1}(1-\alpha)z^p - \sum\limits_{k=p+n}^{\infty}[(p+l)(1-\alpha)+\lambda(k-p)]\xi(m,\lambda,l;\delta,k)a_k z^k}{(p+l)^m z^p - \sum\limits_{k=p+n}^{\infty}\xi(m,\lambda,l;\delta,k)a_k z^k + (-1)^{2m}\sum\limits_{k=p+n-1}^{\infty}\xi(m,\lambda,l;\delta,k)\overline{b_k z^k}}\right.$$

$$\left.-\frac{(-1)^{2m}\sum\limits_{k=p+n-1}^{\infty}[(p+l)(1+\alpha)+\lambda(k-p)]\xi(m,\lambda,l;\delta,k)\overline{b_k z^k}}{(p+l)^m z^p - \sum\limits_{k=p+n}^{\infty}\xi(m,\lambda,l;\delta,k)a_k z^k + (-1)^{2m}\sum\limits_{k=p+n-1}^{\infty}\xi(m,\lambda,l;\delta,k)\overline{b_k z^k}}\right\} \geq 0,$$

where $\xi(m,\lambda,l;\delta,k) = d_{p,k}(m,\lambda,l)C(\delta,k)$.
The above required condition must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0 \leq |z| = r < 1$, we must have

$$\frac{(p+l)^{m+1}(1-\alpha) - \sum\limits_{k=p+n}^{\infty}[(p+l)(1-\alpha)+\lambda(k-p)]\xi(m,\lambda,l;\delta,k)|a_k|r^{k-p}}{(p+l)^m - \sum\limits_{k=p+n}^{\infty}\xi(m,\lambda,l;\delta,k)|a_k|r^{k-p} + \sum\limits_{k=p+n-1}^{\infty}\xi(m,\lambda,l;\delta,k)|b_k|r^{k-p}} - \tag{2.13}$$

$$\frac{\sum\limits_{k=p+n-1}^{\infty}[(p+l)(1+\alpha)+\lambda(k-p)]\xi(m,\lambda,l;\delta,k)|b_k|r^{k-p}}{(p+l)^m - \sum\limits_{k=p+n}^{\infty}\xi(m,\lambda,l;\delta,k)|a_k|r^{k-p} + \sum\limits_{k=p+n-1}^{\infty}\xi(m,\lambda,l;\delta,k)|b_k|r^{k-p}} \geq 0.$$

If the condition (2.12) does not hold, then the numerator in (2.13) is negative for r sufficiently close to 1. Hence there exists a $z_0 = r_o$ in $(0,1)$ for which the quotient in (2.13) is negative. This contradicts the required condition for $f \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ and so the proof is complete. □

3. Distortion bounds

The following theorem gives the distortion bounds for functions in $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ which yields a covering result for this class.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Theorem 3.1. *Let* $f \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$, *with* $0 \le \alpha < 1$, $\lambda n \ge \alpha(p+l)$, $m \in \mathbb{N}_0$, $\lambda \ge 0$. *Then for* $|z| = r < 1$ *one obtains*

$$|f(z)| \le (1+|b_{p+n-1}|r^{n-1})r^p + \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)} \cdot \tag{3.1}$$

$$\cdot \left\{1 - \frac{[(p+l)(1+\alpha)+\lambda(n-1)]d_{p,n+p-1}(m,\lambda,l)C(\delta,n+p-1)}{(p+l)^{m+1}(1-\alpha)}|b_{p+n-1}|\right\} r^{n+p}$$

and

$$|f(z)| \ge (1-|b_{p+n-1}|r^{n-1})r^p - \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)} \cdot$$

$$\cdot \left\{1 - \frac{[(p+l)(1+\alpha)+\lambda(n-1)]d_{p,n+p-1}(m,\lambda,l)C(\delta,n+p-1)}{(p+l)^{m+1}(1-\alpha)}|b_{p+n-1}|\right\} r^{n+p}.$$

Proof. We only prove the left-hand inequality. The proof for the right-hand inequality is similar and will be omitted.

Let $f \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. Taking the absolute value of f we have

$$|f(z)| = |z^p - \sum_{k=p+n}^{\infty} a_k z^k + (-1)^m \sum_{k=p+n-1}^{\infty} b_k \bar{z}^k| \ge (1-|b_{p+n-1}|r^{n-1})r^p - \sum_{k=p+n}^{\infty}(|a_k|+|b_k|)r^{p+n}$$

$$= (1-|b_{p+n-1}|r^{n-1})r^p - \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)} \cdot$$

$$\cdot \sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha)+\lambda n]d_{p,p+n}(m,\lambda,l)C(\delta,p+n)}{(p+l)^{m+1}(1-\alpha)}(|a_k|+|b_k|)r^{p+n} \ge$$

$$(1-|b_{p+n-1}|r^{n-1})r^p - \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)} \cdot$$

$$\cdot \left\{1 - \frac{[(p+l)(1+\alpha)+\lambda(n-1)]d_{p,n+p-1}(m,\lambda,l)C(\delta,n+p-1)}{(p+l)^{m+1}(1-\alpha)}|b_{p+n-1}|\right\} r^{n+p}.$$

The bounds given in Theorem 3.1 for the functions f of the form (1.2) also hold for the functions of the form (1.1) if the coefficient condition (2.8) is satisfied. The upper bound given for $f \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ is sharp and the equality occurs for the function

$$f(z) = z + |b_{p+n-1}|\bar{z}^p + \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)} \cdot$$

$$\cdot \left\{1 - \frac{[(p+l)(1+\alpha)+\lambda(n-1)]d_{p,n+p-1}(m,\lambda,l)C(\delta,n+p-1)}{(p+l)^{m+1}(1-\alpha)}|b_{p+n-1}|\right\} r^{n+p},$$

where $|b_{p+n-1}| \le \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha)+\lambda(n-1)]d_{p,n+p-1}(m,\lambda,l)C(\delta,n+p-1)}$ □

The following covering result follows from the left-hand inequality in Theorem 3.1.

Corollary 3.2. *If the function $f \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$, then*

$$\left\{w\colon |w|<\frac{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)-(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)}-\right.$$
$$\left.-\frac{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)-E^{\alpha}_{p,\delta}(m,\lambda,l)}{[p(1-\alpha)+\lambda n+l]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)}\cdot|b_{p+n-1}|\right\}\subset f(U)$$

where $E^{\alpha}_{p,\delta}(m,\lambda,l)=[(p+l)(1+\alpha)+\lambda(n-1)]d_{p,n+p-1}(m,\lambda,l)C(\delta,n+p-1)$.

References

[1] F. M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Inter. J. of Math. and Mathematical Sci., **27**(2004), 1429-1436.

[2] K. Al-Shaqsi, M. Darus, *An operator defined by convolution involving polylogarithms functions*, Journal of Math. and Statistics, **4**(1)(2008), 46-50.

[3] K. Al-Shaqsi, M. Darus, *On Harmonic Functions Defined by Derivative Operator*, Journal of Inequalities and Applications, vol. 2008, Article ID 263413, doi: 10.1155/2008/263413.

[4] J. Clunie and T. Sheil-Small, *Harmonic Univalent Functions*, Ann. Acad. Sci. Fenn, Ser. A I. Math. **9**(1984), 3-25.

[5] M. Darus, R. W. Ibrahim, *On new classes of univalent harmonic functions defined by generalized differential operator*, Acta Universitatis Apulensis, **18**(2009), 61-69.

[6] J. M. Jahangiri, *Coefficient bounds and univalence criteria for harmonic functions with negative coefficients*, Ann. Univ. Mariae Curie-Sklowdowska Sect. A, **52**(1998), 57-66.

[7] J. M. Jahangiri, G. Murugusundaramoorthy and K. Vijaya *Sălăgean type harmonic univalent functions* South. J. Pure Appl. Math., **2**(2002), 77-82.

[8] Om P. Ahuja and J. M. Jahangiri, *Multivalent harmonic starlike functions with missing coefficients*, Math. Sci. Res. J., **7**(9)(2003), 347-352.

[9] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49**(1975), 109-115.

[10] Gr. Şt. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, Heidelberg and New York, 1013(1983), 362-372.

[11] Sibel Yalcin, *A new class of Sălăgean-type harmonic univalent functions* Appl. Math. Letters, **18**(2005), 191-198.

[1] Department of Mathematics and Computer Sciences, University of Oradea,
Str. Universităţii, No.1, 410087 Oradea, Romania
* *Corresponding author: acatas@gmail.com*

[2] Faculty of Environmental Protection, University of Oradea,
Str. B-dul Gen. Magheru, No.26, 410048 Oradea, Romania
E-mail address: roxana.sendrutiu@gmail.com

GOOD AND SPECIAL WEAKLY PICARD OPERATORS PROPERTIES FOR A CLASS OF DISCRETE LINEAR OPERATORS

LOREDANA-FLORENTINA IAMBOR, ADRIANA CĂTAŞ

ABSTRACT. Based on the results of the weakly Picard operators theory, in this paper we study the good and special convergence of the iterates of a general class of positive linear operators of discrete type introduced by O.Agratini and I.A. Rus ([1]).

2010 AMS Mathematics Subject Classification: 47H10, 41A36.

Keywords and phrases: linear positive operators, weakly Picard operators, good and special Picard operators.

1. Introduction and Preliminaries

The study of the convergence of the sequence of successive approximations is realized in metric spaces. That is, for (X,d) metric space and $A : X \to X$ an operator, for any $x \in X$ can be considered the sequence:

$$(A^m(x))_{m\in\mathbb{N}}\,,\ x \in X \tag{1}$$

where $A^0 = 1_X$ and $A^m = A^{m-1} \circ A$ for $m \in \mathbb{N}^*$.

Investigating the properties of sequence (1), L. d'Apuzzo introduced in 1976 (see [3]) the good and special convergence, giving necessary and sufficient conditions for this kind of convergence (see [2]). In paper [3], she considers the good and special convergence of type M, as a particular case, in which the sequence $(d(A^m(x), A^\infty(x)))_{m\in\mathbb{N}}$, (respectively, $\left(d\left(A^m(x), A^{m-1}(x)\right)\right)_{m\in\mathbb{N}}$) is strictly decreasing for any x. I.A. Rus introduced, in paper (see [8]), the good and special weakly Picard operators .

In what follow, let (X,d) be a metric space and $A : X \to X$ an operator. In this paper we will use the following notations:

$P(X) := \{Y \subset X \mid Y \neq \emptyset\}$;

$F_A := \{x \in X \mid A(x) = x\}$ - the fixed point set of A;

$I(A) := \{Y \in P(X) \mid A(Y) \subset Y\}$ - the family of the nonempty invariant subsets of A.

Definition 1. *(I.A. Rus - [6], [7], [8]) Let (X,d) be a metric space.*
1) An operator $A : X \to X$ is weakly Picard operator (briefly WPO) if the sequence of successive approximations $(A^m(x_0))_{m\in\mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.
2) If the operator $A : X \to X$ is WPO and $F_A = \{x^*\}$, then by definition the operator A is Picard operator (briefly PO).
3) If the operator $A : X \to X$ is WPO, then can be considered the operator A^∞ defined by $A^\infty : X \to X,\ A^\infty(x) := \lim_{m\to\infty} A^m(x)$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

The basic result in the WPO's theory is the following:

Theorem 1. *(Characterization theorem - [6], [7], [8]) An operator $A : X \to X$ is WPO if and only if there exits a partition of X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, such that:*

(a) $X_\lambda \in I(A)$, $\forall\lambda \in \Lambda$;
(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ is PO, $\forall\lambda \in \Lambda$.

Definition 2. *Let (X, d) be a metric space and $A : X \to X$ a WPO.*
1) $A : X \to X$ is good WPO, if the series $\sum_{m=1}^{\infty} d\left(A^{m-1}(x), A^m(x)\right)$ converges, for all $x \in X$ (see [8]). In the case that the sequence $\left(d\left(A^{m-1}(x), A^m(x)\right)\right)_{m\in\mathbb{N}^*}$ is strictly decreasing for all $x \in X$, the operator A is good WPO of type M (see [3]).
2) $A : X \to X$ is special WPO, if the series $\sum_{m=1}^{\infty} d(A^m(x), A^\infty(x))$ converges, for all $x \in X$(see [8]). When the sequence $(d(A^m(x), A^\infty(x)))_{m\in\mathbb{N}^*}$ is strictly decreasing for all $x \in X$, A is special WPO of type M (see [3]).

In 2015, S. Mureşan and L.F. Iambor obtained the following result regarding to good and special weakly Picard operators.

Theorem 2. *([5]) Let (X, d) be a metric space and $A : X \to X$ a WPO. If A is special WPO then A is good WPO.*

In the paper [4], A.Bica and L.F. Galea(Iambor) introduced the notions of uniform good and special weakly Picard operators like this:

Definition 3. *(A.Bica, L.F. Galea - [4]) Let (X, d) be a metric space and $F \subset \{A|\, A : X \to X\}$ a family of operators on X. We say that F is a family of uniform special (good) WPO's if for any $A \in F$, A is special (good) WPO and there exist the functionals $\varphi : X \to \mathbb{R}_+$ and $\psi, \psi' : F \to \mathbb{R}_+$ such that φ is continuous and*

$$\sum_{m=1}^{\infty} d(A^m(x), A^\infty(x)) \leqslant \psi(A)\cdot\varphi(x),\ \forall\ x \in X,\ \forall\ A \in F$$

$$\text{(respectively, } \sum_{m=1}^{\infty} d\left(A^m(x), A^{m-1}(x)\right) \leqslant \psi'(A)\cdot\varphi(x),\ \forall\ x \in X,\ \forall\ A \in F).$$

In what follow, we present the general class of linear positive operators of discrete type and some properties of these operators investigated by O. Agratini and I.A. Rus in [1].

At first they construct an approximation process of discrete type acting on the space $C([a, b])$ endowed with the Chebyshev norm $\|.\|$.

For each integer $n \geqslant 1$ they consider the following:

(i) A net on $[a, b]$ named Δ_n is fixed ($a = x_{n,0} < x_{n,1} < ... < x_{n,n} = b$).

(ii) A system $(\psi_{n,k})_{k=\overline{0,n}}$ is given, where every $\psi_{n,k}$ belongs to $C([a, b])$.

They assume that it is a blending system with a certain connection with Δ_n, more precisely the following conditions hold:

$$\psi_{n,k} \geqslant 0,\ \left(k = \overline{0, n}\right),\ \sum_{k=0}^{n} \psi_{n,k} = e_0,\ \sum_{k=0}^{n} x_{n,k}\psi_{n,k} = e_1$$

Definition 4. *(O.Agratini, I.A. Rus - [1]) The operators $L_n : C([a, b]) \to C([a, b])$ defined by*

$$L_n(f)(x) = \sum_{k=0}^{n} \psi_{n,k}(x) f(x_{n,k})$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

are called ***the operators of discrete type.***

The operators of discrete type L_n, have the following properties:
1) L_n, $n \in \mathbb{N}$ are positive linear operators;
2) $L_n(e_0) = e_0$ and $L_n(e_1) = e_1$.

Theorem 3. *(O.Agratini, I.A. Rus - [1]) Let L_n, $n \in \mathbb{N}$, such that $\psi_{n,0}(a) = \psi_{n,n}(b) = 1$. Let us denote $u_n := \min\limits_{x\in[a,b]} [\Phi_{n,0}(x) + \Phi_{n,n}(x)]$.*

If the $u_n > 0$ the iterates sequence $(L_n^m)_{m\geqslant 1}$ verifies

$$\lim_{m\to\infty} (L_n^m f)(x) = f(a) + \tfrac{f(a)-f(b)}{b-a}(x-a), \ f \in C([a,b])$$

uniformly on $[a,b]$.

Theorem 4. *(O.Agratini, I.A. Rus - [1]) Let L_n, $n \in \mathbb{N}$, such that $\psi_{n,0}(a) = \psi_{n,n}(b) = 1$. Then the operator L_n is weakly Picard operator for every $n \in \mathbb{N}$ and*

$$L_n^\infty(f) = c_1(f)e_1 + c_2(f) \overset{not}{=} f^*(x), \ f \in C([a,b])$$

where $c_1(f) = \frac{f(b)-f(a)}{b-a}$ and $c_2(f) = \frac{bf(a)-af(b)}{b-a}$.

The convergence exists on the space $(C[a,b], \|.\|_\infty)$.

In the application of Characterization theorem of weakly Picard operator, it was considerate the partition of $C([a,b])$:

$$C([a,b]) := \bigcup_{\alpha,\beta\in\mathbb{R}} X_{\alpha,\beta}$$

where $X_{\alpha,\beta} = \{f \in C([a,b]) : f(a) = \alpha, \ f(b) = \beta\}, \ \alpha, \beta \in \mathbb{R}$.

Proposition 5. *(O. Agratini, I.A. Rus - [1]) The operators of discrete type satisfied the following contraction property relative to above partition:*

$$\|L_n(f) - L_n(g)\|_\infty \leqslant (1-u_n)\|f-g\|_\infty, \ \forall \ f,g \in X_{\alpha,\beta}, \ \alpha,\beta \in \mathbb{R} \tag{2}$$

where $u_n = \min\limits_{x\in[a,b]} [\Phi_{n,0}(x) - \Phi_{n,n}(x)]$, $u_n > 0$.

2. Main results

In this section, we will investigate some properties of the iterates of discrete type of operators in sense of good and special convergence.

Theorem 6. *The operators of discrete type L_n, $n \in \mathbb{N}$ are special WPO and good WPO of type M on $C([a,b])$.*

From Theorem 3, we have that L_n, $n \in \mathbb{N}$ is weakly Picard operator.

Let $f \in C([a,b])$. Then $f \in X_{f(a),f(b)}$ and according to (1) we infer that L_n is contraction on $X_{f(a),f(b)}$. So, the operator L_n, $n \in \mathbb{N}$ is special WPO of type M on $X_{f(a),f(b)}$. Finally, we get that L_n, $n \in \mathbb{N}$ is special WPO of type M on $C([a,b])$.

From Theorem 2, any special WPO is good WPO. Then we have that L_n, $n \in \mathbb{N}$ is good WPO of type M on $C([a,b])$.

Theorem 7. *The family of the operators of discrete type $\{L_n : n \in \mathbb{N}^*\}$ is family of uniform special and good WPO's on $C[a,b]$.*

Proof. Using the inequality (1), we obtain the estimation:

$\left|L_n^1(f)(x) - L_n^\infty(f)(x)\right| = \left|L_n^1(f)(x) - L_n^1(L_n^\infty(f))(x)\right| \leqslant$

$\leqslant (1-u_n)|f(x) - L_n^\infty(f)(x)| = (1-u_n)|f(x) - c_1(f)e_1 - c_2(f)| \leqslant$

$\leqslant (1-u_n) \cdot C,\ \forall\ x \in [a,b]$,
where $C = diam\,(\operatorname{Im} f) + 2\max\{|f(a)|, |f(b)|\}$, with
$diam\,(\operatorname{Im} f) = \max\{|f(x) - f(y)| : x, y \in [a,b]\}$.

The constant C was obtained using the following technique:

- If $x = a$ then:
  $|f(x) - L_n^{\infty}(f)(x)| \leqslant \left| f(x) - \frac{f(b)-f(a)}{b-a} \cdot a - \frac{bf(a)-af(b)}{b-a} \right| =$
  $= \left| f(x) - \frac{f(a)(b-a)}{b-a} \right| = |f(x) - f(a)| \leqslant diam\,(\operatorname{Im} f)$
- If $x = b$ then:
  $|f(x) - L_n^{\infty}(f)(x)| \leqslant \left| f(x) - \frac{f(b)-f(a)}{b-a} \cdot b - \frac{bf(a)-af(b)}{b-a} \right| =$
  $= \left| f(x) - \frac{f(b)(b-a)}{b-a} \right| = |f(x) - f(b)| \leqslant diam\,(\operatorname{Im} f)$
- If $x \in [a,b]$ then:
  $|f(x) - L_n^{\infty}(f)(x)| \leqslant \left| f(x) - \frac{f(b)-f(a)}{b-a} \cdot x - \frac{bf(a)-af(b)}{b-a} \right| =$
  $= \left| f(x) - \frac{x-a}{b-a} \cdot f(b) + \frac{b-x}{b-a} \cdot f(a) \right| \leqslant$
  $\leqslant |f(x) - f(b)| + |f(b)| \cdot \left| 1 - \frac{x-a}{b-a} \right| + \left| \frac{b-x}{b-a} \right| \cdot |f(a)| \leqslant$
  $\leqslant diam\,(\operatorname{Im} f) + 2\max\{|f(a)|, |f(b)|\}$.

By induction, for $m \in \mathbb{N}^*$, we have:
$|L_n^m(f)(x) - L_n^{\infty}(f)(x)| = \left| L_n^1\left(L_n^{m-1}(f)\right)(x) - L_n^1\left(L_n^{\infty}(f)\right)(x) \right| \leqslant$
$\leqslant (1-u_n)^m \cdot C,\ \forall\ x \in [a,b]$.

Then, $\sum\limits_{m=1}^{\infty} |L_n^m(f)(x) - L_n^{\infty}(f)(x)| \leqslant$
$\leqslant \lim\limits_{m\to\infty} C\left[(1-u_n) + (1-u_n)^2 + ... + (1-u_n)^m\right] =$
$= \lim\limits_{m\to\infty} \left[C(1-u_n) \cdot \frac{1-(1-u_n)^m}{u_n}\right] \leqslant C \cdot \frac{1-u_n}{u_n}$ (2)

On the other hand, we have:
$\left|L_n^1(f)(x) - L_n^0(f)(x)\right| = \left| \sum\limits_{k=0}^{n} \psi_{n,k}(x) f(x_{n,k}) - f(x) \right| =$
$= \left| \sum\limits_{k=0}^{n} \psi_{n,k}(x)\left[f(x_{n,k}) - f(x)\right] \right| \leqslant C' \sum\limits_{k=0}^{n} \psi_{n,k}(x) = C'e_0,\ \forall\ x \leqslant [a,b]$, where
$C' = diam\,(\operatorname{Im} f) = \max\{|f(x) - f(y)| : x, y \in [a,b]\}$.

By induction, for $m \in \mathbb{N}$, we have:
$\left|L_n^m(f)(x) - L_n^{m-1}(f)(x)\right| = \left| L_n^1\left(L_n^{m-1}(f)\right)(x) - L_n^1\left(L_n^{m-2}(f)\right)(x) \right| \leqslant$
$\leqslant (1-u_n)^{m-1} \cdot C'e_0,\ \forall\ x \in [a,b]$

Then, $\sum\limits_{m=1}^{\infty} \left|L_n^m(f)(x) - L_n^{m-1}(f)(x)\right| \leqslant$
$\leqslant \lim\limits_{m\to\infty} C'e_0 \left[1 + (1-u_n) + (1-u_n)^2 + ... + (1-u_n)^{m-1}\right] \leqslant$
$\leqslant C'e_0 \frac{1}{1-u_n},\ \forall\ f \in C[a,b]$ (3).

Now, the property of uniform and special WPO follows from the estimations (2) and (3). For instance, for the property of uniform special WPO we have:

$$\varphi : C[a,b] \to \mathbb{R}_+,\ \varphi(f) = diam\,(\operatorname{Im} f) + 2\max\{|f(a)|, |f(b)|\} \text{ and}$$

$$\psi : \{L_n : n \in \mathbb{N}^*\} \to \mathbb{R}_+,\ \psi(L_n) = \frac{1-u_n}{u_n},\ \forall\ n \in \mathbb{N}^*$$

and for the property of uniform good WPO we have:

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

$$\varphi' : C[a,b] \to \mathbb{R}_+,\ \varphi'(f) = diam(\operatorname{Im} f) \text{ and}$$
$$\psi' : \{L_n : n \in \mathbb{N}^*\} \to \mathbb{R}_+,\ \psi'(L_n) = \tfrac{1}{1-u_n} e_0,\ \forall\ n \in \mathbb{N}^*.$$

It is easy to prove that $\varphi, \varphi' : C[a,b] \to \mathbb{R}_+$, $\varphi(f) = diam(\operatorname{Im} f) + 2\max\{|f(a)|, |f(b)|\}$ and $\varphi'(f) = diam(Imf)$ are seminorms on $C[a,b]$ and
$\varphi(f-g) \leqslant 2\|f-g\|_C + 2\|f\|_C$, $\varphi'(f-g) \leqslant 2\|f-g\|_C$
since $|\varphi(f) - \varphi(g)| \leqslant \varphi(f-g)$, $\forall\ f,g \in C[a,b]$ and
$|\varphi'(f) - \varphi'(g)| \leqslant \varphi'(f-g)$, $\forall\ f,g \in C[a,b]$
we infer the φ, φ' are continuous. □

References

[1] O. Agratini, I.A. Rus *Iterates of a class of discrete linear operators via contraction principle*, Comment. Math. Univ. Carolinae, 44(3), 2003, 555-563.

[2] L. D'Apuzzo, *On the convergence of the method of succesive approximation in metric spaces*, Ann. Istit. Univ. Navale Napoli, 45/46(1976/1977)(in Italian).

[3] L. D'Apuzzo, *On the notion of good and special convergence of the method of succesive approximations*, Ann. Istit. Univ. Navale Napoli, 45/46(1976/1977), 123-138, (in Italian).

[4] A. Bica, L.F. Galea *On Picard iterative properties of the Bernstein operators and an application to fuzzy numbers*, Comm. in Math. Analysis, 5(1), 2008, 8-19.

[5] S. Mureşan, L.F. Iambor, *On good and special weakly Picard operators*, Analele Universitatii Oradea, Fasc. Matematica, Tom XXII, Issue No.1, pp.5-10, 2015.

[6] I. A. Rus, *Generalized Contractions and Applications*, Cluj University Press, 2001, (in Romanian).

[7] I. A. Rus, *Weakly Picard Operators and Applications*, Seminar on Fixed Point Theory Cluj-Napoca, Volume2, 41-58, 2001.

[8] I. A. Rus, *Picard operators and applications*, Sci. Math.Japon., 58 (1), pp.191-219, 2003.

Department of Mathematics and Computer Science, University of Oradea, Str. Universităţii No.1, 410087, Oradea, Romania,
E-mail address: iambor.loredana@gmail.com, acatas@gmail.com

General Iyengar type Inequalities

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we present general Iyengar type inequalities with respect to L_p norms, with $1 \leq p \leq \infty$. The method is based on the generalized Taylor's formula.

2010 Mathematics Subject Classification: 26D10, 26D15.
Key Words and Phrases: Iyengar inequality, Taylor formula.

1 Introduction

We are motivated by the following famous Iyengar inequality (1938), [2].

Theorem 1 *Let f be a differentiable function on $[a,b]$ and $|f'(x)| \leq M$. Then*

$$\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) \right| \leq \frac{M(b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M}. \quad (1)$$

2 Main Results

We present the following Iyengar type inequalities:

Theorem 2 *Let $n \in \mathbb{N}$, $f \in AC^n([a,b])$ (i.e. $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We assume that $f^{(n)} \in L_\infty([a,b])$. Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \left[(t-a)^{n+1} + (b-t)^{n+1} \right], \quad (2)$$

$\forall\, t \in [a,b]$,

1

ii) at $t = \frac{a+b}{2}$, the right hand side of (2) is minimized, and we get:

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n}, \tag{3}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, ..., n-1$, we obtain

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n}, \tag{4}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \left(\frac{b-a}{N} \right)^{n+1} \left[j^{n+1} + (N-j)^{n+1} \right], \tag{5}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, from (5) we get:

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \left(\frac{b-a}{N} \right)^{n+1} \left[j^{n+1} + (N-j)^{n+1} \right], \tag{6}$$

for $j = 0, 1, 2, ..., N \in \mathbb{N}$,

vi) when $N = 2$ and $j = 1$, (6) turns to

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_\infty([a,b])}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n}, \tag{7}$$

vii) when $n = 1$ (without any boundary conditions), we get from (7) that

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \|f'\|_{\infty,[a,b]} \frac{(b-a)^2}{4}, \tag{8}$$

a similar to Iyengar inequality (1).

2

Proof. Here $n \in \mathbb{N}$ and $f^{(n-1)}$ is absolutely continuous on $[a,b]$. We assumed that

$$\left\| f^{(n)} \right\|_{\infty,[a,b]} := \left\| f^{(n)} \right\|_{L_\infty([a,b])} < +\infty.$$

By [1], we have the following generalized Taylor's formulae:

$$f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f^{(n)}(t)\, dt \tag{9}$$

and

$$f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k = \frac{1}{(n-1)!} \int_b^x (x-t)^{n-1} f^{(n)}(t)\, dt, \tag{10}$$

$\forall\, x \in [a,b]$.

Then we get

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \frac{\left\| f^{(n)} \right\|_{\infty,[a,b]}}{n!} (x-a)^n, \tag{11}$$

$\forall\, x \in [a,b]$,

and

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| = \frac{1}{(n-1)!} \left| \int_b^x (x-t)^{n-1} f^{(n)}(t)\, dt \right| =$$

$$\frac{1}{(n-1)!} \left| \int_x^b (t-x)^{n-1} f^{(n)}(t)\, dt \right| \leq \frac{1}{(n-1)!} \int_x^b (t-x)^{n-1} \left| f^{(n)}(t) \right| dt \leq$$

$$\frac{\left\| f^{(n)} \right\|_{\infty,[a,b]}}{n!} (b-x)^n,$$

that is

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \frac{\left\| f^{(n)} \right\|_{\infty,[a,b]}}{n!} (b-x)^n, \tag{12}$$

$\forall\, x \in [a,b]$.

We call

$$\delta := \frac{\left\| f^{(n)} \right\|_{\infty,[a,b]}}{n!}. \tag{13}$$

So we have

$$-\delta (x-a)^n \leq f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \leq \delta (x-a)^n \tag{14}$$

3

and

$$-\delta(b-x)^n \leq f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!}(x-b)^k \leq \delta(b-x)^n, \tag{15}$$

$\forall\, x \in [a,b]$.

Therefore it holds

$$\sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k - \delta(x-a)^n \leq f(x) \leq \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \delta(x-a)^n \tag{16}$$

and

$$\sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!}(x-b)^k - \delta(b-x)^n \leq f(x) \leq \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!}(x-b)^k + \delta(b-x)^n, \tag{17}$$

$\forall\, x \in [a,b]$.

Let any $t \in [a,b]$, then

$$\sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{(k+1)!}(t-a)^{k+1} - \frac{\delta}{(n+1)}(t-a)^{n+1} \leq \int_a^t f(x)\,dx \leq$$

$$\sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{(k+1)!}(t-a)^{k+1} + \frac{\delta}{(n+1)}(t-a)^{n+1}, \tag{18}$$

and

$$-\sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{(k+1)!}(t-b)^{k+1} - \frac{\delta}{(n+1)}(b-t)^{n+1} \leq \int_t^b f(x)\,dx \leq \tag{19}$$

$$-\sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{(k+1)!}(t-b)^{k+1} + \frac{\delta}{(n+1)}(b-t)^{n+1}.$$

Adding (18) and (19), we obtain:

$$\left\{\sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1} - f^{(k)}(b)(t-b)^{k+1}\right]\right\} -$$

$$\frac{\delta}{(n+1)}\left[(t-a)^{n+1} + (b-t)^{n+1}\right] \leq \int_a^b f(x)\,dx \leq$$

$$\left\{\sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1} - f^{(k)}(b)(t-b)^{k+1}\right]\right\} +$$

$$\frac{\delta}{(n+1)}\left[(t-a)^{n+1} + (b-t)^{n+1}\right], \tag{20}$$

$\forall\, t \in [a,b]$.

4

Consequently we derive:

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\delta}{(n+1)} \left[(t-a)^{n+1} + (b-t)^{n+1} \right], \tag{21}$$

$\forall\ t \in [a,b]$.

Let us consider

$$g(t) := (t-a)^{n+1} + (b-t)^{n+1},\ \forall\ t \in [a,b]. \tag{22}$$

Hence

$$g'(t) = (n+1)\left[(t-a)^n - (b-t)^n\right] = 0,$$

giving $(t-a)^n = (b-t)^n$ and $t-a = b-t$, that is $t = \frac{a+b}{2}$ the only critical number here.

We have $g(a) = g(b) = (b-a)^{n+1}$, and $g\left(\frac{a+b}{2}\right) = \frac{(b-a)^{n+1}}{2^n}$, which is the minimum of g over $[a,b]$.

Consequently the right hand side of (21) is minimized when $t = \frac{a+b}{2}$, with value $\frac{\|f^{(n)}\|_{\infty,[a,b]}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n}$. Assuming $f^{(k)}(a) = f^{(k)}(b) = 0$, for $k = 0, 1, ..., n-1$, then we obtain that

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\|f^{(n)}\|_{\infty,[a,b]}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n}, \tag{23}$$

which is a sharp inequality.

When $t = \frac{a+b}{2}$, then (21) becomes

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\|f^{(n)}\|_{\infty,[a,b]}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n}. \tag{24}$$

Next let $N \in \mathbb{N}$, $j = 0, 1, 2, ..., N$ and $t_j = a + j\left(\frac{b-a}{N}\right)$, that is $t_0 = a$, $t_1 = a + \frac{b-a}{N}, ..., t_N = b$.

Hence it holds

$$t_j - a = j\left(\frac{b-a}{N}\right), \quad (b - t_j) = (N-j)\left(\frac{b-a}{N}\right), \quad j = 0, 1, 2, ..., N. \tag{25}$$

We notice that

$$(t_j - a)^{n+1} + (b - t_j)^{n+1} = \left(\frac{b-a}{N}\right)^{n+1} \left[j^{n+1} + (N-j)^{n+1} \right], \tag{26}$$

$j = 0, 1, 2, ..., N$,

and $(k = 0, 1, ..., n - 1)$

$$\left[f^{(k)}(a)(t_j - a)^{k+1} + (-1)^k f^{(k)}(b)(b - t_j)^{k+1}\right] =$$

$$\left[f^{(k)}(a) j^{k+1} \left(\frac{b-a}{N}\right)^{k+1} + (-1)^k f^{(k)}(b)(N - j)^{k+1} \left(\frac{b-a}{N}\right)^{k+1}\right] = \tag{27}$$

$$\left(\frac{b-a}{N}\right)^{k+1} \left[f^{(k)}(a) j^{k+1} + (-1)^k f^{(k)}(b)(N - j)^{k+1}\right],$$

$j = 0, 1, 2, ..., N$.

By (21) we get

$$\left|\int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N}\right)^{k+1} \left[f^{(k)}(a) j^{k+1} + (-1)^k f^{(k)}(b)(N - j)^{k+1}\right]\right| \leq$$

$$\frac{\left\|f^{(n)}\right\|_{\infty,[a,b]}}{(n+1)!} \left(\frac{b-a}{N}\right)^{n+1} \left[j^{n+1} + (N - j)^{n+1}\right], \tag{28}$$

$j = 0, 1, 2, ..., N$.

If $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n - 1$, then (28) becomes

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{N}\right)[jf(a) + (N - j) f(b)]\right| \leq$$

$$\frac{\left\|f^{(n)}\right\|_{\infty,[a,b]}}{(n+1)!} \left(\frac{b-a}{N}\right)^{n+1} \left[j^{n+1} + (N - j)^{n+1}\right], \tag{29}$$

for $j = 0, 1, 2, ..., N$.

When $N = 2$ and $j = 1$, then (29) becomes

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)[f(a) + f(b)]\right| \leq$$

$$\frac{\left\|f^{(n)}\right\|_{\infty,[a,b]}}{(n+1)!} \left(\frac{b-a}{2}\right)^{n+1} 2 = \frac{\left\|f^{(n)}\right\|_{\infty,[a,b]}}{(n+1)!} \frac{(b-a)^{n+1}}{2^n}. \tag{30}$$

And, if $n = 1$ (without any boundary conditions), we get from (30) that

$$\left|\int_a^b f(x)\,dx - \left(\frac{b-a}{2}\right)(f(a) + f(b))\right| \leq \|f'\|_{\infty,[a,b]} \frac{(b-a)^2}{4}, \tag{31}$$

which a similar inequality to Iyengar inequality (1). ■

We give

6

Theorem 3 *Let $f \in AC^n([a,b])$, $n \in \mathbb{N}$. Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq \tag{32}$$

$$\frac{\left\| f^{(n)} \right\|_{L_1([a,b])}}{n!} \left[(t-a)^n + (b-t)^n \right],$$

$\forall\ t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$, the right hand side of (32) is minimized, and we get:

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_1([a,b])}}{n!} \frac{(b-a)^n}{2^{n-1}}, \tag{33}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0, 1, ..., n-1$, we obtain

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\left\| f^{(n)} \right\|_{L_1([a,b])}}{n!} \frac{(b-a)^n}{2^{n-1}}, \tag{34}$$

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, it holds

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_1([a,b])}}{n!} \left(\frac{b-a}{N} \right)^n \left[j^n + (N-j)^n \right], \tag{35}$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, from (35) we get:

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) \left[j f(a) + (N-j) f(b) \right] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_1([a,b])}}{n!} \left(\frac{b-a}{N} \right)^n \left[j^n + (N-j)^n \right], \tag{36}$$

for $j = 0, 1, 2, ..., N \in \mathbb{N}$,

vi) when $N = 2$ and $j = 1$, (36) turns to

$$\left| \int_a^b f(x)\,dx - \frac{(b-a)}{2} \left(f(a) + f(b) \right) \right| \leq$$

7

$$\frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{n!}\frac{(b-a)^n}{2^{n-1}}, \tag{37}$$

vii) when $n=1$ (without any boundary conditions), we get from (37) that

$$\left|\int_a^b f(x)\,dx-\left(\frac{b-a}{2}\right)(f(a)+f(b))\right|\leq\|f'\|_{L_1([a,b])}(b-a). \tag{38}$$

Proof. Here $n\in\mathbb{N}$ and $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Hence $f^{(n)}$ exists almost everywhere and $f^{(n)}\in L_1([a,b])$. By (9) we get

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right|=\frac{1}{(n-1)!}\left|\int_a^x(x-t)^{n-1}f^{(n)}(t)\,dt\right|\leq$$

$$\frac{1}{(n-1)!}\int_a^x(x-t)^{n-1}\left|f^{(n)}(t)\right|dt\leq\frac{(x-a)^{n-1}}{(n-1)!}\int_a^b\left|f^{(n)}(t)\right|dt \tag{39}$$

$$=\frac{(x-a)^{n-1}}{(n-1)!}\left\|f^{(n)}\right\|_{L_1([a,b])}.$$

That is

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k\right|\leq\frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{(n-1)!}(x-a)^{n-1}, \tag{40}$$

$\forall\ x\in[a,b]$.

By (10) we get

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right|=\frac{1}{(n-1)!}\left|\int_b^x(x-t)^{n-1}f^{(n)}(t)\,dt\right|=$$

$$\frac{1}{(n-1)!}\left|\int_x^b(t-x)^{n-1}f^{(n)}(t)\,dt\right|\leq\frac{1}{(n-1)!}\int_x^b(t-x)^{n-1}\left|f^{(n)}(t)\right|dt\leq \tag{41}$$

$$\frac{(b-x)^{n-1}}{(n-1)!}\int_a^b\left|f^{(n)}(t)\right|dt=\frac{(b-x)^{n-1}}{(n-1)!}\left\|f^{(n)}\right\|_{L_1([a,b])}.$$

That is

$$\left|f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(x-b)^k\right|\leq\frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{(n-1)!}(b-x)^{n-1}, \tag{42}$$

$\forall\ x\in[a,b]$.

Set

$$\rho:=\frac{\left\|f^{(n)}\right\|_{L_1([a,b])}}{(n-1)!}.$$

Hence

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \rho (x-a)^{n-1}, \tag{43}$$

and

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \rho (b-x)^{n-1}, \tag{44}$$

$\forall\, x \in [a,b]$.

As in the proof of Theorem 2 we get:

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\rho}{n} \left[(t-a)^n + (b-t)^n \right], \tag{45}$$

$\forall\, t \in [a,b]$.

The rest of the proof is similar to the proof of Theorem 2. ■

We continue with

Theorem 4 *Let $f \in AC^n([a,b])$, $n \in \mathbb{N}$; $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $f^{(n)} \in L_q([a,b])$. Then*

i)

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[f^{(k)}(a)(t-a)^{k+1} + (-1)^k f^{(k)}(b)(b-t)^{k+1} \right] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \left[(t-a)^{n+\frac{1}{p}} + (b-t)^{n+\frac{1}{p}} \right], \tag{46}$$

$\forall\, t \in [a,b]$,

ii) at $t = \frac{a+b}{2}$, the right hand side of (46) is minimized, and we get:

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \frac{(b-a)^{k+1}}{2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}}, \tag{47}$$

iii) if $f^{(k)}(a) = f^{(k)}(b) = 0$, for all $k = 0,1,...,n-1$, we obtain

$$\left| \int_a^b f(x)\,dx \right| \leq \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)!\left(n+\frac{1}{p}\right)(p(n-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}}, \tag{48}$$

9

which is a sharp inequality,

iv) more generally, for $j = 0, 1, 2, ..., N \in \mathbb{N}$, *it holds*

$$\left| \int_a^b f(x)\,dx - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\frac{b-a}{N} \right)^{k+1} \left[j^{k+1} f^{(k)}(a) + (-1)^k (N-j)^{k+1} f^{(k)}(b) \right] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n + \frac{1}{p} \right) (p(n-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N} \right)^{n+\frac{1}{p}} \left[j^{n+\frac{1}{p}} + (N-j)^{n+\frac{1}{p}} \right], \quad (49)$$

v) if $f^{(k)}(a) = f^{(k)}(b) = 0$, $k = 1, ..., n-1$, *from (49) we get:*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{N} \right) [j f(a) + (N-j) f(b)] \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n + \frac{1}{p} \right) (p(n-1)+1)^{\frac{1}{p}}} \left(\frac{b-a}{N} \right)^{n+\frac{1}{p}} \left[j^{n+\frac{1}{p}} + (N-j)^{n+\frac{1}{p}} \right], \quad (50)$$

for $j = 0, 1, 2, ..., N \in \mathbb{N}$,

vi) when $N = 2$ *and* $j = 1$, *(50) turns to*

$$\left| \int_a^b f(x)\,dx - \frac{(b-a)}{2} (f(a) + f(b)) \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)! \left(n + \frac{1}{p} \right) (p(n-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{n+\frac{1}{p}}}{2^{n-\frac{1}{q}}}, \quad (51)$$

vii) when $n = 1$ *(without any boundary conditions), we get from (51) that*

$$\left| \int_a^b f(x)\,dx - \left(\frac{b-a}{2} \right) (f(a) + f(b)) \right| \leq \frac{\|f'\|_{L_q([a,b])}}{\left(1 + \frac{1}{p} \right)} \frac{(b-a)^{1+\frac{1}{p}}}{2^{\frac{1}{p}}}. \quad (52)$$

Proof. Here $f^{(n)} \in L_q([a,b])$, where $p, q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$. By (9) we get

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| = \frac{1}{(n-1)!} \left| \int_a^x (x-t)^{n-1} f^{(n)}(t)\,dt \right| \leq$$

$$\frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} \left| f^{(n)}(t) \right| dt \leq$$

$$\frac{1}{(n-1)!} \left(\int_a^x (x-t)^{p(n-1)}\,dt \right)^{\frac{1}{p}} \left(\int_a^x \left| f^{(n)}(t) \right|^q dt \right)^{\frac{1}{q}} \leq$$

10

$$\frac{(x-a)^{\frac{p(n-1)+1}{p}}}{(n-1)!\,(p\,(n-1)+1)^{\frac{1}{p}}} \left\| f^{(n)} \right\|_{L_q([a,b])}. \tag{53}$$

That is

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)!\,(p\,(n-1)+1)^{\frac{1}{p}}} (x-a)^{n-\frac{1}{q}}, \tag{54}$$

$\forall\ x \in [a,b]$.

By (10) we get

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| = \frac{1}{(n-1)!} \left| \int_x^b (t-x)^{n-1} f^{(n)}(t)\,dt \right| \leq$$

$$\frac{1}{(n-1)!} \int_x^b (t-x)^{n-1} \left| f^{(n)}(t) \right| dt \leq$$

$$\frac{1}{(n-1)!} \left(\int_x^b (t-x)^{p(n-1)} dt \right)^{\frac{1}{p}} \left(\int_x^b \left| f^{(n)}(t) \right|^q dt \right)^{\frac{1}{q}} \leq$$

$$\frac{(b-x)^{\frac{p(n-1)+1}{p}}}{(n-1)!\,(p\,(n-1)+1)^{\frac{1}{p}}} \left\| f^{(n)} \right\|_{L_q([a,b])}. \tag{55}$$

That is

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)!\,(p\,(n-1)+1)^{\frac{1}{p}}} (b-x)^{n-\frac{1}{q}}, \tag{56}$$

$\forall\ x \in [a,b]$.

Set

$$\gamma := \frac{\left\| f^{(n)} \right\|_{L_q([a,b])}}{(n-1)!\,(p\,(n-1)+1)^{\frac{1}{p}}}, \tag{57}$$

and

$$m := n - \frac{1}{q} > 0. \tag{58}$$

So, we can write

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \gamma\,(x-a)^m, \tag{59}$$

and

$$\left| f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (x-b)^k \right| \leq \gamma\,(b-x)^m, \tag{60}$$

$\forall\ x \in [a,b]$.

As in the proof of Theorem 2 we obtain:

$$\left|\int_a^b f(x)\,dx-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[f^{(k)}(a)(t-a)^{k+1}+(-1)^k f^{(k)}(b)(b-t)^{k+1}\right]\right|\leq$$

$$\frac{\gamma}{(m+1)}\left[(t-a)^{m+1}+(b-t)^{m+1}\right]= \tag{61}$$

$$\frac{\left\|f^{(n)}\right\|_{L_q([a,b])}}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}\left(n+\frac{1}{p}\right)}\left[(t-a)^{n+\frac{1}{p}}+(b-t)^{n+\frac{1}{p}}\right], \tag{62}$$

$\forall\ t\in[a,b]$.

The rest of the proof is similar to the proof of Theorem 2. ∎

References

[1] G.A. Anastassiou, S.S. Dragomir, *On some estimates of the remainder in Taylor's formula*, J. Math. Anal. Appl., 263 (2001), 246-263.

[2] K.S.K. Iyengar, *Note on an inequality*, Math. Student, 6 (1938), 75-76.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

A NOTE ON THE APPROXIMATE SOLUTIONS FOR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY G-BROWNIAN MOTION

F. Faizullah[1], R. Ullah[2], Jihen Majdoubi[3], I. Tlili[4], I. Khan[5]Ghaus ur Rahman

[1]Department of Mathematics, Swansea University, Singleton Park SA2 8PP UK
[2]Department of Mathematics, Women University, Swabi, Pakistan.
[3]Department of computer science college of science and humanities at Alghat Majmaah University, P.O. Box 66 Majmaah 11952, Kingdom of Saudi Arabia
[4]Department of Mechanical and Industrial Engineering, College of Engineering, Majmaah University, P.O. Box 66 Majmaah 11952, Saudi Arabia
[5]Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
Department of Mathematics & Statistics, University of Swat, Khyber Pakhtunkhwa, Pakistan

Abstract. By using the Caratheodory approximation method, the current article presents the analysis of exact and approximate solutions for stochastic differential equations (SDEs) in the framework of G-Brownian motion. In view of the non-linear growth and non-Lipschitz conditions, the boundedness of the Caratheodory approximate solutions $Y^q(t)$, $q \geq 1$ in the space $M_G^2([t_0,T];\mathbb{R}^n)$ has been determined. Estimate for the difference between the exact solution $Y(t)$ and the Caratheodory approximate solutions $Y^q(t)$ has been derived.

Keywords: G-Brownian motion, non-linear growth and non-Lipschitz conditions, Caratheodory approximation procedure, bounded solutions, stochastic differential equations
MSC: 60H20, 60H10, 60H35, 62L20.

1. Introduction

Stochastic differential equations (SDEs) are employed by several and diverse scientific disciplines such as chemistry, statistical physics, biology and engineering. In finance and economics, they are utilized to find out the risk measures and stochastic volatility problems. SDEs describe heavy traffic behavior of communication networks and control systems [16]. Mathematics use the concept of SDEs to incorporate random fluctuations in the model when one investigates the evolution of the number of cells in an organism infected by a virus. The weather and climate can be modeled by these equations. The clarification of fluid through porous structures and water catchment can be modeled by SDEs [17]. They are used to describe the motion of wildlife [4]. SDEs play an important role to study the animal's swarm, such as schooling of fish, flocking of birds or herding of mammals, to find resource of food in noisy and obstacle environment [30]. In physics, SDEs are used to study and model the effect of random variations on distinct physical processes. A large literature is available on the applications of SDEs in numerous fields of engineering such as computer engineering [16, 22], mechanical engineering [26, 28, 29], random vibrations [3, 24], stability theory [25] and wave processes [27]. In general, one can not find the explicit solutions for non-linear SDEs, so we have to present and study the analysis for the solutions of these equations. Moreover, the developments of computational techniques are very important for solving several demanding problems, for instance to find the optimal construction of a design and to determine input data from fundamental principles. Therefore it is valuable to know computational accuracy, which leads us to convergence results and estimates for the difference between exact and approximate solutions.

Corresponding author email (I. Khan): ilyaskhan@tdt.edu.vn

The aim of the current article is to investigate estimates for the difference between exact and approximate solutions for SDEs driven by G-Brownian motion with Caratheodory approximation procedure. In view of growth and Lipschitz conditions, the existence-uniqueness results for G-SDEs was studied by Peng [20, 21] and Gao [15]. Later, Bai and Lin [1] established the existence theory for G-SDEs with integral Lipschitz coefficients. Subject to some discontinuous coefficients, the said theory was generalized by Faizullah [11]. Let $0 \leq t_0 \leq t \leq T < \infty$. Consider the following SDE in the framework of G-Brownian motion

$$
\begin{aligned}
dY(t) =& \kappa(t, Y(t))dt + \lambda(t, Y(t))d\langle W, W\rangle(t) \\
&+ \mu(t, Y(t))dW(t),
\end{aligned} \tag{1.1}
$$

with initial value $Y(t_0) \in \mathbb{R}^n$. The given coefficients $g(.,x), h(.,x)$ and $w(.,x)$ belong to space $M^2_G([t_0,T];\mathbb{R}^n)$, for all $x \in \mathbb{R}^n$. SDE (1.1) in the integral form is expressed as the following

$$
\begin{aligned}
Y(t) = Y(t_0) + \int_{t_0}^{t} \kappa(s, Y(s))ds + \int_{t_0}^{t} \lambda(s, Y(s))d\langle W, W\rangle(s) \\
+ \int_{t_0}^{t} \mu(s, Y(s))dW(s),
\end{aligned} \tag{1.2}
$$

on $t \in [t_0, T]$. Its solution is a process $Y \in M^2_G([t_0,T];\mathbb{R}^n)$ and satisfying SDE (1.2). The rest of the current paper contains three sections. Building on the previous notions of G-expectation, section 2 presents the fundamental definitions and results of G-Browinian motion, sub-expectation, Grownwall's inequality, Doobs martingale inequality, G-Itô's integral and Hölder's inequality etc. Section 3 reveals the Caratheodory approximate solutions procedure for SDEs driven by G-Brownian motion. This section give an important result, which shows that the Caratheodory approximate solutions are bounded. Section 4 derives estimates for the difference between approximate and exact solutions to SDEs driven by G-Brownian motion.

2. Preliminaries

We present some basic results and notions required for the subsequent sections of the current article. We don't give detailed literature on basic notions of G-expectation, so readers are suggested to consult the more depth oriented papers [9, 13, 18, 20, 21]. Let Ω be a given basic non-empty set. Assume $\mathcal{H}$ be a space of linear real functions defined on Ω so that (i) $1 \in \mathcal{H}$ (ii) for every $n \geq 1$, $Y_1, Y_2, ..., Y_n \in \mathcal{H}$ and $\varphi \in C_{b.Lip}(\mathbb{R}^n)$ it satisfies $\varphi(Y_1, Y_2, ..., Y_n) \in \mathcal{H}$ i.e., subject to Lipschitz bounded functions, $\mathcal{H}$ is stable. Then $(\Omega, \mathcal{H}, E)$ is a sub-expectation space, where E is a sub-expectation defined as follows.

Definition 2.1. A functional $E : \mathcal{H} \to \mathbb{R}$ satisfying the below four features is known as a sub-expectation. Let $X, Y \in \mathcal{H}$, then

(**1**) Monotonicity: $E(X) \leq E(Y)$ if $X \leq Y$.
(**2**) Constant preservation: $E(M_1) = M_1$, for all $M_1 \in \mathbb{R}$.
(**3**) Positive homogeneity: $E(N_1Y) = N_1E(Y)$, for all $N_1 \in \mathbb{R}^+$.
(**4**) Sub-additivity: $E(X) + E(Y) \geq E(X + Y)$.

Moreover, let Ω be the space of all $\mathbb{R}^n$-valued continuous paths $(w_t)_{t\geq 0}$ starting from zero. In addition, assume that subject to the below distance, Ω is a metric space

$$\rho(w^1, w^2) = \sum_{i=1}^{\infty} \frac{1}{2^i}(\max_{t\in[0,k]} |w_t^1 - w_t^2| \wedge 1).$$

Fix $T \geq 0$ and set

$$L_{ip}^0(\Omega_T) = \{\phi(W_{t_1}, W_{t_2}, ..., W_{t_m}) : m \geq 1, t_1, t_2, ..., t_m \in [0,T], \phi \in C_{b.Lip}(\mathbb{R}^{m\times n}))\},$$

where W is the canonical process, $L_{ip}^0(\Omega_t) \subseteq L_{ip}^0(\Omega_T)$ for $t \leq T$ and $L_{ip}^0(\Omega) = \cup_{n=1}^{\infty} L_{ip}^0(\Omega_n)$. The completion of $L_{ip}^0(\Omega)$ under the Banach norm $E[|.|^p]^{\frac{1}{p}}$, $p \geq 1$ is denoted by $L_G^p(\Omega)$, where $L_G^p(\Omega_t) \subseteq L_G^p(\Omega_T) \subseteq L_G^p(\Omega)$ for $0 \leq t \leq T < \infty$. Generated by the canonical process $\{W(t)\}_{t\geq 0}$, the filtration is represented as $\mathcal{F}_t = \sigma\{W_s, 0 \leq s \leq t\}$, $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$. Suppose $\pi_T = \{t_0, t_1, ..., t_N\}$, $0 \leq t_0 \leq t_1 \leq ... \leq t_N \leq \infty$ be a division of $[0,T]$. For $p \geq 1$, $M_G^{p,0}(0,T)$ denotes a set of the processes given by

$$\eta_t(w) = \sum_{i=0}^{N-1} \xi_i(w) I_{[t_i,t_{i+1}]}(t), \tag{2.1}$$

where $\xi_i \in L_G^p(\Omega_{t_i})$, $i = 0, 1, ..., N-1$. Furthermore, the completion of $M_G^{p,0}(0,T)$ with the below given norm is indicated by $M_G^p(0,T)$, $p \geq 1$

$$\|\eta\| = \{\int_0^T E[|\eta_s|^p]ds\}^{1/p}.$$

Definition 2.2. An n-dimensional stochastic process $\{W(t)\}_{t\geq 0}$ is called a G-Brownian motion if

(**1**) $W(0) = 0$.
(**2**) For any $t, m \geq 0$, $W_{t+m} - W_t$ is G-normally distributed and independent from $W_{t_1}, W_{t_2}, ...W_{t_n}$, for $n \in N$ and $0 \leq t_1 \leq t_2 \leq, ..., \leq t_n \leq t$,

Definition 2.3. Let $\eta_t \in M_G^{2,0}(0,T)$ having the form (2.1). Then the G-quadratic variation process $\{\langle W\rangle_t\}_{t\geq 0}$ and G-Itô's integral $I(\eta)$ are respectively defined by

$$\langle W\rangle_t = W_t^2 - 2\int_0^t W_s dW(s),$$
$$I(\eta) = \int_0^T \eta_s dW(s) = \sum_{i=0}^{N-1} \xi_i (W_{t_{i+1}} - W_{t_i}).$$

The following two lemmas can be found in the book [19]. They are known as Hölder's and Gronwall's inequalities respectively, .

Lemma 2.4. *Assume $m, n > 1$ such that $\frac{1}{m} + \frac{1}{n} = 1$ and $\beta \in L^2$ then $\alpha\beta \in L^1$ and*

$$\int_a^b \alpha\beta \leq \left(\int_a^b |\alpha|^m\right)^{\frac{1}{m}} \left(\int_a^b |\beta|^n\right)^{\frac{1}{n}}.$$

Lemma 2.5. *Let $\alpha(t) \geq 0$ and $\beta(t)$ be continuous real functions defined on $[a, b]$. If for all $t \in [a, b]$,*

$$\beta(t) \leq K + \int_a^b \alpha(s)\beta(s)ds,$$

where $K \geq 0$, then

$$\beta(t) \leq Ke^{\int_a^t \alpha(s)ds},$$

for all $t \in [a, b]$.

The following lemma, known as Doob's martingale inequality, is borrowed from [15].

Lemma 2.6. *Assume $[c, d]$ be a bounded interval of $\mathbb{R}_+$. Consider an $\mathbb{R}^n$ valued G-martingale $\{X(t) : t \geq 0\}$. Then we have*

$$E(\sup_{c \leq t \leq d} |Y(t)|^p) \leq (\frac{p}{p-1})^p E(|Y(d)|^P),$$

where $p > 1$ and $Y(t) \in L^p_G(\Omega, \mathbb{R}^d)$. In particular, if $p = 2$ then $E(\sup_{c \leq t \leq d} |Y(t)|^2) \leq 4E(|Y(d)|^2)$.

3. Caratheodory approximate solutions

We now present the Caratheodory approximation procedure for equation (1.2). Let $q \geq 1$ be any positive integer. For $t \in [t_0 - 1, t_0]$, we set $Y^q(t) = Y_0$ and for $t \in [t_0, T]$,

$$\begin{aligned} Y^q(t) = Y_0 &+ \int_{t_0}^t \kappa(s, Y^q(s - \frac{1}{q}))ds + \int_{t_0}^t \lambda(s, Y^q(s - \frac{1}{q}))d\langle W, W\rangle(s) \\ &+ \int_{t_0}^t \mu(s, Y^q(s - \frac{1}{q}))dW(s). \end{aligned} \tag{3.1}$$

The approximate solutions $Z^q(.)$ can be determined step-by-step on the intervals $[t_0, t_0 + \frac{1}{q}]$, $(t_0 + \frac{1}{q}, t_0 + \frac{2}{q}]$ and son on with the following procedure. For $t \in [t_0, t_0 + \frac{1}{q}]$, we have

$$\begin{aligned} Y^q(t) = Y_0 &+ \int_{t_0}^t \kappa(s, Y_0)ds + \int_{t_0}^t \lambda(s, Y_0)d\langle W, W\rangle(s) \\ &+ \int_{t_0}^t \mu(s, Y_0)dW(s), \end{aligned}$$

and for $t \in (t_0 + \frac{1}{q}, t_0 + \frac{2}{q}]$,

$$\begin{aligned} Y^q(t) = Y^q(t_0 + \frac{1}{q}) &+ \int_{t_0+\frac{1}{q}}^t \kappa(s, Y^q(s - \frac{1}{q}))ds + \int_{t_0+\frac{1}{q}}^t \lambda(s, Y^q(s - \frac{1}{q}))d\langle W, W\rangle(s) \\ &+ \int_{t_0+\frac{1}{q}}^t \mu(s, Y^q(s - \frac{1}{q}))dW(s), \end{aligned}$$

etc. All through this article, we assume two conditions, described as follows. Let M be a positive constant. For any $t \in [t_0, T]$ and $\kappa(t, 0), \lambda(t, 0), \mu(t, 0) \in L^2$,

$$|\kappa(t, 0)|^2 + |\lambda(t, 0)|^2 + |\mu(t, 0)|^2 \leq M, \tag{3.2}$$

4

which is weakened linear growth condition. Let $t \in [t_0, T]$. For every $u, v \in \mathbb{R}^n$, there exists a concave non-decreasing function $\Psi(.) : \mathbb{R}^+ \to \mathbb{R}^+$ with $\Psi(0) = 0$ and for $s > 0$, $\Psi(s) > 0$ such that

$$|\kappa(t,u) - \kappa(t,v)|^2 + |\lambda(t,u) - \lambda(t,v)|^2 + |\mu(t,u) - \mu(t,v)|^2 \le \Psi(|u-v|^2), \tag{3.3}$$

where $\int_{0+} \frac{ds}{\Psi(s)} = \infty$ and for all $s \ge 0$, $C, D > 0$, $\Psi(s) \le C + Ds$. Assumption (3.3) is a non-uniform Lipschitz condition. Subject to conditions (3.2) and (3.3), we assume that problem (1.1) has a unique solution $Y(t) \in M_G^2([t_0, T]; \mathbb{R}^n)$ [1].

Lemma 3.1. *Let assumptions* (3.2) *and* (3.2) *are satisfied. For every* $q \ge 1$ *and any* $T > 0$,

$$\sup_{t_0 \le t \le T} E(|Y^q(t)|^2) \le N_1, \tag{3.4}$$

where $N_1 = H_1 e^{H_2(T-t_0)}$, $H_1 = 4E|Y_0|^2 + 8T(T+2)(2M+C)$, $H_2 = 8(T+2)D$ *and* M, C, D *are arbitrary positive constants.*

Proof. In view of the inequality $|\sum_{i=1}^4 c_i|^2 \le 7 \sum_{i=1}^4 |c_i|^2$, from (3.1) we derive

$$\begin{aligned}|Y^q(t)|^2 \le 4|Y_0|^2 &+ 4\Big|\int_{t_0}^t \kappa(s, Y^q(s-\frac{1}{q}))ds\Big|^2 + 4\Big|\int_{t_0}^t \lambda(s, Y^q(s-\frac{1}{q}))d\langle W, W\rangle(s)\Big|^2 \\ &+ 4\Big|\int_{t_0}^t \mu(s, Y^q(s-\frac{1}{q}))dW(s)\Big|^2.\end{aligned}$$

Apply G-subexpectation on both sides. Then by virtue of the Doob's martingale, Holder's and BDG [5] inequalities we have

$$\begin{aligned}E(\sup_{t_0 \le s \le t} |Y^q(s)|^2) &\le 4E(|Y_0|^2) + 4T\int_{t_0}^t E|\kappa(s, Y^q(s-\frac{1}{q}))|^2 ds + 4T\int_{t_0}^t E|\lambda(s, Y^q(s-\frac{1}{q}))|^2 ds \\ &+ 16\int_{t_0}^t E|\mu(s, Y^q(s-\frac{1}{q}))|^2 ds \\ &\le 4E(|Y_0|^2) + 8T\int_{t_0}^t E[|\kappa(s, Y^q(s-\frac{1}{q})) - \kappa(s,0)|^2 + |\kappa(s,0)|^2]ds \\ &+ 8T\int_{t_0}^t E[|\lambda(s, Y^q(s-\frac{1}{q})) - \lambda(s,0)|^2 + |\lambda(s,0)|^2]ds \\ &+ 32\int_{t_0}^t E[|\mu(s, Y^q(s-\frac{1}{q})) - \mu(s,0)|^2 + |\mu(s,0)|^2]ds.\end{aligned}$$

Using (3.2) and (3.3), we derive

$$\begin{aligned}E(\sup_{t_0 \le s \le t} |Y^q(s)|^2) &\le 4E(|Y_0|^2) + 16T^2M + 32TM + 8(T+2)\int_{t_0}^t E[\Psi(|Y^q(s-\frac{1}{q})|^2)]ds \\ &\le 4E(|Y_0|^2) + 16T^2M + 32TM + 8(T+2)\int_{t_0}^t [C + DE|Y^q(s-\frac{1}{q})|^2]ds \\ &\le 4E(|Y_0|^2) + 16T^2M + 32TM + 8T(T+2)C + 8(T+2)D\int_{t_0}^t E[\sup_{t_0 \le r \le s} |Y^q(r)|^2]ds\end{aligned}$$

5

By virtue of the Grownwall's inequality, we derive

$$E(\sup_{t_0 \le s \le t} |Y^q(s)|^2) \le H_1 e^{H_2(t-t_0)},$$

where $H_1 = 4E|Y_0|^2 + 8T(T+2)(2M+C)$ and $H_2 = 8(T+2)D$. Consequently, supposing $t = T$, we obtain

$$E(\sup_{t_0 \le s \le T} |Y^q(s)|^2) \le H_1 e^{H_2(T-t_0)} = N_1.$$

The proof stands completed. □

In a similar way as lemma 3.1, we can prove the following result.

Lemma 3.2. *Subject to the growth condition* (3.2)*, for any* $T > 0$,

$$\sup_{t_0 \le t \le T} E(|Y(t)|^2) \le N_1, \tag{3.5}$$

where N_1 *is a positive constant.*

4. Estimates for the difference between exact and Caratheodory approximate solutions

We first give an important result. Then in view of weakened growth and non-uniform Lipschitz conditions, we derive an estimate for the difference between the approximate and exact solutions to problem (1.1).

Lemma 4.1. *Let* $0 \le r < t \le T$*. Suppose that the assumptions of lemma 3.1 are satisfied. For all* $q \ge 1$

$$E[|Z^q(t) - Z^q(u)|^2] \le G_1(t-u), \tag{4.1}$$

where $G_1 = 12(T+2)(M+C+DN_1)$*,* M*,* C*,* D *and* N_1 *are positive constants.*

Proof. In view of the fundamental inequality $|\sum_{i=1}^{3} c_i|^2 \le 7\sum_{i=1}^{3} |c_i|^2$, for any $q \ge 1$ and $0 \le r < t \le T$, from (3.1) we derive

$$|Y^q(t) - Y^q(u)|^2 \le 3|\int_u^t \kappa(s, Y^q(s-\frac{1}{q}))ds|^2 + 3|\int_u^t \lambda(s, Y^q(s-\frac{1}{q}))d\langle W,W\rangle(s)|^2$$
$$+ 3|\int_u^t \mu(s, Y^q(s-\frac{1}{q}))dW(s)|^2.$$

6

Apply G-subexpectation on both sides. Then by virtue of the Doob's martingale, Holder's and BDG [5] inequalities we have

$$\begin{aligned}|Y^q(t)-Y^q(r)|^2 &\leq 3T\int_u^t E|\kappa(s,Y^q(s-\frac{1}{q}))|^2ds+3T\int_{t_0}^t E|\lambda(s,Y^q(s-\frac{1}{q}))|^2ds\\&+12\int_u^t|\mu(s,Y^q(s-\frac{1}{q}))|^2ds\\&\leq 6T\int_u^t E[|\kappa(s,Y^q(s-\frac{1}{q}))-\kappa(s,0)|^2+|\kappa(s,0)|^2]ds\\&+6T\int_u^t E[|\lambda(s,Y^q(s-\frac{1}{q}))-\lambda(s,0)|^2+|\lambda(s,0)|^2]ds\\&+24\int_u^t E[|\mu(s,Y^q(s-\frac{1}{q}))-\mu(s,0)|^2+|\mu(s,0)|^2]ds.\end{aligned}$$

Using (3.2) (3.3), we derive

$$\begin{aligned}|Y^q(t)-Y^q(u)|^2 &\leq 6TM(t-u)+6TM(t-u)+24M(t-u)+12(T+2)\int_u^t E[\Psi(|Y^q(s-\frac{1}{q})|^2)]ds\\&\leq 12TM(t-u)+24M(t-u)+12C(T+2)(t-u)+12D(T+2)\int_u^t E[|Y^q(s-\frac{1}{q})|^2]ds\\&\leq 12TM(t-u)+24M(t-u)+12C(T+2)(t-u)\\&+12D(T+2)\int_u^t E[\sup_{t_0\leq r\leq s}|Y^q(r)|^2]ds\end{aligned}$$

In view of lemma 3.1, we have

$$\begin{aligned}|Y^q(t)-Y^q(u)|^2 &\leq 12TM(t-u)+24M(t-u)+12C(T+2)(t-u)\\&+12D(T+2)N_1(t-u)\end{aligned}$$

Consequently,

$$|Y^q(t)-Y^q(u)|^2\leq G_1(t-u),$$

where $G_1=12(T+2)(M+C+DN_1)$. The proof is complete. □

Next lemma can be proved by using similar arguments as used in lemma 4.1.

Lemma 4.2. *Let $0\leq r<t\leq T$. Subject to conditions* (3.2) *and* (3.3),

$$E[|Z(t)-Z(u)|^2]\leq G_1(t-u),$$

where G_1 is a positive constant.

Theorem 4.3. *Assume* (3.2) *and* (3.3) *holds. Then*

$$E(\sup_{t_0\leq s\leq T}|Y(s)-Y^q(s)|^2)\leq 6T(T+2)[C+\frac{2D}{q}]e^{12(T+2)D(T-t_0)},$$

where C and D are positive constants.

7

Proof. By using the inequality $|\sum_{i=1}^{3} c_i|^2 \leq 7\sum_{i=1}^{3} |c_i|^2$, from (1.2) and (3.1) we obtain

$$|Y(t) - Y^q(t)|^2 \leq 3|\int_{t_0}^{t} [\kappa(s, Y(s)) - \kappa(s, Y^q(s - \frac{1}{q}))]ds|^2 + 3|\int_{t_0}^{t} [\lambda(s, Y(s)) - \lambda(s, Y^q(s - \frac{1}{q}))]d\langle W, W\rangle(s)|^2$$
$$+ 3|\int_{t_0}^{t} [\mu(s, Y(s)) - \mu(s, Y^q(s - \frac{1}{q}))]dW(s)|^2.$$

Apply G-subexpectation on both sides. Then by virtue of the Doob's martingale, Holder's and BDG [5] inequalities we derive

$$E(\sup_{t_0\leq s\leq t} |Y(s) - Y^q(s)|^2) \leq 3T \int_{t_0}^{t} E[|\kappa(s, Y(s)) - \kappa(s, Y^q(s - \frac{1}{q}))|^2]ds$$
$$+ 3T \int_{t_0}^{t} E[|\lambda(s, Y(s)) - \lambda(s, Y^q(s - \frac{1}{q}))|^2]ds$$
$$+ 12 \int_{t_0}^{t} E[|\mu(s, Y(s)) - \mu(s, Y^q(s - \frac{1}{q}))|^2]ds.$$

Using the non-uniform Lipschitz condition we get

$$E(\sup_{t_0\leq s\leq t} |Y(s) - Y^q(s)|^2) \leq 6(T + 2) \int_{t_0}^{t} E[\Psi(|Y(s) - Y^q(s - \frac{1}{q})|^2)]ds$$
$$\leq 6T(T + 2)C + 6(T + 2)D \int_{t_0}^{t} E[|Y(s) - Y^q(s - \frac{1}{q})|^2]ds$$
$$= 6T(T + 2)C + 6(T + 2)D \int_{t_0}^{t} E[|Y(s) - Y^q(s) + Y^q(s) - Y^q(s - \frac{1}{q})|^2]ds$$
$$\leq 6T(T + 2)C + 12(T + 2)D \int_{t_0}^{t} E[|Y(s) - Y^q(s)|^2]ds$$
$$+ 12(T + 2)D \int_{t_0}^{t} E[|Y^q(s) - Y^q(s - \frac{1}{q})|^2]ds$$

Utilizing lemma 4.1, we determine

$$E(\sup_{t_0\leq s\leq t} |Y(s) - Y^q(s)|^2) \leq 6T(T + 2)C + 12(T + 2)D \int_{t_0}^{t} E(\sup_{t_0\leq r\leq s} |Y(r) - Y^q(r)|^2)ds + 12T(T + 2)D\frac{1}{q}$$

Finally, the Grownwall's inequality gives

$$E(\sup_{t_0\leq s\leq t} |Y(s) - Y^q(s)|^2) \leq [6T(T + 2)C + 12T(T + 2)D\frac{1}{q}]e^{12(T+2)D(t-t_0)}.$$

Consequently, by letting $t = T$, we get

$$E(\sup_{t_0\leq s\leq T} |Y(s) - Y^q(s)|^2) \leq 6T(T + 2)[C + \frac{2D}{q}]e^{12(T+2)D(T-t_0)}.$$

The proof stands completed. □

8

5. Conclusion

This paper opens several new research directions with arising the following open problems. What will be the estimates for the difference between exact and Caratheodory approximate solutions to G-SFDEs under non-linear growth and non-Lipschitz conditions? How can one solve the stated problem for G-NSFDEs? Can one gives estimates for the difference between exact and Caratheodory approximate solutions to backward stochastic differential equations in the framework of G-Brownian motion? Under what conditions, can we develop the mentioned theory for stochastic pantograph equations [2, 12, 31, 32]? We hope the current paper will play an essential role to establish a foundation for the concepts briefly discussed.

6. Acknowledgement

This work is supported by the Commonwealth Scholarship Commission in the United Kingdom with project number PKRF-2017-429.

References

[1] X. Bai, Y. Lin, On the existence and uniqueness of solutions to stochastic differential equations driven by G-Brownian motion with Integral-Lipschitz coefficients, Acta Mathematicae Applicatae Sinica, English Series, 3(30) (2014) 589-610.

[2] C. T.H. Baker and E. Buckwar, Continuous Θ-methods for the stochastic pantograph equation, Electronic Transactions on Numerical Analysis, 11 (2000) 131-151.

[3] V.V Bolotin, Random vibrations of elastic systems, Martinus Nijhoff,The Hague (1984).

[4] D. R. Brillinger, H. K. Preisler, A. A. Ager, J. G. Kie, B. S. Stewart, Employing stochastic differential equations to model wildlife motion, Bull. Braz. Math. Soc., 33(3), (2002) 385-408.

[5] L. Denis, M. Hu, S. Peng, Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion paths, Potential Anal., 34 (2010) 139–161.

[6] F. Faizullah, Existence and uniqueness of solutions to SFDEs driven by G-Brownian motion with non-Lipschitz conditions, Journal of Computational Analysis and Applications, 2(23) (2017) 344-354.

[7] F. Faizullah, On the pth moment estimates of solutions to stochastic functional differential equations in the G-framework, SpringerPlus, 5(872) (2016) 1-11.

[8] F. Faizullah, A note on p-th moment estimates for stochastic functional differential equations in the framework of G-Brownian motion, Iranian Journal of Science and Technology, Transaction A: Science, 3(40) (2016) 1-8.

[9] F. Faizullah, Existence results and moment estimates for NSFDEs driven by G-Brownian motion, Journal of Computational and Theoretical Nanoscience, 7(13) (2016) 1-8.

[10] F. Faizullah, Existence of solutions for G-SFDEs with Cauchy-Maruyama Approximation Scheme, Abstract and Applied Analysis, http://dx.doi.org/10.1155/2014/809431, (2014) 1–8.

[11] F. Faizullah, Existence of solutions for stochastic differential equations under G-Brownian motion with discontinuous coefficients, Zeitschrift fr Naturforschung A., 67A (2012) 692–698.

[12] Z. Fan, M. Liu and W. Cao, Existence and uniqueness of the solutions and convergence of semi-implicit Euler methods for stochastic pantograph equations, J. Math. Anal. Appl., 325 (2007) 11421159.

[13] M. Hu, S. Peng, Extended conditional G-expectations and related stopping times, arXiv:1309.3829v1[math.PR] 16 Sep 2013.

[14] M. Hu, S. Ji, S. Peng, Y. Song, Backward stochastic differential equations driven by G-Brownian motion , Stochastic Processes and their Applications, 124 (2014) 759784.

[15] F. Gao, Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-Brownian motion, Stochastic Processes and thier Applications, 2 (2009) 3356–3382.

[16] J.H. Gubner, Probability and random processes for electrical and computer engineering, Cambridge University Press, 2006.

[17] lyas Khan, F. Ali, N. A. Shah, Interaction of magnetic field with heat and mass transfer in free convection flow of a Walters-B fluid, The European Physical Journal Plus, 131(77) (2016) 1-15.

[18] X. Li, S. Peng, Stopping times and related Ito's calculus with G-Brownian motion, Stochastic Processes and thier Applications, 121 (2011) 1492–1508.

[19] X. Mao, *Stochastic Differential Equations and their Applications*, Horwood Publishing Chichester, 1997.

[20] S. Peng, G-expectation, G-Brownian motion and related stochastic calculus of Ito's type, The abel symposium 2005, Abel symposia 2, edit. benth et. al., Springer-vertag., (2006) 541-567.

[21] S. Peng, Multi-dimentional G-Brownian motion and related stochastic calculus under G-expectation, Stochastic Processes and thier Applications, 12 (2008) 2223–2253.

[22] S. Primak, V. Kontorovitch, V. Lyandres, Stochastic methods and their applications to communications: Stochastic differential equations approach, John Wiley & Sons Ltd, Chichester England, 2004.

[23] Y. Ren, Q. Bi, R. Sakthivel, Stochastic functional differential equations with infinite delay driven by G-Brownian motion, Mathematical Methods in the Applied Sciences, 36(13) (2013) 1746–1759.

[24] J.B Roberts, P.D. Spanos, Random vibration and statistical linearization, Dover, New York (2003).

[25] B. Skalmierski, A. Tylikowski, Stability of dynamical systems, Polish Scientific Editors, Warsaw (1973).

[26] B. Skalmierski, A. Tylikowski, Stochastic processes in dynamics, Polish Scientific editors, Warsaw (1982).

[27] K. Sobezyk, Stochastic waive propagation, Polish Scientific editors, Warsaw (1984).

[28] K. Sobezyk, Stochastic differential equations with applications to Physics and Engineering, Kluwer Academic, Dordrecht (1991).

[29] K. Sobezyk, Jr. B.F. Spencer, Random Fatigue: Frome data to theory Academic Press, Boston (1992).

[30] T. V. Ton, N. T. H. Linh, A.Yagi, A stochastic differential equations model for foraging swarms, arXiv:1509.00063v1 [math.PR] 28 Aug 2015.

[31] Y. Xiao, M. Song and M. Liu, Convergence and stability of semi-implicit Euler method with variable stepsize for a linear stochastic pantograph differential equation, International Journal of Numerical Analysis and Modeling, 2(8) (2011) 214225.

[32] Y. Xiao, H.Y. Zhang, A note on convergence of semi-implicit Euler methods for stochastic pantograph equations, Computers and Mathematics with Applications, 59 (2010) 1419-1424.

Behavior of a system of higher-order difference equations

M. A. El-Moneam* A. Q. Khan† E. S. Aly‡ M. A. Aiyashi§

Abstract

We study the local stability about equilibria, periodicity nature of positive solutions and existence of unbounded solutions of higher-order system of rational difference equations. The results presented here are considerably extended and improve some existing results in the literature. Finally theoretical results are verified numerically.

Keywords: difference equations; local stability; periodicity; unbounded solutions

AMS subject classifications: 39A10, 40A05

1 Introduction

In [1], Bajo and Liz have investigated the global behavior of the difference equation: $x_{n+1} = \frac{x_{n-1}}{a+bx_{n-1}x_n}$, where $a,\ b,\ x_0,\ x_{-1} \in \mathbb{R}_+^2$. Aloqeili [2] has investigated the stability and semi-cycle analysis of the difference equation: $x_{n+1} = \frac{x_{n-1}}{a-x_{n-1}x_n}$, $n = 0, 1, \cdots$, where $a,\ x_0,\ x_{-1} \in \mathbb{R}_+^2$. For systemic study of difference equations and systems of difference equations, we refer the reader [3–7] and references cited therein. Motivated by the above studies, our aim in this paper is to investigate the local stability about equilibria, periodicity nature of the positive solutions and existence of unbounded solutions of the following higher-order system of difference equations:

$$x_{n+1} = \frac{\alpha_1 x_{n-k}}{\beta_1 - \gamma_1 \prod_{i=0}^{k} y_{n-i}},\ y_{n+1} = \frac{\alpha_2 y_{n-k}}{\beta_2 - \gamma_2 \prod_{i=0}^{k} x_{n-i}},\ n = 0, 1, \cdots , \tag{1}$$

where $\alpha_i,\ \beta_i,\ \gamma_i$ for $i = 1, 2$ and $x_{-j},\ y_{-j}$ for $j = 0, 1, \cdots , k$ are belong to $\mathbb{R}_+^2$. The rest of the paper is organized as follows: Existence of equilibria and local stability are studied in Section 2. Section 3 deals with the study of periodicity nature and existence of unbounded solutions of system (1). In Section 4, numerical simulations are presented to verify theocratical discussion. A brief conclusion is given in last Section.

*Department of Mathematics, Faculty of Science, Jazan University, Jazan, Kingdom of Saudi Arabia, e-mail: maliahmedibrahim@jazanu.edu.sa

†Department of Mathematics, University of Azad Jammu and Kashmir, Muzaffarabad 13100, Pakistan, e-mail: abdulqadeerkhan1@gmail.com

‡Department of Mathematics, Faculty of Science, Jazan University, Jazan, Kingdom of Saudi Arabia, e-mail: elkhateeb@jazanu.edu.sa

§Department of Mathematics, Faculty of Science, Jazan University, Jazan, Kingdom of Saudi Arabia, e-mail: maiyashi@jazanu.edu.sa

2 Existence of equilibria and local stability

In this section, we will study the existence of equilibria and local stability of system (1). The results about the existence of equilibria are summarized into following Lemma:

Lemma 1. *System (1) has two equilibria in the interior of $\mathbb{R}^2_+$. More precisely*

(i) $\forall$ parametric values, system (1) has a unique boundary equilibrium point $O(0,0)$;

(ii) If $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$, then $A\left(\left(\frac{\beta_2-\alpha_2}{\gamma_2}\right)^{\frac{1}{k+1}}, \left(\frac{\beta_1-\alpha_1}{\gamma_1}\right)^{\frac{1}{k+1}}\right)$ is the unique positive equilibrium point of system (1).

Hereafter we will study the local stability of system (1) about boundary equilibrium $(0,0)$ and the unique positive equilibrium point $A\left(\left(\frac{\beta_2-\alpha_2}{\gamma_2}\right)^{\frac{1}{k+1}}, \left(\frac{\beta_1-\alpha_1}{\gamma_1}\right)^{\frac{1}{k+1}}\right)$ of system (1).

Lemma 2. *For local dynamics about $O(0,0)$ and $A\left(\left(\frac{\beta_2-\alpha_2}{\gamma_2}\right)^{\frac{1}{k+1}}, \left(\frac{\beta_1-\alpha_1}{\gamma_1}\right)^{\frac{1}{k+1}}\right)$, the following statements hold:*

(i) For equilibrium $O(0,0)$, the following holds:

(i.1) $O(0,0)$ is locally asymptotically stable if $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$;

(i.2) $O(0,0)$ is a unstable if $\alpha_1 > \beta_1$ or $\alpha_2 < \beta_2$.

(ii) $A\left(\left(\frac{\beta_2-\alpha_2}{\gamma_2}\right)^{\frac{1}{k+1}}, \left(\frac{\beta_1-\alpha_1}{\gamma_1}\right)^{\frac{1}{k+1}}\right)$ is unstable.

Proof. $(i.1)$ The linearized system of (1) about $(0,0)$ becomes: $X_{n+1} = J_{(0,0)}X_n$ where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_{n-k} \\ y_n \\ y_{n-1} \\ \vdots \\ y_{n-k} \end{pmatrix}, \quad J_{(0,0)} = \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{\alpha_1}{\beta_1} & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \frac{\alpha_2}{\beta_2} \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

. The characteristic equation of $J_(0,0)$ about $(0,0)$ is

$$\lambda^{2k+2} - \left(\frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2}\right)\lambda^{k+1} + \frac{\alpha_1\alpha_2}{\beta_1\beta_2} = 0. \tag{2}$$

If $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$ then all roots of (2) lie inside unit disk. So $O(0,0)$ of system (1) is locally asymptotically stable.

$(i.2)$ It is easy to show that if $\alpha_1 > \beta_1$ or $\alpha_2 > \beta_2$ then $O(0,0)$ is unstable.

(ii). The linearized system of (1) about $A\left(\left(\frac{\beta_2-\alpha_2}{\gamma_2}\right)^{\frac{1}{k+1}}, \left(\frac{\beta_1-\alpha_1}{\gamma_1}\right)^{\frac{1}{k+1}}\right)$ becomes: $X_{n+1} = J_A X_n$ where

$$J_A = E_{(2k+2)\times(2k+2)} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 & \frac{\gamma_1\bar{x}\bar{y}^k}{\alpha_1} & \frac{\gamma_1\bar{x}\bar{y}^k}{\alpha_1} & \dots & \frac{\gamma_1\bar{x}\bar{y}^k}{\alpha_1} & \frac{\gamma_1\bar{x}\bar{y}^k}{\alpha_1} \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{\gamma_2\bar{y}\bar{x}^k}{\alpha_2} & \frac{\gamma_2\bar{y}\bar{x}^k}{\alpha_2} & \dots & \frac{\gamma_2\bar{y}\bar{x}^k}{\alpha_2} & \frac{\gamma_2\bar{y}\bar{x}^k}{\alpha_2} & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Let $\lambda_1,\ \lambda_2,\ \dots,\ \lambda_{2k+2}$ denote the $2k+2$ eigenvalues of matrix E. Let $D = diag(d_1,\ d_2,\ \dots,\ d_{2k+2})$ be a diagonal matrix, where $d_1 = d_{k+2} = 1$, $d_i = d_{k+1+i} = 1 - i\epsilon,\ i = 2,\ 3,\ \cdots,\ k+1$ for $0 < \epsilon < 1$. Clearly, D is invertible. In computing DED^{-1}, we obtain that

$$DED^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & d_1d_{k+1}^{-1} & \frac{\gamma_1\bar{x}\bar{y}^k d_1 d_{k+2}^{-1}}{\alpha_1} & \frac{\gamma_1\bar{x}\bar{y}^k d_1 d_{k+3}^{-1}}{\alpha_1} & \dots & \frac{\gamma_1\bar{x}\bar{y}^k d_1 d_{2k+1}^{-1}}{\alpha_1} & \frac{\gamma_1\bar{x}\bar{y}^k d_1 d_{2k+2}^{-1}}{\alpha_1} \\ d_2d_1^{-1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & d_{k+1}d_k^{-1} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{\gamma_2\bar{y}\bar{x}^k d_{k+2}d_1^{-1}}{\alpha_2} & \frac{\gamma_2\bar{y}\bar{x}^k d_{k+2}d_2^{-1}}{\alpha_2} & \dots & \frac{\gamma_2\bar{y}\bar{x}^k d_{k+2}d_k^{-1}}{\alpha_2} & \frac{\gamma_2\bar{y}\bar{x}^k d_{k+2}d_{k+1}^{-1}}{\alpha_2} & 0 & 0 & \dots & 0 & d_{k+2}d_{2k+2}^{-1} \\ 0 & 0 & \dots & 0 & 0 & d_{k+3}d_{k+2}^{-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & d_{2k+2}d_{2k+1}^{-1} & 0 \end{pmatrix}. \tag{3}$$

From $d_1 > d_2 > \cdots > d_{k+1} > 0$ and $d_{k+2} > d_{k+3} > \cdots > d_{2k+2} > 0$ it implies that $d_2d_1^{-1} < 1,\ d_3d_2^{-1} < 1,\ \cdots,\ d_{k+1}d_k^{-1} < 1$ and $d_{k+3}d_{k+2}^{-1} < 1,\ d_{k+4}d_{k+3}^{-1} < 1,\ \cdots,\ d_{2k+2}d_{2k+1}^{-1} < 1$. Furthermore,

$$\begin{aligned} d_1d_{k+1}^{-1} + \frac{\gamma_1\bar{x}\bar{y}^k d_1 d_{k+2}^{-1}}{\alpha_1} + \frac{\gamma_1\bar{x}\bar{y}^k d_1 d_{k+3}^{-1}}{\alpha_1} + \cdots + \frac{\gamma_1\bar{x}\bar{y}^k d_1 d_{2k+1}^{-1}}{\alpha_1} + \frac{\gamma_1\bar{x}\bar{y}^k d_1 d_{2k+2}^{-1}}{\alpha_1} &= \\ \frac{1}{1-(k+1)\epsilon} + \frac{\gamma_1\bar{x}\bar{y}^k}{\alpha_1}\left(1 + \frac{1}{1-2\epsilon} + \cdots + \frac{1}{1-k\epsilon} + \frac{1}{1-(k+1)\epsilon}\right) > 1. \end{aligned}$$

Also

$$\begin{aligned} \frac{\gamma_2\bar{y}\bar{x}^k d_{k+2}d_1^{-1}}{\alpha_2} + \frac{\gamma_2\bar{y}\bar{x}^k d_{k+2}d_2^{-1}}{\alpha_2} + \cdots + \frac{\gamma_2\bar{y}\bar{x}^k d_{k+2}d_k^{-1}}{\alpha_2} + \frac{\gamma_2\bar{y}\bar{x}^k d_{k+2}d_{k+1}^{-1}}{\alpha_2} + d_{k+2}d_{2k+2}^{-1} &= \\ \frac{\gamma_2\bar{y}\bar{x}^k}{\alpha_2}\left(1 + \frac{1}{1-2\epsilon} + \cdots + \frac{1}{1-k\epsilon} + \frac{1}{1-(k+1)\epsilon}\right) + \frac{1}{1-(k+1)\epsilon} > 1. \end{aligned}$$

It is well-known fact that E has the same eigenvalues as DED^{-1}. Hence, we obtain

$$\begin{aligned}\max_{1\leq m\leq 2k+2}|\lambda_m| \leq \|DED^{-1}\|_\infty \;&=\; \max\{d_2d_1^{-1},\cdots,d_{k+1}d_k^{-1},d_{k+3}d_{k+2}^{-1},\cdots,d_{2k+2}d_{2k+1}^{-1},\frac{1}{1-(k+1)\epsilon}\\ &+\frac{\gamma_1\bar{x}\bar{y}^k}{\alpha_1}\left(1+\frac{1}{1-2\epsilon}+\cdots+\frac{1}{1-k\epsilon}+\frac{1}{1-(k+1)\epsilon}\right),\frac{1}{1-(k+1)\epsilon}\\ &+\frac{\gamma_2\bar{y}\bar{x}^k}{\alpha_2}\left(1+\frac{1}{1-2\epsilon}+\cdots+\frac{1}{1-k\epsilon}+\frac{1}{1-(k+1)\epsilon}\right)\}>1.\end{aligned}$$

This implies that $A\left(\left(\frac{\beta_2-\alpha_2}{\gamma_2}\right)^{\frac{1}{k+1}},\left(\frac{\beta_1-\alpha_1}{\gamma_1}\right)^{\frac{1}{k+1}}\right)$ of system (1) is unstable. □

3 Periodicity nature and existence of unbounded solutions

In this section, we will study the periodicity nature and existence of unbounded solutions of system (1). Let us denote $a_1=\gamma_1y_{-k}y_{1-k}\cdots y_0,\ a_2=\gamma_2x_{-k}x_{1-k}\cdots x_0$ to study the periodicity nature of positive solution of system (1).

Theorem 1. *If $a_1=\beta_1-\alpha_1$ and $a_2=\beta_2-\alpha_2$, then system (1) has prime period-(k+1) solutions.*

Proof. From system (1) and $a_1=\beta_1-\alpha_1,\ a_2=\beta_2-\alpha_2$, we have

$$x_1=\frac{\alpha_1x_{-k}}{\beta_1-\gamma_1\prod_{i=0}^{k}y_{-i}}=\frac{\alpha_1x_{-k}}{\beta_1-a_1}=x_{-k},\ y_1=\frac{\alpha_2y_{-k}}{\beta_2-\gamma_2\prod_{i=0}^{k}x_{-i}}=\frac{\alpha_2y_{-k}}{\beta_2-a_2}=y_{-k}.$$

$$x_2=\frac{\alpha_1x_{1-k}}{\beta_1-\gamma_1\prod_{i=0}^{k}y_{1-i}}=\frac{\alpha_1x_{1-k}}{\beta_1-\gamma_1y_1y_0y_{-1}\cdots y_{1-k}}=\frac{\alpha_1x_{1-k}}{\beta_1-\gamma_1y_0y_{-1}\cdots y_{1-k}y_{-k}}=\frac{\alpha_1x_{1-k}}{\beta_1-a_1}=x_{1-k},$$

$$y_2=\frac{\alpha_2y_{1-k}}{\beta_2-\gamma_2\prod_{i=0}^{k}x_{1-i}}=\frac{\alpha_2y_{1-k}}{\beta_2-\gamma_2x_1x_0x_{-1}\cdots x_{1-k}}=\frac{\alpha_2y_{1-k}}{\beta_2-\gamma_2x_0x_{-1}\cdots x_{1-k}x_{-k}}=\frac{\alpha_2y_{1-k}}{\beta_2-a_2}=y_{1-k}.$$

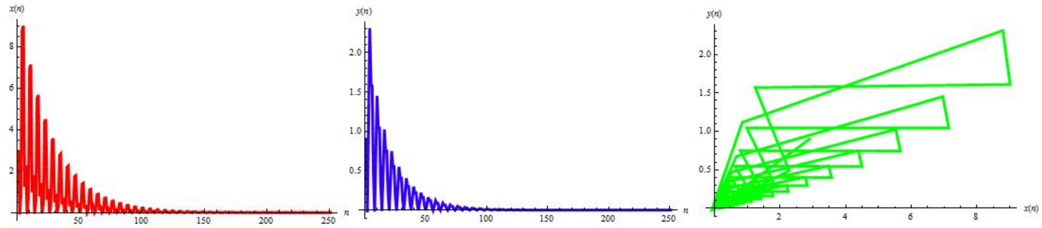
By induction, one has

$$x_{k+2}=\frac{\alpha_1x_1}{\beta_1-\gamma_1\prod_{i=0}^{k}y_{k+1-i}}=\frac{\alpha_1x_1}{\beta_1-\gamma_1y_{k+1}y_ky_{k-1}\cdots y_1}=\frac{\alpha_1x_1}{\beta_1-\gamma_1y_0y_{-1}\cdots y_{1-k}y_{-k}}=\frac{\alpha_1x_1}{\beta_1-a_1}=x_1,$$

$$y_{k+2}=\frac{\alpha_2y_1}{\beta_2-\gamma_2\prod_{i=0}^{k}x_{k+1-i}}=\frac{\alpha_2y_1}{\beta_2-\gamma_2x_{k+1}x_kx_{k-1}\cdots x_1}=\frac{\alpha_2y_1}{\beta_2-\gamma_2x_0x_{-1}\cdots x_{1-k}x_{-k}}=\frac{\alpha_2y_1}{\beta_2-a_2}=y_1.$$

□

Theorem 2. *Assume that $\beta_1<\alpha_1$, $\beta_2<\alpha_2$. Then, every positive solution $\{(x_n,y_n)\}$ of system (1) tends to ∞ as $n\to\infty$.*



(a) (b) (c)

Figure 1: Plots for system (5)

Proof. From system (1), it follows that

$$x_{n+1}=\frac{\alpha_1 x_{n-k}}{\beta_1-\gamma_1\prod_{i=0}^{k} y_{n-i}}\geq\frac{\alpha_1 x_{n-k}}{\beta_1}>x_{n-k},\quad y_{n+1}=\frac{\alpha_2 y_{n-k}}{\beta_2-\gamma_2\prod_{i=0}^{k} x_{n-i}}\geq\frac{\alpha_2 y_{n-k}}{\beta_2}>y_{n-k}. \tag{4}$$

From first equation of (4), we have $x_{(k+1)n+1}>x_{(k+1)n-k}$, and $x_{(k+1)n+(k+2)}>x_{(k+1)n+1}$. Hence, the subsequences $\{x_{(k+1)n+1}\},\cdots,\{x_{(k+1)n+(k+1)}\}$ are increasing, *i.e.*, the sequence $\{x_n\}$ is increasing. So, $x_n\to\infty$ as $n\to\infty$. Similarly, from second equation of (4) one gets: $y_{(k+1)n+1}>y_{(k+1)n-k}$ and $y_{(k+1)n+(k+2)}>y_{(k+1)n+1}$. Hence, the subsequences $\{y_{(k+1)n+1}\},\cdots,\{y_{(k+1)n+(k+1)}\}$ are increasing, *i.e.*, the sequence $\{y_n\}$ is increasing. So, $y_n\to\infty$ as $n\to\infty$. $\square$

4 Numerical simulations

In this section we will present numerical simulations to verify theoretical results.

Example 1. *If* $\alpha_1=50,\beta_1=63,\gamma_1=4,\alpha_2=90,\beta_2=122,\gamma_2=2$ *then system (1) with* $x_{-5}=3.9,x_{-4}=1.5,x_{-3}=12.4,x_{-2}=11.9,x_{-1}=1.6,x_0=2.9,y_{-5}=2.6,y_{-4}=3.8,y_{-3}=5.8,y_{-2}=3.5,y_{-1}=3.1,y_0=0.9$ *can be written as:*

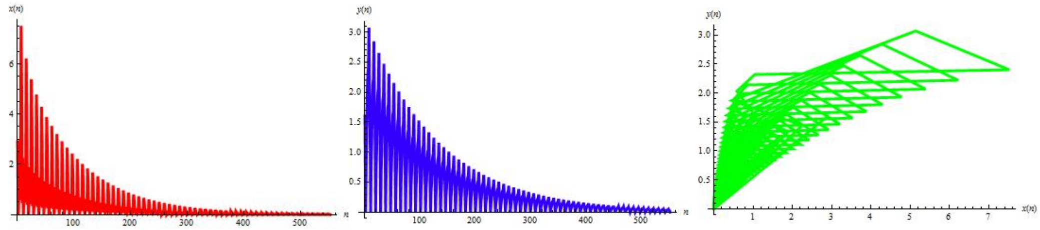
$$x_{n+1}=\frac{50x_{n-5}}{63-4y_ny_{n-1}y_{n-2}y_{n-3}y_{n-4}y_{n-5}},\ y_{n+1}=\frac{90y_{n-5}}{122-2x_nx_{n-1}x_{n-2}x_{n-3}x_{n-4}x_{n-5}}. \tag{5}$$

Moreover, in Fig. 1 the plot of x_n *is shown in Fig. 1a, the plot of* y_n *is shown in Fig. 1b and global attractor of system (5) is shown in Fig. 1c.*

Example 2. *If* $\alpha_1=15.5,\beta_1=17,\gamma_1=27,\alpha_2=11.2,\beta_2=12,\gamma_2=23$*, then system (1) with* $x_{-8}=1.9,x_{-7}=1.7,x_{-6}=2.5,x_{-5}=0.9,x_{-4}=1.5,x_{-3}=10.4,x_{-2}=6.9,x_{-1}=0.6,x_0=2.9,y_{-8}=2.8,y_{-7}=1.6,y_{-6}=1.8,y_{-5}=2.6,y_{-4}=2.8,y_{-3}=2.8,y_{-2}=3.5,y_{-1}=2.1,y_0=1.6$ *can be written as*

$$\begin{aligned} x_{n+1}&=\frac{15.5x_{n-8}}{17-27y_ny_{n-1}y_{n-2}y_{n-3}y_{n-4}y_{n-5}y_{n-6}y_{n-7}y_{n-8}},\\ y_{n+1}&=\frac{11.2y_{n-8}}{12-23x_nx_{n-1}x_{n-2}x_{n-3}x_{n-4}x_{n-5}x_{n-6}x_{n-7}x_{n-8}}. \end{aligned} \tag{6}$$

Moreover, in Fig. 2 the plot of x_n *is shown in Fig. 2a, the plot of* y_n *is shown in Fig. 2b and global attractor of system (6) is shown in Fig. 2c.*



(a) (b) (c)

Figure 2: Plots for system (6)

5 Conclusion and future work

This work is related to the qualitative behavior of a system of higher-order rational difference equations. We have proved that under some restrictions to parameters, system (1) has a boundary equilibrium $O(0,0)$ and the unique positive equilibrium point $A\left(\left(\frac{\beta_2-\alpha_2}{\gamma_2}\right)^{\frac{1}{k+1}}, \left(\frac{\beta_1-\alpha_1}{\gamma_1}\right)^{\frac{1}{k+1}}\right)$ in the closed first quadrant $\mathbb{R}^2_+$. We have analyzed the local stability about equilibria, periodicity nature of positive solutions and existence of unbounded solutions of system (1). Finally, theoretical results are verified numerically. Besides the local properties, the global stability of under consideration system (1), which is our further aim to study.

Acknowledgements

A. Q. Khan research is supported by the Higher Education Commission(HEC) of Pakistan.

References

[1] I. Bajo, E. Liz, Global behaviour of a second-order nonlinear difference equation, *Journal of Difference Equations and Applications*, 17(10)(2011):1471-1486.

[2] M. Aloqeili, Dynamics of a rational difference equation, *Applied Mathematics and Computation*, 176(2)(2006):768-774.

[3] E. A. Grove, G. Ladas, Periodicities in nonlinear difference equations, Chapman and Hall/CRC Press, Boca Raton, (2004).

[4] H. Sedaghat, Nonlinear difference equations:theory with applications to social science models, Kluwer Academic Publishers, Dordrecht, (2003).

[5] V. L. Kocic, G. Ladas, Global behavior of nonlinear difference equations of higher-order with applications, Kluwer Academic Publishers, Dordrecht, (1993).

[6] E. Camouzis, G. Ladas, Dynamics of third-order rational difference equations:with open problems and conjectures, Chapman and Hall/HRC, Boca Raton, (2007).

[7] V. L. Kocic, G. Ladas, Global attractivity in a second order nonlinear difference equations, *Journal of Mathematical Analysis and Applications*, 180(1993):144-150.

ON APPROXIMATING THE GENERALIZED EULER-MASCHERONI CONSTANT*

TI-REN HUANG[1], BO-WEN HAN[2], XIAO-YAN MA[2], AND YU-MING CHU[3,**]

Abstract. In the article, we provide several sharp bounds for the the generalized Euler-Mascheroni constant, which are the generalizations of the previously results on the Euler-Mascheroni constant.

1. Introduction

It is well known that the sequence

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \tag{1.1}$$

is convergent towards the Euler-Mascheroni constant

$$\gamma = 0.57721566490115328\cdots. \tag{1.2}$$

The Euler-Mascheroni constant has been involved in a variety of mathematical formulas and results [1-6], many special functions are closely related to the Euler-Mascheroni constant [7-63]. Recently, the bounds for $\gamma_n - \gamma$ have attracted the attention of many researchers.

Alzer [64] proved that the double inequality

$$\frac{1}{2n+1} \leq \gamma_n - \gamma \leq \frac{1}{2n}$$

holds for $n \geq 1$.

In [65], Tóth proved that the two-sided inequality

$$\frac{1}{2n+\frac{2}{5}} < \gamma_n - \gamma \leq \frac{1}{2n+\frac{1}{3}} \tag{1.3}$$

takes place for $n \geq 1$.

Chen [66] proved that $\alpha = (2\gamma-1)/(1-\gamma)$ and $\beta = 1/3$ are the best possible constants such that the double inequality

$$\frac{1}{2n+\alpha} \leq \gamma_n - \gamma < \frac{1}{2n+\beta} \tag{1.4}$$

holds for $n \geq 1$.

2010 *Mathematics Subject Classification.* Primary: 11Y60; Secondary: 40A05, 33B15.

Key words and phrases. Euler-Mascheroni constant, gamma function, psi function, asymptotic expansion.

*The research was supported by the Natural Science Foundation of China (Grants Nos. 61673169, 11301127, 11701176, 11626101, 11401531, 11601485), the Science and Technology Research Program of Zhejiang Educational Committee (Grant no. Y201635325), the Natural Science Foundation of Zhejiang Province (Grant No. LQ17A010010) and the Science Foundation of Zhejiang Sci-Tech University (Grant No. 14062093-Y).

**Corresponding author: Yu-Ming Chu, Email: chuyuming2005@126.com.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

In [67], Qiu and Vuorinen proved that the double inequality

$$\frac{1}{2n}-\frac{\lambda}{n^2}<\gamma_n-\gamma\leq\frac{1}{2n}-\frac{\mu}{n^2} \tag{1.5}$$

holds for $n\geq 1$ if and only if $\lambda\geq 1/12$ and $\mu\leq\gamma-1/2$.

Let $a>0$. Then the generalized Euler-Mascheroni constant $\gamma(a)$ is defined by

$$\gamma(a)=\lim_{n\to\infty}\left(\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\log\frac{a+n-1}{a}\right), \tag{1.6}$$

which was introduced by Knopp [68]. We clearly see that $\gamma(1)=\gamma$. Recently, the generalized Euler-Mascheroni constant $\gamma(a)$ has been the subject of intensive research [69-71].

In [70], Sîntămărian introduced the sequences

$$x_n=\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\log\frac{a+n}{a}, \tag{1.7}$$

$$y_n=\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\log\frac{a+n-1}{a}, \tag{1.8}$$

and proved that the double inequalities

$$\frac{1}{2(n+a)}\leq\gamma(a)-x_n\leq\frac{1}{2(n+a-1)}, \tag{1.9}$$

$$\frac{1}{2(n+a)}\leq y_n-\gamma(a)\leq\frac{1}{2(n+a-1)} \tag{1.10}$$

hold for $n\geq 1$.

In [71], Berinde and Mortici established Theorems 1.1 and 1.2 as follows.

Theorem 1.1. *The double inequalities*

$$\frac{1}{2(n+a)-\frac{1}{4}}<\gamma(a)-x_n<\frac{1}{2(n+a)-\frac{1}{3}}, \tag{1.11}$$

$$\frac{1}{2(n+a)-\frac{4}{3}}<y_n-\gamma(a)<\frac{1}{2(n+a)-\frac{5}{3}} \tag{1.12}$$

hold for $a>0$ and $n\geq 2$.

Theorem 1.2. *(a) The inequality*

$$\frac{1}{2(n+a)-\frac{1}{3}+\frac{1}{18n}}\leq\gamma(a)-x_n \tag{1.13}$$

holds for $a\geq 13/30$ and any integer $n\geq 1$.

(b) The inequality

$$\frac{1}{2(n+a)-\frac{5}{3}+\frac{1}{18n}}\leq y_n-\gamma(a) \tag{1.14}$$

holds for $a\geq 17/30$ and $n\geq 1$.

The main purpose of this article is to generalize inequalities (1.4) and (1.5) to the generalized Euler-Mascheroni constant $\gamma(a)$. Our main results are the following Theorems 1.3 and 1.4.

Theorem 1.3. *Let $a > 0$, $n \geq 1$. Then one has*
(1) *the double inequality*

$$\frac{1}{2(n+a)-\alpha_1} \leq \gamma(a) - x_n < \frac{1}{2(n+a)-\beta_1} \tag{1.15}$$

holds with the best possible constants

$$\alpha_1 = 2(1+a) - \frac{1}{\psi(1+a)-\log(1+a)}, \quad \beta_1 = \frac{1}{3}; \tag{1.16}$$

(2) *the two-sided inequality*

$$\frac{1}{2(n+a)-\alpha_2} \leq y_n - \gamma(a) < \frac{1}{2(n+a)-\beta_2} \tag{1.17}$$

is valid with the best possible constants

$$\alpha_2 = 2(1-d), \qquad \beta_2 = \frac{5}{3}, \tag{1.18}$$

where

$$d = \max\{\tilde{f}_2(a), \tilde{f}_2(1+a), \tilde{f}_2(2+a)\}, \quad \tilde{f}_2(x) = \frac{1}{2(\psi(x+1)-\log(x))} - x.$$

Theorem 1.4. *Let $a > 0$, $n \geq 1$. Then the double inequalities*

$$\frac{1}{2(n+a)} + \frac{\alpha_3}{(n+a)^2} \leq \gamma(a) - x_n < \frac{1}{2(n+a)} + \frac{\beta_3}{(n+a)^2}, \tag{1.19}$$

$$\frac{1}{2(n+a-1)} + \frac{\alpha_4}{(n+a-1)^2} < y_n - \gamma(a) \leq \frac{1}{2(n+a-1)} + \frac{\beta_4}{(n+a-1)^2} \tag{1.20}$$

hold with the best possible constants

$$\alpha_3 = (1+a)^2[\log(1+a) - \psi(1+a)] - \frac{1+a}{2}, \quad \beta_3 = \frac{1}{12}, \tag{1.21}$$

$$\alpha_4 = -\frac{1}{12}, \quad \beta_4 = a^2[\psi(a) - \log(a)] + \frac{a}{2}. \tag{1.22}$$

2. Lemmas

In order to prove our main results, we need the following formulas and lemmas.

For $x > 0$, the classical gamma function $\Gamma(x)$ and psi function $\psi(x)$ [72-84] are defined as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively.

The psi function $\psi(x)$ has the following recurrence and asymptotic formulas [85]

$$\psi(n+x) = \frac{1}{(n-1)+x} + \frac{1}{(n-2)+x} + \cdots + \frac{1}{2+x} + \frac{1}{1+x} + \frac{1}{x} + \psi(x), \tag{2.1}$$

$$\psi(x) \sim \log(x) - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \cdots \quad (x \to \infty) \tag{2.2}$$

According to (2.1) and the definitions of x_n and y_n given in (1.7) and (1.8), we clearly see that x_n and y_n can be rewritten as

$$x_n = \psi(n+a) - \psi(a) - \log\frac{n+a}{a}, \tag{2.3}$$

$$y_n = \psi(n+a) - \psi(a) - \log\frac{n+a-1}{a}. \tag{2.4}$$

It follows from (1.6) and (2.2) that

$$\gamma(a) = \lim_{n\to\infty} y_n \tag{2.5}$$

$$= \lim_{n\to\infty}\left(\psi(n+a) - \log(n+a-1) + \log(a) - \psi(a)\right) = \log(a) - \psi(a).$$

Therefore,

$$\gamma(a) - x_n = \log(n+a) - \psi(n+a), \tag{2.6}$$

$$y_n - \gamma(a) = \psi(n+a) - \log(n+a-1). \tag{2.7}$$

Lemma 2.1. *The function*

$$f_1(x) = \frac{1}{\log(x) - \psi(x)} - 2x \tag{2.8}$$

is strictly decreasing from $(1,\infty)$ *onto* $(-1/3, 1/\gamma - 2)$.

The function

$$f_2(x) = \frac{1}{\psi(x+1) - \log(x)} - 2x \tag{2.9}$$

is strictly decreasing from $[2,\infty)$ *onto* $(1/3, f_2(2)]$.

Proof. Differentiating $f_1(x)$ gives

$$\left(\log(x) - \psi(x)\right)^2 f_1'(x) = \psi'(x) - \frac{1}{x} - 2(\log(x) - \psi(x))^2.$$

It follows from the inequalities

$$\psi'(x) - \frac{1}{x} < \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7},$$
$$\log(x) - \psi(x) > \frac{1}{2x} + \frac{1}{12x^2} - \frac{1}{120x^4}$$

given in [86] that

$$\left(\log(x) - \psi(x)\right)^2 f_1'(x) < \frac{1}{50400x^8}F_1(x), \tag{2.10}$$

where

$$F_1(x) = -207 - 3840(x-1) - 6580(x-1)^2 - 3640(x-1)^3 - 700(x-1)^4 < 0 \tag{2.11}$$

for $x \in (1,\infty)$.

Therefore, the monotonicity of $f_1(x)$ follows easily from (2.10) and (2.11).

Clearly, $f_1(1) = 1/\gamma - 2$. The limiting value $\lim_{x\to\infty} f_1(x) = -1/3$ follows from the asymptotic formula (2.2).

Differentiating $f_2(x)$ leads to

$$2\left(\psi(x+1) - \log(x)\right)^2 f_2'(x) = \frac{1}{x} + \frac{1}{x^2} - \psi'(x) - 2(\psi(x) + \frac{1}{x} - \log(x))^2.$$

It follows from the inequalities

$$\frac{1}{x}+\frac{1}{x^2}-\psi'(x)<\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5},$$
$$\psi(x)+\frac{1}{x}-\log(x)>\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}-\frac{1}{252x^6}$$

for $x>0$ given in [86] that

$$2\left(\psi(x+1)-\ln(x)\right)^2 f_2'(x)<-\frac{F_2(x)}{3175200x^{12}},$$

where

$$\begin{aligned}F_2(x)&=3217636+17887632(x-2)+39443124(x-2)^2\\&+47009928(x-2)^3+33797841(x-2)^4+15180480(x-2)^5\\&+4189500(x-2)^6+652680(x-2)^7+44100(x-2)^8>0\end{aligned}$$

for $x\geq 2$.

Therefore, $f_2(x)$ is a strictly decreasing function on $[2,\infty)$. The limit $\lim_{x\to\infty} f_2(x)=1/6$ follows from the asymptotic formula (2.2). □

Remark 1. *Qi et. al. [87] proved that the function $f_2(x)$ defined by (2.9) is strictly decreasing on $(12/5,\infty)$.*

The following Lemma 2.2 can be found in [88, 89].

Lemma 2.2. *The function*

$$f_3(x)=x^2(\psi(x)-\log(x))+\frac{x}{2} \tag{2.12}$$

is strictly decreasing from $(0,\infty)$ onto $(-1/12,0)$ and completely monotonic on $(0,\infty)$.

3. Proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3. From (2.6) we clearly see that inequality (1.15) can be rewritten as

$$-\beta<\frac{1}{\log(n+a)-\psi(n+a)}-2(n+a)<-\alpha.$$

It follows from Lemma 2.1 that the sequence

$$f_1(n+a)=\frac{1}{\log(n+a)-\psi(n+a)}-2(n+a)$$

is strictly decreasing, which leads to the conclusion that

$$-\frac{1}{3}=\lim_{n\to\infty} f_1(n)<f_1(n)\leq f_1(1)=\frac{1}{\log(1+a)-\psi(1+a)}-2(1+a).$$

Therefore,

$$\alpha_1=2(1+a)-\frac{1}{\psi(1+a)-\log(1+a)},\quad \beta_1=\frac{1}{3}$$

are the best possible constants such that inequality (1.15) holds.

From (2.7) we clearly see that inequality (1.17) is equivalent to

$$1-\frac{\beta}{2}<\frac{1}{2(\psi(n+a)-\log(n+a-1))}-(n+a-1)\leq 1-\frac{\alpha}{2}.$$

It follows from Lemma 2.1 that the sequence

$$\tilde{f}_2(n+a-1) = \frac{1}{2(\psi(n+a) - \log(n+a-1))} - (n+a-1)$$

is strictly decreasing for $n \geq 2$, which leads to the conclusion that

$$\frac{1}{6} = \lim_{n\to\infty} \tilde{f}_2(n) < \tilde{f}_2(n) \leq \max\left\{\tilde{f}_2(a), \tilde{f}_2(1+a), \tilde{f}_2(2+a)\right\} = d.$$

Therefore,

$$\alpha_2 = 2(1-d), \qquad \beta_2 = \frac{5}{3} \tag{3.1}$$

are the best possible constants such that inequality (1.17) holds.

Proof of Theorem 1.4. From (2.6) and (2.7) we know that inequalities (1.19) and (1.20) can be rewritten as

$$\alpha_3 \leq (n+a)^2 \left(\log(n+a) - \psi(n+a)\right) - \frac{(n+a)}{2} < \beta_3,$$

$$\alpha_4 < (n+a-1)^2 \left(\psi(n+a-1) - \log(n+a-1)\right) + \frac{(n+a-1)}{2} \leq \beta_4,$$

respectively.

It follows from Lemma 2.2 that the sequence

$$\tilde{f}_3(n+a-1) = (n+a-1)^2 \left(\psi(n+a-1) - \log(n+a-1)\right) + \frac{(n+a-1)}{2}$$

is strictly decreasing for $n \in \mathbb{N}$.

Note that

$$\lim_{n\to\infty} f_3(n) = -\frac{1}{12}.$$

Therefore,

$$\alpha_3 = (1+a)^2[\log(1+a) - \psi(1+a)] - \frac{1+a}{2}, \quad \beta_3 = \frac{1}{12},$$

$$\alpha_4 = -\frac{1}{12}, \qquad \beta_4 = (a)^2[\psi(a) - \log(a)] + \frac{a}{2}$$

are the best possible constants such that inequalities (1.19) and (1.20) hold.

Remark 2. (1) *Let $a = 1$. Then Theorem 1.3(2) leads to inequality (1.4) with the best possible constants $\alpha = (2\gamma - 1)/(1-\gamma)$ and $\beta = 1/3$.*

(2) *Let $a = 1$. Then inequality (1.20) becomes inequality (1.5) with the best possible constants $\alpha = 1/12$ and $\beta = \gamma - 1/2$.*

(3) *From Theorem 1.3 we know that both the upper bounds $1/[2(n+a) - 1/3]$ for $\gamma(a) - x_n$ and $1/[2(n+a) - 5/3]$ for $y_n - \gamma(a)$ given in (1.11) and (1.12) are sharp for any $a > 0$.*

References

[1] P. A. Panzone, Formulas for the Euler-Mascheroni constant, Rev. Un. Mat. Argentina, 2009, **50**(1), 161–164.

[2] J. Choi and H. M. Srivastava, Integral representations for the Euler-Mascheroni constant γ, Integral Transforms Spec. Funct., 2010, **21**(9-10), 675–690.

[3] T. Burić and N. Elezović, Approximants of the Euler-Mascheroni constant and harmonic numbers, Appl. Math. Comput., 2013, **222**, 604–611.

[4] P. A. Panzone, An approximation formula for Euler-Mascheroni constant, Rev. Un. Mat. Argentina, 2016, **57**(1), 9–22.

[5] T.-R. Huang, B.-W. Han, X.-Y. Ma and Y.-M. Chu, Optimal bounds for the generalized Euler-Mascheroni constant, J. Inequal. Appl., 2018, **2018**, Article 118, 9 pages.

[6] T.-R. Huang, S.-Y. Tan, X.-Y. Ma and Y.-M. Chu, Monotonicity properties and bounds for the complete p-elliptic integrals, J. Inequal. Appl., 2018, **2018**, Article 239, 11 pages.

[7] J. C. Sampedro, On a new constant related to Euler's constant, Ramanujan J., 2018, **46**(1), 77–89.

[8] N. Kurokawa and Y. Taguchi, A p-analogue of Euler's constant and congruence zeta functions, Proc. Japan Acad Ser. A Math. Sci., 2018, **94**(2), 13–16.

[9] O. Furdni, Multiple fractional part integrals and Euler's constant, Miskolc Math. Notes, 2016, **17**(1), 255–266.

[10] A. W. Addison, A series representation for Euler's constant, Amer. Math. Monthly, 1967, **74**, 823–824.

[11] W.-D. Jiang, M.-K. Wang, Y.-M. Chu, Y.-P. Jiang and F. Qi, Convexity of the generalized sine function and the generalized hyperbolic sine function, J. Approx. Theory, 2013, **174**, 1–9.

[12] M.-K. Wang, Y.-M. Chu, S.-L. Qiu and Y.-P. Jiang, Convexity of the complete elliptic integrals of the first kind with respect to Hölder means, J. Math. Anal. Appl., 2012, **388**(2), 1141–1146.

[13] Y.-M. Chu, M.-K. Wang, Y.-P. Jiang and S.-L. Qiu, Monotonicity, convexity, and inequalities invoving the Agard distortion function, Abstr. Appl. Anal., 2011, **2011**, Article ID 671765, 8 pages.

[14] X.-M. Zhang and Y.-M. Chu, Convexity of the integral arithmetic mean of a convex function, Rocky Mountain J. Math., 2010, **40**(3), 1061–1068.

[15] Y.-M. Chu and T.-H. Zhao, Concavity of the error functions with respect to Hölder means, Math. Inequal. Appl., 2016, **19**(2), 589–595.

[16] Y.-M. Chu, M.-K. Wang, Y.-P. Jiang and S.-L. Qiu, Concavity of the complete elliptic integrals of the second kind with respect to Hölder means, J. Math. Anal. Appl., 2012, **395**(2), 637–642.

[17] Y.-M. Chu and X.-M. Zhang, Multiplicative concavity of the integral of multiplicatively concave functions, J. Inequal. Appl., 2010, **2010**, Article ID 845930, 8 pages.

[18] X.-M. Zhang, Y.-M. Chu and X.-H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its applications, J. Inequal. Appl., 2010, **2010**, Article ID 507560, 11 pages.

[19] Y.-M. Chu, G.-D. Wang and X.-H. Zhang, Schur convexity and Hadamard's inequality, Math. Inequal. Appl., 2010, **13**(4), 725–731.

[20] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the error function, Math. Inequal. Appl., 2018, **21**(2), 469–479.

[21] Zh.-H. Yang, W. Zhang and Y.-M. Chu, Sharp Gautschi inequality for parameter $0 < p < 1$ with applications, Math. Inequal. Appl., 2017, **20**(4), 1107–1120.

[22] Zh.-H. Yang, Y.-M. Chu and W. Zhang, Accurate approximations for the complete elliptic of the second kind, J. Math. Anal. Appl., 2016, **438**(2), 875–888.

[23] Y.-M. Chu, M.-K. Wang, S.-L. Qiu and Y.-P. Jiang, Bounds for complete elliptic integrals of the second kind with applications, Comput. Math. Appl., 2012, **63**(7), 1177–1184.

[24] M.-K. Wang, S.-L. Qiu, Y.-M. Chu and Y.-P. Jiang, Generalized Hersch-Pfluger distortion function and complete elliptic integrals, J. Math. Anal. Appl., 2012, **385**(1), 221–229.

[25] M.-K. Wang, Y.-M. Li and Y.-M. Chu, Inequalities and infinite product formula for Ramanujan generalized modular equation function, Ramanujan J., 2018, **46**(1), 189–200.

[26] M.-K. Wang, Y.-M. Chu and Y.-Q. Song, Asymptotical formulas for Gaussian and generalized hypergeometric functions, Appl. Math. Comput., 2016, **276**, 44–60.

[27] M.-K. Wang, Y.-M. Chu and Y.-P. Jiang, Ramanujan's cubic transformaation inequalities for zero-balanced hypergeometric functions, Rocky Mountain J. Math., 2016, **46**(2), 679–691.

[28] Y.-M. Chu, M.-K. Wang and Y.-F. Qiu, On Alzer and Qiu's conjecture for complete elliptic integral and inverse hyperbolic tangent function, Abstr. Appl. Anal., 2011, **2011**, Article ID 697547, 7 pages.

[29] M.-K. Wang and Y.-M. Chu, Asymptotical bounds for complete elliptic integrals of the second kind, J. Math. Anal. Appl., 2013, **402**(1), 119–126.

[30] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, Hölder mean inequalities for the complete elliptic integrals, Integral Transforms Spec. Funct., 2012, **23**(7), 521–527.

[31] Y.-M. Chu, M.-K. Wang and S.-L. Qiu, Optimal combinations bounds of root-square and arithmetic means for Toader mean, Proc. Indian Acad. Sci. Math. Sci., 2012, **122**(1), 41–51.

[32] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind, J. Math. Anal. Appl., 2018, **462**(2), 1714–1726.

[33] W.-M. Qian and Y.-M. Chu, Sharp bounds for a special quasi-arithmetic mean in terms of arithmetic and geometric means with two parameters, J. Inequal. Appl., 2017, **2017**, Article 274, 10 pages.

[34] Y.-M. Chu and M.-K. Wang, Optimal Lehmer mean bounds for the Toader mean, Results Math., 2012, **61**(3-4), 223–229.

[35] M.-K. Wang, Y.-M. Chu, Y.-F. Qiu and S.-L. Qiu, An optimal power mean inequality for the complete elliptic integrals, Appl. Math. Lett., 2011, **24**(6), 887–890.

[36] Zh.-H. Yang, W.-M. Qian and Y.-M. Chu, Monotonicity properties and bounds involving the complete elliptic integrals of the first kind, Math. Inequal. Appl., 2018, **21**(4), 1185–1199.

[37] T.-H. Zhao, M.-K. Wang, W. Zhang and Y.-M. Chu, Quadratic transformation inequalities for Gaussian hypergeometric function, J. Inequal. Appl., 2018, **2018**, Article 251, 15 pages.

[38] M.-K. Wang, S.-L. Qiu and Y.-M. Chu, Infinite series formula for Hübner upper bound function with applications to Hersch-Pfluger distortion function, Math. Inequal. Appl., 2018, **21**(3), 629–648.

[39] M.-K. Wang and Y.-M. Chu, Landen inequalities for a class of hypergeometric functions with applications, Math. Inequal. Appl., 2018, **21**(2), 521–537.

[40] Zh.-H. Yang and Y.-M. Chu, A monotonicity property involving the generalized elliptic integral of the first kind, Math. Inequal. Appl., 2017, **20**(3), 729–735.

[41] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, Monotonicity rule for the quotient of two function and its applications, J. Inequal. Appl., 2017, **2017**, Article 106, 13 pages.

[42] H.-Z. Xu, Y.-M. Chu and W.-M. Qian, Sharp bounds for the Sándor-Yang means in terms of arithmetic and contra-harmonic means, J. Inequal. Appl., 2018, **2018**, Article 127, 13 pages.

[43] W.-M. Gong, Y.-Q. Song, M.-K. Wang and Y.-M. Chu, A sharp double inequality between Seiffert, arithmetic, and geometric means, Abstr. Appl. Anal., 2012, **2012**, Article ID 684834, 7 pages.

[44] Y.-M. Chu and M.-K. Wang, Inequalities between arithmetic-geometric, Gini, and Toader means, Abstr. Appl. Anal., 2012, **2012**, Article ID 830585, 11 pages.

[45] Y.-M. Chu, Y.-F. Qiu, M.-K. Wang and G.-D. Wang, The optimal convex combination bounds of arithmetic and harmonic means for the Seiffert's mean, J. Inequal. Appl., 2010, **2010**, Article ID 436457, 7 pages.

[46] W.-F. Xia, Y.-M. Chu and G.-D. Wang, The optimal upper and lower power mean bounds for a convex combiantion of the arithmetic and logarithmic means, Abstr. Appl. Anal., 2010, **2010**, Article ID 604804, 9 pages.

[47] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, Best possible inequalities among harmonic, geometric, logarithmic and Seiffert means, Math. Inequal. Appl., 2012, **15**(2), 415–422.

[48] Y.-M. Chu, S.-S. Wang and C. Zong, Optimal lower power mean bound for the convex combination of harmonic and logarithmic means, Abstr. Appl. Anal., 2011, **2011**, Article ID 520648, 9 pages.

[49] Y.-F. Qiu, M.-K. Wang, Y.-M. Chu and G.-D. Wang, Two sharp inequalities for Lehmer mean, identric mean and logarithmic mean, J. Math. Inequal., 2011, **5**(3), 301–306.

[50] Y.-M. Chu and W.-F. Xia, Two double inequalities between power mean and logarithmic mean, Comput. Math. Appl., 2010, **60**(1), 83–89.

[51] Y.-M. Chu and B.-Y. Long, Best possible inequalities between generalized logarithmic mean and classical means, Abstr. Appl. Anal., 2010, **2010**, Article ID 303286, 13 pages.
[52] W.-M. Qian and Y.-M. Chu, Best possible bounds for Yang mean using generalized logarithmic mean, Math. Probl. Eng., 2016, **2016**, Article ID 8901258, 7 pages.
[53] Y.-M. Li, B.-Y. Long and Y.-M. Chu, Sharp bounds for the Neuman-Sándor mean in terms of generalized logarithmic mean, J. Math. Inequal., 2012, **6**(4), 567–577.
[54] Y.-M. Chu, M.-K. Wang and G.-D. Wang, The optimal generalized logarithmic mean bounds for Seiffert's mean, Acta Math. Sci., 2012, **32B**(4), 1619–1626.
[55] Y.-M. Chu, Y.-M. Li, W.-F. Xia and X.-H. Zhang, Best possible inequalities for the harmonic mean of error function, J. Inequal. Appl., 2014, **2014**, Article 525, 9 pages.
[56] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, A best-possible double inequality between Seiffert and harmonic means, J. Inequal. Appl., 2011, **2011**, Article 94, 7 pages.
[57] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, A sharp double inequality between harmonic and identric means, Abstr. Appl. Anal., 2011, **2011**, Article ID 657935, 7 pages.
[58] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, Sharp power mean bounds for the combination of Seiffert and geoemtric means, Abstr. Appl. Anal., 2010, **2010**, Article ID 108920, 12 pages.
[59] B.-Y. Long and Y.-M. Chu, Optimal power mean bounds for the weighted geometric mean of classical means, J. Inequal. Appl., 2010, **2010**, Article ID 905697, 6 pages.
[60] M.-K. Wang, Z.-K. Wang and Y.-M. Chu, An double inequality between geometric and identric means, Appl. Math. Lett., 2012, **25**(3), 471–475.
[61] Y.-M. Chu, B.-Y. Long and B.-Y. Liu, Bounds of the Neuman-Sándor mean using power and identric means, Abstr. Appl. Anal., 2013, **2013**, Article ID 832591, 6 pages.
[62] M.-K. Wang, Y.-M. Chu and Y.-F. Qiu, Some comparison inequalities for generalized Muirhead and identric means, J. Inequal. Appl., 2010, **2010**, Article ID 295620, 10 pages.
[63] W.-M. Qian, X.-H. Zhang and Y.-M. Chu, Sharp bounds for the Toader-Qi mean in terms of harmonic and geometric means, J. Math. Inequal., 2017, **11**(1), 121–127.
[64] H. Alzer, Inequalities for the gamma and polygamma functions, Abh. Math. Sem. Univ. Hamburg, 1998, **68**, 363–372.
[65] L. Tóth, Problem E3432, Amer. Math. Monthly, 1991, **98**(3), 264–264.
[66] C.-P. Chen, Inequalities for the Euler-Mascheroni constant, Appl. Math. Lett., 2010, **23**(2), 161–164.
[67] S.-L. Qiu and M. Vuorinen, Some properties of the gamma and psi functions, with applications, Math. Comput., 2005, **74**(250), 723–742.
[68] K. Knopp, Theory and Applications of Infinite Series, Dover Publications, New York, 1990.
[69] C. Mortici, Improved convergence towards generalized Euler-Mascheroni constant, Appl. Math. Comput., 2010, **215**(9), 3443-3448.
[70] A. Sîntămărian, A generalization of Euler's constant, Numer. Algorithms, 2007, **46**(2), 141–151.
[71] V. Berinde and C. Mortici, New sharp estimates of the generalized Euler-Mascheroni constant, Math. Inequal. Appl., 2013, **16**(1), 279–288.
[72] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On rational bounds for the gamma function, J. Inequal. Appl., 2017, **2017**, Article 210, 17 pages.
[73] T.-H. Zhao and Y.-M. Chu, A class of logarithmically completely monotonic functions associated with gamma function, J. Inequal. Appl., 2010, **2010**, Article ID 392431, 11 pages.
[74] T.-H. Zhao, Y.-M. Chu and H. Wang, Logarithmically complete monotonicity properties relating to the gamma function, Abstr. Appl. Anal., 2011, **2011**, Article ID 896483, 13 pages.
[75] M. Adil Khan, A. Iqbal, M. Suleman and Y.-M. Chu, Hermite-Hadamard typ inequalities for fractional integrals via Green's function, J. Inequal. Appl., 2018, **2018**, Article 161, 15 pages.
[76] M. Adil Khan, Y.-M. Chu, A. Kashuri, R. Liko and G. Ali, Conformable fractional integrals version of Hermite-Hadamard inequalities and their applications, J. Funct. Spaces, 2018, **2018**, Article ID 6928130, 9 pages.
[77] M. Adil Khan, Y. Khurshid, T.-S. Du and Y.-M. Chu, Generalization of Hermite-Hadamard type inequalities via conformable fractional integrals, J. Funct. Spaces, 2018, **2018**, Article ID 5357463, 12 pages.
[78] A. Iqbal, M. Adil Khan, S. Ullah, Y.-M. Chu and A. Kashuri, Hermite-Hadamard type inequalities pertaining conformable fractional integrals and their applications, AIP Advances, 2018, **8**, Article ID 075101, 18 pages, DOI: 10.1063/1.5031954.

[79] M. Adil Khan, Z. M. Al-sahwi and Y.-M. Chu, New estimations for Shannon and Zipf-Mandelbrot entropies, Entropy, 2018, **20**, Article 608, 10 pages, DOI: 10.3390/e2008608.

[80] M. Adil Khan, Y.-M. Chu, A. Kashuri and R. Liko, Hermite-Hadamard type fractional integral inequalities for $\mathrm{MT}_{(r;g,m,\varphi)}$-preinvex functions, J. Comput. Anal. Appl., 2019, **26**(8), 1487–1503.

[81] M. Adil Khan, Y.-M. Chu, T. U. Khan and J. Khan, Some new inequalities of Hermite-Hadamard type for s-convex functions with applications, Open Math., 2017, **15**, 1414–1430.

[82] Y.-M. Chu, M. Adil khan, T. Ali and S. S. Dragomir, Inequalities for α-fractional differentiable functions, J. Inequal. Appl, 2017, **2017**, Article 93, 12 pages.

[83] Y.-Q. Song, M. Adil Khan, S. Zaheer Ullah and Y.-M. Chu, Integral inequalities involving strongly convex functions, J. Funct. Spaces, 2018, **2018**, Article ID 6595921, 8 pages.

[84] M. Adil Khan, S. Begum, Y. Khurshid and Y.-M. Chu, Ostrowski type inequalities involving conformable fractional integrals, J. Inequal. Appl., 2018, **2018**, Article 70, 14 pages.

[85] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, U. S. Government Printing Office, Washington, 1964.

[86] C.-P. Chen and F. Qi, The best lower and upper bounds of harmonic sequence, RGMIA Res. Rep. Coll., 2003, **6**(2), Article 14, 5 pages.

[87] F. Qi, R.-Q. Cui, C.-P. Chen and B.-N. Guo, Some completely monotonic functions involving polygamma functions and an applications, J. Math. Anal. Appl., 2005, **310**(1), 303–308.

[88] S.-L. Qiu and M. Vuorinen, Some properties of the gamma and psi functions, with applications, Math. Comput., 2005, **74**(250), 723–742.

[89] C.-P. Chen, F. Qi and H. M. Srivastava, Some properties of functions related to the gamma and psi functions, Integral Transforms Spec. Funct., 2010, **21**(1-2), 153–166.

Ti-Ren Huang, College of Science, Hunan City University, Yiyang 413000, China
E-mail address: huangtiren@163.com

Bo-Wen Han, Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, China
E-mail address: 2648440793@qq.com

Xiao-Yan Ma, Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, China
E-mail address: mxymath@126.com

Yu-Ming Chu (Corresponding author), Department of Mathematics, Huzhou University, Huzhou 313000, China
E-mail address: chuyuming2005@126.com

General study on Volterra integral equations of the second kind in space with weight function

M. E. Nasr [1,2] and M. F. Jabbar[1]

[1]Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt

[2]Department of Mathematics, Collage of Science and Arts -Al Qurayyat, Jouf University, Kingdom of Saudi Arabia

mohamed.naser@fsc.bu.edu.eg[1], and masar.lefta@gmail.com[1]

Abstract

This paper is devoted to present a new and simple algorithm to prove that the function $\varphi_n(x)$ is a good approximation to the solution $\varphi(x)$ for Volterra integral equations (VIEs) of the second kind in the space $L^2_{p(x)}[0, 2\pi]$ with weight function $p(x)$. This approximation is discussed in details with help of the Valleé-Poussin's and Fèjer's, operators. Special attention is given to study the convergence analysis and estimation of an upper bound for the error of the approximated solution.

Key-Words: Volterra integral equations; Valleé-Poussin's and Fèjer's operators; Convergence analysis;

1. Introduction

In this paper, we present the approximate solution for Volterra integral equations (VIEs) of the second kind in the space $L^2_{p(x)}[0, 2\pi]$ with weight function $p(x) \geq 1$ where $p(x)$ is a summable function on $[0, 2\pi]$

$$\varphi(x) = f(x) + \lambda \int_0^x k(x, y)\varphi(y)dy, \qquad 0 \leq x, y \leq 2\pi, \tag{1}$$

where the functions $f(x)$, $k(x, y)$ belong to $L^2_{p(x)}[0, 2\pi]$ and are 2π-periodic functions, $\frac{1}{\lambda}$ is a regular value of the kernel $k(x, y)$ and the kernel $k(x, y)$ satisfies the following conditions

1. $\{\int_0^x p(y)|k(x, y)|^2 dy\}^{\frac{1}{2}} = \chi(x) \in L^2_{p(x)}[0, 2\pi]$;
2. $|\lambda| \|k(x, y)\|_{L^2_p} < 1$,

where

$$\|k(x, y)\|_{L^2_p} = \|k(x, y)\|_{L^2_{p(x)}[0,2\pi]} = \left[\int_0^{2\pi} \int_0^x p(x)p(y)|k(x, y)|^2 dydx\right]^{\frac{1}{2}}.$$

The simplicity of finding a solution for Fredholm integral equations (FIEs) of the second kind with degenerate kernel naturally leads one to think of replacing the given equation (1) by FIE with degenerate kernel, see [1, 2, 8, 9]. The solution of the new equation is taken as an approximate solution of the original equation. The study employs Dzyadyk's method which is based on the linear polynomial operator ([3]-[5]).

Eq.(1) can be written in the new form

$$\varphi(x) = f(x) + \lambda \int_0^x \tilde{k}(x, y)\varphi(y)dy, \tag{2}$$

where

$$\tilde{k}(x,y) = e(x,y)k(x,y), \qquad e(x,y) = \begin{cases} 1, & \text{for} \quad y \le x, \\ 0, & \text{for} \quad y > x. \end{cases} \tag{3}$$

From (3), it is found that the kernel $\tilde{k}(x,y)$ in (2) satisfies the following conditions (A^*)

1. $\{\int_0^{2\pi} p(y)|\tilde{k}(x,y)|^2 dy\}^{\frac{1}{2}} = \rho(x) \in L^2_{p(x)}[0,2\pi]$;

2. $|\lambda| \|\tilde{k}(x,y)\|_{L^2_p} < 1$,

where

$$\|\tilde{k}(x,y)\|_{L^2_p[0,2\pi]} = \left[\int_0^{2\pi}\int_0^{2\pi} p(x)p(y)|\tilde{k}(x,y)|^2 dydx\right]^{\frac{1}{2}}.$$

Now, instead of Eqs.(1) and (2), let us solve the following equations

$$\varphi_n(x) = U_n(f;x) + \lambda \int_0^{2\pi} U_n[\tilde{k}(.,y);x]\varphi_n(y)dy, \qquad 0 \le x, y \le 2\pi, \tag{4}$$

The notation $U_n[\tilde{k}(.,y);x]$ will mean that the operator U_n acts on $\tilde{k}(x,y)$ as a function of x and at the same time, the variable y plays the role of the parameter.

Now, since the functions $U_n(f;x)$ and $U_n[\tilde{k}(.,y);x]$ are both trigonometric polynomials of order n with respect to x, the solution $\varphi_n(x)$ of the Eq.(4) will also be trigonometric polynomial of order n in x. It is well known that the problem of determination of the solution of Fredholm integral equation of the second kind with degenerate kernel is reduced to the solution of corresponding system of algebraic equations [11]. In this study, it will be proved that the function $\varphi_n(x)$ is a good approximation to the solution $\varphi(x)$ of Eq.(1) on the space $L^2_{p(x)}[0,2\pi]$. This approximation is discussed in details for Valleé-Poussin's and Fèjer's operators.

2. Preliminaries

Starting from the known linear polynomial operators $U_n(g;x)$ which are good approximation to the function $g(x)$ in the space $L^2_{p(x)}$, and have the form:

$$U_n(g;x) = \frac{1}{\pi}\int_0^{2\pi} g(t)U_n(x-t)dt = \frac{1}{\pi}\int_0^{2\pi} g(x-t)U_n(t)dt, \tag{5}$$

where

$$U_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \lambda_k^{(n)} \cos(kx), \tag{6}$$

$\lambda_k^{(n)}$ are constants which define the method of approximation.

Theorem 1. *[6]*

For $k(x,y)$ belongs to $L^2_p[0,2\pi]$, such that $|\lambda|\|\tilde{k}(x,y)\|_{L^2_p} < 1$, and $f(x)$ belongs to $L^2_{p(x)}$, then the integral equation

$$\varphi(x) = f(x) + \lambda \int_0^{2\pi} k(x,y)\varphi(y)dy,$$

has an unique solution $\varphi(x)$ in $L^2_{p(x)}[0,2\pi]$.

Now, with the help of the following theorem we will find the condition by which the equation (4) has an unique solution.

Theorem 2. *[6]*

If A and B are two bounded linear operators in Banach space E, while A has an inverse and $\|B\|_E\|A^{-1}\|_E < 1$, then the operator $(A+B)$ has also an inverse and

$$\|(A+B)^{-1}\|_E \leq \|A^{-1}\|_E(1-\|B\|_E\|A^{-1}\|_E)^{-1}.$$

To find this condition, we write both of Eqs.(2) and (4) in the operator form

$$(I-\lambda\tilde{K})\varphi = f, \qquad (I-\lambda U_n(\tilde{K}))\varphi_n = f_n,$$

where

$$\tilde{K}\varphi = \int_{-\pi}^{\pi}\tilde{k}(x,y)\varphi(y)dy, \qquad U_n(\tilde{K})\varphi_n = \int_{-\pi}^{\pi}U_n[\tilde{k}(.,y);x]\varphi_n(y)dy.$$

It is obvious that $I-\lambda\tilde{K} = A, \quad \lambda(\tilde{K}-U_n(\tilde{K})) = B$, are two bounded linear operators in the space $L^2_{p(x)}$.

It is well-known that the operator $I-\lambda\tilde{K}$ has an inverse for each λ such that $\frac{1}{\lambda}$ is a regular value of $\tilde{K}$ [6]. So Eq.(2) has an unique solution and we can write

$$\varphi = (I+\lambda R)f = f+\lambda Rf,$$

where $(I-\lambda\tilde{K})^{-1} = (I+\lambda R)$ and R is the resolvent of the operator $\tilde{K}$. From theorem 2 if $|\lambda|\|(I-\lambda\tilde{K})^{-1}\|_E\|\tilde{K}-U_n(\tilde{K})\|_E < 1$, then $(I-\lambda U_n(\tilde{K}))$ has also an inverse, thereby Eq.(4) has an unique solution and can be written in the form

$$\varphi_n = (I+\lambda R_n)f_n = f_n+\lambda R_n f_n,$$

where $(I-\lambda U_n(\tilde{K}))^{-1} = I+\lambda R_n$ and R_n is the resolvent of the operator $U_n(\tilde{K})$.

Now, we return to the functional representation of resolvents $R(x,y;\lambda); R_n(x,y;\lambda)$ and equations (2) and (4). Knowing the resolvent $R(x,y;\lambda)$, we at once obtain the solution of the original equation (2) with an arbitrary right hand side $f(x)$ in the following form

$$\varphi(x) = f(x)+\lambda\int_0^{2\pi}R(x,y;\lambda)f(y)dy.$$

Also, the solution of Eq.(4) can be represented through the resolvent as follows

$$\varphi_n(x) = f_n(x)+\lambda\int_0^{2\pi}R_n(x,y;\lambda)f_n(y)dy.$$

Theorem 3.

For any kernel $k(x,y) \in L_p^2[0,2\pi]$, if the linear polynomial operator U_n of order n is defined in $L^2_{p(x)}$ and if the function $f(x) \in L^2_{p(x)}$, then

$$U_n\left[\int_a^b k(.,y)f(y)dy;x\right] = \int_a^b U_n[k(.,y);x]f(y)dy.$$

The proof of this theorem is very similar to the proof of a theorem in [4].

3. Auxiliary definitions and theorems

Definition 1.

The averaged-modulus of continuity of the kernel $k(x,y) \in L_p^2[0,2\pi]$ is defined as follows

$$w_{L_p^2}(k;t) = w_{L_p^2}(t) = \frac{1}{2\pi}\sup_{|s|\le t}\left[\int_0^{2\pi}\int_0^{x-s} p(x)p(y)[k(x-s,y)-k(x,y)]^2dxdy\right]^{\frac{1}{2}}. \tag{7}$$

Lemma 1.

The function $w_{L_p^2}(t)$ has the following properties:

1. $w_{L_p^2}(t) \to 0$ for $t \to 0$;
2. $w_{L_p^2}(t)$ is positive and monotonic increasing;
3. $w_{L_p^2}(t_1+_2) \le w_{L_p^2}(t_1) + w_{L_p^2}(t_2)$;
4. $w_{L_p^2}(t)$ is continuous;
5. for any positive real number η, the following inequality holds $w_{L_p^2}(\eta t) \le (1+\eta)w_{L_p^2}(t)$.

Also, by the averaged-modulus of continuity with respect to x and y of a function $\tilde{k}(x,y) = e(x,y)k(x,y)$ defined in $[0,2\pi]$, we mean the following function $\Omega_{L_p^2}(t)$

$$\Omega_{L_p^2}(k;t) = \Omega_{L_p^2}(t) = \frac{1}{2\pi}\sup_{|s|\le t}\left[\int_0^{2\pi}\int_0^{2\pi} p(x)p(y)[k(x,y)[e(x-s,y)-e(x,y)]]^2dxdy\right]^{\frac{1}{2}}. \tag{8}$$

It is evident that the function $\Omega_{L_p^2}(t)$ satisfies the above properties of the modulus of continuity (1-5).

Definition 2.

The value of the following norm

$$\delta_n(\tilde{k}) = \delta(\tilde{k};U_n) = \|U_n(\tilde{k}(.,y);x) - \tilde{k}(x,y)\|_{L_p^2} = \left[\int_0^{2\pi}\int_0^{2\pi} p(x)p(y)[U_n(\tilde{k}(.,y);x) - \tilde{k}(x,y)]^2dxdy\right]^{\frac{1}{2}}, \tag{9}$$

will play an important role in estimating the error arising from replacement of Eq.(1) by Eq.(4). The following theorem provides an estimate of $\delta(\tilde{k},U_n)$.

Theorem 4.

For any kernel $\tilde{k}(x,y) \in L_p^2[0,2\pi]$, and for any linear polynomial operator $U_n(g;x)$, we always have the following inequality

$$\delta_n(\tilde{k}) \leq 2\left[w_{L_p^2}(\frac{1}{n}) + \Omega_{L_p^2}(\frac{1}{n})\right]\int_{-\pi}^{\pi}[n|t|+1]|U_n(t)|dt. \tag{10}$$

Proof. Using Minkowski inequality and equalities (5) and (7), we obtain

$$\begin{aligned}
\delta_n(\tilde{k}) &= \|U_n(\tilde{k}(.,y);x) - \tilde{k}(x,y)\|_{L_p^2} = \\
&= \frac{1}{\pi}\left\|\int_{-\pi}^{\pi}[\tilde{k}(x-t,y) - \tilde{k}(x,y)]U_n(t)dt\right\|_{L_p^2} \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}p(x)p(y)\left[\int_{-\pi}^{\pi}U_n(t)(\tilde{k}(x-t,y) - \tilde{k}(x,y))\right]^2 dydx\right]^{\frac{1}{2}} \\
&\leq \frac{1}{\pi}\int_{-\pi}^{\pi}|U_n(t)|\left[\int_0^{2\pi}\int_0^{2\pi}p(x)p(y)[\tilde{k}(x-t,y) - \tilde{k}(x,y)]^2 dydx\right]^{\frac{1}{2}} dt \\
&\leq \frac{1}{\pi}\int_{-\pi}^{\pi}|U_n(t)|\left[\int_0^{2\pi}\int_0^{2\pi}p(x)p(y)[e(x-t,y)k(x-t,y) - e(x,y)k(x,y)]^2 dydx\right]^{\frac{1}{2}} dt \\
&\leq \frac{1}{\pi}\int_{-\pi}^{\pi}|U_n(t)|\left[\int_0^{2\pi}\int_0^{2\pi}p(x)p(y)k(x,y)[e(x-t,y) - e(x,y)]^2 dydx\right]^{\frac{1}{2}} \\
&\quad + \left[\left[\int_0^{2\pi}\int_0^{2\pi}p(x)p(y)e(x-t,y)[k(x-t,y) - k(x,y)]^2 dydx\right]^{\frac{1}{2}} dt\right] \\
&\leq \frac{1}{\pi}\int_{-\pi}^{\pi}|U_n(t)|\left[\int_0^{2\pi}\int_0^{2\pi}p(x)p(y)k(x,y)[e(x-t,y) - e(x,y)]^2 dydx\right]^{\frac{1}{2}} \\
&\quad + \left[\left[\int_0^{2\pi}\int_0^{2\pi}p(x)p(y)e(x-t,y)[k(x-t,y) - k(x,y)]^2 dydx\right]^{\frac{1}{2}} dt\right] \\
&\leq 2\int_{-\pi}^{\pi}|U_n|[w_{L_p^2}(t) + \Omega_{L_p^2}(t)]dt \leq \\
&\leq 2\left[w_{L_p^2}(\frac{1}{n}) + \Omega_{L_p^2}(\frac{1}{n})\right]\int_{-\pi}^{\pi}[n|t|+1]|U_n(t)|dt.
\end{aligned}$$

□

Definition 3.

We define the error of approximation of $\tilde{k}(x,y)$ as follows

$$\begin{aligned}
E^*_{n,m}(\tilde{k})_{L_p^2} &= \|\tilde{k}(x,y) - T^*_{n,m}(x,y)\|_{L_p^2} \\
&= \inf_{T_{n,m}(x,y)}\left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}p(x)p(y)[\tilde{k}(x,y) - T_{n,m}(x,y)]^2 dxdy\right]^{\frac{1}{2}}, \\
E^*_{n,\infty}(\tilde{k})_{L_p^2} &= \|\tilde{k}(x,y) - T^*_{n,\infty}(x,y)\|_{L_p^2} \\
&= \inf_{T_{n,\infty}(x,y)}\left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}p(x)p(y)[\tilde{k}(x,y) - T_{n,\infty}(x,y)]^2 dxdy\right]^{\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned}E^*_{\infty,m}(\tilde{k})_{L^2_p} &= \|\tilde{k}(x,y)-T^*_{\infty,m}(x,y)\|_{L^2_p}\\ &= \inf_{T_{\infty,m}(x,y)}\left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}p(x)p(y)[\tilde{k}(x,y)-T_{\infty,m}(x,y)]^2dxdy\right]^{\frac{1}{2}},\end{aligned}$$

where $T^*_{n,m}(x,y)$ denotes the trigonometric polynomial in x of order n and in y of order m of best approximation of $\tilde{k}(x,y)$ in the metric $L^2_p[0,2\pi]$, $T^*_{n,\infty}(x,y)$ denotes the trigonometric polynomial in x of order n of best approximation of $\tilde{k}(x,y)$ in the metric $L^2_p[0,2\pi]$, $T^*_{\infty,m}(x,y)$ denotes the trigonometric polynomial in y of order m of best approximation of $\tilde{k}(x,y)$ in the metric $L^2_p[0,2\pi]$. The estimates of how rapidly the quantities $E^*_{n,m}(\tilde{k})_{L^2_p}$, $E^*_{n,\infty}(\tilde{k})_{L^2_p}$ and $E^*_{\infty,m}(\tilde{k})_{L^2_p}$ tend to zero as $n\to\infty, m\to\infty$ are given in [10], where

$$E^*_{n,m}(\tilde{k})_{L^2_p}\to 0, \qquad n,m\to\infty,$$

$$E^*_{n,m}(\tilde{k})_{L^2_p}\geq E^*_{n,\infty}(\tilde{k})_{L^2_p}, \qquad E^*_{n,m}(\tilde{k})_{L^2_p}\geq E^*_{\infty,m}(\tilde{k})_{L^2_p}$$

then

$$E^*_{n,\infty}(\tilde{k})_{L^2_p}\to 0, \quad as \quad n\to\infty, \tag{11}$$

$$E^*_{\infty,m}(\tilde{k})_{L^2_p}\to 0, \quad as \quad m\to\infty. \tag{12}$$

Now, we will mention the bounds of the norm (9) for various linear polynomial operators U_n as the following cases:

Case 1: Valleè-Poussin's method [5]:

$U_n=V_n$, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|V_n(t)|dt\leq\frac{1}{3}+\frac{2\sqrt{3}}{\pi}, \tag{13}$$

from Eq.(10) and definition 3, we get

$$E^*_{n,\infty}(\tilde{k})_{L^2_p}\leq 12\pi\left[w_{L^2_p}(\frac{1}{n})+\Omega_{L^2_p}(\frac{1}{n})\right]. \tag{14}$$

By using the inequality (13) and considering that the method of Valleè-Poussin's V_n leaves trigonometric polynomial of order n invariant, then

$$\begin{aligned}\delta_n(\tilde{k};V_n) &= \|\tilde{k}(x,y)-V_n(\tilde{k}(.,y);x)\|_{L^2_p}\\ &= \|\tilde{k}(x,y)-T^*_{n,\infty}(x,y)-V_n[\tilde{k}(.,y)-T^*_{n,\infty}(.,y);x]\|_{L^2_p}\\ &\leq E^*_{n,\infty}(\tilde{k})_{L^2_p}+\frac{1}{\pi}\int_{-\pi}^{\pi}|V_n(t)|\Big[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}p(x)p(y)[\tilde{k}(x-t,y)-T^*_{n,\infty}(x-t,y)]^2dydx\Big]^{\frac{1}{2}}dt\\ &\leq\left[1+\frac{1}{\pi}\int_{-\pi}^{\pi}|V_n(t)|dt\right]E^*_{n,\infty}(\tilde{k})_{L^2_p}\simeq 2.436E^*_{n,\infty}(\tilde{k})_{L^2_p},\end{aligned} \tag{15}$$

and from (14) we get

$$\delta_n(\tilde{k};V_n)\leq 29.232\pi[w_{L^2_p}(\frac{1}{n})+\Omega_{L^2_p}(\frac{1}{n})]. \tag{16}$$

Case 2: Féjer's method [5]:

$U_n = F_n$, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |F_n(t)|dt = 1, \tag{17}$$

$$\int_{-\pi}^{\pi} (1+n|t|)|F_n(t)|dt < 6(1+\ln n), \quad \forall n \geq 3. \tag{18}$$

We let $n' = \frac{\sqrt{n}}{2}, a_i(y), b_i(y), a_i^*(y)$ and $b_i^*(y)$ denote the corresponding coefficients of Fourier series in the variable x of the functions $\tilde{k}(x,y)$ and $V_{n'}[\tilde{k}(.,y);x]$. Then,

$$\begin{aligned}
&\|V_{n'}(\tilde{k}(.,y);x) - F_n[V_{n'}(\tilde{k}(.,y);x)]\|_{L_p^2} \\
&= \left\|\sum_{i=1}^{2n'} \frac{i}{n}[a_i^*(y)\cos ix + b_i^*(y)\sin ix]\right\|_{L_p^2} \\
&\leq \left\|\left[\sum_{i=1}^{2n'}\left(\frac{i}{n}\right)^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{2n'}[a_i^*(y)\cos ix + b_i^*(y)\sin ix]^2\right]^{\frac{1}{2}}\right\|_{L_p^2} \\
&\leq \left[\sum_{i=1}^{2n'}\left(\frac{i}{n}\right)^2\right]^{\frac{1}{2}}\left\|\left[\sum_{i=1}^{2n'}[a_i^{*2}(y) + b_i^{*2}(y)]\right]^{\frac{1}{2}}\right\|_{L_p^2} \\
&\leq \frac{1}{\sqrt{\pi}n}\left[\sum_{i=1}^{2n'} i^2\right]^{\frac{1}{2}}\|\tilde{k}(x,y)\|_{L_p^2} \leq \frac{1}{\sqrt{\pi}\,n}(2n')^{\frac{3}{2}}\|\tilde{k}(x,y)\|_{L_p^2} \\
&\leq \frac{1}{\sqrt{\pi}\,n^{\frac{1}{4}}}\|\tilde{k}(x,y)\|_{L_p^2}.
\end{aligned}$$

Thereby

$$\begin{aligned}
\delta(\tilde{k};F_n) &= \|\tilde{k}(x,y) - F_n(\tilde{k}(.,y);x)\|_{L_p^2} \\
&= \|\tilde{k}(x,y) - V_{n'}(\tilde{k}(.,y);x) + V_{n'}(\tilde{k}(.,y);x) - F_n(V_{n'}(\tilde{k}(.,y);x)) + F_n(V_{n'} - \tilde{k});x)\|_{L_p^2} \\
&\leq \|\tilde{k}(x,y) - V_{n'}(\tilde{k}(.,y);x)\|_{L_p^2} + \|F_n(V_{n'} - \tilde{k});x)\|_{L_p^2} + \|V_{n'}(\tilde{k}(.,y);x) - F_n(V_{n'}(\tilde{k}(.,y);x))\|_{L_p^2} \\
&\leq \left(1 + \frac{1}{\pi}\int_{-\pi}^{\pi}|F_n(t)|dt\right)(2.5)E_{n',\infty}^*(\tilde{k})_{L_p^2} + \frac{1}{\sqrt{\pi}\,n^{\frac{1}{4}}}\|\tilde{k}(x,y)\|_{L_p^2},
\end{aligned} \tag{19}$$

from Eqs.(17) and (19), we get

$$\delta(\tilde{k};F_n) \leq 5E_{n',\infty}^*(\tilde{k})_{L_p^2} + \frac{1}{\sqrt{\pi}\,n^{\frac{1}{4}}}\|\tilde{k}(x,y)\|_{L_p^2}. \tag{20}$$

Also, from Eqs.(18) and (10), we have

$$\delta(\tilde{k};F_n) \leq 12(1+\ln n)[w_{L_p^2}(\frac{1}{n}) + \Omega_{L_p^2}(\frac{1}{n})]. \tag{21}$$

Now from (16), (20) and (21) it is clear that $\delta_n(\tilde{k}) \to 0$ as $n \to \infty$ for Valleé-Poussin's and Féjer's methods for every periodic function $\tilde{k}(x,y) \in L_p^2[0,2\pi]$, $w_{L_p^2}(\frac{1}{n}) = o(1/\ln n)$ and $\Omega_{L_p^2}(\frac{1}{n}) = o(1/\ln n)$.

Definition 4.

The following quantities will play an important role in estimating the error of our approximation

$$\xi(\tilde{k};U_n;\varphi)=\xi_n=\left\|\int_0^{2\pi}\tilde{k}(x,y)[\varphi(y)-U_n(\varphi;y)]dy\right\|_{L_p^2}, \tag{22}$$

$$\gamma_m=\gamma_m(U_n;\varphi)=\sum_{i=1}^{m}|1-\lambda_i^{(n)}|E_{i-1}(\varphi)_{L_p^2}, \tag{23}$$

where

$$E_n(\varphi)_{L_p^2}=\inf_{T_n}\|\varphi(x)-T_n(x)\|_{L_p^2},$$

$T_n(x)$ is a trigonometric polynomial of order n in x, $m\leq n$.

Theorem 5.

For any kernel $\tilde{k}(x,y)\in L_p^2[0,2\pi]$ and for linear polynomial operator $U_n(g;x)$ the following inequality holds

$$\begin{aligned}\xi_n(\tilde{k})=\xi_n(\tilde{k};U_n;\varphi)&=\left\|\int_0^{2\pi}\tilde{k}(x,y)[\varphi(y)-U_n(\varphi;y)]dy\right\|_{L_p^2}\\ &\leq E^*_{\infty,m}(\tilde{k})_{L_p^2}\|\varphi(y)-U_n(\varphi;y)\|_{L_p^2}+\sqrt{\frac{2}{\pi}}\gamma_m(U_n;\varphi)\left[\int_0^{2\pi}p(x)dx\right]^{\frac{1}{2}}\left[\|\tilde{k}(x,y)\|_{L_p^2}+E^*_{\infty,m}(\tilde{k})_{L_p^2}\right],\end{aligned} \tag{24}$$

for any positive integer $m\leq n$.

Proof. For any function $\varphi(x)\in L_p^2$ with Fourier coefficients c_i and d_i in view of Bunyakovskii inequality and $p(x)\geq 1$, we obtain

$$\begin{aligned}|c_i\cos ix+d_i\sin ix|&=\inf_{T_{i-1}(t)}\frac{1}{\pi}\left|\int_{-\pi}^{\pi}[\varphi(t)-T_{i-1}(t)]\cos(i(x-t))dt\right|\\ &\leq\frac{1}{\pi}\inf_{T_{i-1}(t)}\left[\int_{-\pi}^{\pi}p(t)[\varphi(t)-T_{i-1}(t)dt]^2\right]^{\frac{1}{2}}.\left[\int_{-\pi}^{\pi}\frac{[\cos(i(x-t))]^2}{p(t)}dt\right]^{\frac{1}{2}}\\ &\leq\sqrt{\frac{2}{\pi}}\inf_{T_{i-1}(t)}\left[\int_{-\pi}^{\pi}p(t)|\varphi(t)-T_{i-1}(t)|^2dt\right]^{\frac{1}{2}}\\ &\leq\sqrt{\frac{2}{\pi}}E^*_{i-1}(\varphi)_{L_p^2},\end{aligned}$$

therefore

$$\|c_i\cos ix+d_i\sin ix\|_{L_p^2}\leq\sqrt{\frac{2}{\pi}}E^*_{i-1}(\varphi)_{L_p^2}\left(\int_{-\pi}^{\pi}p(x)dx\right)^{\frac{1}{2}}.$$

Letting

$$T_{\infty,m}(x,y)=\sum_{i=0}^{m}a_i(x)\cos iy+b_i(x)\sin iy,$$

$$E^*_{\infty,m}(\tilde{k})_{L^2_p} = \inf_{a_i,b_i} \left\| \tilde{k}(x,y) - \sum_{i=0}^{m} a_i(x)\cos iy + b_i(x)\sin iy \right\|_{L^2_p},$$

and taking into consideration (23) and using Bunyakovskii inequality, we obtain

$$\begin{aligned}
\xi_n = \xi_n(\tilde{k}; U_n; \varphi) &= \left\| \int_0^{2\pi} \tilde{k}(x,y)[\varphi(y) - U_n(\varphi;y)]dy \right\|_{L^2_p} \\
&= \left[\int_0^{2\pi} p(x) \left[\int_0^{2\pi} \tilde{k}(x,y)[\varphi(y) - U_n(\varphi;y)]dy \right]^2 dx \right]^{\frac{1}{2}} \\
&\le \Big[\int_0^{2\pi} p(x) \inf_{T_{\infty,m}(x,y)} \Big[\int_0^{2\pi} |\tilde{k}(x,y) - T_{\infty,m}(x,y)||\varphi(y) - U_n(\varphi;y)|dy \\
&+ \left| \int_0^{2\pi} (\tilde{k}(x,y) + T_{\infty,m}(x,y) - \tilde{k}(x,y))(\varphi(y) - U_n(\varphi;y))dy \right| \Big]^2 dx \Big]^{\frac{1}{2}} \\
&\le \left[\int_0^{2\pi} p(x) \inf_{T_{\infty,m}(x,y)} \left[\int_0^{2\pi} |\tilde{k}(x,y) - T_{\infty,m}(x,y)||\varphi(y) - U_n(\varphi;y)|dy \right]^2 dx \right]^{\frac{1}{2}} \\
&+ \Big[\int_0^{2\pi} p(x) \inf_{T_{\infty,m}(x,y)} \Big[\int_0^{2\pi} (\tilde{k}(x,y) + T_{\infty,m}(x,y) - \tilde{k}(x,y)). \\
&\left(\sum_{i=1}^{m} (1 - \lambda_i^{(n)})(c_i\cos iy + d_i\sin iy) \right) dy \Big]^2 dx \Big]^{\frac{1}{2}} \\
&\le E^*_{\infty,m}(\tilde{k})_{L^2_p} \|\varphi(y) - U_n(\varphi;y)\|_{L^2_p} \\
&+ \sqrt{\frac{2}{\pi}} \gamma_m(U_n;\varphi) \left[\int_0^{2\pi} p(x)dx \right]^{\frac{1}{2}} \left[\|\tilde{k}(x,y)\|_{L^2_p} + E^*_{\infty,m}(\tilde{k})_{L^2_p} \right].
\end{aligned}$$

$\square$

4. The approximate solution and its error bounds

The following theorem shows that for sufficiently good linear methods $U_n(g;x)$, the difference between the polynomials $\varphi_n(x)$ and the original solution $\varphi(x)$ is sufficiently small.

Theorem 6.

If the kernel $\tilde{k}(x,y)$ in Eq.(2) satisfies the assumptions (A^*), all functions appearing in (2) are $2\pi-$periodic in x and y, then any linear polynomial operator $U_n(g;x)$, if $|\lambda|R\delta(\tilde{k};U_n) < 1$ and if Eq.(1) is replaced by Eq.(4), the following inequality holds

$$\|\varphi(x) - \varphi_n(x)\|_{L^2_p} \le (1 + \alpha_n(\tilde{k}))\|\varphi(x) - U_n(\varphi;x)\|_{L^2_p}, \tag{25}$$

in which

$$\alpha_n(\tilde{k}) = |\lambda|R \left[\delta(\tilde{k};U_n) + \frac{\xi(\tilde{k};U_n;\varphi)}{\|\varphi(x) - U_n(\varphi;x)\|_{L^2_p}} \right] / [1 - |\lambda|R\delta(\tilde{k};U_n)], \tag{26}$$

where $\delta(\tilde{k};U_n)$ and $\xi(\tilde{k};U_n;\varphi)$ are defined in (9) and (22), respectively, and $R = 1 + |\lambda|\|R(x,y)\|_{L^2_p}$, where $R(x,y)$ denotes the resolvent of the kernel $\tilde{k}(x,y)$.

Proof. Using theorem 3, and Eq.(2), we represent the solution $\varphi_n(x)$ of Eq.(4) in the form

$$\begin{aligned}\varphi_n(x) &= U_n(f;x) + \lambda U_n\left[\int_0^{2\pi}\tilde{k}(.,y)\varphi_n(y)dy;x\right]\\ &= U_n(f;x) + \lambda U_n\left[\int_0^{2\pi}\tilde{k}(.,y)[\varphi_n(y)-\varphi(y)]dy + \int_0^{2\pi}\tilde{k}(.,y)\varphi_n(y)dy;x\right]\\ &= \lambda\int_0^{2\pi}U_n[\tilde{k}(.,y);x][\varphi_n(y)-\varphi(y)]dy + U_n\left[f(.) + \lambda\int_0^{2\pi}\tilde{k}(.,y)\varphi(y)dy;x\right]\\ &= \lambda\int_0^{2\pi}U_n[\tilde{k}(.,y);x][\varphi_n(y)-\varphi(y)]dy + U_n(\varphi;x),\end{aligned} \tag{27}$$

it follows that

$$\varphi_n(x) - U_n(\varphi;x) = \lambda\int_0^{2\pi}\tilde{k}(x,y)[\varphi_n(y) - U_n(\varphi;y)]dy + g_n(x), \tag{28}$$

where

$$g_n(x) = \lambda\int_0^{2\pi}[U_n(\tilde{k}(.,y);x) - \tilde{k}(x,y)][\varphi_n(y)-\varphi(y)]dy + \lambda\int_{-\pi}^{\pi}\tilde{k}(x,y)[U_n(\varphi;y)-\varphi(y)]dy.$$

Thus, by Eqs.(9), (10) and (22) we get the estimate

$$\begin{aligned}\|g_n(x)\|_{L_p^2} \le{}& |\lambda|\left\|\int_0^{2\pi}[U_n(\tilde{k}(.,y);x) - \tilde{k}(x,y)][\varphi_n(y)-\varphi(y)]dy\right\|_{L_p^2}\\ &+ |\lambda|\left\|\int_0^{2\pi}\tilde{k}(x,y)[U_n(\varphi;y)-\varphi(y)]dy\right\|_{L_p^2}\\ &\le |\lambda|\delta(\tilde{k};U_n)\left[\|\varphi_n(x) - U_n(\varphi;x)\|_{L_p^2} + \|U_n(\varphi;x)-\varphi(x)\|_{L_p^2}\right] + |\lambda|\xi(\tilde{k};U_n;\varphi).\end{aligned} \tag{29}$$

In view of $|\lambda|\|\tilde{k}(x,y)\|_{L_p^2} < 1$, Eq.(28) has an unique solution given by

$$\varphi_n(x) - U_n(\varphi;x) = g_n(x) + \lambda\int_0^{2\pi}R(x,y)g_n(y)dy.$$

Therefore

$$\begin{aligned}\|\varphi_n(x) - U_n(\varphi;x)\|_{L_p^2} &\le \|g_n(x)\|_{L_p^2}\left[1 + |\lambda|\|R(x,y)\|_{L_p^2}|\right] = R\|g_n(x)\|_{L_p^2}\\ &\le R|\lambda|\left[\delta(\tilde{k};U_n)[\|\varphi(x) - U_n(\varphi;x)\|_{L_p^2} + \|\varphi_n(x) - U_n(\varphi;x)\|_{L_p^2}] + \xi(\tilde{k};U_n;\varphi)\right].\end{aligned}$$

Taking into consideration $|\lambda|R\delta(\tilde{k};U_n) < 1$, we obtain

$$\|\varphi_n(x) - U_n(\varphi;x)\|_{L_p^2} \le \frac{|\lambda|R[\delta(\tilde{k};U_n)\|U_n(\varphi;x)-\varphi(x)\|_{L_p^2} + \xi(\tilde{k};U_n;\varphi)]}{1 - |\lambda|R\delta(\tilde{k};U_n)}$$

Therefore

$$\begin{aligned}\|\varphi(x) - \varphi_n(x)\|_{L_p^2} &\le \|\varphi(x) - U_n(\varphi;x)\|_{L_p^2} + \|\varphi_n(x) - U_n(\varphi;x)\|_{L_p^2}\\ &\le \|\varphi(x) - U_n(\varphi;x)\|_{L_p^2} + \frac{|\lambda|R[\delta(\tilde{k};U_n)\|U_n(\varphi;x)-\varphi(x)\|_{L_p^2} + \xi(\tilde{k};U_n;\varphi)]}{1 - |\lambda|R\delta(\tilde{k};U_n)}\\ &\le (1 + \alpha_n(\tilde{k}))\|\varphi(x) - U_n(\varphi;x)\|_{L_p^2},\end{aligned} \tag{30}$$

where α_n is given by (26). Thus, the inequality (25) is proved. □

5. The results

It is well-known that in [7], one cannot achieve an error less than the corresponding to the best approximation. The error estimate in (25) with rate of convergence $\alpha_n(\tilde{k})$, means that, the rate of convergence of $\varphi_n(x)$ to $\varphi(x)$ is comparable with the rate of convergence of the best approximate, which means that the error estimate (25) is optimal. Applying theorem 6, and also the corresponding results from section 3, we obtain the following results:

In the case of the application of Valleé-Poussin's method:

From [10] and (25) we obtain

$$\|\varphi(x)-\varphi_n(x)\|_{L_p^2} \leq (1+\alpha_n(\tilde{k}))(\frac{4}{3}+\frac{2\sqrt{3}}{\pi})E_n^*(\varphi)_{L_p^2} \leq (1+\alpha_n(\tilde{k}))(2.5)E_n^*(\varphi)_{L_p^2},$$

where by (15) we have

$$\alpha_n(\tilde{k}) \leq |\lambda| R \frac{2.5E_{n,\infty}^*(\tilde{k})_{L_p^2}+E_{\infty,m}^*(\tilde{k})_{L_p^2}}{1-\lambda R(2.5)E_{n,\infty}^*(\tilde{k})_{L_p^2}},$$

then $\alpha_n(\tilde{k}) \to 0$ as $n \to \infty$ for all $\varphi(x) \in L_{p(x)}^2$, $\tilde{k}(x,y) \in L_{p(x)}^2[0,2\pi]$.

In the case of the application of Féjer's method:

The quantity $\alpha_n(\tilde{k})$ in the relation (25) will not tend to zero for any solution $\varphi(x)$, but will tend to zero only under the condition that "the solution $\varphi(x)$ belongs to some subclasses of integrable functions". Restricting ourselves to the Holder classes $W^{(r)}H^{\beta}(L_p^2)$ where r is a non-negative integer and $0<\beta\leq 1$, we obtain the following case:

In order that $\alpha_n(\tilde{k}) \to 0$ as $n\to\infty$ considering (20), (21) and [10], it is sufficient that the following conditioned be satisfied

$$\varphi(x) \in W^{(0)}H^{\beta}(L_p^2), \quad \text{i.e. } r=0, \quad 0<\beta\leq 1, \quad w(\frac{1}{n})_{L_p^2}=o(1/\ln n), \quad \Omega(\frac{1}{n})_{L_p^2}=o(1/\ln n).$$

6. Conclusion and remarks

In this article, we presented the approximate solutions of the Volterra integral equations of the second kind in the space $L_{p(x)}^2[0,2\pi]$ with weight function $p(x)$ with the help of the Valleé-Poussin's and Fèjer's operators. In the same time, we proved that the function $\varphi_n(x)$ is a good approximation to the exact solution $\varphi(x)$ for the Volterra integral equations. From the obtained approximate solutions using ADM, we can conclude that the proposed approach is easy to implement and computationally very attractive. A good agreement between the theoretical study with the obtained approximate solutions have been obtained.

References

[1] M. A. Abdou, M. E. Nasr and M. A. Abdel-Aty, Study of the Normality and Continuity for the Mixed Integral Equations with Phase-Lag Term, Inter. J. of Math. Analysis, **11** (2017), 787–799. https://doi.org/10.12988/ijma.2017.7798

[2] M. A. Abdou, M. E. Nasr and M. A. Abdel-Aty, A study of normality and continuity for mixed integral equations, J. of Fixed Point Theory Appl., **20**(1) (2018). https://doi.org/10.1007/s11784-018-0490-0

[3] V. K. Dzyadyk, V. T. Gavrilyuk and O. I. Stepanets, On the best approximations of Holder class functions by Rogozinski polynomials, Dokl. Akad. Nauk Ukr. SSR. Ser. A, **3**, 1969.

[4] V. K. Dzyadyk, On the approximations of linear methods to the approximations by polynomials of functions which are solution of Fredholm integral equation of the second kind I and II, Urain. Matem. Zh. **22**, 1970.

[5] V. K. Dzyadyk, Approximation Methods for Solutions of Differential and Integral Equations, The Netherlands, VSP, 1995.

[6] A. N. Kolmogorov and S. V. Fomin, Elements of the Theory of Functions and Functional Analysis, Dover Pubns, 1999.

[7] P. P. Korovkin, Linear Operators and Approximation Theory, Hindustan (India), 1960 (Translated from the Russian).

[8] M. E. Nasr and M. F. Jabbar, An Approximate Solution for Volterra Integral Equations of the Second Kind in Space with Weight Function, Inter. J. of Math. Analysis, 11 (2017), 849–861.

[9] M. E. Nasr and M. A. Abdel-Aty, Analytical discussion for the mixed integral equations, J. of Fixed Point Theory Appl., 20(3) (2018). https://doi.org/10.1007/s11784-018-0589-3

[10] A. F. Timan, Theory of Approximations of Functions of a Real Variable, Dover, New York, 1994.

[11] F. G. Tricomi, Integral Equations, Dover, New York, 1985.

A Modified SSDP Method for Nonlinear Semidefinie Programming*

Jianling Li† Chunting Lu Hui Zhang

College of Mathematics and Information Science, Guangxi University,

Nanning, Guangxi, 530004, China

Abstract In this paper, we investigate nonlinear semidefinite programming and propose a modified sequential semidefinite programming (SSDP for short) algorithm without a penalty function or a filter. At each iteration, the search direction is yielded by solving a linear semidefinite programming subproblem and a quadratic semidefinite programming subproblem. The nonmonotone line search ensures that the objective function or constraint violation function is sufficiently reduced. Under some appropriate conditions, the global convergence of the proposed algorithm is shown. Some preliminary numerical results are reported.

Key words nonlinear semidefinite programming; sequential semidefinite programming; nonmonotone line search; global convergence

1 Introduction

Consider the following nonlinear semidefinite programming (NLSDP):

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & G(x) \preceq 0, \end{array} \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is assumed to be a smooth and real value function, $G : \mathbb{R}^n \to \mathbb{S}^m$ is a smooth and matrix value function. $\mathbb{S}^m$ represents the set of all real symmetric matrices. The symbol $A \preceq B$ means that $A - B$ is a negative semidefinite matrix.

Nonlinear semidefinite programming has many real-world applications, such as engineering design, optimal structure design, optimal robust control and robust feedback control design (see [1]-[4]). In recent years, the investigation of NLSDP has attracted much attention. The main solution methods for NLSDP are augmented Lagrange method [5]-[10], interior point method [11]-[15], SSDP method [16]-[21]. In this paper, our focus is on SSDP method. Correa and Ramirez in [16] proposed an SSDP algorithm. At each iteration, the search direction is generated by solving a traditional quadratic semidefinite programming (QSDP for short) subproblem. A subdifferentiable penalty function is used as a merit function to design line search. Under some conditions, the algorithm is globally convergent. However, it is not easy for the choice of an appropriate penalty parameter. Gomez in [17] proposed a filter-type SSDP algorithm for nonlinear semidefinite programming problem. For each iteration point, by solving a trust-region type QSDP subproblem

*Project supported by the National Natural Science Foundation (No. 11561005), the National Science Foundation of Guangxi (No. 2016GXNSFAA380248

† Corresponding author. E-mail: jianlingli@126.com

to get search direction. When objective function value or the constraint violation function is improved, the trial point is accepted by filter. Chen in [21] proposed a trust region SSDP method without a penalty function or a filter. The search direction is obtained by solving trust region QSDP subproblem. Whether the trial point is accepted or not depends on the decline of the objective function or constraint violation function.

In all above SSDP algorithms, the traditional QSDP subproblem, which generated the search direction, may be incompatible. Motivated by the idea of modified SQP methods for nonlinear programming, in this paper, we proposed a modified SSDP algorithm for NLSDP (1.1). At each iteration, the search direction is yielded by solving a linear semidefinite programming (LSDP for short) subproblem and a modified QSDP subproblem. Nonmonotone line search technique is used to determine step size.

The paper is organized as follows. In the next section, the algorithm is described in detail. The global convergence is shown in Section 3. Some preliminary numerical results are reported in Section 4 and some concluding remarks are given in the final section.

2 Description of Algorithm

In this section, we first restate some concepts and notations about nonlinear semidefinite programming, and then describe the proposed algorithm.

Let $G(x): \mathbb{R}^n \to \mathbb{S}^m$ be a matrix value function, we use the notation

$$DG(x) = \left(\frac{\partial G(x)}{\partial x_1}, \cdots, \frac{\partial G(x)}{\partial x_n}\right)^{\mathrm{T}} \tag{2.1}$$

for its differential operator evaluated at x. For any $d = (d_1, \cdots, n) \in \mathbb{R}^n$, $DG(x)d$ is defined by

$$DG(x)d = \sum_{i=1}^{n} d_i \frac{\partial G(x)}{\partial x_i}. \tag{2.2}$$

The adjoint operator $DG(x)^*$ of the linear operator $DG(x)$ satisfies

$$DG(x)^*Y = \left(< \frac{\partial G(x)}{\partial x_1}, Y >, < \frac{\partial G(x)}{\partial x_2}, Y >, \cdots, < \frac{\partial G(x)}{\partial x_n}, Y >\right)^{\mathrm{T}}, \quad \forall\, Y \in \mathbb{S}^m. \tag{2.3}$$

where $< A, B >$ means the inner product of the matrix A and B.

Definition 2.1 [16] Let $\widetilde{x} \in \mathbb{R}^n$ be a feasible point of NLSDP (1.1), if there exists $\widetilde{Y} \in \mathbb{S}^m$ satisfying the following KKT conditions

$$\nabla_x L(\tilde{x}, \tilde{Y}) = \nabla f(\tilde{x}) + DG(\tilde{x})^* \tilde{Y} = 0, \tag{2.4}$$

$$\tilde{Y} \succeq 0, \qquad < G(\tilde{x}), \tilde{Y} >= 0, \tag{2.5}$$

where $L : \mathbb{R}^n \times \mathbb{S}^m \to \mathbb{R}$ is the Lagrangian function of NLSDP (1.1), that is,

$$L(x, \lambda, Y) = f(x) + < Y, G(x) >,$$

then $\widetilde{x}$ is called a KKT point of NLSDP (1.1), the matrix $\widetilde{Y}$ is called a Lagrangian multiplier associated with $\widetilde{x}$.

Let $x^k \in \mathbb{R}^n$ be the current iterate point. In order to generate search directions, we borrow the ideas in [22] and construct the following linear semidefinite programming (LSDP (x^k) for short):

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & G(x^k) + DG(x^k)d \preceq zI_m, \\ & z \geq 0, \end{aligned} \tag{2.6}$$

where I_m is the m order identity. Obviously, the feasible set of LSDP(x^k)(2.6) is not empty, so there exists an optimal solution of (2.6). Let $(\hat{d^k}^{\mathrm{T}},\ z_k)^{\mathrm{T}}$ be an optimal solution of (2.6), then we construct a quadratic semidefinite programming (QSDP (x^k, H_k) for short) as follows:

$$\begin{aligned} \min_{d\in\mathbb{R}^n} \ & \nabla f(x^k)^{\mathrm{T}}d + \tfrac{1}{2}d^{\mathrm{T}}H_k d \\ \text{s.t.} \quad & G(x^k) + DG(x^k)d \preceq z_k I_m. \end{aligned} \tag{2.7}$$

If H_k is a symmetric positive definite matrix, then the solution of QSDP(x^k, H_k) (2.7) is unique.

To measure the degree of feasibility at the iterate point, we define the degree of constraint violation as follows:

$$h(x) = (\lambda_1(G(x)))_+, \tag{2.8}$$

where $\lambda_1(\cdot)$ is the largest eigenvalue of a matrix, $(\alpha)_+ = \max\{0, \alpha\}$. Obviously, $h(x) = 0$ is equivalent with that x is a feasible point of NLSDP (1.1).

Let d^k be the solution of QSDP(x^k, H_k) (2.7). Similar to the idea of filter method, we hope that the search direction d^k can improve the feasibility of the iterate point or the value of the objective function. In other words, if d^k satisfies

$$\nabla f(x^k)^{\mathrm{T}}d^k \leq -\frac{1}{2}(d^k)^{\mathrm{T}}H_k d^k, \tag{2.9}$$

and t satisfies

$$f(x^k + td^k) \leq \max_{0\leq j\leq m(k)}\{f(x^{k-j})\} - t\alpha(d^k)^{\mathrm{T}}H_k d^k, \tag{2.10}$$

$$h(x^k + td^k) \leq \beta \max_{0\leq j\leq m(k)}\{h(x^{k-j})\}, \tag{2.11}$$

where $\alpha \in (0, \frac{1}{2})$, $m(0) = 0$, $m(k) = \min\{m(k-1)+1,\ M\}$, M is a positive integer, then the corresponding trial step $x^k + td^k$ is accepted.

If d^k does not satisfy (2.9), that is,

$$\nabla f(x_k)^{\mathrm{T}}d^k > -\tfrac{1}{2}(d^k)^{\mathrm{T}}H_k d^k, \tag{2.12}$$

then let $t = 1$. If the following inequality

$$h(x^k + d^k) \leq \beta \max_{0\leq j\leq m(k)}\{h(x^{k-j})\} \tag{2.13}$$

hold, then the corresponding trial step $x^k + d^k$ is accepted.

Based on the above strategy, we now present the new algorithm in detail.

Algorithm A

S0. Given $x^0 \in \mathbb{R}^n,\ H_0 = I_m,\ \alpha \in (0, \frac{1}{2}),\ \sigma \in (0,1),\ \beta \in (\frac{1}{2}, 1),\ m(0) = 0,\ apositiveinteger\ M$. Let $k := 0$.
S1. Solve LSDP(x^k) (2.6) to get a solution $(\hat{d^k}^{\mathrm{T}}, z_k)^{\mathrm{T}}$. If $\hat{d^k} = 0$ and $z_k \neq 0$, stop.
S2. Solve QSDP (x^k, H_k) (2.7) to get the solution d^k. If $d^k = 0$, stop.
S3. If d^k satisfies (2.9), then let t_k be the first number in the sequence of $\{1, \sigma, \sigma^2, \cdots\}$ satisfying the following inequality

$$f(x^k + td^k) \leq \max_{0 \leq j \leq m(k)} \{f(x^{k-j})\} - t\alpha(d^k)^{\mathrm{T}} H_k d^k, \tag{2.14}$$

and go to S4; otherwise, let $t_k = 1$ and go to S4.
S4. Let $x^{k+1} = x^k + t_k d^k$. If the following inequality

$$h(x^{k+1}) \leq \beta \max_{0 \leq j \leq m(k)} \{h(x^{k-j})\}, \tag{2.15}$$

holds, then set $m(k+1) = \min\{m(k) + 1,\ M\}$. Update H_k such that H_{k+1} is a positive definite matrix. Let $k = k + 1$ and go to S1; otherwise, go into the restoration phase to obtain a new point x^{k+1}. Let $k = k + 1$ and go to S1.

Remark. In the restoration phase, our aim is to decrease the value of $h(x)$. The restoration algorithm is similar to the one given by Long et al. [23].

3 Global Convergence

In this section, we first show that Algorithm A is well-defined, and then show the global convergence. To this end, the following assumptions are necessary.

***A* 1** The iterate $\{x^k\}$ remains in a closed, bounded subset $\mathcal{X}$.

***A* 2** The objective function $f(x)$ and the constraint function $G(x)$ are twice continuously differentiable in $\mathbb{R}^n$.

***A* 3** There exist two constants $0 < a \leq b$ such that $a\|d\|^2 \leq d^{\mathrm{T}} H_k d \leq b\|d\|^2$ for any $d \in \mathbb{R}^n$.

In what follows, we analyze the feasibility of Algorithm A. To this end, it is necessary to extend the definition of infeasible stationary point for nonlinear programming [24] to nonlinear semidefinite programming.

Definition 3.1 Let $\widetilde{x} \in \mathbb{R}^n$ be an infeasible point of $NLSDP$ (1.1), if

$$\min_{d \in \mathbb{R}^n} \max\{\lambda_1(G(\widetilde{x}) + DG(\widetilde{x})d), 0\} = \max\{\lambda_1(G(\widetilde{x})), 0\} = h(\widetilde{x}), \tag{3.1}$$

then $\widetilde{x}$ is called an infeasible sationary of $NLSDP$ (1.1).

Lemma 3.1 Supposed that the assumptions A1-A3 hold, if Algorithm A terminates at x^k, then x^k is either an infeasible stationary point or a KKT point of $NLSDP$ (1.1).

Proof. The proof is divided into two cases.

Case A. If Algorithm A terminates in S1, then $\hat{d}^k = 0$ and $z_k \neq 0$. We know from $LSDP(x^k)$ (2.6) that $z_k = h(x^k)$, so $h(x^k) \neq 0$, which implies x^k is an infeasible point of $NLSDP$ (1.1).

In the following, we prove that x^k is an infeasible stationary point of $NLSDP$ (1.1), namely, x^k satisfies:

$$\min_{d\in\mathbb{R}^n} \max\{\lambda_1(G(x^k) + DG(x^k)d), 0\} = \max\{\lambda_1(G(x^k)), 0\} = h(x^k).$$

By contradiction, suppose that the conclusion is not true. So there exists $d^{k,0} \in \mathbb{R}^n$ such that

$$\widehat{z} := \max\{\lambda_1(G(x^k) + DG(x^k)d^{k,0}), 0\} < h(x^k). \tag{3.2}$$

Clearly, $({d^{k,0}}^{\mathrm{T}}, \hat{z})^{\mathrm{T}}$ is a feasible solution of $LSDP(x^k)$ (2.6). Note that z_k is a solution of $LSDP(x^k)$ (2.6), so we obtain

$$z_k \le \widehat{z} < h(x^k), \tag{3.3}$$

this contradicts $z_k = h(x^k)$. Therefore, x^k is an infeasible stationary point of $NLSDP$ (1.1).

Case B. If Algorithm A terminates in S2, then the solution d^k of $QSD(x^k, H_k)$ (2.7) is zero, i.e., $d^k = 0$. Further, $d^k = 0$ satisfies KKT condition of $QSDP(x^k, H_k)$ (2.7), that is to say, there exists $Y_k \in \mathbb{S}^m$, such that

$$\nabla f(x^k) + DG(x^k)^* Y_k = 0, \tag{3.4}$$

$$G(x^k) \preceq z_k I_m, \tag{3.5}$$

$$Y_k \succeq 0, \qquad < G(x^k) - z_k I_m, Y_k >= 0. \tag{3.6}$$

In what follows, we prove that $z_k = 0$. By contradiction, supposed that $z_k \neq 0$, obviously, $(0^{\mathrm{T}}, z_k)^{\mathrm{T}}$ is a solution of $LSDP(x^k)$ (2.6) from (3.5). Therefore, x^k is an infeasible point of $NLSDP$ (1.1). Since Algorithm A does not stop in S1, $z_k < h(x^k)$.

On the other hand, it follows from (3.5) that

$$\lambda_1(G(x^k)) \le z_k.$$

In view of $z_k > 0$, we obtain $h(x^k) = \max\{\lambda_1(G(x^k)), 0\} \le z_k$. This contradict $z_k < h(x^k)$. Therefore, $z_k = 0$.

Substituting $z_k = 0$ into (3.5), and conbining with (3.4) and (3.6), we know that x^k is a KKT point of $NLSDP$ (1.1). □

Lemma 3.2 If d^k satisfies the inequality (2.9), then the line search (2.14) is performed.

Proof. It is sufficient to show that there exsits $t \in (0, 1)$ such that (2.14) hold.

In view of $\nabla f(x^k)^{\mathrm{T}} d^k \leq -\frac{1}{2}(d^k)^{\mathrm{T}} H_k d^k$, so in combination with the positive definiteness of H_k, we know that there exists $d^k \neq 0$ such that $\nabla f(x^k)^{\mathrm{T}} d^k < 0$. For convinence, denote

$$f(x^{l(k)}) = \max_{0 \leq j \leq m(k)} \{f(x^{k-j})\}. \tag{3.7}$$

By contradiction, if the conclusion is not true, then for all $t \in (0,1)$, we have

$$f(x^k + td^k) - f(x^{l(k)}) > -t\alpha (d^k)^{\mathrm{T}} H_k d^k \geq 2t\alpha \nabla f(x^k)^{\mathrm{T}} d^k. \tag{3.8}$$

From (3.7), it is obvious that $f(x^{l(k)}) \geq f(x^k)$, so combining with (3.8), we have

$$f(x^k + td^k) - f(x^k) \geq f(x^k + td^k) - f(x^{l(k)}) > 2t\alpha \nabla f(x^k)^{\mathrm{T}} d^k, \tag{3.9}$$

equivalently,

$$\tfrac{[f(x^k + td^k) - f(x^k)]}{t} > 2\alpha \nabla f(x^k)^{\mathrm{T}} d^k. \tag{3.10}$$

Let $t \to 0^+$, taking the limit for the both sides, it follows that

$$\nabla f(x^k)^{\mathrm{T}} d^k \geq 2\alpha \nabla f(x^k)^{\mathrm{T}} d^k.$$

This implies $\alpha \in [\frac{1}{2}, \propto)$ due to $\nabla f(x^k)^{\mathrm{T}} d^k < 0$,. This contradicts $\alpha \in (0, \frac{1}{2})$. Hence, the desired result holds. $\square$

Lemma 3.3 Supposed that the assumptions A1-A3 hold, then there exists $\bar{t} > 0$ such that $t_k \geq \bar{t}$ for k sufficiently large,.

Proof. According to Algorithm A, without loss of generality, suppose that the search direction d^k satisfies (2.10), that is,

$$\nabla f(x^k)^{\mathrm{T}} d^k \leq -\frac{1}{2}(d^k)^{\mathrm{T}} H_k d^k.$$

By Taylor expansion, (3.7) and the assumptions A1-A3, we have

$$\begin{aligned}
& f(x^k + t_k d^k) - f(x^{l(k)}) + t_k \alpha (d^k)^{\mathrm{T}} H_k d^k \\
&= f(x^k) + t_k \nabla f(x^k)^{\mathrm{T}} d^k + \tfrac{1}{2} t_k^2 (d^k)^{\mathrm{T}} \nabla^2 f(y^k) d^k - f(x^{l(k)}) + t_k \alpha (d^k)^{\mathrm{T}} H_k d^k \\
&\leq f(x^k) + t_k \nabla f(x^k)^{\mathrm{T}} d^k + \tfrac{1}{2} t_k^2 (d^k)^{\mathrm{T}} \nabla^2 f(y^k) d^k - f(x^k) + t_k \alpha (d^k)^{\mathrm{T}} H_k d^k \\
&= t_k \nabla f(x^k)^{\mathrm{T}} d^k + \tfrac{1}{2} t_k^2 (d^k)^{\mathrm{T}} \nabla^2 f(y^k) d^k + t_k \alpha (d^k)^{\mathrm{T}} H_k d^k \\
&\leq -\tfrac{1}{2} t_k (d^k)^{\mathrm{T}} H_k d^k + \tfrac{1}{2} t_k^2 (d^k)^{\mathrm{T}} \nabla^2 f(y^k) d^k + t_k \alpha (d^k)^{\mathrm{T}} H_k d^k \\
&\leq -a t_k (\tfrac{1}{2} - \alpha) \|d^k\|^2 + \tfrac{1}{2} t_k^2 M \|d^k\|^2,
\end{aligned} \tag{3.11}$$

where y^k is between x^k and $x^k + t_k d^k$, M is a positive integer such that $\|\nabla^2 f(x)\| \leq M$.

Let $\bar{t} = \frac{a(1-2\alpha)}{M} > 0$, so (2.10) holds for $t_k \geq \bar{t}$ and $\alpha \in (0, \frac{1}{2})$. $\square$

Lemma 3.4 Supposed that the assumptions A1-A3 hold, $\{x^k\}$ is an infinite sequence generated by Algorithm A, then $\lim\limits_{k \to \infty} h(x^k) = 0$.

Proof. Since $m(k+1) \leq m(k)+1$, we have

$$h(x^{l(k+1)}) = \max_{0\leq j\leq m(k+1)}\{h(x^{k+1-j})\} \leq \max_{0\leq j\leq m(k)+1}\{h(x^{k+1-j})\} = \max\{h(x^{k+1}),\ h(x^{l(k)})\} = h(x^{l(k)}),$$

this implies that the sequence $\{h(x^{l(k)})\}$ is not increasing for k. Combining with $h(x^{l(k)}) \geq 0$, we conclude that $\{h(x^{l(k)})\}$ is convergent.

By Algorithm A, we have

$$h(x^{k+1}) \leq \beta \max_{0\leq j\leq m(k)}\{h(x^{k-j})\} = \beta h(x^{l(k)}). \tag{3.12}$$

Replace k by $l(k)-1$. we obtain

$$h(x^{l(k)}) \leq \beta h(x^{l(l(k)-1)}), \tag{3.13}$$

which together with $\beta \in (\frac{1}{2}, 1)$ gives $\lim\limits_{k\to\infty} h(x^{l(k)}) = 0$. Further, by (3.12), we can conclude that $\lim\limits_{k\to\infty} h(x^k) = 0$. □

Theorem 3.1 Supposed that the assumptions A1-A3 hold, $\{x^k\}$ is an infinite sequence generated by Algorithm A, d^k is the solution of $QSDP(x^k, H_k)$ (2.7). If the multiplier corresponding to d^k is uniform bounded, then there exists $\widetilde{\mathcal{K}} \subseteq \{1, 2, \cdots\}$ such that $\lim\limits_{k\in\widetilde{\mathcal{K}}} d^k = 0$.

Proof. By the assumption A1, we know that $\{x^k\}$ is bounded, so there exists an infinite index set $\mathcal{K} \subseteq \{1, 2, \cdots\}$, such that $\{x^k\}_{\mathcal{K}}$ is convergent. Let $\lim\limits_{k\in\mathcal{K}} x^k = x^*$.

We consider the following two cases:

Case 1. The index set $\mathcal{K}_0 = \{k \in \mathcal{K} \mid \nabla f(x^k)^{\mathrm{T}} d^k \leq -\frac{1}{2}(d^k)^{\mathrm{T}} H_k d^k\}$ is infinite.

By (2.14), we obtain

$$f(x^{k+1}) = f(x^k + t_k d^k) \leq f(x^{l(k)}) - t_k\alpha(d^k)^{\mathrm{T}} H_k d^k \leq f(x^{l(k)}), \quad \forall\ k \in \mathcal{K}_0. \tag{3.14}$$

Since $m(k+1) \leq m(k)+1$, we obtain

$$f(x^{l(k+1)}) \leq \max_{0\leq j\leq m(k)+1}\{f(x^{k+1-j})\} = \max\{f(x^{k+1}), f(x^{l(k)})\} = f(x^{l(k)}). \tag{3.15}$$

This implies that the sequence $\{f(x^{l(k)})\}$ is not increasing. Combining with the boundedness of $\{f(x^{l(k)})\}$, it follows that $\{f(x^{l(k)})\}_{\mathcal{K}_0}$ is convergent.

For $\{l(k)-1,\ \ k \in \mathcal{K}_0\}$, we obtain

$$f(x^{l(k)}) \leq f(x^{l(l(k)-1)}) - t_{l(k)-1}\alpha(d^{l(k)-1})^{\mathrm{T}} H_{l(k)-1} d^{l(k)-1}. \tag{3.16}$$

Since $\{f(x^{l(k)})\}$ is convergent, we have

$$\lim_{\mathcal{K}_0} t_{l(k)-1}\alpha(d^{l(k)-1})^{\mathrm{T}} H_{l(k)-1} d^{l(k)-1} = 0,$$

By Lemma 3.3, we know that there exists $\bar{t} > 0$ such that $t_{l(k)-1} \geq \bar{t} > 0$, so by the assumption A3, we obtain

$$\lim_{\mathcal{K}_0} d^{l(k)-1} = 0. \tag{3.17}$$

The uniform continuity of $f(x)$ implies that

$$\lim_{\mathcal{K}_0} f(x^{l(k)-1}) = \lim_{\mathcal{K}_0} f(x^{l(k)}). \tag{3.18}$$

Let $\hat{l}(k) = l(k+M+2)$, it is not difficult to prove by induction that for any given $j \geq 1$,

$$\lim_{\mathcal{K}_0} \|d^{\hat{l}(k)-j}\| = 0, \tag{3.19}$$

$$\lim_{\mathcal{K}_0} f(x^{\hat{l}(k)-j}) = \lim_{\mathcal{K}_0} f(x^{l(k)}). \tag{3.20}$$

For any $k \in \mathcal{K}_0$, we obtain $x^{k+1} = x^{\hat{l}(k)} - \sum\limits_{j=1}^{\hat{l}(k)-k-1} t_{\hat{l}(k)-j} d^{\hat{l}(k)-j}$. Note that $\hat{l}(k) - k - 1 \leq M + 1$ and (3.19), we get $\lim\limits_{\mathcal{K}_0} \|x^{k+1} - x^{\hat{l}(k)}\| = 0$. So it follows from the convergence of $\{f(x^{l(k)})\}$ and the uniform continuity of $f(x)$ that

$$\lim_{\mathcal{K}_0} f(x^{k+1}) = \lim_{\mathcal{K}_0} f(x^{l(k)}).$$

So let $k\ (\in \mathcal{K}_0) \to \infty$, taking the limit in (3.14), we have

$$\lim_{\mathcal{K}_0} t_k \alpha (d^k)^{\mathrm{T}} H_k d^k = 0. \tag{3.21}$$

Similar to the proof of (3.17), we obtain $\lim\limits_{\mathcal{K}_0} d^k = 0$. Hence, let $\widetilde{\mathcal{K}} = \mathcal{K}_0$ and the conclusion follows.

Case 2. The index set $\mathcal{K}_0 = \{k \in \mathcal{K} \mid \nabla f(x^k)^{\mathrm{T}} d^k \leq -\frac{1}{2}(d^k)^{\mathrm{T}} H_k d^k\}$ is finite, which implies that $\mathcal{K}_1 = \{k \in \mathcal{K} \mid \nabla f(x^k)^{\mathrm{T}} d^k > -\frac{1}{2}(d^k)^{\mathrm{T}} H_k d^k\}$ is infinite.

By contradiction, supposed that the conclusion is not true, then $\lim\limits_{\mathcal{K}_1} d^k \neq 0$. So there exist $\mathcal{K}_2 \subseteq \mathcal{K}_1$ and a constant $\varepsilon > 0$, such that $\|d^k\| > \varepsilon$ for $k \in \mathcal{K}_2$.

Since d^k is the solution of $QSDP(x^k, H_k)$ (2.7), by KKT condition of $QSD(x^k, H_k)$ (2.7) , it follows that there exists a positive semidefinite matrix Y_k such that

$$\nabla f(x^k) + H_k d^k + DG(x^k)^* Y_k = 0, \tag{3.22}$$

$$\mathrm{Tr}((G(x^k) + DG(x^k) d^k - z_k I_m) Y_k) = 0, \tag{3.23}$$

According to the assumption of Theorem 3.1, there exists $\widetilde{M} > 0$ such that $\|Y_k\|_F \leq \widetilde{M}$.

By Lemma 3.4, we know $\lim\limits_{k\to\infty} h(x^k) = 0$, hence there exists $k_0 > 0$, such that

$$h(x^k) \leq \frac{1}{2\widetilde{M} m} a\varepsilon^2, \quad \text{for } k\ (\in \mathcal{K}_2) > k_0, \tag{3.24}$$

combining with $\|d^k\| > \varepsilon$ and the assumption A3, we obtain

$$h(x^k) \leq \frac{1}{2\tilde{M} m} (d^k)^{\mathrm{T}} H_k d^k. \tag{3.25}$$

It follows from (2.2) that

$$\mathrm{Tr}(DG(x^k) d^k Y_k) = \mathrm{Tr}((\sum_{i=1}^{n} d_i^k \frac{\partial G(x^k)}{\partial x_i}) Y_k) = \sum_{i=1}^{n} \mathrm{Tr}(\frac{\partial G(x^k)}{\partial x_i} Y_k) d_i^k = \sum_{i=1}^{n} < \frac{\partial G(x^k)}{\partial x_i},\ Y_k > d_i^k. \tag{3.26}$$

It follows from (3.23) that

$$\mathrm{Tr}((DG(x^k)d^k)Y_k) = \mathrm{Tr}((G(x^k) - z_k I_m)Y_k), \tag{3.27}$$

so (3.26) and (3.27) give rise to

$$\sum_{i=1}^{n} < \frac{\partial G(x^k)}{\partial x_i},\ Y_k > d_i^k = \mathrm{Tr}((G(x^k) - z_k I_m)Y_k). \tag{3.28}$$

By (3.22) and (3.28), we have

$$\begin{aligned}\nabla f(x^k)^{\mathrm{T}} d^k &= -(d^k)^{\mathrm{T}} H_k d^k - (DG(x^k)^* Y_k)^{\mathrm{T}} d^k \\ &= -(d^k)^{\mathrm{T}} H_k d^k - \sum_{i=1}^{n} < \frac{\partial G(x^k)}{\partial x_i}, Y_k > d_i^k \\ &= -(d^k)^{\mathrm{T}} H_k d^k + \mathrm{Tr}((G(x^k) - z_k I_m)Y_k). \end{aligned} \tag{3.29}$$

By $Neumann$ Inequality, we obtain

$$\begin{aligned}\mathrm{Tr}((G(x^k) - z_k I_m)Y_k) &\le \sum_{i=1}^{m} \lambda_i(G(x^k) - z_k I_m)\lambda_i(Y_k) \\ &\le \sum_{i=1}^{m} \lambda_i(G(x^k) - z_k I_m)\|Y_k\|_F \\ &\le \sum_{i=1}^{m} \lambda_i(G(x^k) - z_k I_m)\widetilde{M} \\ &\le \sum_{i=1}^{m} \lambda_i(G(x^k))\widetilde{M}, \end{aligned} \tag{3.30}$$

the last inequality above is due to $z_k \ge 0$. According to the definition (2.8) of $h(x^k)$ and (3.30), we obtain

$$\mathrm{Tr}((G(x^k) - z_k I_m)Y_k) \le \widetilde{M} m h(x^k) \le \frac{1}{2}(d^k)^{\mathrm{T}} H_k d^k. \tag{3.31}$$

Substituting (3.31) into (3.29), it follows that

$$\nabla f(x^k)^{\mathrm{T}} d^k \le -\frac{1}{2}(d^k)^{\mathrm{T}} H_k d^k,$$

which contradicts the definition of $\mathcal{K}_1$. Hence, the conclusion is true. □

Theorem 3.2 Supposed that $\{x^k\}$ is an infinite sequence generated by Algorithm A, and the assumptions in Theorem 3.1 hold, then any accumulation point of $\{x^k\}$ is a KKT point of $NLSDP$ (1.1).

Proof. Supposed that x^* is an accumulation point of $\{x^k\}$, then there exists $\mathcal{K} \subseteq \{1, 2, \cdots\}$, such that $\lim\limits_{k\in\mathcal{K}} x^k = x^*$. In view of the assumption A3, without loss of generality, we suppose that $\lim\limits_{k\in\mathcal{K}} H_k = H_*$.

By Lemma 3.4, we have $\lim_{k\in\mathcal{K}} h(x^k) = h(x^*) = 0$, which means that x^* is a feasible point of $NLSDP$ (1.1).

By Theorem 3.1, there exists $\widetilde{\mathcal{K}} \subseteq \{1, 2, \cdots\}$ such that $\lim_{\widetilde{\mathcal{K}}} d^k = d^* = 0$. By the proof of Theorem 3.1, we know that $\widetilde{\mathcal{K}} \subseteq \mathcal{K}$.

According to KKT conditions of $QSDP$ (2.7), we obtain

$$\begin{aligned} &\nabla f(x^k) + H_k d^k + DG(x^k)^* Y_k = 0, \\ &Y_k \succeq 0, \quad \mathrm{Tr}((G(x^k) + DG(x^k)d^k - z_k I_m)Y_k) = 0. \end{aligned}$$

Let $k(\in \widetilde{\mathcal{K}}) \to \infty$, taking the limit, we obtain

$$\begin{aligned} &\nabla f(x^*) + DG(x^*)^* Y_* = 0, \\ &Y_* \succeq 0, \quad < G(x^*), Y_* >= 0. \end{aligned}$$

This implies that x^* is a KKT point of $NLSDP$ (1.1). □

4 Numerical experiments

In this section, preliminary numerical experiments of Algorithm A is implemented. Algorithm A was coded by Matlab (2014a) and run on the computer with Windows 7 (64 bite), Intel(R) Core(TM) i7-4790 CPU @ 3.60GHz 3.60GHz, RAM: 4.00GB.

In the numerical experiments, the parameters are chosen as follows: $\alpha = 0.25$, $\beta = 0.85$, $\sigma = 0.5$, $M = 3$. And the termination criteria of Algorithm A is: $\parallel d^k \parallel \leq 10^{-4}$.

The test problem is chosen from [11].

Problem 1. Nearest Correlation Matrix (NCM) Problem:

$$\begin{array}{ll} \min & f(X) = \frac{1}{2}\|X - C\|_F^2 \\ \text{s.t} & X \preceq \epsilon I, \\ & X_{ii} = 1, i = 1, 2, ..., m, \end{array} \tag{4.1}$$

where $C \in \mathbb{S}^m$ is a given matrix, $X \in \mathbb{S}^m$, ϵ is a scalar.

In the implementation, $\epsilon = 10^{-3}$, C is generated randomly, which diagonal elements are 1. We test ten times for every fixed dimensionality.

We compare Algorithm A with the ones in [11] (denoted by Algo. YYH) and [14] (denoted by Algo. YYY).

The numerical results are listed in Table 1. The meaning of the notations in Table 1 are described as follows:

$$\begin{array}{rl} n: & \text{the dimensionality of independent variable;} \\ m: & \text{the dimensionality of} \mathcal{G}(x); \\ \mathrm{A}-Iter: & \text{the average number of evaluation of iterations.} \end{array}$$

Table 1. Numerical results of NCM

n	m	x^0	Algorithm	A-$Iter$
			Algorithm A	15
10	5	$(0.5, ..., 0.5)^{\mathrm{T}}$	Algo. YYY	8
			Algo. YYH	9
			Algorithm A	15
45	10	$(0.5, ..., 0.5)^{\mathrm{T}}$	Algo. YYY	8
			Algo. YYH	10
			Algorithm A	17
105	15	$(0.5, ..., 0.5)^{\mathrm{T}}$	Algo. YYY	10
			Algo. YYH	11
			Algorithm A	17
190	20	$(0.5, ..., 0.5)^{\mathrm{T}}$	Algo. YYY	11
			Algo. YYH	12

5 Concluding remarks

In this paper, we have presented a new SSDP algorithm for nonlinear semidefinite programming. Two subproblems, which are constructed skillfully, are solved to generate the search directions. The nonmonotone line search ensures that the objective function or constraint violation function is sufficiently reduced. The global convergence of the proposed algorithm is shown under some mild conditions. The preliminary numerical results show that the proposed algorithm is effective.

References

[1] Konno, H., Kawadai, N. and Wu, D.: Estimation of failure probability using semi-definite logitmodel. Computational Management Science, 1: 59-73 (2003)

[2] Apkarian, P., Noll, D and Tuan, D.: Fixed-order HI control design via partially argmented Lagrangian method. International Journal of Control, 11: 1137-1148 (2003)

[3] Kanno, Y. and Takewaki, I.: Sequential semidefinite program for maximum robustness design of structures under load uncertainty. Journal of Optimization Theroy and Applications, 130: 265-287 (2006)

[4] Freund, R. W., Jarre, F. and Vogelbusch, C. H.: Nonlinear semidefinite programming: sensitivity, convergence and an application in passive reduce-order modeling. Mathematical Programming, 109: 581-611 (2007)

[5] Sun, J., Zhang, L. W. and Wu, Y.: Properties of the augmented Lagrangian in nonlinear semidefinite optimization. Journal of Optimization Theory and Applications, 129: 437-456 (2006)

[6] Noll, D.: Local convergence of an augmented Lagrangian method for matrix inequality constrained programming. Optimization Methods and Software, 22: 777 – 802 (2007)

[7] Sun, D. F., Sun, J. and Zhang, L. W.: The rate of convergence of the augmented Lagrangian method for nonlinear semidefinite programming. Mathwmatical Programming, 114: 349-391 (2008)

[8] Luo, H. Z., Wu, H. X. and Chen, G. T.: On the convergence of augmented Lagrangian methods for nonlinear semidefinite programming. Journal of Global Optimization, 54: 599-618 (2012)

[9] Wu, H. X., Luo, H. Z. and Ding, X. D.: Global convergence of modified augmented Lagrangian methods for nonlinear semidefinite programming[J]. Computational Optimization and Applications, 56: 531-558 (2013)

[10] Wu, H. X., Luo, H. Z. and Yang, J. F.: Nonlinear separation approach for the augmented Lagrangianin nonlinear semidenite programming. Journal of Global Optimization, 59(4): 695-727 (2014)

[11] Yamashita, H., Yabe, H. and Harada, K.: A primal – dual interior point method for nonlinear semidefinite programming. Mathematical Programming, 135: 89-121 (2012)

[12] Yamashita, H. and Yabe, H.: Local and superlinear convergence of a primal-dual interior point method for nonlinear semidefinite programming. Mathematical Programming, 132: 1-30 (2012)

[13] Aroztegui, M., Herskovits, J. and Roche, J. R.: A feasible direction interior point algorithm for nonlinear semidefinite programming. Structral and Multidisciplinary Optimization, 50(6): 1019-1035 (2014)

[14] Yamakawa, Y., Yamashita, N. and Yabe, H.: A differentiable merit function for the shifted perturbed Karush-Kuhn-Tucker conditions of the nonlinear semidefinite programming. Pacific Journal of Optimization, 11: 557-579 ((2015))

[15] Li, J. L., Yang, Z. P. and Jian, J. B.: A globally convergent QP-free algorithm for nonlinear semidefinite programming. Journal of Inequalities and Applications, 145: 1-21 (2017),

[16] Correa, R. and Ramirez, H.: A global algorithm for nonlinear semidefinite programming. SIAM Journal on Optimization, 15: 303-318 (2004)

[17] Gomez, W. and Ramirez, H.: A filter algorithm for nonlinear semidefinite programming. Computation and Applied Mathematics, 29: 297-32 (2010)

[18] Zhu, Z. B. and Zhu, H. L.: A filter method for nonlinear semidefinite programming with global convergence. Acta Mathematica Sinica (English Series), 30: 1810-1826 (2014)

[19] Li, C. J. and Sun, W. Y.: On filter-successive linearization methods for nonlinear semidefinite programming. Science in China (Series A), Mathematics, 52: 2341-2361 (2009)

[20] Zhao, Q.: Global convergence of new method for semidefinite programming. South East Asian Journal of Mathematics and Mathematical Sciences, 3: 189-198 (2013)

[21] Chen, Z. W. and Miao, S. C.: A penalty-free method with trust region for nonlinear semidefinite programming. Asia-Pacific Journal of Operational Reasearch, 32(1): 1-24 (2015)

[22] Zhang, J. L. and Zhang, X. S.: A robust SQP method for optimization with inequality constraints. Journal of Computational Mathematics, 21(2): 247-256 (2003)

[23] Long, J., Ma, C. F. and Nie, P. Y.: A new filter method for solving nonlinear complementarity problems. Application Mathematics and Computation, 185: 705-718 (2007)

[24] Liu, X. W. and Yuan, Y. X.: A robust algorithm for optimization with general equality and inequality constraints. SIAM Journal on Optimization, 22(2): 517-534 (2000)

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Approximation by Sublinear and Max-product Operators using Convexity

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we consider quantitatively using convexity the approximation of a function by general positive sublinear operators with applications to Max-product operators. These are of Bernstein type, of Favard-Szász-Mirakjan type, of Baskakov type, of Meyer-Köning and Zeller type, of sampling type, of Lagrange interpolation type and of Hermite-Fejér interpolation type. Our results are both: under the presence of smoothness and without any smoothness assumption on the function to be approximated which fulfills a convexity property.

2010 AMS Mathematics Subject Classification: 41A17, 41A25, 41A36.
Keywords and Phrases: positive sublinear operators, Max-product operators, modulus of continuity, convexity.

1 Background

We make

Remark 1 *Let* $f \in C([a,b])$, $x_0 \in (a,b)$, $0 < h \leq \min(x_0 - a, b - x_0)$, *and* $|f(t) - f(x_0)|$ *is convex in* $t \in [a,b]$.

By Lemma 8.1.1, p. 243 of [1] we have that

$$|f(t) - f(x_0)| \leq \frac{\omega_1(f,h)}{h} |t - x_0|, \quad \forall\ t \in [a,b], \tag{1}$$

where

$$\omega_1(f,h) := \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq h}} |f(x) - f(y)|, \tag{2}$$

the first modulus of continuity of f.

1

We also make

Remark 2 *Let $f \in C^n([a,b])$, $n \in \mathbb{N}$, $x_0 \in (a,b)$, $0 < h \leq \min(x_0 - a, b - x_0)$, and $\left|f^{(n)}(t) - f^{(n)}(x_0)\right|$ is convex in $t \in [a,b]$. We have that*

$$f(t) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(t - x_0)^k + I_t, \tag{3}$$

where

$$I_t = \int_{x_0}^{t} \left(\int_{x_0}^{t_1} \dots \left(\int_{x_0}^{t_{n-1}} \left(f^{(n)}(t_n) - f^{(n)}(x_0) \right) dt_n \right) \dots \right) dt_1. \tag{4}$$

Assuming $f^{(k)}(x_0) = 0$, $k = 1, ..., n$, we get

$$f(t) - f(x_0) = I_t. \tag{5}$$

By Lemma 8.1.1, p. 243 of [1] we have

$$\left|f^{(n)}(t) - f^{(n)}(x_0)\right| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h}|t - x_0|, \quad \forall\ t \in [a,b]. \tag{6}$$

Furthermore it holds

$$|I_t| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h}\frac{|t - x_0|^{n+1}}{(n+1)!}, \quad \forall\ t \in [a,b]. \tag{7}$$

Hence we derive that

$$|f(t) - f(x_0)| \overset{(5)}{\leq} \frac{\omega_1\left(f^{(n)}, h\right)}{h}\frac{|t - x_0|^{n+1}}{(n+1)!}, \quad \forall\ t \in [a,b]. \tag{8}$$

We have proved the following results:

Theorem 3 *Let $f \in C([a,b])$, $x \in (a,b)$, $0 < h \leq \min(x - a, b - x)$, and $|f(\cdot) - f(x)|$ is convex over $[a,b]$. Then*

$$|f(\cdot) - f(x)| \leq \frac{\omega_1(f, h)}{h}|\cdot - x|, \quad \text{over } [a,b]. \tag{9}$$

Theorem 4 *Let $f \in C^n([a,b])$, $n \in \mathbb{N}$, $x \in (a,b)$, $0 < h \leq \min(x - a, b - x)$, and $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ is convex over $[a,b]$. Assume that $f^{(k)}(x) = 0$, $k = 1, ..., n$. Then*

$$|f(\cdot) - f(x)| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h}\frac{|\cdot - x|^{n+1}}{(n+1)!}, \quad \text{over } [a,b]. \tag{10}$$

We give

Definition 5 *Call* $C_+([a,b]) := \{f : [a,b] \to \mathbb{R}_+ \text{ and continuous}\}$. *Let* L_N *from* $C_+([a,b])$ *into* $C_+([a,b])$ *be a sequence of operators satisfying the following properties (see also [6], p. 17):*

(i) (positive homogeneous)

$$L_N(\alpha f) = \alpha L_N(f), \ \forall \ \alpha \geq 0, \ \forall \ f \in C_+([a,b]), \tag{11}$$

(ii) (Monotonicity)
if $f, g \in C_+([a,b])$ *satisfy* $f \leq g$, *then*

$$L_N(f) \leq L_N(g), \ \ \forall \ N \in \mathbb{N}, \tag{12}$$

(iii) (Subadditivity)

$$L_N(f+g) \leq L_N(f) + L_N(g), \ \ \forall \ f, g \in C_+([a,b]). \tag{13}$$

We call L_N *positive sublinear operators.*

We make

Remark 6 *As in [6], p. 17, we get that for* $f, g \in C_+([a,b])$

$$|L_N(f)(x) - L_N(g)(x)| \leq L_N(|f-g|)(x), \ \ \forall \ x \in [a,b]. \tag{14}$$

From now on we assume that $L_N(1) = 1$, $\forall \ N \in \mathbb{N}$. *Hence it holds*

$$|L_N(f)(x) - f(x)| \leq L_N(|f(\cdot) - f(x)|)(x), \ \ \forall \ x \in [a,b], \ \forall \ N \in \mathbb{N}, \tag{15}$$

see also [6], p. 17.

We obtain the following results:

Theorem 7 *Let* $f \in C_+([a,b])$, $x \in (a,b)$, $0 < h \leq \min(x-a, b-x)$, *and* $|f(\cdot) - f(x)|$ *is a convex function over* $[a,b]$. *Let* $\{L_N\}_{N \in \mathbb{N}}$ *positive sublinear operators from* $C_+([a,b])$ *into itself, such that* $L_N(1) = 1$, $\forall \ N \in \mathbb{N}$. *Then*

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1(f,h)}{h} L_N(|\cdot - x|)(x), \ \ \forall \ N \in \mathbb{N}. \tag{16}$$

Proof. By (9) and (15). ∎

Theorem 8 *Let* $f \in C^n([a,b], \mathbb{R}_+)$, $n \in \mathbb{N}$, $x \in (a,b)$, $0 < h \leq \min(x-a, b-x)$, *and* $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ *is convex over* $[a,b]$. *Assume that* $f^{(k)}(x) = 0$, $k = 1, ..., n$. *Let* $\{L_N\}_{N \in \mathbb{N}}$ *positive sublinear operators from* $C_+([a,b])$ *into itself, such that* $L_N(1) = 1$, $\forall \ N \in \mathbb{N}$. *Then*

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} L_N\left(|\cdot - x|^{n+1}\right)(x), \ \ \forall \ N \in \mathbb{N}. \tag{17}$$

3

Proof. By (10) and (15). ■

We continue with

Theorem 9 *Let $f \in C_+([a,b])$, $x \in (a,b)$, $0 < L_N(|\cdot - x|)(x) \leq \min(x-a, b-x)$, $\forall\ N \in \mathbb{N}$, and $|f(\cdot) - f(x)|$ is a convex function over $[a,b]$. Here L_N are positive sublinear operators from $C_+([a,b])$ into itself, such that $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. Then*

$$|L_N(f)(x) - f(x)| \leq \omega_1(f, L_N(|\cdot - x|)(x)), \quad \forall\ N \in \mathbb{N}. \tag{18}$$

If $L_N(|\cdot - x|)(x) \to 0$, then $L_N(f)(x) \to f(x)$, as $N \to +\infty$.

Proof. By (16). ■

Theorem 10 *Let $f \in C^n([a,b], \mathbb{R}_+)$, $n \in \mathbb{N}$, $x \in (a,b)$, $0 < L_N\left(|\cdot - x|^{n+1}\right)(x) \leq \min(x-a, b-x)$, $\forall\ N \in \mathbb{N}$, and $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ is convex over $[a,b]$. Assume that $f^{(k)}(x) = 0$, $k = 1, ..., n$. Here $\{L_N\}_{N \in \mathbb{N}}$ are positive sublinear operators from $C_+([a,b])$ into itself, such that $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. Then*

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1\left(f^{(n)}, L_N\left(|\cdot - x|^{n+1}\right)(x)\right)}{(n+1)!}, \quad \forall\ N \in \mathbb{N}. \tag{19}$$

If $L_N\left(|\cdot - x|^{n+1}\right)(x) \to 0$, then $L_N(f)(x) \to f(x)$, as $N \to +\infty$.

Proof. By (17). ■

Next we combine both Theorems 7, 8:

Theorem 11 *Let $f \in C^n([a,b], \mathbb{R}_+)$, $n \in \mathbb{Z}_+$, $x \in (a,b)$, $0 < h \leq \min(x-a, b-x)$, and $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ is convex over $[a,b]$. Assume that $f^{(k)}(x) = 0$, $k = 1, ..., n$. Let $\{L_N\}_{N \in \mathbb{N}}$ positive sublinear operators from $C_+([a,b])$ into itself, such that $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. Then*

$$|L_N(f)(x) - f(x)| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} L_N\left(|\cdot - x|^{n+1}\right)(x), \quad \forall\ N \in \mathbb{N};\ n \in \mathbb{Z}_+. \tag{20}$$

The initial conditions $f^{(k)}(x) = 0$, $k = 1, ..., n$, are void when $n = 0$.

In this article we study under convexity quantitatively the approximation properties of Max-product operators to the unit. These are special cases of positive sublinear operators. We present also results regarding the convergence to the unit of general positive sublinear operators under convexity. Special emphasis is given to our study about approximation under the presence of smoothness. Our work is inspired from [6].

Under our convexity conditions the derived convergence inequalities are elegant and compact with very small constants.

2 Main Results

Here we apply Theorem 11 to Max-product operators.

We make

Remark 12 *We start with the Max-product Bernstein operators ([6], p. 10)*

$$B_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} p_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} p_{N,k}(x)}, \quad \forall\ N \in \mathbb{N}, \tag{21}$$

$p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-1}$, *$x \in [0,1]$, $\bigvee$ stands for maximum, and* $f \in C_+([0,1]) = \{f : [0,1] \to \mathbb{R}_+$ *is continuous*$\}$, *where* $\mathbb{R}_+ := [0,\infty)$.

Clearly $B_N^{(M)}$ *is a positive sublinear operators from* $C_+([0,1])$ *into itself with* $B_N^{(M)}(1) = 1$.

By [6], p. 31, we have

$$B_N^{(M)}(|\cdot - x|)(x) \leq \frac{6}{\sqrt{N+1}}, \quad \forall\ x \in [0,1],\ \forall\ N \in \mathbb{N}. \tag{22}$$

And by [2] we get:

$$B_N^{(M)}(|\cdot - x|^m)(x) \leq \frac{6}{\sqrt{N+1}}, \quad \forall\ x \in [0,1],\ m, N \in \mathbb{N}. \tag{23}$$

Denote by

$$C_+^n([0,1]) = \{f : [0,1] \to \mathbb{R}_+, n\text{-times continuously differentiable}\}, \quad n \in \mathbb{Z}_+.$$

We present

Theorem 13 *Let* $f \in C_+^n([0,1])$, $n \in \mathbb{Z}_+$, $x \in (0,1)$ *and* $N^* \in \mathbb{N} : 0 < \frac{1}{\sqrt{N^*+1}} \leq \min(x, 1-x)$, *and* $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ *is convex over* $[0,1]$. *Assume that* $f^{(k)}(x) = 0$, $k = 1, ..., n$. *Then*

$$\left|B_N^{(M)}(f)(x) - f(x)\right| \leq \frac{6\omega_1\left(f^{(n)}, \frac{1}{\sqrt{N+1}}\right)}{(n+1)!}, \quad \forall\ N \in \mathbb{N} : N \geq N^*. \tag{24}$$

It holds $\lim_{N \to +\infty} B_N^{(M)}(f)(x) = f(x)$.

Proof. By (20) we get

$$\left|B_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} B_N^{(M)}\left(|\cdot - x|^{n+1}\right)(x) \overset{(23)}{\leq}$$

$$\frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} \frac{6}{\sqrt{N+1}} =$$

(setting $h := \frac{1}{\sqrt{N+1}}$)

$$\frac{6\omega_1\left(f^{(n)}, \frac{1}{\sqrt{N+1}}\right)}{(n+1)!}, \tag{25}$$

proving the claim. ■

We make

Remark 14 *Here we focus on the truncated Favard-Szász-Mirakjan operators*

$$T_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} s_{N,k}(x)\, f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} s_{N,k}(x)}, \quad x \in [0,1], \ N \in \mathbb{N}, \ f \in C_+([0,1]), \tag{26}$$

$s_{N,k}(x) = \frac{(Nx)^k}{k!}$, *see also [6], p. 11.*

By [6], p. 178-179 we have

$$T_N^{(M)}(|\cdot - x|)(x) \leq \frac{3}{\sqrt{N}}, \quad \forall\, x \in [0,1], \ \forall\, N \in \mathbb{N}. \tag{27}$$

And by [2] we get

$$T_N^{(M)}(|\cdot - x|^m)(x) \leq \frac{3}{\sqrt{N}}, \quad \forall\, x \in [0,1], \ \forall\, m, N \in \mathbb{N}. \tag{28}$$

The operators $T_N^{(M)}$ are positive sublinear from $C_+([0,1])$ into itself with $T_N^{(M)}(1) = 1$, $\forall\, N \in \mathbb{N}$.

We give

Theorem 15 *Let $f \in C_+^n([0,1])$, $n \in \mathbb{Z}_+$, $x \in (0,1)$ and $N^* \in \mathbb{N} : 0 < \frac{1}{\sqrt{N^*}} \leq \min(x, 1-x)$, and $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ is convex over $[0,1]$. Assume that $f^{(k)}(x) = 0$, $k = 1, ..., n$. Then*

$$\left|T_N^{(M)}(f)(x) - f(x)\right| \leq \frac{3\omega_1\left(f^{(n)}, \frac{1}{\sqrt{N}}\right)}{(n+1)!}, \quad \forall\, N \in \mathbb{N} : N \geq N^*. \tag{29}$$

It holds $\lim_{N \to +\infty} T_N^{(M)}(f)(x) = f(x)$.

Proof. By (20) we get

$$\left|T_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} T_N^{(M)}\left(|\cdot - x|^{n+1}\right)(x) \overset{(28)}{\leq}$$

$$\frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} \frac{3}{\sqrt{N}} =$$

6

(setting $h := \frac{1}{\sqrt{N}}$)

$$\frac{3\omega_1\left(f^{(n)}, \frac{1}{\sqrt{N}}\right)}{(n+1)!}, \tag{30}$$

proving the claim. ■

We make

Remark 16 *Next we study the truncated Max-product Baskakov operators (see [6], p. 11)*

$$U_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} b_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} b_{N,k}(x)}, \quad x \in [0,1], \ f \in C_+([0,1]), \ N \in \mathbb{N}, \tag{31}$$

where

$$b_{N,k}(x) = \binom{N+k-1}{k} \frac{x^k}{(1+x)^{N+k}}.$$

From [6], pp. 217-218, we get ($x \in [0,1]$)

$$\left(U_N^{(M)}(|\cdot - x|)\right)(x) \leq \frac{12}{\sqrt{N+1}}, \quad N \geq 2, \ N \in \mathbb{N}. \tag{32}$$

And as in [2], we obtain ($m \in \mathbb{N}$)

$$\left(U_N^{(M)}(|\cdot - x|^m)\right)(x) \leq \frac{12}{\sqrt{N+1}}, \quad N \geq 2, \ N \in \mathbb{N}, \ \forall \ x \in [0,1]. \tag{33}$$

Also it holds $U_N^{(M)}(1)(x) = 1$, and $U_N^{(M)}$ are positive sublinear operators from $C_+([0,1])$ into itself, $\forall \ N \in \mathbb{N}$.

We give

Theorem 17 *Let $f \in C_+^n([0,1])$, $n \in \mathbb{Z}_+$, $x \in (0,1)$ and $N^* \in \mathbb{N} - \{1\} : 0 < \frac{1}{\sqrt{N^*+1}} \leq \min(x, 1-x)$, and $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ is convex over $[0,1]$. Assume that $f^{(k)}(x) = 0$, $k = 1, ..., n$. Then*

$$\left|U_N^{(M)}(f)(x) - f(x)\right| \leq \frac{12\omega_1\left(f^{(n)}, \frac{1}{\sqrt{N+1}}\right)}{(n+1)!}, \quad \forall \ N \in \mathbb{N} : N \geq N^*. \tag{34}$$

It holds $\lim_{N \to +\infty} U_N^{(M)}(f)(x) = f(x)$.

Proof. By (20) we get

$$\left|U_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} U_N^{(M)}\left(|\cdot - x|^{n+1}\right)(x) \overset{(33)}{\leq}$$

7

$$\frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} \frac{12}{\sqrt{N+1}} =$$

(setting $h := \frac{1}{\sqrt{N+1}}$)

$$\frac{12\omega_1\left(f^{(n)}, \frac{1}{\sqrt{N+1}}\right)}{(n+1)!}, \tag{35}$$

proving the claim. ■

We make

Remark 18 *Here we study Max-product Meyer-Köning and Zeller operators (see [6], p. 11) defined by*

$$Z_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} s_{N,k}(x) f\left(\frac{k}{N+k}\right)}{\bigvee_{k=0}^{\infty} s_{N,k}(x)}, \quad \forall\ N \in \mathbb{N},\ f \in C_+([0,1]), \tag{36}$$

$s_{N,k}(x) = \binom{N+k}{k} x^k$, $x \in [0,1]$.

By [6], p. 253, we get that

$$Z_N^{(M)}(|\cdot - x|)(x) \leq \frac{8\left(1+\sqrt{5}\right)}{3} \frac{\sqrt{x}(1-x)}{\sqrt{N}}, \ \ \forall\ x \in [0,1],\ N \geq 4. \tag{37}$$

And by [2], we derive that

$$Z_N^{(M)}(|\cdot - x|^m)(x) \leq \frac{8\left(1+\sqrt{5}\right)}{3} \frac{\sqrt{x}(1-x)}{\sqrt{N}}, \tag{38}$$

$\forall\ x \in [0,1]$, $N \geq 4$, $\forall\ m \in \mathbb{N}$.

The ceiling $\left\lceil \frac{8(1+\sqrt{5})}{3} \right\rceil = 9$, *and using a basic calculus technique (see [4]) we get that* $g(x) := (1-x)\sqrt{x}$ *has an absolute maximum over* $(0,1] : g\left(\frac{1}{3}\right) = \frac{2}{3\sqrt{3}}$. *That is* $(1-x)\sqrt{x} \leq \frac{2}{3\sqrt{3}}$, $\forall\ x \in [0,1]$.

Consequently it holds

$$Z_N^{(M)}(|\cdot - x|^m)(x) \leq \frac{6}{\sqrt{3}\sqrt{N}}, \tag{39}$$

$\forall\ x \in [0,1]$, $\forall\ N \in \mathbb{N} : N \geq 4$, $\forall\ m \in \mathbb{N}$.

Also it holds $Z_N^{(M)}(1) = 1$, *and* $Z_N^{(M)}$ *are positive sublinear operators from* $C_+([0,1])$ *into itself,* $\forall\ N \in \mathbb{N}$.

We give

8

Theorem 19 *Let $f \in C_{+}^{n}([0,1])$, $n \in \mathbb{Z}_{+}$, $x \in (0,1)$ and $N^{*} \in \mathbb{N} : N^{*} \geq 4$ with $0 < \frac{1}{\sqrt{N^{*}}} \leq \min(x, 1-x)$, and $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ is convex over $[0,1]$. Assume that $f^{(k)}(x) = 0$, $k = 1, ..., n$. Then*

$$\left|Z_{N}^{(M)}(f)(x) - f(x)\right| \leq \left(\frac{6}{\sqrt{3}(n+1)!}\right) \omega_{1}\left(f^{(n)}, \frac{1}{\sqrt{N}}\right), \quad \forall\, N \in \mathbb{N} : N \geq N^{*}. \tag{40}$$

It holds $\lim_{N \to +\infty} Z_{N}^{(M)}(f)(x) = f(x)$.

Proof. By (20) we get

$$\left|Z_{N}^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_{1}\left(f^{(n)}, h\right)}{h(n+1)!} Z_{N}^{(M)}\left(|\cdot - x|^{n+1}\right)(x) \overset{(39)}{\leq}$$

$$\frac{\omega_{1}\left(f^{(n)}, h\right)}{h(n+1)!} \frac{6}{\sqrt{3}\sqrt{N}} =$$

(setting $h := \frac{1}{\sqrt{N}}$)

$$\left(\frac{6}{\sqrt{3}(n+1)!}\right) \omega_{1}\left(f^{(n)}, \frac{1}{\sqrt{N}}\right), \tag{41}$$

proving the claim. ■

We make

Remark 20 *Here we mention the Max-product truncated sampling operators (see [6], p. 13) defined by*

$$W_{N}^{(M)}(f)(x) := \frac{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi}}, \quad x \in [0,\pi], \tag{42}$$

$f : [0,\pi] \to \mathbb{R}_{+}$, continuous,

and

$$K_{N}^{(M)}(f)(x) := \frac{\bigvee_{k=0}^{N} \frac{\sin^{2}(Nx-k\pi)}{(Nx-k\pi)^{2}} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin^{2}(Nx-k\pi)}{(Nx-k\pi)^{2}}}, \quad x \in [0,\pi], \tag{43}$$

$f : [0,\pi] \to \mathbb{R}_{+}$, continuous.

By convention we take $\frac{\sin(0)}{0} = 1$, which implies for every $x = \frac{k\pi}{N}$, $k \in \{0, 1, ..., N\}$ that we have $\frac{\sin(Nx-k\pi)}{Nx-k\pi} = 1$.

We define the Max-product truncated combined sampling operators (see also [5])

$$M_{N}^{(M)}(f)(x) := \frac{\bigvee_{k=0}^{N} \rho_{N,k}(x) f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \rho_{N,k}(x)}, \quad x \in [0,\pi], \tag{44}$$

9

$f \in C_+([0,\pi])$, where

$$M_N^{(M)}(f)(x) := \begin{cases} W_N^{(M)}(f)(x), \text{ if } \rho_{N,k}(x) := \frac{\sin(Nx-k\pi)}{Nx-k\pi}, \\ K_N^{(M)}(f)(x), \text{ if } \rho_{N,k}(x) := \left(\frac{\sin(Nx-k\pi)}{Nx-k\pi}\right)^2. \end{cases} \tag{45}$$

By [6], p. 346 and p. 352 we get

$$\left(M_N^{(M)}(|\cdot - x|)\right)(x) \leq \frac{\pi}{2N}, \tag{46}$$

and by [3] ($m \in \mathbb{N}$) we have

$$\left(M_N^{(M)}(|\cdot - x|^m)\right)(x) \leq \frac{\pi^m}{2N}, \ \forall \ x \in [0,\pi], \forall \ N \in \mathbb{N}. \tag{47}$$

Also it holds $M_N^{(M)}(1) = 1$, and $M_N^{(M)}$ are positive sublinear operators from $C_+([0,\pi])$ into itself, $\forall \ N \in \mathbb{N}$.

We give

Theorem 21 *Let $f \in C^n([0,\pi],\mathbb{R}_+)$, $n \in \mathbb{Z}_+$, $x \in (0,\pi)$ and $N^* \in \mathbb{N} : 0 < \frac{1}{N^*} \leq \min(x, \pi - x)$, and $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ is convex over $[0,\pi]$. Assume that $f^{(k)}(x) = 0$, $k = 1, ..., n$. Then*

$$\left|M_N^{(M)}(f)(x) - f(x)\right| \leq \left(\frac{\pi^{n+1}}{2(n+1)!}\right)\omega_1\left(f^{(n)}, \frac{1}{N}\right), \tag{48}$$

$\forall \ N \in \mathbb{N} : N \geq N^*$; $n \in \mathbb{Z}_+$.

It holds $\lim_{N\to+\infty} M_N^{(M)}(f)(x) = f(x)$.

Proof. By (20) we have:

$$\left|M_N^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} M_N^{(M)}\left(|\cdot - x|^{n+1}\right)(x) \overset{(47)}{\leq}$$

$$\frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!}\frac{\pi^{n+1}}{2N} =$$

(setting $h := \frac{1}{N}$)

$$\left(\frac{\pi^{n+1}}{2(n+1)!}\right)\omega_1\left(f^{(n)}, \frac{1}{N}\right), \tag{49}$$

proving the claim. ■

We make

Remark 22 *The Chebyshev knots of first kind* $x_{N,k} := \cos\left(\frac{2(N-k)+1}{2(N+1)}\pi\right) \in (-1,1)$, $k \in \{0,1,...,N\}$, $-1 < x_{N,0} < x_{N,1} < ... < x_{N,N} < 1$, *are the roots of the first kind Chebyshev polynomial* $T_{N+1}(x) := \cos((N+1)\arccos x)$, $x \in [-1,1]$.

Define ($x \in [-1,1]$)

$$h_{N,k}(x) := (1 - x \cdot x_{N,k}) \left(\frac{T_{N+1}(x)}{(N+1)(x - x_{N,k})} \right)^2, \tag{50}$$

the fundamental interpolation polynomials.

The Max-product interpolation Hermite-Fejér operators on Chebyshev knots of the first kind (see p. 12 of [6]) are defined by

$$H_{2N+1}^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} h_{N,k}(x) f(x_{N,k})}{\bigvee_{k=0}^{N} h_{N,k}(x)}, \quad \forall\ N \in \mathbb{N}, \tag{51}$$

for $f \in C_+([-1,1])$, $\forall\ x \in [-1,1]$.

By [6], p. 287, we have

$$H_{2N+1}^{(M)}(|\cdot - x|)(x) \leq \frac{2\pi}{N+1}, \quad \forall\ x \in [-1,1], \forall\ N \in \mathbb{N}. \tag{52}$$

And by [3], we get that

$$H_{2N+1}^{(M)}(|\cdot - x|^m)(x) \leq \frac{2^m \pi}{N+1}, \quad \forall\ x \in [-1,1], \forall\ m, N \in \mathbb{N}. \tag{53}$$

Notice $H_{2N+1}^{(M)}(1) = 1$, *and* $H_{2N+1}^{(M)}$ *maps* $C_+([-1,1])$ *into itself, and it is a positive sublinear operator. Furthermore it holds* $\bigvee_{k=0}^{N} h_{N,k}(x) > 0$, $\forall\ x \in [-1,1]$. *We also have* $h_{N,k}(x_{N,k}) = 1$, *and* $h_{N,k}(x_{N,j}) = 0$, *if* $k \neq j$, *and* $H_{2N+1}^{(M)}(f)(x_{N,j}) = f(x_{N,j})$, *for all* $j \in \{0,1,...,N\}$, *see [6], p. 282.*

We give

Theorem 23 *Let* $f \in C^n([-1,1], \mathbb{R}_+)$, $n \in \mathbb{Z}_+$, $x \in (-1,1)$ *and let* $N^* \in \mathbb{N} : 0 < \frac{1}{N^*+1} \leq \min(x+1, 1-x)$, *and* $\left|f^{(n)}(\cdot) - f^{(n)}(x)\right|$ *is convex over* $[-1,1]$. *Assume that* $f^{(k)}(x) = 0$, $k = 1,...,n$. *Then*

$$\left|H_{2N+1}^{(M)}(f)(x) - f(x)\right| \leq \left(\frac{2^{n+1}\pi}{(n+1)!}\right) \omega_1\left(f^{(n)}, \frac{1}{N+1}\right), \tag{54}$$

$\forall\ N \geq N^*$, $N \in \mathbb{N}$; $n \in \mathbb{Z}_+$.

It holds $\lim_{N \to +\infty} H_{2N+1}^{(M)}(f)(x) = f(x)$.

Proof. By (20) we get

$$\left|H_{2N+1}^{(M)}(f)(x) - f(x)\right| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} H_{2N+1}^{(M)}\left(|\cdot - x|^{n+1}\right)(x) \overset{(53)}{\leq}$$

$$\frac{\omega_1\left(f^{(n)},h\right)}{h\,(n+1)!}\left(\frac{2^{n+1}\pi}{N+1}\right)=$$

(setting $h:=\frac{1}{N+1}$)

$$\left(\frac{2^{n+1}\pi}{(n+1)!}\right)\omega_1\left(f^{(n)},\frac{1}{N+1}\right), \tag{55}$$

proving the claim. ■

We make

Remark 24 *Let $f\in C_+([-1,1])$. Let the Chebyshev knots of second kind $x_{N,k}=\cos\left(\left(\frac{N-k}{N-1}\right)\pi\right)\in[-1,1]$, $k=1,...,N$, $N\in\mathbb{N}-\{1\}$, which are the roots of $\omega_N(x)=\sin(N-1)t\sin t$, $x=\cos t\in[-1,1]$. Notice that $x_{N,1}=-1$ and $x_{N,N}=1$.*

Define

$$l_{N,k}(x):=\frac{(-1)^{k-1}\omega_N(x)}{(1+\delta_{k,1}+\delta_{k,N})(N-1)(x-x_{N,k})}, \tag{56}$$

$N\geq 2$, $k=1,...,N$, and $\omega_N(x)=\prod_{k=1}^{N}(x-x_{N,k})$ and $\delta_{i,j}$ denotes the Kronecher's symbol, that is $\delta_{i,j}=1$, if $i=j$, and $\delta_{i,j}=0$, if $i\neq j$.

The Max-product Lagrange interpolation operators on Chebyshev knots of second kind, plus the endpoints ±1, are defined by ([6], p. 12)

$$L_N^{(M)}(f)(x)=\frac{\bigvee_{k=1}^{N}l_{N,k}(x)f(x_{N,k})}{\bigvee_{k=1}^{N}l_{N,k}(x)},\quad x\in[-1,1]. \tag{57}$$

By [6], pp. 297-298 and [3], we get that

$$L_N^{(M)}(|\cdot-x|^m)(x)\leq\frac{2^{m+1}\pi^2}{3(N-1)}, \tag{58}$$

$\forall\ x\in(-1,1)$ and $\forall\ m\in\mathbb{N}$; $\forall\ N\in\mathbb{N}$, $N\geq4$.

We see that $L_N^{(M)}(f)(x)\geq0$ is well defined and continuous for any $x\in[-1,1]$. Following [6], p. 289, because $\sum_{k=1}^{N}l_{N,k}(x)=1$, $\forall\ x\in[-1,1]$, for any x there exists $k\in\{1,...,N\}:l_{N,k}(x)>0$, hence $\bigvee_{k=1}^{N}l_{N,k}(x)>0$. We have that $l_{N,k}(x_{N,k})=1$, and $l_{N,k}(x_{N,j})=0$, if $k\neq j$. Furthermore it holds $L_N^{(M)}(f)(x_{N,j})=f(x_{N,j})$, all $j\in\{1,...,N\}$, and $L_N^{(M)}(1)=1$.

By [6], pp. 289-290, $L_N^{(M)}$ are positive sublinear operators.

Finally we present

Theorem 25 *Let $f\in C^n([-1,1],\mathbb{R}_+)$, $n\in\mathbb{Z}_+$, $x\in(-1,1)$ and let $N^*\in\mathbb{N}$: $N^*\geq4$, with $0<\frac{1}{N^*-1}\leq\min(x+1,1-x)$, and $\left|f^{(n)}(\cdot)-f^{(n)}(x)\right|$ is convex over $[-1,1]$. Assume that $f^{(k)}(x)=0$, $k=1,...,n$. Then*

$$\left|L_N^{(M)}(f)(x)-f(x)\right|\leq\left(\frac{2^{n+2}\pi^2}{3(n+1)!}\right)\omega_1\left(f^{(n)},\frac{1}{N-1}\right), \tag{59}$$

12

$\forall\ N \in \mathbb{N} : N \geq N^* \geq 4;\ n \in \mathbb{Z}_+$.

It holds $\lim\limits_{N \to +\infty} L_N^{(M)}(f)(x) = f(x)$.

Proof. Using (20) we get:

$$\left| L_N^{(M)}(f)(x) - f(x) \right| \leq \frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} L_N^{(M)}\left(|\cdot - x|^{n+1}\right)(x) \overset{(58)}{\leq}$$

$$\frac{\omega_1\left(f^{(n)}, h\right)}{h(n+1)!} \left(\frac{2^{n+2}\pi^2}{3(N-1)} \right) =$$

(setting $h := \frac{1}{N-1}$)

$$\left(\frac{2^{n+2}\pi^2}{3(n+1)!} \right) \omega_1\left(f^{(n)}, \frac{1}{N-1}\right), \tag{60}$$

proving the claim. ■

References

[1] G. Anastassiou, *Moments in probability and approximation theory*, Pitman Research Notes in Mathematics Series, Longman Group UK, New York, NY, 1993.

[2] G. Anastassiou, *Approximation by Sublinear Operators*, Acta Mathematica Universitatis Comenianae, 87(2018)(2), 237-250.

[3] G. Anastassiou, *Approximation by Max-Product Operators*, Fasc. Math. 60(2018), 5-28.

[4] G. Anastassiou, *Approximation of Fuzzy numbers by Max-product operators*, Transylvanian Journal of Mathematics and Mechanics, 9(2017)(2),117-123.

[5] G. Anastassiou, *Approximations by Multivariate Sublinear and Max-product Operators under Convexity*, Demonstratio Mathematica, 51(2018), 85-105.

[6] B. Bede, L. Coroianu, S. Gal, *Approximation by Max-Product type Operators*, Springer, Heidelberg, New York, 2016.

Symmetric identities for Carlitz's generalized twisted q-Bernoulli numbers and polynomials associated with p-adic invariant integral on $\mathbb{Z}_p$

Cheon Seoung Ryoo

Department of Mathematics, Hannam University, Daejeon 34430, Korea

Abstract : In this paper, we study the symmetry for the Carlitz's generalized twisted q-Bernoulli polynomials $\beta_{n,\chi,q,\zeta}(x)$. We obtain some interesting identities of the power sums and the Carlitz's generalized twisted q-Bernoulli polynomials $\beta_{n,\chi,q,\zeta}(x)$ using the symmetric properties for the p-adic invariant integral on $\mathbb{Z}_p$.

Key words : Symmetric properties, power sums, Bernoulli numbers and polynomials, Carlitz's generalized twisted q-Bernoulli numbers and polynomials, p-adic invariant integral on $\mathbb{Z}_p$.

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Bernoulli polynomials, q-Bernoulli polynomials, the second kind Bernoulli polynomials, Euler polynomials, tangent polynomials, and Bell polynomials were studied by many authors(see [1, 3, 4, 5, 6, 7, 8, 9, 10]). Recently, Y. He obtained several identities of symmetry for Carlitz's q-Bernoulli numbers and polynomials in complex field(see [1]). D. Kim *et al.*[3] studied some identities of symmetry for generalized Carlitz's q-Bernoulli numbers and polynomials by using the p-adic integrals on $\mathbb{Z}_p$ in p-adic field. The purpose of this paper is to obtain some interesting identities of the power sums and Carlitz's generalized twisted q-Bernoulli polynomials $\beta_{n,\chi,q,\zeta}(x)$ using the symmetric properties for the p-adic invariant integral on $\mathbb{Z}_p$.

Let p be a fixed prime number. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q} \text{ (cf. [1-4]) }.$$

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \le 1$ in the present p-adic case. Let

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\}.$$

For $g \in UD(\mathbb{Z}_p)$ the p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_1(g) = \int_{\mathbb{Z}_p} g(x)d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} g(x), \text{ (cf. [2, 3, 4]) }. \tag{1.1}$$

Let a fixed positive integer d with $(p, d) = 1$, set

$$\begin{aligned} X &= X_d = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \quad X_1 = \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p, \\ a + dp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$. It is easy to see that

$$\int_X g(x)d\mu_q(x) = \int_{\mathbb{Z}_p} g(x)d\mu_q(x). \tag{1.2}$$

Let $T_p = \cup_{N\geq 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \{\zeta | \zeta^{p^N} = 1\}$ is the cyclic group of order p^N. For $\zeta \in T_p$, we denote by $\phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \zeta^x$(cf. [6, 10]).

2. Symmetric identities for Carlitz's generalized twisted q-Bernoulli numbers and polynomials

Mathematicians investigated interesting properties of symmetry for special polynomials using p-adic invariant integral on $\mathbb{Z}_p$(see [1, 3, 4, 5]). If we take $\chi^0 = 1$, then [5] is the special case of this paper. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $(d,p) = 1$. For $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$, the twisted q-Bernoulli polynomials $\beta_{n,q,\zeta}(x)$ are defined by

$$\beta_{n,q,\zeta}(x) = \int_{\mathbb{Z}_p} \phi_\zeta(y)q^y[x+y]_q^n d\mu_1(y).$$

We introduce the Carlitz's generalized twisted q-Bernoulli polynomials $\beta_{n,\chi,q,\zeta}(x)$ attached to χ. The Carlitz's generalized twisted q-Bernoulli polynomials $\beta_{n,\chi,q,\zeta}(x)$ attached to χ are defined by

$$\beta_{n,\chi,q,\zeta}(x) = \int_X \chi(y)\phi_\zeta(y)q^y[x+y]_q^n d\mu_1(y).$$

When $x = 0$, $\beta_{n,\chi,q,\zeta} = \beta_{n,\chi,q,\zeta}(0)$ is called the n-th Carlitz's generalized twisted q-Bernoulli numbers $\beta_{n,\chi,q,\zeta}$. We note that

$$\sum_{n=0}^{\infty} \beta_{n,\chi,q,\zeta}\frac{t^n}{n!} = \int_X \chi(y)\zeta^y q^y e^{[x+y]_q t}d\mu_1(x).$$

Let w_1 and w_2 be natural numbers. Then, by (1.1) and (1.2), we obtain

$$\begin{aligned}
&\frac{1}{w_1}\int_X \chi(y)\zeta^{w_1y}q^{w_1y}e^{[w_1w_2x+w_2j+w_1y]_q t}d\mu_1(y)\\
&= \lim_{N\to\infty}\frac{1}{w_1}\frac{1}{dw_2p^N}\sum_{y=0}^{dw_2p^N-1}\chi(y)\zeta^{w_1y}q^{w_1y}e^{[w_1w_2x+w_2j+w_1y]_q t}q^{w_1y}\\
&= \lim_{N\to\infty}\frac{1}{dw_1w_2p^N}\sum_{i=0}^{dw_2-1}\chi(i)q^{w_1i}\zeta^{w_1i}\sum_{y=0}^{p^N-1}\zeta^{dw_1w_2y}q^{dw_1w_2y}e^{[w_1w_2x+w_2j+w_1i+dw_1w_2y]_q t}.
\end{aligned} \tag{2.1}$$

From (2.1), we can derive the following equation (2.2):

$$\begin{aligned}
&\frac{1}{w_1}\sum_{j=0}^{dw_1-1}\chi(j)\zeta^{w_2j}q^{w_2j}\int_X \chi(y)\zeta^{w_1y}q^{w_1y}e^{[w_1w_2x+w_2j+w_1y]_q t}d\mu_1(y)\\
&= \lim_{N\to\infty}\frac{1}{dw_1w_2p^N}\sum_{j=0}^{dw_1-1}\sum_{i=0}^{dw_2-1}\sum_{y=0}^{p^N-1}\chi(i)\chi(j)\zeta^{w_2j}\zeta^{w_1i}q^{w_2j}q^{w_1i}\\
&\qquad\times e^{[w_1w_2x+w_2j+w_1i+dw_1w_2y)]_q t}\zeta^{dw_1w_2y}q^{dw_1w_2y}.
\end{aligned} \tag{2.2}$$

By the same method as (2.2), we obtain

$$\begin{aligned}
&\frac{1}{w_2}\sum_{j=0}^{dw_2-1}\chi(j)\zeta^{w_1j}q^{w_1j}\int_X \chi(y)\zeta^{w_2y}q^{w_2y}e^{[w_1w_2x+w_1j+w_2y]_q t}d\mu_1(y)\\
&= \lim_{N\to\infty}\frac{1}{dw_1w_2p^N}\sum_{j=0}^{dw_2-1}\sum_{i=0}^{dw_1-1}\sum_{y=0}^{p^N-1}\chi(i)\chi(j)\zeta^{w_1i}\zeta^{w_2i}q^{w_1j}q^{w_2i}\\
&\qquad\times e^{[w_1w_2x+w_1j+w_2i+dw_1w_2y]_q t}\zeta^{dw_1w_2y}q^{dw_1w_2y}.
\end{aligned} \tag{2.3}$$

Therefore, by (2.2) and (2.3), we have the following theorem.

Theorem 1. For $w_1, w_2 \in \mathbb{N}$, we have

$$\begin{aligned}&\frac{1}{w_1}\sum_{j=0}^{dw_1-1}\chi(j)\zeta^{w_2j}q^{w_2j}\int_X \chi(y)\zeta^{w_1y}q^{w_1y}e^{[w_1w_2x+w_2j+w_1y]_q t}d\mu_1(y)\\&=\frac{1}{w_2}\sum_{j=0}^{dw_2-1}\chi(j)\zeta^{w_1j}q^{w_1j}\int_X \chi(y)\zeta^{w_2y}q^{w_2y}e^{[w_1w_2x+w_1j+w_2y]_q t}d\mu_1(y).\end{aligned}\tag{2.4}$$

By substituting Taylor series of e^{xt} into (2.4) and after calculations, we obtain the following corollary.

Corollary 2. For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have

$$\begin{aligned}&\frac{[w_1]_q^n}{w_1}\sum_{j=0}^{dw_1-1}\chi(j)\zeta^{w_2j}q^{w_2j}\int_X \chi(y)\zeta^{w_1y}q^{w_1y}\left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}^n d\mu_1(y)\\&=\frac{[w_2]_q^n}{w_2}\sum_{j=0}^{dw_2-1}\chi(j)\zeta^{w_1j}q^{w_1j}\int_X \chi(y)\zeta^{w_2y}q^{w_2y}\left[w_1x+\frac{w_1}{w_2}j+y\right]_{q^{w_2}}^n d\mu_1(y).\end{aligned}$$

By Corollary 2, we have the following theorem.

Theorem 3. For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have

$$\begin{aligned}&\frac{[w_1]_q^n}{w_1}\sum_{j=0}^{dw_1-1}\chi(j)\zeta^{w_2j}q^{w_2j}\beta_{n,\chi,q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{w_2}{w_1}j\right)\\&=\frac{[w_2]_q^n}{w_2}\sum_{j=0}^{dw_2-1}\chi(j)\zeta^{w_1j}q^{w_1j}\beta_{n,\chi,q^{w_2},\zeta^{w_2}}\left(w_1x+\frac{w_1}{w_2}j\right).\end{aligned}$$

By (2.5), we can derive the following equation:

$$\begin{aligned}&\int_X \chi(y)\zeta^{w_1y}q^{w_1y}\left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}^n d\mu_1(y)\\&=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^i [j]_{q^{w_2}}^i q^{w_2(n-i)j}\int_X \chi(y)\zeta^{w_1y}q^{w_1y}\left[w_2x+y\right]_{q^{w_1}}^{n-i}d\mu_1(y)\\&=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^i [j]_{q^{w_2}}^i q^{w_2(n-i)j}\beta_{n-i,\chi,q^{w_1},\zeta^{w_1}}(w_2x).\end{aligned}\tag{2.5}$$

Again, by (2.5), and Theorem 3, we have

$$\begin{aligned}&\frac{[w_1]_q^n}{w_1}\sum_{j=0}^{dw_1-1}\chi(j)\zeta^{w_2j}q^{w_2j}\int_X \chi(y)\zeta^{w_1y}q^{w_1y}\left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}^n d\mu_1(y)\\&=\sum_{j=0}^{dw_1-1}\chi(j)\zeta^{w_2j}q^{w_2j}\sum_{i=0}^{n}\binom{n}{i}\frac{[w_2]_q^i[w_1]_q^{n-i}}{w_1}[j]_{q^{w_2}}^i q^{w_2(n-i)j}\beta_{n-i,\chi,q^{w_1},\zeta^{w_1}}(w_2x)\\&=\sum_{i=0}^{n}\binom{n}{i}\frac{[w_2]_q^i[w_1]_q^{n-i}}{w_1}\beta_{n-i,\chi,q^{w_1},\zeta^{w_1}}(w_2x)\sum_{j=0}^{dw_1-1}\zeta^{w_2j}q^{w_2(n-i+1)j}[j]_{q^{w_2}}^i\\&=\sum_{i=0}^{n}\binom{n}{i}\frac{[w_2]_q^i[w_1]_q^{n-i}}{w_1}\beta_{n-i,\chi,q^{w_1},\zeta^{w_1}}(w_2x)\,S_{n,i}(dw_1,\zeta^{w_2},q^{w_2}|\chi),\end{aligned}\tag{2.6}$$

where

$$S_{n,i}(w_1,\zeta,q|\chi)=\sum_{j=0}^{w_1-1}\chi(j)\zeta^j q^{(n-i+h)j}[j]_q^i.$$

By the same method as (2.6), we get

$$\begin{aligned}&\frac{[w_2]_q^n}{w_2}\sum_{j=0}^{dw_2-1}\chi(j)\zeta^{w_1 j}q^{w_1 j}\int_X \chi(y)\zeta^{w_2 y}q^{w_2 y}\left[w_1 x+\frac{w_1}{w_2}j+y\right]_{q^{w_2}}^n d\mu_1(y)\\&=\sum_{i=0}^{n}\binom{n}{i}\frac{[w_1]_q^i[w_2]_q^{n-i}}{w_2}\beta_{n-i,\chi,q^{w_2},\zeta^{w_2}}\left(w_1 x\right)S_{n,i}(dw_2,\zeta^{w_1},q^{w_1}|\chi).\end{aligned}\tag{2.7}$$

Therefore, by (2.6) and (2.7), we have the following theorem.

Theorem 4. For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}\frac{[w_2]_q^i[w_1]_q^{n-i}}{w_1}\beta_{n-i,\chi,q^{w_1},\zeta^{w_1}}\left(w_2 x\right)S_{n,i}(dw_1,\zeta^{w_2},q^{w_2}|\chi)\\&=\sum_{i=0}^{n}\binom{n}{i}\frac{[w_1]_q^i[w_2]_q^{n-i}}{w_2}\beta_{n-i,\chi,q^{w_2},\zeta^{w_2}}\left(w_1 x\right)S_{n,i}(dw_2,\zeta^{w_1},q^{w_1}|\chi).\end{aligned}$$

Remark 5. Let $w_1, w_2 \in \mathbb{N}, n \geq 0$, and χ be the trivial character. Then we have

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}\frac{[w_2]_q^i[w_1]_q^{n-i}}{w_1}\beta_{n-i,q^{w_1},\zeta^{w_1}}\left(w_2 x\right)S_{n,i}(w_1|\zeta^{w_2},q^{w_2})\\&=\sum_{i=0}^{n}\binom{n}{i}\frac{[w_1]_q^i[w_2]_q^{n-i}}{w_2}\beta_{n-i,q^{w_2},\zeta^{w_2}}\left(w_1 x\right)S_{n,i}(w_2|\zeta^{w_1}q^{w_1}).\end{aligned}$$

REFERENCES

1. Yuan He, *Symmetric identities for Carlitz's q-Bernoulli numbers and polynomials*, Advances in Difference Equations, 246 (2013), 10 pages.
2. T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys., 9 (2002), 288-299.
3. D. Kim, T. Kim, Dmitry V. Dolgy, J.-J. Seo, *A note on identities of symmetry for generalized Carlitz's q-Bernoulli polynomials,* Int. Journal of Math. Analysis, 8 (2014), 907-914.
4. B. A. Kupershmidt, *Reflection symmetries of q-Bernoulli polynomials,* J. Nonlinear Math. Phys., 12 (2005), 412-422.
5. C. S. Ryoo, *Symmetric identities for Carlitz's twisted q-Bernoulli numbers and polynomials associated with p-adic invariant integral on $\mathbb{Z}_p$*, Global Journal of Pure and Applied Mathematics, 11 (2015), pp. 2413-2417.
6. C. S. Ryoo, *On the symmetric properties for the generalized twisted q-tangent numbers and polynomials*, Advanced Studies in Theoretical Physics, 9 (2015), pp. 63-69.
7. C. S. Ryoo, (2011). *On the alternating sums of powers of consecutive odd integers*, *Journal of Computational Analysis and Applications*, 13 (2011), pp. 1019-1024.
8. C. S. Ryoo, *Calculating zeros of q-extension of the second kind Bernoulli polynomials*, *Journal of Computational Analysis and Applications*, 15 (2013), pp. 248-254.
9. C. S. Ryoo, *Differential equations associated with generalized Bell polynomials and their zeros*, *Open Mathematics*, 14(2016), pp. 807-815.
10. Y. Simsek, *Theorem on twisted L-function and twisted Bernoulli numbers*, Advan. Stud. Contemp. Math. 12 (2006), pp. 237-246.

An efficient optimal algorithm for high frequency in wavelet based image reconstruction

Jingjing Liu,* Guoxi Ni†

LCP, Institute of Applied Physics and Computational Mathematics,Beijing,China

Abstract

Wavelet algorithms for high-resolution image reconstruction has been shown effectively, it relies on the decomposition of low/high frequency, and hard/soft thresholding arguments are often used to denoise for high frequency. In this paper, instead of using this kind of thresholding arguments,, we apply the gradient based shrinkage thresholding optimization for high-frequency, in this way, we can keep the useful information in the original signal as much as possible, coupling the shrinkage thresholding optimization with the wavelet algorithm, we get an efficient reconstruction algorithm. Numerical results show we obtain a higher resolution,better peak signal-to-noise ratios and lower relative errors.

Key words: Wavelet; high-resolution; image reconstruction; shrinkage thresholding; high frequency.

1 Introduction

Increasing the resolution is important and necessary for many applications, lots of studies have been done on the high-resolution image reconstruction [13, 14, 18, 20, 21, 22, 23, 24, 27].

Among the methods in image processing, wavelet method is a well developed technology [6, 9, 10, 12, 26]. In this method, global patterns are represented by densely distributed coefficients obtained from low-pass filtering, while local features are represented by coefficients obtained from high-pass filtering. This makes it easy for us to distinct between smooth and sharp image components. In this way wavelet frames can effectively separate smooth image components and non smooth ones, and the wavelet-based procedure is essentially to approximate iteratively the densely distributed coefficients folded by the given low-pass filter. To overcome the incompatibility of symmetry and exact reconstruction, bi-orthogonal wavelet system is thus proposed in image processing, see [1, 8, 25].

*E-mail: liujingjing0618@126.com

†E-mail: gxni@iapcm.ac.cn

The relatively complex hard/soft thresholding methods [11] are often used to denoise for high frequency information, but some useful information will lose in the processing because of its cut off action. Preserving useful high frequency part while removing noise is the main goal in image denoising, some techniques developed in the past years has shown their advantage than the hard- and soft-thresholding in the wavelet field, for example, the wavelet packet method,it is based on the further decomposition of wavelet coefficients by packets, and this leads to an essentially translation invariant wavelet packet system.

To get an efficient algorithm while keep useful information in high frequency as much as possible, we consider the optimization strategy instead of hard/soft thresholding method for the high frequency components, this strategy is based on the classic variation technology, and has been previously used in image reconstruction, because of its computational complex, a fast iterative shrinkage-thresholding algorithms are proposed in [2, 4], this kind of method , which can be viewed as an extension of the classical gradient algorithm, is attractive due to its simplicity, it is adequate for solving large-scale problems even with dense matrix data in image reconstruction,to improve the convergence rate, a more fast iterative shrinkage-thresholding algorithm with a significantly better global convergence rate is introduced in [3, 28], this algorithm improves the convergence rate from $O(1/k)$ to $O(1/k^2)$, it relies on computing the next iteration based on the values not only in the previous one, but also in two previously computed steps.

In this paper, we are intent to improve the wavelet algorithm in image reconstruction. We begin with the bi-orthogonal wavelet systems, and obtain the decomposition formula, which represent a perfect reconstruction equation for the symbols of the low-pass and the high-pass filters,theoretical analysis shows that the noise is contains in high-frequency part, and the hard/soft thresholding argument will inevitably delete some useful information, instead of using this kind of thresholding argument for high-frequency components of the original image, we take advantage of shrinkage thresholding algorithm for the optimization of high-frequency, it has been proved that it has notable effect in image denoising, to get the algorithm more efficient, we apply some accelerating iteration argument in shrinkage thresholding algorithm.

The outline of the paper is as follows. the algorithms are derived in section 2. Numerical examples are given in section 3 to illustrate the effectiveness of the algorithms. Some concluding remarks are given in section 4.

2 Reconstruction algorithm

In this section, we construct a shrinkage thresholding optimization coupling with the wavelet based algorithm for high resolution image reconstruction.

2.1 Iterative scheme

Refer to [5], we obtain that using the periodic boundary condition and ordering the discretized values of f and g in a row-by-row fashion, we obtain $M1M2 \times M1M2$ linear system of the form:

$$Lf = g \tag{1}$$

where f is original image, g low-resolution image, $L = L^x \otimes L^y$ is the blurring matrix which is made up from each sensor, and L^x, L^y have circulant structure as follow:

$$L^x = \frac{1}{L} \cdot circulant(a),$$

where

$$a = [1, \cdots, 1, \frac{1}{2}, 0, \cdots, 0, \frac{1}{2}, 1, \cdots, 1]^t,$$

where $circulant(a)$ represents circulant matrix, and the first $L/2$ entries in a are equal to 1, the last $L/2-1$ entries are equal to 1. The matrix L^y can be define similarly, these matrix are circulant matrices, then we get that the matrix L is a block-circulant-circulant-block (BCCB)matrix [17].

By the biorthogonal wavelet theory [7, 19], the symbols of the refinement masks and wavelet masks satisfy the following equation

$$\overline{\hat{a}^d}\hat{a} + \sum_{\nu \in Z_K^2 \setminus \{(0,0)\}} \overline{\hat{b}_\nu^d}\hat{b}_\nu^d = 1 \tag{2}$$

where K is sensor size.

The equation (2) is not only for the reconstruction of function but also for image reconstruction, the matrix representation of the perfect reconstruction from biorthogonal system can be written as

$$L^d L + \sum_{\nu \in Z_K^2 \setminus \{(0,0)\}} M_\nu^d M_\nu = I, \tag{3}$$

here denote by L, L^d, M_ν^d, M_ν the matrices generated by the symbols of the refinement and wavelet masks $\hat{a}, \overline{\hat{a}^d}, \hat{b_\nu}, \overline{\hat{b_\nu^d}}$, respectively.

Since $g = Lf$ is just the observed high-resolution image, and the other $M_\nu f, \nu \neq (0,0)$, represent the high-frequency components of f, from equation (3) we obtain an iterative algorithm

$$f_{n+1} = L^d g + (\sum_{\nu \in Z_K^2 \setminus \{(0,0)\}} M_\nu^d M_\nu) f_n \tag{4}$$

In the usual denoising procedure, the high frequency components are often penalized by a factor, this smoothes the original signals,so a nonlinear denoising scheme can be built into equation (4), and thus obtain an iterative algorithm

$$f_{n+1} = L^d g + \sum_{\nu \in Z_2^2 \setminus \{(0,0)\}} M_\nu^d T(M_\nu f_n). \tag{5}$$

3

where T is a denoising operator, a hard/soft thresholding wavelet denoising algorithm is presented in [7], in which a further decomposition by the translation invariant wavelet packets is used, this can remedy the smoothing effect on the original signals in some sense, some useful information in high frequency is still lost, this motivates us to consider an efficient optimal method to keep information as much as possible.

2.2 Shrinkage thresholding optimization for high frequency

Let $b = M_\nu f_n$,we consider the following formulation:

$$x^* = \min_x F(x), \quad F(x) = ||Ax - b||^2, \tag{6}$$

where $A = L^d L$,and L is the blurring matrix in the last section, the norm $||\cdot||$ is the inner product, and x is the vector we are looking for, this is the classical least square problem. The optimization problem (6) can be cast as a second order cone programming problem and thus could be solved via interior point methods.

Usually this problem is not only in large scale but also involves dense matrix data, which often precludes the use and potential advantage of sophisticated interior point methods. This motivated a simpler gradient-based algorithms for solving it, the gradient algorithm generates a sequence $\{x_k\}$ via

$$x_k = x_{k-1} - t_k \nabla F(x_{k-1}),$$

where $t_k > 0$ is a suitable stepsize. It can be viewed as an approximal regularization of the linearized function F at x_{k-1}, and can be written equivalently as

$$x_k = \min_x \{F(x_{k-1}) + (x - x_{k-1}, \nabla F(x_{k-1})) + \frac{1}{2t_k}||x - x_{k-1}||^2\}.$$

After ignoring constant terms, this can be rewritten as

$$x_k = \min_x \{\frac{1}{2t_k}||x - (x_{k-1} - t_k \nabla F(x_{k-1}))||^2\},$$

the computation of x_k reduces to solving a one-dimensional minimization problem for each of its components, which produces

$$x_k = \mathcal{T}_{\lambda t_k}(x_{k-1} - t_k \nabla F(x_{k-1})),$$

considering the expression of $F(x)$ in (6), we get:

$$x_k = \mathcal{T}_{\lambda t_k}(x_{k-1} - 2t_k A^T(Ax_{k-1} - b)), \tag{7}$$

where $\mathcal{T}_\alpha : R^2 \to R^2$ is the thresholding operator defined by

$$\mathcal{T}_\alpha(x_i) = (|x_i| - \alpha)_+ sgn(x_i). \tag{8}$$

This algorithm (7) is a kind of iterative shrinkage thresholding algorithm similar as that in [15].

It has been proved that for large-scale problems this first order methods are only practical option, and the sequence x_k converges quite slowly to its solution, that is

$$F(x_k) - F(x^*) = O(1/k),$$

namely, it shares a sublinear global rate of convergence.

To improve the efficiency of the iterative shrinkage thresholding algorithm (ISTA) with better global rate of convergence, Beck etc.[3] consider an improved fast iteration, that is the x_k in (7) is not dependent on the previous point x_{k-1}, but rather on the point y_k which is a linear combination of the previous two point $\{x_{k-1}, x_{k-2}\}$, with this modification, they get a fast ISTA, and the convergence rate is

$$F(x_k) - F(x^*) = O(1/k^2).$$

In this way, we get x^* from b according to the above iterative shrinkage thresholding algorithm.

2.3 Summary of Algorithms

For convenience,we call our shrinkage thresholding algorithm in wavelet based reconstruction algorithm as STWL,to compare with the hard/soft thresholding wavelet reconstructed algorithm (abbr. TWL) in [7]. Now we embed the Shrinkage thresholding optimization algorithm into the iteration scheme (5), denote the previous two iterations as $\{f_{n-1}, f_{n-2}\}$, then our algorithm for the model equation (5) can be summarized as following:

(1) Choose an initial approximation f_0 (e.g., $f_0 = L^d g$);

(2) Iterate until convergence:

Outer circulation:

$$f_{n+1} = L^d g + \sum_{\nu \in Z_2^2 \setminus \{(0,0)\}} M_\nu^d \tilde{T}(M_\nu f_n). \tag{9}$$

Begin inner loop:

To get optimal high frequency part $\tilde{T}(M_\nu f_n)$. Let $b_\nu = M_\nu f_n$, $y_{1,\nu} = M_\nu f_n, h_{0,\nu} = 1, t_1 = 1$, then a fast iteration for (7) is

$$h_{k,\nu} = \mathcal{T}_{\lambda t_k}(y_{k,\nu} - 2t_k A^T(Ay_{k,\nu} - b_\nu)),$$

$$t_{k+1} = \frac{1 + \sqrt{1 + 4t_k^2}}{2},$$

$$y_{k+1,\nu} = h_{k,\nu} + (\frac{t_k - 1}{t_{k+1}})(h_{k,\nu} - h_{k-1,\nu}),$$

5

where $\mathcal{T}_{\lambda t_k}$ is defined as in (8), λ is estimated by the method given in [11] which uses the median of the absolute value of the entries in the vector $M_\nu f_n$.

End inner loop.

Return the optimized results $y^*_{n,\nu}$ for high frequency $\tilde{T}(M_\nu f_n)$, that is

$$f_{n+1} = L^d g + \sum_{\nu \in Z_2^2 \setminus \{(0,0)\}} M_\nu^d y^*_{n,\nu}.$$

End outer circulation.

Remark: When the operator $\tilde{T}$ in (9) is realized by soft/hard thresholding operator T,then this reduces to the soft/hard thresholding wavelet reconstruction algorithm as that in [7].

3 Experimental results

In this part, we present the efficiency and accuracy of our shrinkage thresholding wavelet based reconstruction algorithm(abbr. STWL),and compare with the hard/soft thresholding wavelet reconstructed method (abbr. TWL). As usual,we evaluate the methods using the peak signal-to-noise ratio (PSNR),relative error (RE) and cpu time cost they are defined by

$$RE = \frac{\|f - f_c\|_2}{\|f\|_2},$$

and

$$PSNR = 10 \log_{10} \frac{\|f\|_2^2}{\|f - f_c\|_2^2}$$

for 1D signals, while as

$$PSNR = 10 \log_{10} \frac{255^2 NM}{\|f - f_c\|_2^2}$$

for 2D images, respectively, with the size of the signals (images) is $N \times M$. Where f is original image, and f_c is restored image.

In our tests, $N = 1$ for 1D signals while $N = M$ for 2D images. Here we take 2×2 and 4×4 sensor arrays in 2D.

3.1 1D denoisy signal recovery

We take the original signal data from the WaveLab toolbox at *http://statweb. stanford.edu/ wavelab/* developed by Donoho's research group. Fig. 1(a) shows the original signal f. Fig. 1(b) depicts the contaminated signal with white noise at signal-to-noise ratio ($SNR = 25$), here we use matlab function *awgn* to add noise to the original signal. The results of denoising by the above two algorithms with periodic conditions are shown in Fig. 1(c) and (d), respectively. From the data results of experiments in table 1, it shows that our algorithm STWL has a better performance, and has a better time efficiency than TWL.

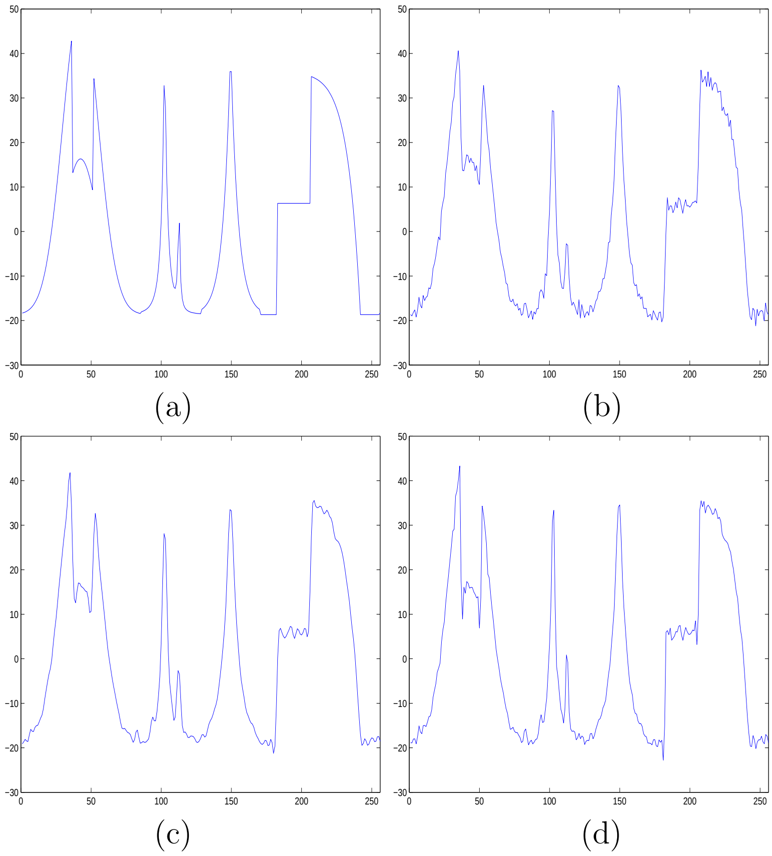


Fig. 1: (a) Original signal; (b) Contaminated by white noise at $SNR = 25$; (c) Reconstructed signal from the algorithm TWL; (d) Reconstructed signal from our algorithm $STWL$.

Table 1: Corresponding PSNR, RE and timecost values using algorithm TWL in [7] and our algorithm STWL.

Algorithms	$SNR = 25$			$SNR = 30$		
	PSNR	RE	timecost	PSNR	RE	timecost
TWL	47.1560	0.0946	0.7956	48.0624	0.0904	0.9984
$STWL$	53.3382	0.0695	0.6396	57.5931	0.0562	0.7332

3.2 High-resolution image reconstruction

In this section, we use the classical "boat", the "Lena" and the "cameraman" images with size of 256×256 as the original images for our tests, and consider 2×2 sensor arrays and 4×4 sensor arrays respectively, and the Gaussian white noise is added to these original images.

7

In the algorithm TWL, the thresholding value λ is chosen to be $\sigma_{n,\nu}\sqrt{2\log(M_1M_2)}$, where the variance $\sigma_{n,\nu}$ is estimated by the median of the absolute value of the entries in the vector $M_\nu f_n$ named high frequency term. In our algorithm STWL, we have decomposed the image data into low and high frequency parts, and denoise the high frequency information by shrinkage thresholding method, the Lipschitz constant is computable in the examples since the eigenvalues of the matrix A^TA can be easily calculated using the two-dimensional cosine transform [17]. For simplicity, we will only consider the matrices for the periodic case.

3.2.1 2×2 sensor array

For 2×2 sensor arrays, the corresponding refinement mask m is the piecewise linear spine,

$$m(-1)=\frac{1}{4}, m(0)=\frac{1}{2}, m(1)=\frac{1}{4},$$

and $m(\alpha)=0$ for all other α. The nonzero terms of the dual mask of m used in this paper are

$$m^d(-2)=-\frac{1}{8}, m^d(-1)=\frac{1}{4}, m^d(0)=\frac{3}{4}, m^d(1)=\frac{1}{4}, m^d(2)=-\frac{1}{8}.$$

The dual pair of the wavelet masks are $r_\alpha:=(-1)^\alpha m^d(1-\alpha)$ and $r^d(\alpha):=(-1)^\alpha m(1-\alpha)$, see [12] for details.

The tensor product dual pair of the refinement symbols are given by $\hat{a}(\omega)=\hat{m}(\omega_1)\hat{m}(\omega_2)$, $\hat{a}^d(\omega)=\hat{m}^d(\omega_1)\hat{m}^d(\omega_2)$, and the corresponding wavelet symbols are $\hat{b}_{(0,1)}(\omega)=\hat{m}(\omega_1)\hat{r}(\omega_2)$, $\hat{b}^d_{(0,1)}(\omega)=\hat{m}^d(\omega_1)\hat{r}^d(\omega_2)$, $\hat{b}_{(1,0)}(\omega)=\hat{r}(\omega_1)\hat{m}(\omega_2)$, $\hat{b}^d_{(1,0)}(\omega)=\hat{r}^d(\omega_1)\hat{m}^d(\omega_2)$, $\hat{b}_{(1,1)}(\omega)=\hat{r}(\omega_1)\hat{r}(\omega_2)$, $\hat{b}^d_{(1,1)}(\omega)=\hat{r}^d(\omega_1)\hat{r}^d(\omega_2)$, where $\omega=(\omega_1,\omega_2)$.

Although we give here only the details of the refinable functions and their corresponding wavelets with dilation $2I$, the whole theory can be carried over to the general isotropic integer dilation matrices.

The wavelet matrices are formed by the tensor product,and we consider

$$Z_2^2=\{(0,0),(0,1),(1,0),(1,1)\}.$$

In particular, we have

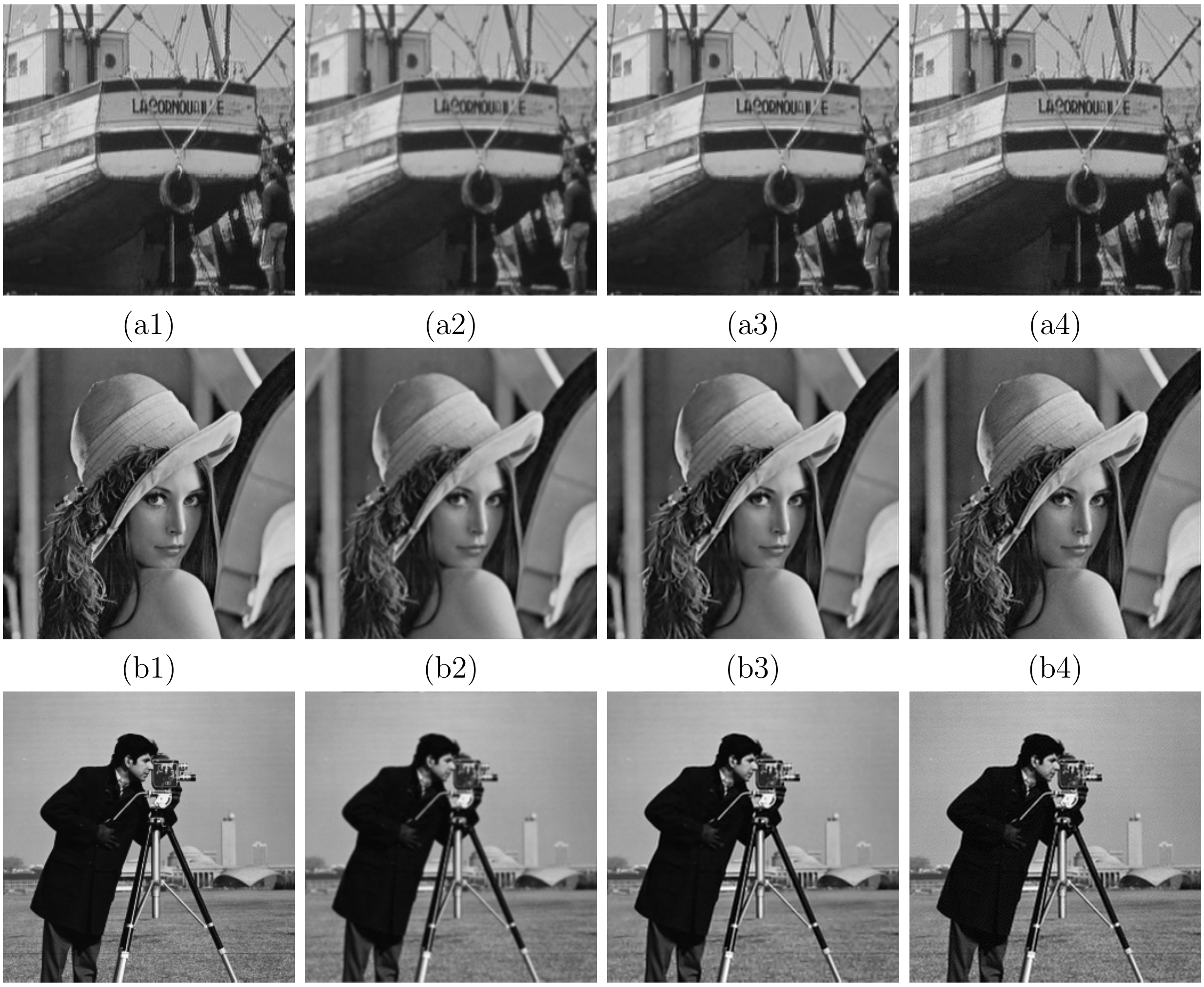
$$L=L_2\otimes L_2,\quad M_{(0,1)}=L_2\otimes M_2,\ M_{(1,0)}=M_2\otimes L_2,\quad M_{(1,1)}=M_2\otimes M_2,$$
$$L^d=L_2^d\otimes L_2^d,\quad M^d_{(0,1)}=L_2^d\otimes M_2^d,\ M^d_{(1,0)}=M_2^d\otimes L_2^d,\quad M^d_{(1,1)}=M_2^d\otimes M_2^d.$$

where

$$L_2=circulant(\frac{1}{2},\frac{1}{4},0,\cdots,0,\frac{1}{4}),\ L_2^d=circulant(\frac{3}{4},\frac{1}{4},-\frac{1}{8},0,\cdots,0,-\frac{1}{8},\frac{1}{4})$$

$$M_2=circulant(\frac{1}{4},-\frac{3}{4},\frac{1}{4},\frac{1}{8},0,\cdots,0,\frac{1}{8}),\ M_2^d=circulant(\frac{1}{4},0,\cdots,0,\frac{1}{4},-\frac{1}{2}).$$

Fig.2 demonstrate the reconstructed high-resolution image for the "boat" , the "Lena" and the "cameraman" images respectively, in these figures, (a1)-(c1) are the original images, (a2)-(c2) are with noise $PSNR = 40dB$, (a3)-(c3) are the denoisy images with the algorithm TWL, and (a4)-(c4) are obtained by our algorithm STWL. Table 2 gives the PSNR, RE, and the cputime of the reconstructed images for different levels of Gaussian noise, our algorithm shows less RE, less cputime and better PSNR, we can conclude that our algorithm STWL is better than the original algorithm TWL in [7].



(a1) (a2) (a3) (a4)

(b1) (b2) (b3) (b4)

(c1) (c2) (c3) (c4)

Fig. 2: (a1)-(c1) the original images; (a2)-(c2) the noisy images; (a3)-(c3) the reconstructed image by algorithm TWL; (a4)-(c4) the reconstructed images by our algorithm STWL.

Table 2: Comparison of PSNR, RE and cputime values using algorithm TWL and STWL for 2×2 sensor arrays with different kinds noise level.

Image	Evaluation	TWL		$STWL$	
		$SNR = 35$	$SNR = 40$	$SNR = 35$	$SNR = 40$
boat	PSNR	81.9938	82.2142	82.4965	82.4724
	RE	0.0166	0.0164	0.0162	0.0162
	timecost	10.3585	9.9529	5.9280	5.8344
Lena	PSNR	84.0785	84.1040	84.6254	85.0012
	RE	0.0149	0.0149	0.0145	0.0143
	timecost	11.3881	11.7157	5.9436	5.5380
cameraman	PSNR	85.8451	86.1629	86.1727	86.8056
	RE	0.0137	0.0135	0.0135	0.0130
	timecost	11.2165	9.7033	5.9124	5.6472

3.2.2 4×4 sensor array

In this case, we give the refinable and wavelet masks with dilation $4I$ that used to generate the matrices for 4×4 sensor arrays.

For 4×4 sensor arrays, the corresponding mask is

$$m(\alpha) = \frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \alpha = -2, \cdots, 2,$$

with $m(\alpha) = 0$ for all other α. The nonzero terms of a dual refinement mask of m is

$$m^d(\alpha) = -\frac{1}{16}, \frac{1}{8}, \frac{5}{16}, \frac{1}{4}, \frac{5}{16}, \frac{1}{8}, -\frac{1}{16}, \alpha = -3, \cdots, 3.$$

The nonzero terms of the corresponding wavelet masks are

$$r_1(\alpha) = -\frac{1}{8}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{8}, \alpha = -2, \cdots, 2,$$

$$r_2(\alpha) = -\frac{1}{16}, -\frac{1}{8}, \frac{5}{16}, -\frac{1}{4}, \frac{5}{16}, -\frac{1}{8}, -\frac{1}{16}, \alpha = -2, \cdots, 4,$$

$$r_3(\alpha) = \frac{1}{16}, \frac{1}{8}, -\frac{7}{16}, 0, \frac{7}{16}, -\frac{1}{8}, -\frac{1}{16}, \alpha = -2, \cdots, 4.$$

The dual wavelet masks are

$$r_1^d(\alpha) = (-1)^{1-\alpha} r_3(1-\alpha), r_2^d(\alpha) = (-1)^{1-\alpha} m(1-\alpha), r_3^d(\alpha) = (-1)^{1-\alpha} r_1(1-\alpha)$$

The observed high-resolution image g is generated by applying the bivariate lowpass filter on the true image f,again, we consider periodic boundary condition. The matrices

$$L, L^d, M_\nu, M_\nu^d, \nu \in Z_4^2 \setminus \{(0,0)\}$$

can be generated by the corresponding filters.

Fig.3 shows the reconstructed high-resolution image for the "boat" , the "Lena" and the "cameraman" images, (a1)-(c1) are blurred with noise $PSNR = 40dB$, (a2)-(c2) are obtained from the algorithm TWL, and (a1)-(c1) are obtained from our algorithm STWL. From Table 3, we can also find that our algorithm shows less RE, less cputime and better PSNR, since the problem is more difficult than the 2×2 sensor case, we need more cputime consuming, we can see that the performance of our algorithm STWL is much better than the original algorithm TWL.

Table 3: Comparison of PSNR, RE and cputime values using algorithm TWL and STWL for 4×4 sensor arrays with different kinds noise level.

Image	Evaluation	TWL		$STWL$	
		$SNR = 30$	$SNR = 40$	$SNR = 30$	$SNR = 40$
boat	PSNR	67.3135	67.4297	68.3053	68.5968
	RE	0.0345	0.0343	0.0329	0.0324
	timecost	19.5157	20.4673	15.9277	15.8809
Lena	PSNR	69.9279	69.9769	71.3131	71.4538
	RE	0.0303	0.0302	0.0283	0.0281
	timecost	20.9041	20.0773	16.1773	15.9745
cameraman	PSNR	72.9596	73.2392	73.6651	73.9353
	RE	0.0260	0.0257	0.0251	0.0248
	timecost	19.8589	20.9041	14.8981	15.6313

4 Conclusions

In this paper, we constructed a shrinkage thresholding algorithm in wavelet based image reconstruction, instead of using the hard/soft thresholding algorithm we apply the iterative shrinkage thresholding algorithm for the optimization for high frequency. Our new algorithm works effectively both in one-dimensional and two-dimensional situations, numerical tests show that this algorithm gives higher resolution, larger signal-to noise ratios,lower relative errors and less cputime.

Fig. 3: (a1)-(c1) is the noisy images,(a2)-(c2) is reconstructed from the algorithm TWL; (a3)-(c3) is reconstructed from our algorithm STWL.

Acknowledgments

This work is supported by China Natural Science Fund (No.91130020) and Defense Industrial Technology Development Program (No.B1520133015).

References

[1] Chenglong Bao, Bin Dong, Likun Hou, Zuowei Shen,Image restoration by minimizing zero norm of wavelet frame coefficients, Inverse Problems, 32(11), (2016),115004.

[2] A. Beck, M. Teboulle, Fast gradient-based algorithms for constrained total variation image denoising and deblurring problems, IEEE Trans. Image Process. 18, 2419-2434(2009).

[3] A. Beck, M. Teboulle, A fast iterative shrinkage-thresholding algorithm for linear inverse problems, SIAM Journal on Imaging Sciences. 2(1), 183-202(2009).

[4] J. Bioucas-Dias, M. Figueiredo, A new TWIST: Two-step iterative shrinkage/thresholding algorithms for image restoration, IEEE Trans. Image Process.,16, 2992-3004(2007).

[5] N. Bose, K. Boo, High-resolution image reconstruction with multisensors, Int. J. Imag. Syst. Technol. 9, 294-304(1998).

[6] A. Chambolle, R. A. DeVore, N. Y. Lee, and B. J. Lucier, Nonlinear wavelet image processing: Variational problems, compression, and noise removal through wavelet shrinkage, IEEE Trans. Image Proces., 7, 319-335 (1998).

[7] R.H. Chan, T. Chan, L. Shen, Z. Shen, Wavelet algorithms for high-resolution image reconstruction, SIAM J. Sci. Comput. 24(4), 1408-1432(2003).

[8] R.H. Chan, T. Chan, L. Shen, Z. Shen, Wavelet deblurring algorithms for spatially varying blur from high-resolution image reconstruction, Linear Algebra Appl. 366, 139-155(2003).

[9] R. H. Chan, S. D. Riemenschneider, L. Shen, Z. Shen, Tight frame: an efficient way for high-resolution image reconstruction, Applied and Computational Harmonic Analysis. 17, 91-115(2004).

[10] T. Chan and J. Shen, Variational image inpainting, Commun. Pure Appl. Math, vol. 58, 579-619 (2005).

[11] D. Donoho, De-noising by soft-thresholding, IEEE Trans, Inform. Theory, 41, 613-627(1995).

[12] I. Daubechies, Ten lectures on wavelets, CBMS-NSF Regional Conf. Ser. in Appl. Math., 61. SIAM, Philadelphia, 1992.

[13] M. Elad, A. Feuer, Restoration of a single superresolution image from several blurred, noisy and undersampled measured images, IEEE Trans. Image Process. 6, 1646-1658(1997).

[14] M. Elad, A. Feuer, Superresolution restoration of an image sequence: adaptive filtering approach, IEEE Trans. Image Process. 8(3), 387-395(1999).

[15] M. A. T. Figueiredo and R. D. Nowak, An EM algorithm for wavelet-based image restoration, IEEE Trans. Image Process., 12, 906-916(2003).

[16] T. M. Stadtmiller, J. C. Gillete, R. C. Hardie, Aliasing reduction in staring infrared images utilizing subpixel techniques, Aerospace and Electronics Conference, 2(11), 874-880(1995).

[17] P. C. Hansen, J. G. Nagy and D. P. O'Leary, Deblurring Images: Matrices, Spectra and Filtering, SIAM, Philadelphia, 2006.

[18] T. Huang, R. Tsay, Multiple frame image restoration and registration, in: T. S. Huanf(Ed.), Advances in Computer Vision and Image Processing, vol. 1, 317-339(1984).

[19] H. Ji, S. Riemenschneider, and Z. Shen, Multivariate compactly supported fundamental refinable functions, dual and biorthogonal wavelets, Stud. Appl. Math.,102, 862-874(1999).

[20] M. Ng, R. Chan, T. Chan, A. Yip, Cosine transform preconditioners for high resolution image reconstruction, Linear Algebra Appl. 316, 89-104(2000).

[21] M. Ng, R. Chan, W. Tang, A fast algorithm for deeblurring models with Neumann boundary conditions, SIAM J.Sci. Comput. 21, 851-866(2000).

[22] M. K. Ng, N, Bose, Analysis of displacement errors in high-resolution image reconstruction with multisensors, IEEE Trans. Circuits Systems I Fund. Theory Appl. 49(6), 806-813(2002).

[23] A. Patti, M. Sezan, A. Tekalp, Superresolution video reconstruction with arbitrary sampling lattices and nonzero aperture time, IEEE Trans. Image Process. 6, 1064-1076(1997).

[24] R. Schultz, R. Stevenson, Extraction of high-resolution frames from video sequences, IEEE Trans. Image Process. 5, 996-1011(1996).

[25] Z. Shen, Extension of matrices with Laurent polynomial entries, in proceedings of the 15th IMACS World Congress 1997 on Scientific Computation, Modelling and Applied Mathematics, A. Syclow, ed., 57-61(1997).

[26] L. Shen, Q. Sun, Bi-orthogonal wavelet system for high-resolution image reconstruction, IEEE Trans. Signal Process. 52(7),1997-2011(2003).

[27] P. Vandewalle, J. Kovacevic, and M. Vetterli, Reproducible research in signal processing What, why, and how, IEEE Signal Process., 26, 37-47(2009).

[28] M. Zulfiquar, A. Bhotto, M. O. Ahmad, M. N. S. Swamy, An improved fast iterative shrinkage thresholding algorithm for image deblurring, SIAM Journal on Imaging Sciences. 8(3),1640-1657(2015).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.5, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Negative Domain Local fractional Inequalities

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

This research is about inequalities in a local fractional environment over a negative domain. The author presents the following types of analytic local fractional inequalities: Opial, Hilbert-Pachpatte, comparison of means, Poincare and Sobolev. The results are with respect to uniform and L_p norms, involving left and right Riemann-Liouville fractional derivatives.

2010 Mathematics Subject Classification: 26A33, 26D10, 26D15.
Keywords and phrases: Local fractional derivative, Riemann-Liouville fractional derivative, Opial inequality, Hilbert Pachpatte, Poincare inequality, fractional inequalities.

1 Introduction

Many sources motivate us to write this work. The first one comes next. It is the famous Opial inequality ([13]):

$$\int_0^a |y'(x)\, y(x)|\, dx \leq \frac{a}{2} \int_0^a |y'(x)|^2\, dx, \tag{1}$$

where $y(x)$ is absolutely continuous function and $y(0) = 0$. The above inequality is proved sharp.

The well known Ostrowski ([14]) inequality also motivates this work and has as follows:

$$\left| \frac{1}{b-a} \int_a^b f(y)\, dy - f(x) \right| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) (b-a) \|f'\|_\infty, \tag{2}$$

where $f \in C^1([a,b])$, $x \in [a,b]$, and it is a sharp inequality.

Next $D_{*a}^{\rho} f$ indicates the left Caputo fractional derivative of order $\rho > 0$, anchored at $a \in \mathbb{R}$, see [10], p. 50.

1

The author in [7], pp. 82-83, proved the following left Caputo fractional Landau inequality: Let $0 < \nu \leq 1$, $f \in AC^2([0, b])$ (i.e. $f' \in AC([0, b])$, absolutely continuous functions), $\forall\ b > 0$. Suppose $\|f\|_{\infty,\mathbb{R}_+} < +\infty$, $D_{*0}^{\nu+1} f \in L_\infty(\mathbb{R}_+)$, and

$$\left\|D_{*a}^{\nu+1} f\right\|_{\infty,[a,+\infty)} \leq \left\|D_{*0}^{\nu+1} f\right\|_{\infty,\mathbb{R}_+}, \ \ \forall\ a \geq 0. \tag{3}$$

Then

$$\|f'\|_{\infty,\mathbb{R}_+} \leq (\nu+1)\left(\frac{2}{\nu}\right)^{\frac{\nu}{\nu+1}} (\Gamma(\nu+2))^{-\frac{1}{\nu+1}} \left(\|f\|_{\infty,\mathbb{R}_+}\right)^{\frac{\nu}{\nu+1}} \left(\left\|D_{*0}^{\nu+1} f\right\|_{\infty,\mathbb{R}_+}\right)^{\frac{1}{\nu+1}}, \tag{4}$$

that is $\|f'\|_{\infty,\mathbb{R}_+}$ is finite.

The last inequality is another inspiration.

The author's monographs [2], [3], [4], [5], [6], [8], motivate and support largely this work too. See also [1].

Under the point of view of local fractional differentiation the author examines the broad area of analytic inequalities and produces a variety of well-known inequalities in a local fractional setting over a negative domain to all possible directions.

2 Background

We mention

Definition 1 *([11]) Let $x, x' \in [a, b]$, $f \in C([a, b])$. The Riemann-Liouville (R-L) fractional derivative of a function f of order q $(0 < q < 1)$ is defined as*

$$D_x^q f(x') = \left\{ \begin{array}{l} D_{x+}^q f(x'), \quad x' > x, \\ D_{x-}^q f(x'), \quad x' < x \end{array} \right\} =$$

$$\frac{1}{\Gamma(1-q)} \left\{ \begin{array}{l} \frac{d}{dx'} \int_x^{x'} (x'-t)^{-q} f(t)\, dt, \quad x' > x, \\ -\frac{d}{dx'} \int_{x'}^{x} (t-x')^{-q} f(t)\, dt, \quad x' < x, \end{array} \right. \tag{5}$$

the left and right R-L fractional derivatives, respectively.

We need

Definition 2 *([11], [12]) The local fractional derivative of order q $(0 < q < 1)$ of a function $f \in C([a, b])$ is defined as*

$$D^q f(x) = \lim_{x' \to x} D_x^q (f(x') - f(x)). \tag{6}$$

More generally we define

2

Definition 3 *([9]) Let $N \in \mathbb{Z}_+$, $0 < q < 1$, the local fractional derivative of order $(N+q)$ of a function $f \in C^N([a,b])$ is defined by*

$$D^{N+q} f(x) = \lim_{x' \to x} D_x^q \left(f(x') - \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} (x'-x)^n \right). \tag{7}$$

If $N = 0$, then Definition 3 collapses to Definition 2.
We need

Definition 4 *(related to Definition 3) Let $f \in C^N([a,b])$, $N \in \mathbb{Z}_+$. Set*

$$F(x, x'-x; q, N) := D_x^q \left(f(x') - \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} (x'-x)^n \right). \tag{8}$$

Let $x' - x := t$, then $x' = x + t$, and

$$F(x, t; q, N) = D_x^q \left(f(x+t) - \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} t^n \right). \tag{9}$$

We make

Remark 5 *Here $x', x \in [a,b]$, and $a \leq x + t \leq b$, equivalently $a - x \leq t \leq b - x$. From $a \leq x \leq b$, we get $a - x \leq 0 \leq b - x$. We assume here that $F(x, \cdot; q, N) \in C^1([a-x, b-x])$. Clearly, then it holds*

$$D^{N+q} f(x) = F(x, 0; q, N), \tag{10}$$

and $D^{N+q} f(x)$ exists in $\mathbb{R}$.

We would need:

Theorem 6 *([9]) Let $f \in C^N([a,b])$, $N \in \mathbb{Z}_+$. Here $x, x' \in [a,b]$, and $F(x, \cdot; q, N) \in C^1([a-x, b-x])$. Then*

$$f(x') = \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!} (x'-x)^n + \frac{D^{N+q} f(x)}{\Gamma(q+1)} |x'-x|^q + \tag{11}$$

$$\frac{1}{\Gamma(q+1)} \int_0^{x'-x} \frac{dF(x,t;q,N)}{dt} |(x'-x) - t|^q \, dt.$$

Corollary 7 *(to Theorem 6, $N = 0$) Let $f \in C([a,b])$, $x, x' \in [a,b]$, and $F(x, \cdot; q, 0) \in C^1([a-x, b-x])$. Then*

$$f(x') = f(x) + \frac{D^q f(x)}{\Gamma(q+1)} |x'-x|^q + \tag{12}$$

$$\frac{1}{\Gamma(q+1)} \int_0^{x'-x} \frac{dF(x,t;q,0)}{dt} |(x'-x) - t|^q \, dt.$$

We make

Remark 8 *Let* $f \in C^N([a,b])$, $N \in \mathbb{Z}_+$. *Here* $x, x' \in [a,b] : x' < x$, *and* $F(x, \cdot; q, N) \in C^1([a-x, b-x])$, $0 < q < 1$. *By Theorem 6 we get*

$$f(x') = \sum_{n=0}^{N} \frac{f^{(n)}(x)}{n!}(x'-x)^n + \frac{D^{N+q}f(x)}{\Gamma(q+1)}(x-x')^q -$$

$$\frac{1}{\Gamma(q+1)} \int_{x'-x}^{0} \frac{dF(x,t;q,N)}{dt}(t-x'+x)^q \, dt. \tag{13}$$

Clearly then we get:

Let $f \in C^N([a,0])$, $a < 0$, $N \in \mathbb{Z}_+$, $F(0,\cdot;q,N) \in C^1([a,0])$, $0 < q < 1$. *Then, for any* $x \in [a,0]$, *we derive*

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}x^n + \frac{D^{N+q}f(0)}{\Gamma(q+1)}(-x)^q -$$

$$\frac{1}{\Gamma(q+1)} \int_{x}^{0} \frac{dF(0,t;q,N)}{dt}(t-x)^q \, dt. \tag{14}$$

In this article we will use a lot (14).

Remark 9 *Let* $f \in C^N([a,0])$, $N \in \mathbb{Z}_+$, $a < 0$, $x \in [a,0]$; $F(0,\cdot;q,N) \in C^1([a,0])$, $0 < q < 1$. *Then, by (14), we have*

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}x^n + \frac{D^{N+q}f(0)}{\Gamma(q+1)}(-x)^q \tag{15}$$

$$-\frac{1}{\Gamma(q+1)} \int_{x}^{0} \frac{dF(0,t;q,N)}{dt}(t-x)^q \, dt.$$

Assume that $f^{(n)}(0) = 0$, $n = 0, 1, ..., N$, *and* $D^{N+q}f(0) = 0$ *(*$= F(0,0;q,N) = D_0^q f(0)$*).*

Then

$$-f(x) = \frac{1}{\Gamma(q+1)} \int_{x}^{0} \frac{dF(0,t;q,N)}{dt}(t-x)^q \, dt, \tag{16}$$

$\forall \ x \in [a,0]$.

Here it is

$$F(0,t;q,N) = D_0^q(f(t)) \in C^1([a,0]),$$

where D_0^q *is the right Riemann-Liouville fractional derivative.*

Let $a \le x \le w \le 0$, *then*

$$-f(w) = \frac{1}{\Gamma(q+1)} \int_{w}^{0} \frac{dF(0,t;q,N)}{dt}(t-w)^q \, dt. \tag{17}$$

4

Consider $p_1, q_1 > 1 : \frac{1}{p_1} + \frac{1}{q_1} = 1$. *Then*

$$|f(w)| = \frac{1}{\Gamma(q+1)} \int_w^0 \left| \frac{dF(0,t;q,N)}{dt} \right| (t-w)^q \, dt \leq$$

$$\frac{1}{\Gamma(q+1)} \left(\int_w^0 \left| \frac{dF(0,t;q,N)}{dt} \right|^{q_1} dt \right)^{\frac{1}{q_1}} \left(\int_w^0 (t-w)^{qp_1} \, dt \right)^{\frac{1}{p_1}} =$$

$$\frac{1}{\Gamma(q+1)} \frac{(-w)^{\frac{qp_1+1}{p_1}}}{(qp_1+1)^{\frac{1}{p_1}}} \left(\int_w^0 \left| \frac{dF(0,t;q,N)}{dt} \right|^{q_1} dt \right)^{\frac{1}{q_1}} =$$

$$\frac{1}{\Gamma(q+1)} \frac{(-w)^{\frac{qp_1+1}{p_1}}}{(qp_1+1)^{\frac{1}{p_1}}} (z(w))^{\frac{1}{q_1}}, \tag{18}$$

where

$$z(w) := \int_w^0 \left| \frac{dF(0,t;q,N)}{dt} \right|^{q_1} dt, \tag{19}$$

all $a \leq x \leq w \leq 0$, *and* $z(0) = 0$.

From

$$-z(w) = \int_0^w \left| \frac{dF(0,t;q,N)}{dt} \right|^{q_1} dt,$$

we get

$$-z'(w) = (-z(w))' = \left| \frac{dF(0,w;q,N)}{dw} \right|^{q_1}, \tag{20}$$

and

$$\left| \frac{dF(0,w;q,N)}{dw} \right| = (-z'(w))^{\frac{1}{q_1}}. \tag{21}$$

Therefore we obtain

$$|f(w)| \left| \frac{dF(0,w;q,N)}{dw} \right| \leq \tag{22}$$

$$\frac{1}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}} (-w)^{\frac{qp_1+1}{p_1}} (z(w))^{\frac{1}{q_1}} (-z'(w))^{\frac{1}{q_1}}.$$

Hence it holds

$$\int_x^0 |f(w)| \left| \frac{dF(0,w;q,N)}{dw} \right| dw \leq \tag{23}$$

$$\frac{1}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}} \int_x^0 (-w)^{\frac{qp_1+1}{p_1}} (z(w)(-z'(w)))^{\frac{1}{q_1}} dw \leq$$

$$\frac{1}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}} \left(\int_x^0 (-w)^{qp_1+1} dw \right)^{\frac{1}{p_1}} \left(\int_x^0 z(w)(-z'(w)) \, dw \right)^{\frac{1}{q_1}} =$$

$$\frac{1}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}} \left(\frac{(-x)^{qp_1+2}}{qp_1+2} \right)^{\frac{1}{p_1}} \left(-\frac{z^2(w)}{2} \Big|_x^0 \right)^{\frac{1}{q_1}} = \tag{24}$$

5

$$\frac{1}{\Gamma\left(q+1\right)\left(qp_{1}+1\right)^{\frac{1}{p_{1}}}\left(qp_{1}+2\right)^{\frac{1}{p_{1}}}}\left(-x\right)^{\frac{qp_{1}+2}{p_{1}}}\frac{\left(z\left(x\right)\right)^{\frac{2}{q_{1}}}}{2^{\frac{1}{q_{1}}}}.$$

We have proved that

$$\int_{x}^{0}\left|f\left(w\right)\right|\left|\frac{dF\left(0,w;q,N\right)}{dw}\right|dw\leq$$

$$\frac{\left(-x\right)^{q+\frac{2}{p_{1}}}}{2^{\frac{1}{q_{1}}}\Gamma\left(q+1\right)\left[\left(qp_{1}+1\right)\left(qp_{1}+2\right)\right]^{\frac{1}{p_{1}}}}\left(\int_{x}^{0}\left|\frac{dF\left(0,w;q,N\right)}{dw}\right|^{q_{1}}dw\right)^{\frac{2}{q_{1}}}. \tag{25}$$

We have established the following negative domain L_p-Opial type local right fractional inequality:

Theorem 10 *Let $p_1, q_1 > 1 : \frac{1}{p_1} + \frac{1}{q_1} = 1$; $f \in C^N([a,0])$, $N \in \mathbb{Z}_+$, $a < 0$, $x \in [a,0]$; $F(0,\cdot;q,N) \in C^1([a,0])$, $0 < q < 1$. Assume that $f^{(n)}(0) = 0$, $n = 0,1,...,N$, and $D^{N+q}f(0) = 0$ ($= F(0,0;q,N) = D_0^q f(0)$). [Here it is $F(0,t;q,N) = D_0^q(f(t)) \in C^1([a,0])$, where D_0^q is the right Riemann-Liouville fractional derivative]. Then*

$$\int_{x}^{0}\left|f\left(t\right)\right|\left|\frac{dF\left(0,t;q,N\right)}{dt}\right|dt\leq$$

$$\frac{\left(-x\right)^{q+\frac{2}{p_{1}}}}{2^{\frac{1}{q_{1}}}\Gamma\left(q+1\right)\left[\left(qp_{1}+1\right)\left(qp_{1}+2\right)\right]^{\frac{1}{p_{1}}}}\left(\int_{x}^{0}\left|\frac{dF\left(0,t;q,N\right)}{dt}\right|^{q_{1}}dt\right)^{\frac{2}{q_{1}}}, \tag{26}$$

$\Leftrightarrow$

it holds

$$\int_{x}^{0}\left|f\left(t\right)\right|\left|\frac{dD_{0}^{q}\left(f\left(t\right)\right)}{dt}\right|dt\leq$$

$$\frac{\left(-x\right)^{q+\frac{2}{p_{1}}}}{2^{\frac{1}{q_{1}}}\Gamma\left(q+1\right)\left[\left(qp_{1}+1\right)\left(qp_{1}+2\right)\right]^{\frac{1}{p_{1}}}}\left(\int_{x}^{0}\left|\frac{dD_{0}^{q}\left(f\left(t\right)\right)}{dt}\right|^{q_{1}}dt\right)^{\frac{2}{q_{1}}}, \tag{27}$$

$\forall\ x \in [a,0]$.

The case $p_1 = q_1 = 2$ follows:

Corollary 11 *All as in Theorem 10, with $p_1 = q_1 = 2$. Then*

$$\int_{x}^{0}\left|f\left(t\right)\right|\left|\frac{dF\left(0,t;q,N\right)}{dt}\right|dt\leq \tag{28}$$

$$\frac{\left(-x\right)^{q+1}}{2\Gamma\left(q+1\right)\sqrt{\left(q+1\right)\left(2q+1\right)}}\left(\int_{x}^{0}\left|\frac{dF\left(0,t;q,N\right)}{dt}\right|^{2}dt\right),$$

$\Leftrightarrow$

6

it holds

$$\int_x^0 |f(t)| \left| \frac{dD_0^q (f(t))}{dt} \right| dt \leq \tag{29}$$

$$\frac{(-x)^{q+1}}{2\Gamma(q+1)\sqrt{(q+1)(2q+1)}} \left(\int_x^0 \left(\frac{dD_0^q (f(t))}{dt} \right)^2 dt \right),$$

$\forall\ x \in [a, 0]$.

We make

Remark 12 *Let f_1, f_2 according to the assumptions of Theorem 10. Then*

$$-f_1(x_1) = \frac{1}{\Gamma(q+1)} \int_{x_1}^0 \frac{dF_1(0, t_1; q, N)}{dt_1} (t_1 - x_1)^q dt_1, \tag{30}$$

$\forall\ x_1 \in [a_1, 0]$, $a_1 < 0$;

$$-f_2(x_2) = \frac{1}{\Gamma(q+1)} \int_{x_2}^0 \frac{dF_2(0, t_2; q, N)}{dt_2} (t_2 - x_2)^q dt_2, \tag{31}$$

$\forall\ x_2 \in [a_2, 0]$, $a_2 < 0$.

Here it is

$$F_i(0, t_i; q, N) = D_0^q (f_i(t_i)) \in C^1([a_i, 0]), \quad i = 1, 2;$$

where D_0^q is the right Riemann-Liouville fractional derivative.

Consider $p_1, q_1 > 1 : \frac{1}{p_1} + \frac{1}{q_1} = 1$.

Hence

$$|f_i(x_i)| \leq \frac{1}{\Gamma(q+1)} \int_{x_i}^0 \left| \frac{dF_i(0, t_i; q, N)}{dt_i} \right| (t_i - x_i)^q dt_i, \tag{32}$$

$i = 1, 2$; $\forall\ x_i \in [a_i, 0]$.

We get by Hölder's inequality:

$$|f_1(x_1)| \leq$$

$$\frac{1}{\Gamma(q+1)} \left(\int_{x_1}^0 (t_1 - x_1)^{qp_1} dt_1 \right)^{\frac{1}{p_1}} \left(\int_{x_1}^0 \left| \frac{dF_1(0, t_1; q, N)}{dt_1} \right|^{q_1} dt_1 \right)^{\frac{1}{q_1}} \leq \tag{33}$$

$$\frac{1}{\Gamma(q+1)} \frac{(-x_1)^{\frac{qp_1+1}{p_1}}}{(qp_1+1)^{\frac{1}{p_1}}} \left\| \frac{dF_1(0, t_1; q, N)}{dt_1} \right\|_{q_1, [a_1, 0]},$$

$\forall\ x_1 \in [a_1, 0]$.

Similarly, we obtain

$$|f_2(x_2)| \leq \frac{1}{\Gamma(q+1)} \frac{(-x_2)^{\frac{qq_1+1}{q_1}}}{(qq_1+1)^{\frac{1}{q_1}}} \left\| \frac{dF_2(0, t_2; q, N)}{dt_2} \right\|_{p_1, [a_2, 0]}, \tag{34}$$

7

$\forall\ x_2 \in [a_2, 0]$.

Therefore we have

$$|f_1(x_1)| |f_2(x_2)| \leq \frac{1}{(\Gamma(q+1))^2 (qp_1+1)^{\frac{1}{p_1}} (qq_1+1)^{\frac{1}{q_1}}} \tag{35}$$

$$(-x_1)^{\frac{qp_1+1}{p_1}} (-x_2)^{\frac{qq_1+1}{q_1}} \left\| \frac{dF_1(0,t_1;q,N)}{dt_1} \right\|_{q_1,[a_1,0]} \left\| \frac{dF_2(0,t_2;q,N)}{dt_2} \right\|_{p_1,[a_2,0]} \leq$$

(using Young's inequality for $a,b \geq 0$, $a^{\frac{1}{p_1}} b^{\frac{1}{q_1}} \leq \frac{a}{p_1} + \frac{b}{q_1}$*)*

$$\frac{1}{(\Gamma(q+1))^2 (qp_1+1)^{\frac{1}{p_1}} (qq_1+1)^{\frac{1}{q_1}}} \left[\frac{(-x)^{qp_1+1}}{p_1} + \frac{(-x_2)^{qq_1+1}}{q_1} \right]$$

$$\left\| \frac{dF_1(0,t_1;q,N)}{dt_1} \right\|_{q_1,[a_1,0]} \left\| \frac{dF_2(0,t_2;q,N)}{dt_2} \right\|_{p_1,[a_2,0]}, \tag{36}$$

$\forall\ x_i \in [a_i, 0]$, $i = 1, 2$.

So far we have established

$$\frac{|f_1(x_1)| |f_2(x_2)|}{\left[\frac{(-x)^{qp_1+1}}{p_1} + \frac{(-x_2)^{qq_1+1}}{q_1} \right]} \leq \frac{1}{(\Gamma(q+1))^2 (qp_1+1)^{\frac{1}{p_1}} (qq_1+1)^{\frac{1}{q_1}}} \tag{37}$$

$$\left\| \frac{dF_1(0,t_1;q,N)}{dt_1} \right\|_{q_1,[a_1,0]} \left\| \frac{dF_2(0,t_2;q,N)}{dt_2} \right\|_{p_1,[a_2,0]},$$

$\forall\ x_i \in [a_i, 0]$, $i = 1, 2$.

The denominator of left hand side of (37) can be zero only when $x_1 = 0$ *and* $x_2 = 0$. *By integrating (37) over* $[a_1, 0] \times [a_2, 0]$ *we get*

$$\int_{a_1}^{0} \int_{a_2}^{0} \frac{|f_1(x_1)| |f_2(x_2)| \, dx_1 dx_2}{\left[\frac{(-x)^{qp_1+1}}{p_1} + \frac{(-x_2)^{qq_1+1}}{q_1} \right]} \leq \frac{a_1 a_2}{(\Gamma(q+1))^2 (qp_1+1)^{\frac{1}{p_1}} (qq_1+1)^{\frac{1}{q_1}}} \tag{38}$$

$$\left\| \frac{dF_1(0,t_1;q,N)}{dt_1} \right\|_{q_1,[a_1,0]} \left\| \frac{dF_2(0,t_2;q,N)}{dt_2} \right\|_{p_1,[a_2,0]}.$$

We have proved the following negative domains local right fractional Hilbert-Pachpatte inequality:

Theorem 13 *Let* $p_1, q_1 > 1 : \frac{1}{p_1} + \frac{1}{q_1} = 1$*;* $i = 1, 2$ *for* $f_i \in C^N([a_i, 0])$, $N \in \mathbb{Z}_+$, $a_i < 0$*;* $F_i(0, \cdot; q, N) \in C^1([a_i, 0])$, $0 < q < 1$. *Assume that* $f_i^{(n)}(0) = 0$, $n = 0, 1, ..., N$, *and* $D^{N+q} f_i(0) = 0$, $i = 1, 2$ *(i.e.* $F_i(0, 0; q, N) = D_0^q f_i(0) = 0$*). [Here it is* $F_i(0, t_i; q, N) = D_0^q(f_i(t_i)) \in C^1([a_i, 0])$, *where* D_0^q *is the right Riemann-Liouville fractional derivative]. Then*

$$\int_{a_1}^{0} \int_{a_2}^{0} \frac{|f_1(x_1)| |f_2(x_2)| \, dx_1 dx_2}{\left[\frac{(-x)^{qp_1+1}}{p_1} + \frac{(-x_2)^{qq_1+1}}{q_1} \right]} \leq \frac{a_1 a_2}{(\Gamma(q+1))^2 (qp_1+1)^{\frac{1}{p_1}} (qq_1+1)^{\frac{1}{q_1}}} \tag{39}$$

8

$$\left\| \frac{dF_1(0,t_1;q,N)}{dt_1} \right\|_{q_1,[a_1,0]} \left\| \frac{dF_2(0,t_2;q,N)}{dt_2} \right\|_{p_1,[a_2,0]},$$

$\Leftrightarrow$

it holds

$$\int_{a_1}^{0} \int_{a_2}^{0} \frac{|f_1(x_1)| |f_2(x_2)| dx_1 dx_2}{\left[\frac{(-x)^{qp_1+1}}{p_1} + \frac{(-x_2)^{qq_1+1}}{q_1} \right]} \leq \frac{a_1 a_2}{(\Gamma(q+1))^2 (qp_1+1)^{\frac{1}{p_1}} (qq_1+1)^{\frac{1}{q_1}}} \tag{40}$$

$$\left\| \frac{dD_0^q(f_1(t_1))}{dt_1} \right\|_{q_1,[a_1,0]} \left\| \frac{dD_0^q(f_2(t_2))}{dt_2} \right\|_{p_1,[a_2,0]}.$$

We make

Remark 14 *Let $f \in C^N([a,0])$, $a < 0$, $N \in \mathbb{Z}_+$, $F(0,\cdot;q,N) \in C^1([a,0])$, $0 < q < 1$. Then for any $x \in [a,0]$, we have*

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n + \frac{D^{N+q}f(0)}{\Gamma(q+1)} (-x)^q \tag{41}$$

$$-\frac{1}{\Gamma(q+1)} \int_x^0 \frac{dF(0,t;q,N)}{dt} (t-x)^q dt.$$

Assume that $f^{(n)}(0) = 0$, $n = 0,1,...,N$. Here $D^{N+q}f(0) = F(0,0;q,N) = D_0^q f(0)$, where D_0^q is the right Riemann-Liouville fractional derivative.

So far we have

$$f(x) = \frac{D^{N+q}f(0)}{\Gamma(q+1)} (-x)^q + R(x), \tag{42}$$

where

$$R(x) := -\frac{1}{\Gamma(q+1)} \int_x^0 \frac{dF(0,t;q,N)}{dt} (t-x)^q dt. \tag{43}$$

We also assume that $D_0^q f \in C^1([a,0])$.

We can rewrite

$$R(x) = -\frac{1}{\Gamma(q+1)} \int_x^0 \left(\frac{d}{dt} D_0^q f(t) \right) (t-x)^q dt. \tag{44}$$

We notice that

$$|R(x)| \leq \frac{1}{\Gamma(q+1)} \int_x^0 \left| \frac{d}{dt} D_0^q f(t) \right| (t-x)^q dt \leq$$

$$\frac{1}{\Gamma(q+1)} \left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty,[a,0]} \frac{(-x)^{q+1}}{q+1}.$$

That is

$$|R(x)| \leq \frac{(-x)^{q+1}}{\Gamma(q+2)} \left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty,[a,0]}, \tag{45}$$

$\forall\ x \in [a,0]$.

Hence, it holds

$$\int_a^0 f(x)\,dx = \frac{D^{N+q} f(0)}{\Gamma(q+1)} \int_a^0 (-x)^q\,dx + \int_a^0 R(x)\,dx = \tag{46}$$

$$\frac{D^{N+q} f(0)}{\Gamma(q+1)} \frac{(-a)^{q+1}}{q+1} + \int_a^0 R(x)\,dx = \frac{D^{N+q} f(0)}{\Gamma(q+2)} (-a)^{q+1} + \int_a^0 R(x)\,dx.$$

Therefore, we get

$$\int_a^0 f(x)\,dx - \frac{D^{N+q} f(0)}{\Gamma(q+2)} (-a)^{q+1} = \int_a^0 R(x)\,dx. \tag{47}$$

Consequently, we derive

$$\left| \int_a^0 f(x)\,dx - \frac{(D_0^q f)(0)}{\Gamma(q+2)} (-a)^{q+1} \right| \leq \int_a^0 |R(x)|\,dx \leq \tag{48}$$

$$\frac{\left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty,[a,0]}}{\Gamma(q+2)} \int_a^0 (-x)^{q+1}\,dx = \frac{\left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty,[a,0]}}{\Gamma(q+2)} \frac{(-a)^{q+2}}{q+2}$$

$$= \frac{\left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty,[a,0]} (-a)^{q+2}}{\Gamma(q+3)}. \tag{49}$$

We have proved the following negative domain local right fractional comparison of means results:

Theorem 15 *Let $f \in C^N([a,0])$, $a < 0$, $N \in \mathbb{Z}_+$, $D_0^q f \in C^1([a,0])$, $0 < q < 1$. Assume $f^{(n)}(0) = 0$, $n = 0,1,...,N$. Then*

$$\left| \int_a^0 f(x)\,dx - \frac{(D_0^q f)(0)}{\Gamma(q+2)} (-a)^{q+1} \right| \leq \frac{\left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty,[a,0]} (-a)^{q+2}}{\Gamma(q+3)}, \tag{50}$$

$\Leftrightarrow$

$$\left| \frac{1}{(-a)} \int_a^0 f(x)\,dx - \frac{(D_0^q f)(0)}{\Gamma(q+2)} (-a)^q \right| \leq \frac{\left\| \frac{d}{dt} D_0^q f(t) \right\|_{\infty,[a,0]} (-a)^{q+1}}{\Gamma(q+3)}. \tag{51}$$

We make

Remark 16 *All as in Theorem 10. Then*

$$-f(x) = \frac{1}{\Gamma(q+1)} \int_x^0 \frac{dF(0,t;q,N)}{dt} (t-x)^q\,dt, \tag{52}$$

$\forall\ x \in [a, 0]$.

Let $p_1, q_1 > 1 : \frac{1}{p_1} + \frac{1}{q_1} = 1$. *Thus*

$$|f(x)| \leq \frac{1}{\Gamma(q+1)} \int_x^0 \left| \frac{dF(0,t;q,N)}{dt} \right| (t-x)^q \, dt \leq$$

$$\frac{1}{\Gamma(q+1)} \left(\int_x^0 \left| \frac{dF(0,t;q,N)}{dt} \right|^{q_1} dt \right)^{\frac{1}{q_1}} \left(\int_x^0 (t-x)^{qp_1} \, dt \right)^{\frac{1}{p_1}} \leq$$

$$\frac{1}{\Gamma(q+1)} \left\| \frac{dF(0,t;q,N)}{dt} \right\|_{q_1,[a,0]} \frac{(-x)^{\frac{qp_1+1}{p_1}}}{(qp_1+1)^{\frac{1}{p_1}}}. \tag{53}$$

That is

$$|f(x)| \leq \frac{(-x)^{\frac{qp_1+1}{p_1}}}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}} \left\| \frac{dF(0,t;q,N)}{dt} \right\|_{q_1,[a,0]}, \tag{54}$$

$\forall\ x \in [a, 0]$.

Therefore

$$|f(x)|^{q_1} \leq \frac{(-x)^{q_1(q+1)-1}}{(\Gamma(q+1))^{q_1} (qp_1+1)^{\frac{q_1}{p_1}}} \left\| \frac{dF(0,t;q,N)}{dt} \right\|_{q_1,[a,0]}^{q_1}. \tag{55}$$

Consequently, it holds

$$\int_a^0 |f(x)|^{q_1} \, dx \leq$$

$$\frac{(-a)^{q_1(q+1)}}{\left[\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}\right]^{q_1} q_1(q+1)} \left\| \frac{dF(0,t;q,N)}{dt} \right\|_{q_1,[a,0]}^{q_1}. \tag{56}$$

That is

$$\|f\|_{q_1,[a,0]} \leq \frac{(-a)^{(q+1)}}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}} (q_1(q+1))^{\frac{1}{q_1}}} \left\| \frac{dF(0,t;q,N)}{dt} \right\|_{q_1,[a,0]}. \tag{57}$$

We have proved the following negative domain local right fractional Poincare inequality:

Theorem 17 *All as in Theorem 10. Then*

$$\|f\|_{q_1,[a,0]} \leq \frac{(-a)^{(q+1)}}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}} (q_1(q+1))^{\frac{1}{q_1}}} \left\| (D_0^q(f))' \right\|_{q_1,[a,0]}. \tag{58}$$

We make

Remark 18 *All as in Theorem 10, plus $r > 0$. By (54) we have*

$$|f(x)| \leq \frac{(-x)^{\frac{qp_1+1}{p_1}}}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}} \left\| \frac{dF(0,t;q,N)}{dt} \right\|_{q_1,[a,0]}, \tag{59}$$

$\forall\, x \in [a,0]$.

Hence it holds

$$|f(x)|^r \leq \frac{(-x)^{r\left(q+\frac{1}{p_1}\right)}}{\left[\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}\right]^r} \left\| \left(D_0^q(f)\right)' \right\|_{q_1,[a,0]}^r. \tag{60}$$

Consequently, we get

$$\int_a^0 |f(x)|^r\, dx \leq \frac{(-a)^{r\left(q+\frac{1}{p_1}\right)+1}}{\left[\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}}\right]^r \left[r\left(q+\frac{1}{p_1}\right)+1\right]} \left\| \left(D_0^q(f)\right)' \right\|_{q_1,[a,0]}^r. \tag{61}$$

We have proved the following negative domain local ritgh fractional Sobolev type inequality:

Theorem 19 *All as in Theorem 10, plus $r > 0$. Then*

$$\|f\|_{r,[a,0]} \leq \frac{(-a)^{q+\frac{1}{p_1}+\frac{1}{r}}}{\Gamma(q+1)(qp_1+1)^{\frac{1}{p_1}} \left[r\left(q+\frac{1}{p_1}\right)+1\right]^{\frac{1}{r}}} \left\| \left(D_0^q(f)\right)' \right\|_{q_1,[a,0]}. \tag{62}$$

References

[1] F.B. Adda, J. Cresson, *Fractional differentiation equations and the Schrödinger equation*, Applied Math. & Computation, 161 (2005), 323-345.

[2] G.A. Anastassiou, *Quantitative Approximations*, CRC press, Boca Raton, London, New York, 2001.

[3] G.A. Anastassiou, *Fractional differentiation inequalities*, Springer, Heidelberg, New York, 2009.

[4] G.A. Anastassiou, *Probabilistic Inequalities*, World Scientific, Singapore, New York, 2010.

[5] G.A. Anastassiou, *Advanced Inequalities*, World Scientific, Singapore, New York, 2010.

[6] G.A. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Springer, Heidelberg, New York, 2011.

[7] G.A. Anastassiou, *Advances on Fractional Inequalities*, Springer, Heidelberg, New York, 2011.

[8] G.A. Anastassiou, *Intelligent Comparisons: Analytic Inequalities*, Springer, Heidelberg, New York, 2016.

[9] G.A. Anastassiou, *Local Fractional Taylor Formula*, J. of Computational Analysis and Applications, accepted, 2018.

[10] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Heidelberg, New York, 2010.

[11] K.M. Kolwankar, *Local fractional calculus: a Review*, arXiv: 1307:0739v1 [nlin.CD] 2 Jul. 2013.

[12] K.M. Kolwankar and A.D. Gangal, *Local fractional calculus: a calculus for fractal space-time*, Fractals: theory and applications in engineering, 171-181, London, New York, Springer, 1999.

[13] Z. Opial, *Sur une inégalité*, Ann. Polon. Math. 8 (1960), 29-32.

[14] A. Ostrowski, *Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmittelwert*, Comment. Math. Helv., 10 (1938), 226-227.

Approximate controllability for semilinear integro-differential control equations in Hilbert spaces

Yong Han Kang[1], Jin-Mun Jeong[2,*] and Ah-ran Park[3]

[1]Institute of Liberal Education, Catholic University of Daegu
Gyeongsan, 712-702, South Korea
E-mail:yonghann@cu.ac.kr

[2,3]Department of Applied Mathematics, Pukyong National University
Busan 48513, South Korea
E-mail: *jmjeong@pknu.ac.kr(Corresponding author), alanida@naver.com

Abstract

This paper deals with the approximate controllability for a class of semilinear integro-differential functional control equations, which is provided under general sufficient conditions on the system operator, controller and nonlinear terms. Our used tool is applying results similar to Fredholm alternative for nonlinear operators under restrictive assumptions. Finally, a simple example to which our main result can be applied is given.

Keywords: approximate controllability, semilinear control equations, integro-differential control equations, controller, Fredholm alternative.

AMS Classification: Primary 93B05, 35F25

1 Introduction

In this paper, we deal with the approximate controllability for semilinear integro-differential functional control equations in the form

$$\begin{cases} \frac{d}{dt}x(t) &= Ax(t) + \int_0^t k(t-s)g(s,x(s),u(s))ds + Bu(t), \quad 0 < t \leq T, \\ x(0) &= x_0 \end{cases} \tag{1.1}$$

Email: alanida@naver.com(A. park), *jmjeong@pknu.ac.kr(J. jeong), nabifly@hanmail.net(E. Y. Ju)
This work was supported by a Research Grant of Pukyong National University(2021 Year).

in a Hilbert space H, where k belongs to $L^2(0,T)(T>0)$ and g is a nonlinear mapping as detailed in Section 2. The principal operator A generates an analytic semigroup $(S(t))_{t\geq 0}$ and B is a bounded linear operator from another Hilbert space U to H.

The controllability problem is a question of whether is possible to steer a dynamic system from an initial state to an arbitrary final state using the set of admissible controls. Naito [13] was the first to deal with the range condition argument of controller in order to obtain the approximate controllability of a semilinear control system. In [3, 9, 17, 18], they have studied continuously about controllability of semilinear systems dominated by linear parts(in case $g \equiv 0$) by assuming that $S(t)$ is compact operator for each $t>0$ as matters connected with [13]. Another approach used to obtain sufficient conditions for approximate solvability of nonlinear equations is a fixed point theorem combined with technique of operator transformations by configuring the resolvent as seen in [2]

The controllability for various nonlinear equations has been studied by many authors, for example, see [5, 6, 12] for local controllability of neutral functional differential systems with unbounded delay, [10, 14] for neutral evolution integrodifferential systems with state dependent delay.

Sukavanam and Tomar [15] studied the approximate controllability for the general retarded initial value problem by assuming that the Lipschitz constant of the nonlinear term is less then 1, and Wang [17] for general retarded semilinear equations assuming the growth condition of the nonlinear term and the compactness of the semigroup.

In this paper, authors want to use a different method than the previous one. Our used tool is the theorems similar to the Fredholm alternative for nonlinear operators under restrictive assumption, which is on the solution of nonlinear operator equations $\lambda T(x) - F(x) = y$ in dependence on the real number λ, where T and F are nonlinear operators defined a Banach space X with values in a Banach space Y. In order to obtain the approximate controllability for a class of semilinear integro-differential functional control equations, it is necessary to suppose that T acts as the identity operator while F related to the nonlinear term of (1.1) is completely continuous

In Section 2, we introduce regularity properties for (1.1). Since we apply the Fredholm theory in the proof of the main theorem, we assume some compactness of the embedding between intermediate spaces. Then by virtue of Aubin [1], we can show that the solution mapping of a control space to the terminal state space is completely continuous. Based on Section 2, it is shown the sufficient conditions on the controller and nonlinear terms for approximate controllability for (1.1) by using the Fredholm theory. Finally, a simple example to which our main result can be applied is given.

2 Semilinear functional equations

Let V and H be complex Hilbert spaces forming a Gelfand triple

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$$

by identifying the antidual of H with H. Therefore, for the brevity, we may regard that $||u||_* \leq |u| \leq ||u||$ for all $u \in V$, where the notations $|\cdot|$, $||\cdot||$ and $||\cdot||_*$ denote the norms of H, V and V^*, respectively as usual. Let $a(u,v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding's inequality

$$\mathrm{Re}\ a(u,u) \geq c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \geq 0.$$

Let A be the operator associated with this sesquilinear form:

$$(Au, v) = -a(u,v), \quad u,\ v \in V.$$

Then A is a bounded linear operator from V to V^*. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by A. For the sake of simplicity we assume that $c_1 = 0$ and hence the closed half plane $\{\lambda : \mathrm{Re}\,\lambda \geq 0\}$ is contained in the resolvent set of A. It is known that A generates an analytic semigroup $S(t)$ in both H and V^*. As seen in Lemma 3.6.2 of [16], there exists a constant $M > 0$ such that

$$|S(t)x| \leq M|x| \quad \text{and} \quad ||S(t)x||_* \leq M||x||_*, \tag{2.1}$$

The following initial value problem for the abstract linear parabolic equation

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + k(t), \quad 0 < t \leq T, \\ x(0) = x_0. \end{cases} \tag{2.2}$$

By virtue of Theorem 3.3 of [4](or Theorem 3.1 of [9]), we have the following result on the corresponding linear equation (2.2).

Proposition 2.1. *Suppose that the assumptions for the principal operator A stated above are satisfied. Then the following properties hold:*
1) For $x_0 \in V$ and $k \in L^2(0,T;H)$, $T > 0$, there exists a unique solution x of (2.2) belonging to

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset C([0,T];V)$$

and satisfying

$$||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \leq C_1(||x_0|| + ||k||_{L^2(0,T;H)}), \tag{2.3}$$

where C_1 is a constant depending on T.
2) Let $x_0 \in H$ and $k \in L^2(0,T;V^*)$, $T > 0$. Then, there exists a unique solution x of (2.2) belonging to

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

and satisfying

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_1(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{2.4}$$

where C_1 is a constant depending on T.

By virtue of Proposition 2.1, we have the following lemma.

Lemma 2.1. *Suppose that* $k \in L^2(0,T;H)$ *and* $x(t) = \int_0^t S(t-s)k(s)ds$ *for* $0 \le t \le T$. *Then there exists a constant* C_2 *such that*

$$||x||_{L^2(0,T;H)} \le C_2 T ||k||_{L^2(0,T;H)}, \tag{2.5}$$

and

$$||x||_{L^2(0,T;V)} \le C_2 \sqrt{T} ||k||_{L^2(0,T;H)}. \tag{2.6}$$

Consider the following initial value problem for the abstract semilinear parabolic equation

$$\begin{cases} \frac{d}{dt}x(t) & = Ax(t) + \int_0^t k(t-s)g(s,x(s),u(s))ds + Bu(t), \\ x(0) & = x_0. \end{cases} \tag{2.7}$$

Let U be a Hilbert space and the controller operator B be a bounded linear operator from U to H.

Let $g : \mathbb{R}^+ \times V \times U \to H$ be a nonlinear mapping satisfying the following:

Assumption (F).

(i) For any $x \in V$, $u \in U$ the mapping $g(\cdot, x, u)$ is strongly measurable;

(ii) There exist positive constants L_0, L_1, L_2 such that

  (a) $u \mapsto g(t,x,u)$ is an odd mapping $(g(\cdot, x, -u) = -g(\cdot, x, u))$;

  (b) for all $t \in \mathbb{R}^+$, $x, \hat{x} \in V$, and $u, \hat{u} \in U$,

$$|g(t,x,u) - g(t,\hat{x},\hat{u})| \le L_1||x-\hat{x}|| + L_2||u-\hat{u}||_U,$$
$$|g(t,0,0)| \le L_0.$$

For $x \in L^2(0,T;V)$, we set

$$f(t,x,u) = \int_0^t k(t-s)g(s,x(s),u(s))ds$$

where k belongs to $L^2(0,T)$.

Lemma 2.2. *Let Assumption (F) be satisfied. Assume that* $x \in L^2(0,T;V)$ *for any* $T > 0$. *Then* $f(\cdot, x, u) \in L^2(0,T;H)$ *and*

$$\begin{aligned} ||f(\cdot,x,u))||_{L^2(0,T;H)} &\le L_0||k||_{L^2(0,T)} T/\sqrt{2} \\ &+ ||k||_{L^2(0,T)} \sqrt{T}(L_1||x||_{L^2(0,T;V)} + L_2||u||_{L^2(0,T;U)}). \end{aligned} \tag{2.8}$$

Moreover if $x,\ \hat{x} \in L^2(0,T;V)$, *then*

$$\begin{aligned} &||f(\cdot,x,u) - f(\cdot,\hat{x},\hat{u})||_{L^2(0,T;H)} \\ &\le ||k||_{L^2(0,T)} \sqrt{T}(L_1||x-\hat{x}||_{L^2(0,T;V)} + L_2||u-\hat{u}||_{L^2(0,T;U)}). \end{aligned} \tag{2.9}$$

4

The proof is easily from Assumption (F), and using the Hölder inequality.

By virtue of Theorem 2.1 of [8], we have the following result on (2.7).

Proposition 2.2. *Let Assumption (F) be satisfied. Then there exists a unique solution x of (2.7) such that*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

for any $x_0 \in H$. Moreover, there exists a constant C_3 such that

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_3(|x_0| + ||u||_{L^2(0,T;U)}). \tag{2.10}$$

Corollary 2.1. *Assume that the embedding $D(A) \subset V$ is completely continuous. Let Assumption (F) be satisfied, and x_u be the solution of equation (2.7) associated with $u \in L^2(0,T;U)$. Then the mapping $u \mapsto x_u$ is completely continuous from $L^2(0,T;U)$ to $L^2(0,T;V)$.*

Proof. If u is bounded in $L^2(0,T;U)$, then so is x_u in $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$ by (2.8). Since $D(A)$ is compactly embedded in V by assumption, the embedding

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V)$$

is completely continuous in view of Theorem 2 of [1], the mapping $u \mapsto x_u$ is completely continuous from $L^2(0,T;U)$ to $L^2(0,T;V)$. □

3 Approximate controllability

Throughout this section, we assume that $D(A)$ is compactly embedded in V. Let $x(T;f,u)$ be a state value of the system (2.7) at time T corresponding to the nonlinear term f and the control u. We define the reachable sets for the system (2.7) as follows:

$$R_T(f) = \{x(T;f,u) : u \in L^2(0,T;U)\},$$
$$R_T(0) = \{x(T;0,u) : u \in L^2(0,T;U)\}.$$

Definition 3.1. *The system (2.7) is said to be approximately controllable in the time interval $[0,T]$ if for every desired final state $x_1 \in H$ and $\epsilon > 0$ there exists a control function $u \in L^2(0,T;U)$ such that the solution $x(T;f,u)$ of (2.7) satisfies $|x(T;f,u) - x_1| < \epsilon$, that is, if $\overline{R_T(f)} = H$ where $\overline{R_T(f)}$ is the closure of $R_T(f)$ in H, then the system (2.9) is called approximately controllable at time T.*

Let us introduce the theory of the degree for completely continuous perturbations of the identity operator, which is the infinite dimensional version of Borsuk's theorem. Let $0 \in D$ be a bounded open set in a Banach space X, $\overline{D}$ its closure and ∂D its boundary. The number $d[I-T;D,0]$ is the degree of the mapping $I-T$ with respect to the set D and the point 0 (see Fučik et al. [7] or Lloid [11]).

5

Theorem 3.1. *(Borsuk's theorem) Let D be a bounded open symmetric set in a Banach space X, $0 \in D$. Suppose that $T : \overline{D} \to X$ be odd completely continuous operator satisfying $T(x) \neq x$ for $x \in \partial D$. Then $d[I - T; D, 0]$ is odd integer. That is, there exists at least one point $x_0 \in D$ such that $(I - T)(x_0) = 0$.*

Definition 3.2. *Let T be a mapping defined by on a Banach space X with value in a real Banach space Y. The mapping T is said to be a (K, L, α)-homeomorphism of X onto Y if*

(i) *T is a homeomorphism of X onto Y;*

(ii) *there exist real numbers $K > 0$, $L > 0$, and $\alpha > 0$ such that*

$$L||x||_X^\alpha \leq ||T(x)||_Y \leq K||x||_X^\alpha, \quad \forall x \in X.$$

Lemma 3.1. *Let T be an odd (K, L, α)-homeomorphism of X onto Y and $F : X \to Y$ a continuous operator satisfying*

$$\limsup_{||x||_X \to \infty} \frac{||F(x)||_Y}{||x||_X^\alpha} = N \in \mathbb{R}^+.$$

Then if $|\lambda| \notin [\frac{N}{K}, \frac{N}{L}] \cup \{0\}$ then

$$\lim_{||x||_X \to \infty} ||\lambda T(x) - F(x)||_Y = \infty.$$

Proof. Suppose that there exist a constant $M > 0$ and a sequence $\{x_n\} \subset X$ such that

$$||\lambda T(x_n) - F(x_n)||_Y \leq M \tag{3.1}$$

as $x_n \to \infty$. From (3.1) it follows that

$$\frac{\lambda T(x_n)}{||x_n||_X^\alpha} - \frac{F(x_n)}{||x_n||_X^\alpha} \to 0.$$

Hence, we have

$$\limsup_{n \to \infty} \frac{|\lambda|||T(x_n)||_Y}{||x_n||_X^\alpha} = N,$$

and so, $|\lambda|K \geq N \geq |\lambda|L$. It is a contradiction with $|\lambda| \notin [\frac{N}{K}, \frac{N}{L}]$. □

Proposition 3.1. *Let T be an odd (K, L, α)-homeomorphism of X onto Y and $F : X \to Y$ an odd completely continuous operator. Suppose that for $\lambda \neq 0$,*

$$\lim_{||x||_X \to \infty} ||\lambda T(x) - F(x)||_Y = \infty. \tag{3.2}$$

Then $\lambda T - F$ maps X onto Y.

Proof. We follow the proof Theorem 1.1 in Chapter II of Fučik et al. [7]. Suppose that there exists $y \in Y$ such that $\lambda T(x) = y$. Then from (3.2) it follows that $FT^{-1} : Y \to Y$ is an odd completely continuous operator and

$$\lim_{||y||_Y \to \infty} ||y - FT^{-1}(\frac{y}{\lambda})||_Y = \infty.$$

Let $y_0 \in Y$. There exists $r > 0$ such that

$$||y - FT^{-1}(\frac{y}{\lambda})||_Y > ||y_0||_Y \geq 0$$

for each $y \in Y$ satisfying $||y||_Y = r$. Let $Y_r = \{y \in Y : ||y||_Y < r\}$ be a open ball. Then by view of Theorem 3.1, we have $d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, 0]$ is an odd number. For each $y \in Y$ satisfying $||y||_Y = r$ and $t \in [0, 1]$, there is

$$||y - FT^{-1}(\frac{y}{\lambda}) - ty_0||_Y \geq ||y - FT^{-1}(\frac{y}{\lambda})||_Y - ||y_0||_Y > 0$$

and hence, by the homotopic property of degree, we have

$$d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, y_0] = d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, 0] \neq 0.$$

Hence, by the existence theory of the Leray-Schauder degree, there exists a $y_1 \in Y_r$ such that

$$y_1 - FT^{-1}(\frac{y_1}{\lambda}) = y_0.$$

We can choose $x_0 \in X$ satisfying $\lambda T(x_0) = y_1$, and so, $\lambda T(x_0) - F(x_0) = y_0$. Thus, it implies that $\lambda T - F$ is a mapping of X onto Y. □

Combining Lemma 3.1. and Proposition 3,1, we have the following results.

Corollary 3.1. *Let T be an odd (K, L, α)-homeomorphism of X onto Y and $F : X \to Y$ an odd completely continuous operator satisfying*

$$\limsup_{||x||_X \to \infty} \frac{||F(x)||_Y}{||x||_X^\alpha} = N \in \mathbb{R}^+.$$

Then if $|\lambda| \notin [\frac{N}{K}, \frac{N}{L}] \cup \{0\}$ then $\lambda T - F$ maps X onto Y. Therefore, if $N = 0$, then for all $\lambda \neq 0$ the operator $\lambda T - F$ maps X onto Y.

First we consider the approximate controllability of the system (2.7) in case where the controller B is the identity operator on H under Assumption (F) on the nonlinear operator f in Section 2. Hence, noting that $H = U$, we consider the linear system given by

$$\begin{cases} \frac{d}{dt}y(t) &= Ay(t) + u(t), \\ y(0) &= x_0, \end{cases} \tag{3.3}$$

and the following semilinear control system

$$\begin{cases} \frac{d}{dt}x(t) &= Ax(t) + f(t, x(t), v(t)) + v(t), \\ x(0) &= x_0. \end{cases} \tag{3.4}$$

Theorem 3.2. *Assume that*

$$\limsup_{||u|| \to \infty} \frac{||f(\cdot, x_u, u)||_{L^2(0,T;H)}}{||u||_{L^2(0,T;H)}} < 1. \tag{3.5}$$

Under the Assumption (F) we have

$$R_T(0) \subset R_T(f).$$

Therefore, if the linear system (3.3) with $f = 0$ is approximately controllable, then so is the semilinear system (3.4).

Proof. Let $x(t)$ be solution of (3.4) corresponding to a control u. First, we show that there exist a $v \in L^2(0, T; H)$ such that

$$\begin{cases} v(t) &= u(t) - f(t, x(t), v(t)), \quad 0 < t \le T, \\ v(0) &= u(0). \end{cases}$$

Let us define an operator $F : L^2(0, T; H) \to L^2(0, T; H)$ as

$$Fv = -f(\cdot, x_v, v).$$

Then by Corollary 2.1, F is a compact mapping from $L^2(0, T; H)$ to itself, and we have

$$\lim_{||v|| \to \infty} ||\lambda I(v) - F(v)||_{L^2(0,T;H)} = \infty,$$

where the identity operator I on $L^2(0, T; H)$ is an odd $(1, 1, 1)$-homeomorphism. Thus, from (3.5) and Corollary 3.1, if $\lambda \ge 1$ then $\lambda I - F$ maps $L^2(0, T; H)$ onto itself. Hence, we have showed that there exists a $v \in L^2(0, T; H)$ such that $v(t) = u(t) - f(t, y(t), v(t))$. Let y and x be solutions of (3.3) and (3.4) corresponding to controls u and v, respectively. Then, equation (3.4) is rewritten as

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + f(t, x(t), v(t)) + v(t), \ 0 < t \le T \\ &= Ax(t) + f(t, x(t), v(t)) + u(t) - f(t, y(t), v(t)) \\ &= Ax(t) + u(t) \end{aligned}$$

with $x(0) = x_0$, which means

$$\begin{aligned} x(t) =& S(t)x_0 + \int_0^t S(t-s)\{f(s,x(s),v(s)) + v(s)\}ds \\ =& S(t)x_0 + \int_0^t S(t-s)u(s)ds = y(t), \end{aligned}$$

where y be solution of (3.3) corresponding to a control u. Therefore, we have proved that $R_T(0) \subset R_T(f)$. □

Corollary 3.2. *Let us assume that*

$$||k||_{L^2(0,T)}\sqrt{T}\big(L_1C_3 + L_2\big) < 1,$$

where C_3 is the constant in Proposition 2.2. Under the Assumption (F), we have

$$R_T(0) \subset R_T(f)$$

in case where $B \equiv I$.

Proof. By Lemma 2.2 and Proposition 2.2, we have

$$\begin{aligned} &||Fu||_{L^2(0,T;H)} = ||f(\cdot, x_u, u)||_{L^2(0,T;H)} \\ &\le L_0||k||_{L^2(0,T)}T/\sqrt{2} + ||k||_{L^2(0,T)}\sqrt{T}(L_1||x||_{L^2(0,T;V)} + L_2||u||_{L^2(0,T;U)}) \\ &\le L_0||k||_{L^2(0,T)}T/\sqrt{2} + ||k||_{L^2(0,T)}\sqrt{T}\big\{L_1C_3\big(|x_0| + ||u||_{L^2(0,T;U)}\big) + L_2||u||_{L^2(0,T;U)}\big\}. \end{aligned}$$

Hence, we have

$$\limsup_{||u||\to\infty} \frac{||F(u)||_{L^2(0,T;H)}}{||u||_{L^2(0,T;U)}} \le ||k||_{L^2(0,T)}\sqrt{T}\big(L_1C_3 + L_2\big).$$

Thus, from Theorem 3.2, it follows that if $\lambda \ge 1$ then $\lambda I - F$ maps $L^2(0,T;H)$ onto itself, and so, by the same argument as in the proof of theorem it holds that $R_T(0) \subset R_T(f)$. □

From now on, we consider the initial value problem for the semilinear parabolic equation (2.7). Let U be some Hilbert space and the controller operator B be a bounded linear operator from U to H.

Assumption (B) There exists a constant $\beta > 0$ such that $\mathrm{R}(f) \subset \mathrm{R}(B)$ and

$$||Bu|| \ge \beta||u||, \quad \forall u \in L^2(0,T;U).$$

Consider the linear system given by

$$\begin{cases} \frac{d}{dt}y(t) &= Ay(t) + Bu(t), \\ y(0) &= x_0. \end{cases} \tag{3.6}$$

Theorem 3.3. *Under the Assumptions (3.5), (B) and (F), we have*

$$R_T(0) \subset R_T(f).$$

Therefore, if the linear system (3.6) with $f = 0$ is approximately controllable, then so is the semilinear system (2.7).

Proof. Let y be a solution of the linear system (3.6) with $f = 0$ corresponding to a control u, and let x be a solutions of the semilinear system (3.4) corresponding to a control v. Set $v(t) = u(t) - B^{-1}f(t, x(t), v(t))$. Then, system (2.9) is rewritten as

$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) + f(t, x(t), v(t)) + Bv(t),\ 0 < t \le T\\ &= Ax(t) + f(t, x(t), v(t)) + Bu(t) - f(t, x(t), v(t))\end{aligned}$$

with $x(0) = x_0$. Hence, we have

$$\begin{aligned}x(t) =&S(t)x_0 + \int_0^t S(t-s)\{f(s, x(s), v(s)) + v(s))\}ds\\ =&S(t)x_0 + \int_0^t S(t-s)u(s))ds = y(t).\end{aligned}$$

Thus, we obtain that $R_T(0) \subset R_T(f)$. □

References

[1] Aubin, J. P., Un thèoréme de compacité, C. R. AQcad. Sci.,, 256 (1963), 5042–5044.

[2] K. Balachandran and J. P. Dauer, Controllability of nonlinear systems in Banach spaces; a survey, J. optim. Theory Appl., 115 (2002), 7–28.

[3] J. P. Dauer and N. I. Mahmudov, *Approximate controllability of semilinear functional equations in Hilbert spaces*, J. Math. Anal., 273 (2002), 310–327.

[4] G. Di Blasio, K. Kunisch and E. Sinestrari, L^2-*regularity for parabolic partial integrodifferential equations with delay in the highest-order derivatives*, J. Math. Anal. Appl., 102 (1984), 38–57.

[5] X. Fu,*Controllability of neutral functional differential systems in abstract space*, Appl. Math. Comput., 141 (2003), 281–296.

[6] X. Fu, J. Lu and Y. You, *Approximate controllability of a semilinear neutral evolution systems with delay*, Inter. J. Control, 87 (2014), 665-681.

[7] Fučik, S., Nečas, J., Souček, J., & Souček, V.(1973), *lecture Notes in Mathematics 346*, Springer-verlag, Belin-Heidelberg-NewYork.

[8] J. M. Jeong and H. Kim, *Controllability for semilinear functional integrodifferential equations*, Bull. Korean Math. Soc. 46(3) (2009), 463–475.

[9] J. M. Jeong, Y. C. Kwun and J. Y. Park, *Approximate controllability for semilinear retarded functional differential equations*, J. Dynamics and Control Systems, 5 (1999), no. 3, 329–346.

[10] Y. C. Kwun, S. H. Park, D. K. Park and S. J. Park, *Controllability of semilinear neutral functional differential evolution equations with nonlocal conditions*, J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math., 15 (2008), 245-??57.

[11] N. G. Lloid, Degree Theory, Cambridge Univ. Press. 1978.

[12] F. Z. Mokkedem and X. Fu, *Approximate controllability of a semi-linear neutral evolution systems with infinite delay* Internat. J. Robust Nonlinear Control, 27, (2017), 1122-1146.

[13] K. Naito, *Controllability of semilinear control systems dominated by the linear part*, SIAM J. Control Optim., 25 (1987), 715–722.

[14] B. Radhakrishnan and K. Balachandran, *Controllability of neutral evolution integrodifferential systems with state dependent delay*, J. Optim. Theory Appl., 153 (2012), 85-97.

[15] N. Sukavanam and N. K. Tomar, *Approximate controllability of semilinear delay control system*, Nonlinear Func. Anal.Appl., 12 (2007), 53–59.

[16] H. Tanabe, *Equations of Evolution*, Pitman-London, 1979.

[17] L. Wang, *Approximate controllability for integrodifferential equations and multiple delays*, J. Optim. Theory Appl., 143 (2009), 185–206.

[18] H. X. Zhou, Approximate controllability for a class of semilinear abstract equations, SIAM J. Control Optim., 21 (1983), 551–565.

Convergence theorems and approximating endpoints for multivalued Suzuki mappings in hyperbolic spaces

Preeyalak Chuadchawna[1], Ali Farajzadeh[2] and Anchalee Kaewcharoen[3]*
[2]Department of Mathematics, Razi University, Kermanshah, 67149, Iran
E-mail: farajzadehali@gmail.com
[1,3] Department of Mathematics, Faculty of Science, Naresuan University
Phitsanulok 65000, Thailand
[3] Center of Excellence in Nonlinear Analysis and Optimization, Faculty of Science
Naresuan University Phitsanulok 65000, Thailand
E-mails: Chuadchawna@hotmail.com; anchaleeka@nu.ac.th

November 13, 2018

Abstract

The objective of this paper is to determine a modified SP-iteration process for multi-valued mappings and to establish the convergence theorems for sequences generated by modified SP-iteration processes involving multi-valued Suzuki mappings converging to endpoints in uniformly convex hyperbolic spaces. The numerical example for supporting our main result is also presented.

Keywords: modified SP-iteration; Δ-convergence theorem; strong convergence theorem; endpoint; hyperbolic space.

MSC: Primary 47H10; Seconday 54H25.

1 Introduction

The distance from u in a metric space (X, d) to a nonempty subset E of X is defined by

$$\operatorname{dist}(u, E) := \inf\{d(u, v) : v \in E\}.$$

It is denoted by $K(E)$ the family of nonempty compact subsets of E. The Hausdorff distance on $K(E)$ is defined by

$$H(U, V) := \max\{\sup_{u \in U} \operatorname{dist}(u, V), \sup_{v \in V} \operatorname{dist}(v, U)\} \quad \text{for all } U, V \in K(E).$$

*Corresponding author.

1

For an element x in E, if $x \in T(x)$, then x is said to be a fixed point of T. Moreover, if $\{x\} = T(x)$, then x is said to be an endpoint of T. It denote by $Fix(T)$ the set of all fixed points of T and by $End(T)$ the set of all endpoints of T. We can see that for every a multi-valued mapping T, $End(T) \subset Fix(T)$ and whenever t is a single-valued mapping, $End(T) = Fix(T)$.

The notion of endpoints for multi-valued mappings is significant notion which put between the notion of fixed points for single-valued mappings and the notion of fixed points for multi-valued mappings.

Aubin and Siegel [3] were first studied the existence of endpoints for special kind of contractive mappings on complete metric spaces. The endpoint results for several types of contractive mappings have been quickly developed and many of papers have showed (see, e.g.,[9],[18],[20],[21]).

On the other hand, Panyanak [15] presented the existence of endpoints for multi-valued nonexpansive mappings in uniformly convex Banach spaces. Next, Kudtha and Panyanak [13] proved the existence of endpoints for Suzuki mappings in uniformly convex hyperbolic spaces.

Recently, Panyanak [16] established the convergence theorems to an endpoint for modified Ishikawa iteration of multi-vaued nonexpansive mappings in uniformly convex Banach spaces.

Motivated and inspired by above mention, we prove the convergence results to an endpoint for modified SP-iteration of multi-valued Suzuki mappings in uniformly convex hyperbolic spaces. The numerical example for supporting our main result is also presented.

2 Preliminaries

For this paper, we work in the setting of a hyperbolic space which is defined by Kohlenbach [12].

Definition 2.1 A hyperbolic space [12] is a metric space (X, d) together with a mapping $W : X^2 \times [0,1] \to X$ satisfying the following statements:

(W1) $d(u, W(x, y, \alpha)) \leq (1-\alpha)d(u, x) + \alpha d(u, y)$;

(W2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$;

(W3) $W(x, y, \alpha) = W(y, x, (1-\alpha))$;

(W4) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1-\alpha)d(x, y) + \alpha d(z, w)$,

for all $x, y, u, z, w \in X$ and $\alpha, \beta \in [0,1]$.

If $x, y \in X$ and $\alpha \in [0,1]$, then we use the notion $(1-\alpha)x \oplus \alpha y$ for $W(x, y, \alpha)$. A hyperbolic space (X, d, W) is said to be *uniformly convex* [14] if for any $r > 0$ and $\varepsilon \in (0, 2]$ there exists a $\delta \in (0, 1]$ such that for all $u, x, y \in X$, we have

$$d(W(x, y, \frac{1}{2}), u) \leq (1-\delta)r,$$

provided $d(x, u) \leq r$, $d(y, u) \leq r$ and $d(x, y) \geq \varepsilon r$.

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$ is well known as a modulus of uniformly convexity of

X. We call η monotone if it decreases with r (for a fixed ε), i.e., for any given $\varepsilon > 0$ and for any $r_2 \geq r_1 > 0$, we have $\eta(r_2, \varepsilon) \leq \eta(r_1, \varepsilon)$.

A nonempty subset E of a hyperbolic space X is *convex* if $W(x, y, \alpha) \in E$ for any $x, y \in E$ and $\alpha \in [0, 1]$.

Obviously, uniformly convex Banach spaces are uniformly convex hyperbolic spaces, CAT(0) spaces are also uniformly convex hyperbolic spaces, [14].

Definition 2.2 [7] A multi-valued mapping $T : E \to CB(E)$ is called to be a Suzuki mapping if

$$\frac{1}{2}\mathrm{dist}(x, T(x)) \leq d(x, y) \text{ implies } H(T(x), T(y)) \leq d(x, y) \tag{1}$$

for all $x, y \in X$.

Definition 2.3 [1] A multi-valued mapping $T : E \to CB(E)$ is said to satisfy condition (E_μ) provided that

$$\mathrm{dist}(x, T(y)) \leq \mu \mathrm{dist}(x, T(x)) + d(x, y), \text{ for all } x, y \in E.$$

We say that T satisfies condition (E) whenever T satisfies condition (E_μ) for some $\mu \geq 1$.

Lemma 2.4 *[6] If E is a nonempty closed convex subset of X and $T : E \to CB(E)$ is a multi-valued Suzuki mapping, then T satisfies the condition (E_3).*

We need the following definition of convergence in hyperbolic spaces [5] which is called Δ-convergence.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. Define a function $r(\cdot, \{x_n\}) : X \to [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n), \text{ for all } x \in X.$$

The asymptotic radius of a bounded sequence $\{x_n\}$ with respect to a nonempty subset K of X is defined and denoted by

$$r_K(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\}.$$

The asymptotic center of a bounded sequence $\{x_n\}$ with respect to a nonempty subset K of X is defined and denoted by

$$AC_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \leq r(y, \{x_n\}), \text{ for all } y \in K\}.$$

Recall that a sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write Δ-$\lim_{n\to\infty} x_n = x$ and call x the Δ-lim of $\{x_n\}$.

The sequence $\{x_n\}$ is called to be *regular* relative to E if $r(E, \{x_n\}) = r(E, \{x_{n_j}\})$ for every subsequence $\{x_{n_j}\}$ of $\{x_n\}$. It is known that every bounded sequence in a Banach space has a regular subsequence (see [8]). The proof is metric in nature and carries over to the present setting without change.

Lemma 2.5 *[4] Let (X, d, W) be a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity η and E is a closed convex subset of X if $\{x_n\}$ is a bounded sequence in E, then the asymptotic center of $\{x_n\}$ is in E.*

Lemma 2.6 *[10] Let (X, d, W) be a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq c$, $\limsup_{n\to\infty} d(y_n, x) \leq c$, $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = c$ for some $c \geq 0$, then*

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Lemma 2.7 *[11] Every bounded sequence in a complete CAT(0)(and hence hyperbolic) space has a Δ-convergent subsequence.*

Lemma 2.8 *[7] If $\{x_n\}$ is a bounded sequence in complete uniformly convex hyperbolic space (X, d, W) with $A(\{x_n\}) = \{p\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $p = u$.*

Definition 2.9 [8] Let E be a nonempty subset of a metric space (X, d) and $x \in X$. The *radius* of E relative to x is defined by

$$r_x(E) := \sup\{d(x, y) : y \in E\}.$$

The *diameter* of E is defined by

$$\operatorname{diam}(E) := \sup\{d(x, y) : x, y \in E\}.$$

Definition 2.10 [2] Let $T : E \to CB(E)$ be a multi-valued mapping. A sequence $\{x_n\}$ in E is called an *approximate fixed point sequence* (resp. *an approximate endpoint sequence*) for T if $\lim_{n\to\infty} \operatorname{dist}(x_n, T(x_n)) = 0$ (resp. $\lim_{n\to\infty} r_{x_n}(T(x_n)) = 0$). A mapping T is said to have *the approximate fixed point property* (resp. *the approximate endpoint property*) if it has an approximate fixed point sequence (resp. an approximate endpoint sequence) in E.

Lemma 2.11 *[15] Let E be a nonempty subset of X, $\{x_n\}$ be a sequence in E and $T : E \to K(E)$ be a multi-valued mapping. Then $r_{x_n}(T(x_n)) \to 0$ if and only if dist$(x_n, T(x_n)) \to 0$ and diam$(T(x_n)) \to 0$.*

Lemma 2.12 *[13] Let E be a nonempty bounded closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity and $T : E \to K(E)$ be a multi-valued Suzuki mapping. Then T has an endpoint if and only if T has the approximate endpoint property.*

Next, we also need the following definitions that will be used in the next section.

A sequence $\{x_n\}$ in E is said to be Fejér monotone with respect to E if

$$d(x_{n+1}, q) \leq d(x_n, q) \text{ for all } q \in E \text{ and } n \in \mathbb{N}.$$

4

Definition 2.13 [13] Let E be a nonempty subset of a hyperbolic space X. A mapping $T : E \to K(E)$ is said to satisfy condition (J) if there exists a nondecreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$, $h(r) > 0$ for $r \in (0, \infty)$ such that

$$r_x(T(x)) \geq h(\text{dist}(x, End(T))) \text{ for all } x \in E.$$

The mapping T is called *semicompact* if for any sequence $\{x_n\}$ in E such that

$$\lim_{n\to\infty} r_{x_n}(T(x_n)) = 0,$$

there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $q \in E$ such that $\lim_{n\to\infty} x_{n_j} = q$.

3 Main results

For this part, we start by introducing the notion of the modified SP-iteration process for multi-valued mappings. Notice that it is an improvement of the one so called the SP-iteration process given in Phuengrattana and Suantai [17]. They [17] also showed that SP-iteration process is a generalized version and the sequence generated by the SP-iteration process converges faster than Ishikawa for the class of nondecreasing and continuous functions.

Let X be a hyperbolic space and E be a nonempty convex subset of X, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $[0, 1]$ and $T : E \to K(E)$ be a multi-valued mapping. The sequence generated by the modified SP-iteration is defined by $z_1 \in E$,

$$\begin{cases} y_n = W(u_n, z_n, \gamma_n) \\ w_n = W(v_n, y_n, \beta_n) \\ z_{n+1} = W(x_n, w_n, \alpha_n), \end{cases} \tag{2}$$

where $u_n \in T(z_n)$ such that $d(z_n, u_n) = r_{z_n}(T(z_n))$, $v_n \in T(y_n)$ such that $d(v_n, y_n) = r_{y_n}(T(y_n))$ and $x_n \in T(w_n)$ such that $d(x_n, w_n) = r_{w_n}(T(w_n))$.

We need the following important Lemmas that will be used in the sequel.

Lemma 3.1 *Let E be a nonempty bounded closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity and $T : E \to K(E)$ be a multi-valued Suzuki mapping. If $\{z_n\}$ is a sequence in E, then the following holds:*

$$z_n \xrightarrow{\triangle} z,\ dist(z_n, T(z_n)) \to 0 \text{ and } diam(T(z_n)) \to 0 \text{ imply } z \in End(T).$$

Proof. From Lemma 2.5, we obtain that $z \in E$. For each $n \in \mathbb{N}$, we can choose $w_n \in T(z_n)$ such that $d(z_n, w_n) = \text{dist}(z_n, T(z_n))$. By passing throught a subsequence, we may assume that $\{z_n\}$ is regular relative to E. Let $A(E, \{z_n\}) = \{z\}$ and $r = r(E, \{z_n\})$. By similar way in the proof of Lemma 2.12, we obtain that $z \in End(T)$. ■

Lemma 3.2 *Let E be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity and $T : E \to K(E)$ be a multi-valued Suzuki mapping with $End(T) \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by the modified SP-iteration process (2). Then $\lim_{n\to\infty} d(z_n, q)$ exists for each $q \in End(T)$.*

Proof. Let T be a multi-valued Suzuki mapping and $q \in End(T)$. Therefore,

$$\frac{1}{2}\text{dist}(q, T(q)) = 0 \leq d(q, y_n), \tag{3}$$

$$\frac{1}{2}\text{dist}(q, T(q)) = 0 \leq d(q, w_n), \tag{4}$$

and

$$\frac{1}{2}\text{dist}(q, T(q)) = 0 \leq d(q, z_n), \tag{5}$$

for all $n \in \mathbb{N}$. This implies that

$$H(T(q), T(y_n)) \leq d(q, y_n), \tag{6}$$

$$H(T(q), T(w_n)) \leq d(q, w_n), \tag{7}$$

and

$$H(T(q), T(z_n)) \leq d(q, z_n). \tag{8}$$

Using (2) and (8), we obtain that

$$\begin{aligned} d(y_n, q) &= d(W(u_n, z_n, \gamma_n), q) \\ &\leq (1-\gamma_n)d(u_n, q) + \gamma_n d(z_n, q) \\ &= (1-\gamma_n)\text{dist}(u_n, T(q)) + \gamma_n d(z_n, q) \\ &\leq (1-\gamma_n)H(T(z_n), T(q)) + \gamma_n d(z_n, q) \\ &\leq (1-\gamma_n)d(z_n, q) + \gamma_n d(z_n, q) \\ &\leq d(z_n, q). \end{aligned} \tag{9}$$

Next, using (2), (6) and (9)

$$\begin{aligned} d(w_n, q) &= d(W(v_n, y_n, \beta_n), q) \\ &\leq (1-\beta_n)d(v_n, q) + \beta_n d(y_n, q) \\ &= (1-\beta_n)\text{dist}(v_n, T(q)) + \beta_n d(y_n, q) \\ &\leq (1-\beta_n)H(T(y_n), T(q)) + \beta_n d(y_n, q) \\ &\leq (1-\beta_n)d(y_n, q) + \beta_n d(y_n, q) \\ &\leq d(y_n, q) \leq d(z_n, q). \end{aligned} \tag{10}$$

Again, using (2), (7) and (10)

$$\begin{aligned} d(z_{n+1},q) &= d(W(x_n,w_n,\alpha_n),q) \\ &\leq (1-\alpha_n)d(x_n,q)+\alpha_n d(w_n,q) \\ &= (1-\alpha_n)\mathrm{dist}(x_n,T(q))+\alpha_n d(w_n,q) \\ &\leq (1-\alpha_n)H(T(w_n),T(q))+\alpha_n d(w_n,q) \\ &\leq (1-\alpha_n)d(w_n,q)+\alpha_n d(w_n,q) \\ &\leq d(w_n,q) \\ &\leq d(z_n,q). \end{aligned} \tag{11}$$

This shows that sequence $\{d(z_n,q)\}$ is decreasing and bounded below. Thus $\lim_{n\to\infty} d(z_n,q)$ exists for each $q \in End(T)$.
■

Next, we prove Δ-convergence theorem for a multi-valued mapping in hyperbolic spaces.

Theorem 3.3 *Let E be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity and $T : E \to K(E)$ be a multi-valued Suzuki mapping with $End(T) \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by the modified SP-iteration process (2). Then $\{z_n\}$ Δ-converges to an endpoint of T.*

Proof. First we will prove that $r_{z_n}(T(z_n)) \to 0$. Let $q \in End(T)$. Since T is a multi-valued Suzuki mapping and

$$\frac{1}{2}\mathrm{dist}(q,T(q)) = 0 \leq d(q,z_n)$$

for all $n \in \mathbb{N}$, then

$$H(T(q),T(z_n)) \leq d(q,z_n).$$

From Lemma 3.2, we know that for each $q \in End(T)$, $\lim_{n\to\infty} d(z_n,q)$ exists. Let $\lim_{n\to\infty} d(z_n,q) = t \geq 0$. If $t = 0$, then

$$\begin{aligned} d(z_n,u_n) &\leq d(z_n,q)+d(q,u_n) \\ &= d(z_n,q)+\mathrm{dist}(T(q),u_n) \\ &\leq d(z_n,q)+H(T(q),T(z_n)) \\ &\leq d(z_n,q)+d(z_n,q). \end{aligned}$$

Taking $n \to \infty$ on above inequality, we have

$$\lim_{n\to\infty} r_{z_n}(T(z_n)) = \lim_{n\to\infty} d(z_n,u_n) = 0.$$

If $t > 0$, then

$$\begin{aligned} d(y_n, q) &= d(W(u_n, z_n, \gamma_n), q) \\ &\leq (1-\gamma_n)d(u_n, q) + \gamma_n d(z_n, q) \\ &= (1-\gamma_n)\mathrm{dist}(u_n, T(q)) + \gamma_n d(z_n, q) \\ &\leq (1-\gamma_n)H(T(z_n), T(q)) + \gamma_n d(z_n, q) \\ &\leq (1-\gamma_n)d(z_n, q) + \gamma_n d(z_n, q) \\ &\leq d(z_n, q). \end{aligned}$$

Letting limsup as $n \to \infty$ on the both sides of above inequality, we have

$$\limsup_{n\to\infty} d(y_n, q) \leq \limsup_{n\to\infty} d(z_n, q) \leq t. \tag{12}$$

From (11), we have $d(z_{n+1}, q) \leq d(w_n, q)$.
Then we obtain that

$$t \leq \liminf_{n\to\infty} d(z_{n+1}, q) \leq \liminf_{n\to\infty} d(w_n, q). \tag{13}$$

From the proof in (10), we have $d(w_n, q) \leq d(y_n, q)$.
Taking liminf as $n \to \infty$ on above inequality and using (13),

$$t \leq \liminf_{n\to\infty} d(y_n, q). \tag{14}$$

Combine (12) and (14), we obtain that

$$\lim_{n\to\infty} d(W(u_n, z_n, \gamma_n), q) = \lim_{n\to\infty} d(y_n, q) = t. \tag{15}$$

Since

$$\begin{aligned} d(u_n, q) &= dist(u_n, T(q)) \\ &\leq H(T(z_n), T(q)) \leq d(z_n, q), \end{aligned}$$

this implies that

$$\limsup_{n\to\infty} d(u_n, q) \leq t. \tag{16}$$

By (15), (16), $\lim_{n\to\infty} d(z_n, q) = t$ together with Lemma 2.6, we have

$$\lim_{n\to\infty} d(u_n, z_n) = 0. \tag{17}$$

From the condition of the modified SP-iteration, so

$$\lim_{n\to\infty} r_{z_n}(T(z_n)) = \lim_{n\to\infty} d(u_n, z_n) = 0. \tag{18}$$

Hence by the both cases we can conclude that $r_{z_n}(T(z_n)) \to 0$. It follows from Lemma 2.11, we have $\mathrm{dist}(z_n, T(z_n)) \to 0$ and $\mathrm{diam}(T(z_n)) \to 0$.

To show that $\{z_n\}$ Δ-converges to an endpoint of T. Now we prove that $W_\omega(z_n) := \cup_{\{s_n\}\subset\{z_n\}} AC(E,\{s_n\}) \subset End(T)$ and $W_\omega(z_n)$ consists of exactly one point. Let $s \in W_\omega(z_n)$. Therefore there exists a subsequence $\{s_n\}$ of $\{z_n\}$ such that $AC(E,\{s_n\}) = \{s\}$. From Lemma 2.5 and Lemma 2.7, there exists a subsequence $\{t_n\}$ of $\{s_n\}$ such that $\Delta\text{-}\lim_{n\to\infty} t_n = t \in E$. Since $\text{dist}(t_n, T(t_n)) \to 0$ and $\text{diam}(T(t_n)) \to 0$ and it follows from Lemma 3.1, we have $t \in End(T)$ and $\lim_{n\to\infty} d(z_n,t)$ exists by Lemma 3.2. Thus by Lemma 2.8 we have $s = t \in End(T)$. This shows that $W_\omega(z_n) \subset End(T)$. Next, we prove that $W_\omega(z_n)$ consists of exactly one point. Let $\{s_n\}$ be a subsequence of $\{z_n\}$ such that $AC(E,\{s_n\}) = \{s\}$ and $AC(E,\{z_n\}) = \{z\}$. Since $s \in W_\omega(z_n) \subset End(T)$ and from Lemma 3.2, we know that $\{d(z_n,s)\}$ exists. By Lemma 2.8, $z = s$. Therefore the proof is completed.
■

Next, we present the following key lemma for proving the strong convergence theorem.

Lemma 3.4 *Let E be a nonempty closed subset of a complete hyperbolic space X and $\{w_n\}$ be a Fejér monotone sequence with respect to E. Then $\{w_n\}$ converges strongly to an element of E if and only if* $\lim_{n\to\infty}$ *dist*$(w_n, E) = 0$.

Proof. Assume that $\{w_n\}$ converges strongly to $q \in E$. Thus $\lim_{n\to\infty} d(w_n,q) = 0$. Because $0 \leq \text{dist}(w_n,E) \leq d(w_n,q)$, therefore $\lim_{n\to\infty} \text{dist}(w_n,E) = 0$.
Conversely, suppose that $\lim_{n\to\infty} \text{dist}(w_n,E) = 0$. Since $\{w_n\}$ is a Fejér monotone sequence with respect to E, we have

$$d(w_{n+1},q) \leq d(w_n,q) \ \text{ for all } \ q \in E.$$

Thus $\inf_{q\in E} d(w_{n+1},q) \leq \inf_{q\in E} d(w_n,q)$, which means that $\text{dist}(w_{n+1},E) \leq \text{dist}(w_n,E)$. Therefore $\lim_{n\to\infty} \text{dist}(w_n,E)$ exists. By hypothesis, we obtain that $\lim_{n\to\infty} \text{dist}(w_n,E) = 0$. Next, we show that $\{w_n\}$ is a Cauchy sequence in E. Let $r > 0$. Since $\lim_{n\to\infty} \text{dist}(w_n,E) = 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\text{dist}(w_n,E) < \frac{r}{2} \ \text{ for all } \ n \geq n_0.$$

Inparticular, $\inf\{d(w_{n_0},q) : q \in E\} < \frac{r}{2}$.
Therefore there exists $q_0 \in E$ such that $d(w_{n_0},q_0) < \frac{r}{2}$. For any $n, m \geq n_0$, we have

$$\begin{aligned} d(w_{n+m},w_n) &\leq d(w_{n+m},q_0) + d(q_0,w_n) \\ &\leq d(w_{n_0},q_0) + d(q_0,w_{n_0}) \\ &\leq \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

This means that a sequence $\{w_n\}$ is a Cauchy sequence in E. Since E is a closed subset of a complete hyperbolic space X, we have E is also complete. Then $\{w_n\}$ must be convergent to a point in E. ■

9

Theorem 3.5 *Let E be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity and $T : E \to K(E)$ be a multi-valued Suzuki mapping with $End(T) \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by the modified SP-iteration process (2). If T satisfies condition (J), then $\{z_n\}$ converges strongly to an endpoint of T.*

Proof. First, we will show that $End(T)$ is closed. Let $\{z_n\} \subseteq End(T)$ such that $z_n \to z \in E$. We will prove that $z \in End(T)$. Since T is a multi-valued Suzuki mapping, therefore T satisfies condition (E_3). Then

$$\mathrm{dist}(z_n, Tz) \leq 3\mathrm{dist}(z_n, T(z_n)) + d(z_n, z) \to 0 \text{ as } n \to \infty.$$

This implies that $z \in T(z)$. Next, we show that $\{z\} = T(z)$. Take any point $w \in T(z)$. Since T is a multi-valued Suzuki mapping,

$$\frac{1}{2}\mathrm{dist}(z_n, T(z_n)) = 0 \leq d(z_n, z) \text{ implies that } H(T(z_n), T(z)) \leq d(z_n, z).$$

Since $z_n \in End(T)$, we have

$$\begin{aligned} d(w, z) &\leq d(w, z_n) + d(z_n, z) \\ &= \mathrm{dist}(w, T(z_n)) + d(z_n, z) \\ &\leq H(T(z), T(z_n)) + d(z_n, z) \\ &\leq d(z_n, z) + d(z_n, z) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $w = z$. Because $w \in T(z)$ is arbitrary, then $T(z) = \{z\}$, so $z \in End(T)$. Thus $End(T)$ is closed. Next, as in the proof of Theorem 3.3, we have $r_{z_n}(T(z_n)) \to 0$ and it follows from T satisfies condition (J),

$$h(\mathrm{dist}(z_n, End(T))) \leq r_{z_n}(T(z_n)) \to 0.$$

This implies that $\lim_{n\to\infty} h(\mathrm{dist}(z_n, End(T))) = 0$. Since $h : [0, \infty) \to [0, \infty)$ is nondecreasing with $h(0) = 0$, $h(r) > 0$ for $r \in (0, \infty)$, we obtain that $\lim_{n\to\infty} \mathrm{dist}(z_n, End(T)) = 0$. As in the proof of Lemma 3.2 implies that $\{z_n\}$ is Fejér monotone with respect to $End(T)$. By applying Lemma 3.4, we obtain the desired result. ■

Theorem 3.6 *Let E be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity and $T : E \to K(E)$ be a multi-valued Suzuki mapping with $End(T) \neq \emptyset$. Let $\{z_n\}$ be a sequence generated by the modified SP-iteration process (2). If T is semicompact, then $\{z_n\}$ converges strongly to an endpoint of T.*

Proof. As in the proof of Theorem 3.3, $r_{z_n}(T(z_n)) \to 0$ and T is semicompact, we may assume a subsequence $z_{n_k} \to z$ for some $z \in E$. Again, as in the proof of Theorem 3.3, we obtain that $r_{z_{n_k}}(T(z_{n_k})) \to 0$. By Lemma 2.11, we also get

$\text{dist}(z_{n_k}, T(z_{n_k})) \to 0$ as $k \to \infty$. Since T is a multi-valued Suzuki mapping, therefore T satisfies condition(E_3). Because of

$$\begin{aligned} \text{dist}(z, T(z)) &\leq d(z, z_{n_k}) + \text{dist}(z_{n_k}, T(z)) \\ &\leq d(z, z_{n_k}) + 3\text{dist}(z_{n_k}, T(z_{n_k})) + d(z_{n_k}, z) \to 0 \text{ as } k \to \infty, \end{aligned}$$

we obtain that $z \in T(z)$. Next, we show that $\{z\} = T(z)$.
Notice that $\frac{1}{2}\text{dist}(z, T(z)) = 0 \leq d(z_{n_k}, z)$ for all $k \in \mathbb{N}$. Since T is a multi-valued Suzuki mapping, we have

$$H(T(z_{n_k}, T(z))) \leq d(z_{n_k}, z).$$

We now let $u \in T(z)$ and choose $w_{n_k} \in T(z_{n_k})$ so that $d(u, w_{n_k}) = \text{dist}(u, T(z_{n_k}))$. For all $k \in \mathbb{N}$, we obtain that

$$\begin{aligned} d(z, u) &\leq d(z, z_{n_k}) + d(z_{n_k}, w_{n_k}) + d(w_{n_k}, u) \\ &\leq d(z, z_{n_k}) + r_{z_{n_k}}(T(z_{n_k})) + \text{dist}(u, T(z_{n_k})) \\ &\leq d(z, z_{n_k}) + r_{z_{n_k}}(T(z_{n_k})) + H(T(z), T(z_{n_k})) \\ &\leq d(z, z_{n_k}) + r_{z_{n_k}}(T(z_{n_k})) + d(z, z_{n_k}). \end{aligned}$$

Taking limit as $k \to \infty$, we get that $z = u$ for all $u \in T(z)$ and so $\{z\} = T(z)$. Hence $z \in End(T)$. By Lemma 3.2, $\lim_{n\to\infty} d(z_n, q)$ exists for each $q \in End(T)$, it follows that $z_n \to z$ as $n \to \infty$. This completes the proof. ■

4 Numerical example

In this section, we give an example shows that there exists a mapping which is a multi-valued Suzuki mapping but is not a nonexpansive mapping. Furthermore, we illustrate that a sequence generated by the modified SP-iteration process (2) converges to an endpoint of the multi-valued Suzuki mapping.

Example 4.1 Let $X = \mathbb{R}$ with metric defined by $d(x, y) = |x-y|$ and $E = [0, 3]$. Define $W : X^2 \times [0, 1] \to X$ by $W(x, y, \alpha) := \alpha x + (1 - \alpha)y$ for all $x, y \in X$ and $\alpha \in [0, 1]$. Then (X, d, W) is a complete uniformly hyperbolic space with a monotone modulus of uniform convexity and E is a nonempty compact convex subset of X. Let $T : E \to K(E)$ defined by

$$Tz = \begin{cases} \{0\}, & z \neq 3; \\ \{1\}, & z = 3. \end{cases}$$

By [19] showed that the mapping T is a Suzuki mapping. But T is not a nonexpansive mapping if we take $x = 2.9$ and $y = 3$. Moreover, $End(T) = \{0\}$. For initial point $z_0 = 0.1$ and $\alpha_n = \beta_n = \gamma_n = \dfrac{1}{\sqrt{3n+7}}$. Therefore $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq [0, 1]$. Set stop parameter to $|z_n - 0| \leq 10^{-12}$, where 0 is an endpoint of T. By using MATLAB, we compute the sequence generated by the modified SP-iteration process (2) converging to 0 as in Table 1 and Figure 1.

iterate	the modified SP-iteration process
z_0	0.1
z_1	0.024068308483
z_2	0.006367770608
z_3	0.001780515413
z_4	0.000516698713
:	:
z_{20}	0.000000000008
z_{21}	0.000000000003
z_{22}	0.000000000000

Table 1: Sequences generated by SP-iteration process

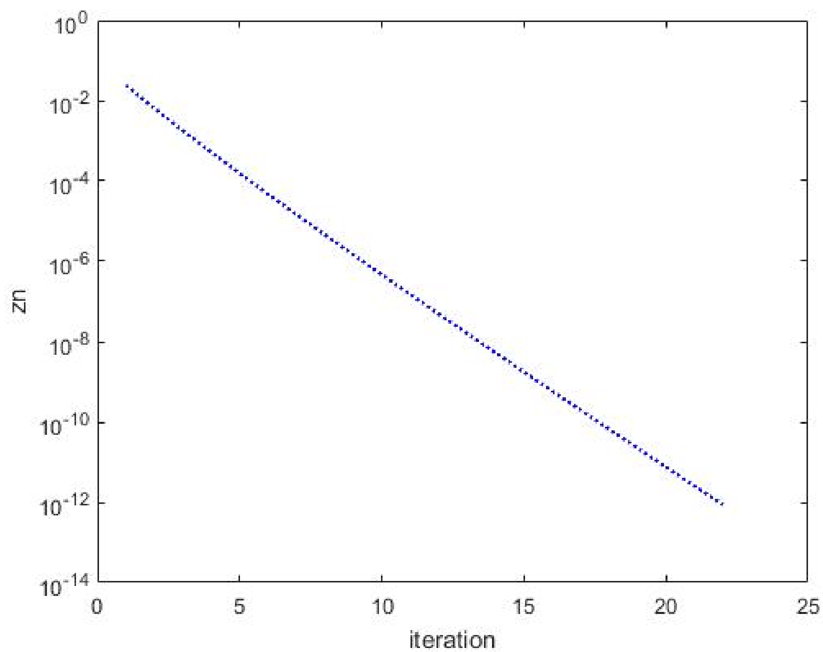


Figure 1 Convergence of iterative sequences generated by SP-iteration process

Acknowledgement

The first and the third authors would like to express their deep thanks to Naresuan University for supporting this research.

12

References

[1] A. Abkar, M.Eslamian, Common fixed point results in CAT(0) spaces, *Nonlinear Anal.*, 74, 1835-1840(2011).

[2] A. Amini-Harandi, Endpoints of set-valued contractions in metric spaces, *Nonlinear Anal.*, 72, 132-134(2010).

[3] J.P. Aubin, J. Siegel, Fixed points and stationary points of dissipative multi-valued maps, *Proc. Amer. Math. Soc.*, 78, 391-398(1980).

[4] S. Dhompongsa, W. Kirk, B. Panyanak, Nonexpansive set-valued mappings in metric and Banach spaces, *Nonlinear Convex Anal.*, 8,35-45(2007).

[5] S. Dhompongsa, B. Panyanak, On Δ-convergence theorem in CAT(0) spaces, *Comput. Math. Appl.*, 56(10), 2572-2579(2008).

[6] R. Espinola, P. Lorenzo, A. Nicolae, Fixed points selections and common fixed points for nonexpansive-type mappings, *Math. Anal. Appl.*, 382,503-515(2011).

[7] J. García-Falset, E. Llorens-Fuster, T. Suzuki, Fixed point theory for a class of generalized nonexpansive mappings, *Math. Anal. Appl.*, 375, 185-195(2011).

[8] K. Goebel, W.A. Kirk, Topics in metric fixed point theory, *Cambridge Univ.Press, Cambridge* (1990).

[9] M.S. Kahn, K.R. Rao, Y.J. Cho, Common stationary points for set-valued mappings, *Internat. Math. Math. Sci.*, 16, 733-736(1993).

[10] A.R. Khan, H. Fukhar-ud-din, M.A.A. Kuan, An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces, *Fixed Point Theory Appl.*, 2012, 54(2012).

[11] W.A. Kirk, B. Panyanak, A concept of convergence in geodesic spaces, *Nonlinear anal.*, 68, 3689-3696(2008).

[12] U. Kohlenbach, Some logical metatheorems with applications in functional analysis, *Trans. Am. Math. Soc.*, 357, 89-128(2005).

[13] A. Kudtha, B. Panyanak, Common endpoints for Suzuki mappings in uniformly convex hyperbolic spaces, *Thai.J. Math*, 159-168, A.M.M.(2017).

[14] L. Leustean, A quadratic rate of asymptotic regularity for CAT(0) spaces, *Math. Anal. Appl.*, 235, 386-399(2007).

[15] B. Panyanak, Endpoints of multivalued nonexpansive mappings in geodesic spaces, *Fixed Point Theory Appl.*, 2015, 147(2015).

[16] B. Panyanak, Approximating endpoints of multi-valued nonexpansive mappings in Banach spaces, *Fixed Point Theory Appl.*(2018).

[17] W. Phuengrattana, S. Suantai, On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval, *Comput. Appl. Math.*, 235,3006-3014(2011).

[18] S.L. Singh, S.N. Mishra, Coincidence points, hybrid fixed and stationary points of orbitally weakly dissipative maps, *Math. Japan.*, 39,451-459(1994).

[19] T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, *Math. Anal. Appl.*, 340, 1088-1095(2008).

[20] Y. Yamamoto, A path following algorithm for stationary point problems, *Oper. Res. Soc. Japan*, 30,181-199(1987).

[21] Y. Yamamoto, Fixed point algorithms for stationary point problems, *Mathematical Programming*(Tokyo, 1988), *Math. Appl.* (Japanese Ser.), *SCIPRESS*, Tokyo 6, 283-307(1989).

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO. 5, 2020

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO. 5, 2020

(continued)

Volume 28, Number 6 December, 2020
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of

Computational

Analysis and

Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(six times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, St.Martin Univ.,Olympia,WA,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

Journal Article

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus homepage.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

SHARP INEQUALITIES BETWEEN TOADER AND NEUMAN MEANS*

WEI-MAO QIAN[1], ZAI-YIN HE[2], AND YU-MING CHU[3,**]

ABSTRACT. In the article, we prove that the double inequalities

$$\alpha_1 Q(a,b)+(1-\alpha_1)N_{GA}(a,b)<T(a,b)<\beta_1 Q(a,b)+(1-\beta_1)N_{GA}(a,b),$$
$$\alpha_2 Q(a,b)+(1-\alpha_2)N_{QA}(a,b)<T(a,b)<\beta_2 Q(a,b)+(1-\beta_2)N_{QA}(a,b),$$
$$\alpha_3 C(a,b)+(1-\alpha_3)N_{GA}(a,b)<T(a,b)<\beta_3 C(a,b)+(1-\beta_3)N_{GA}(a,b),$$
$$\alpha_4 C(a,b)+(1-\alpha_4)N_{QA}(a,b)<T(a,b)<\beta_4 C(a,b)+(1-\beta_4)N_{QA}(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leq 5/8$, $\beta_1\geq(16-\pi^2)/[(4\sqrt{2}-\pi)\pi]=0,7758\cdots$, $\alpha_2\leq 1/4$, $\beta_2\geq 1-2(\sqrt{2}\pi-4)/[(\sqrt{2}-\log(1+\sqrt{2}))\pi]=0.4708\cdots$, $\alpha_3\leq 5/14=0.3571\cdots$, $\beta_3\geq(16-\pi^2)/[(8-\pi)\pi]=0.4016\cdots$, $\alpha_4\leq 1/10$ and $\beta_4\geq 1-4(\pi-2)/[(4-\sqrt{2}-\log(1+\sqrt{2}))\pi]=0.1472\cdots$, where $Q(a,b)$, $C(a,b)$ and $T(a,b)$ are respectively the quadratic, contra-harmonic and Toader means, and $N_{GA}(a,b)$ and $N_{QA}(a,b)$ are the Neuman means.

1. Introduction

Let $p\in\mathbb{R}$, $r\in(0,1)$ and $a,b>0$ with $a\neq b$. Then the complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ [1-32] of the first and second kinds, geometric mean $G(a,b)$, arithmetic mean $A(a,b)$, quadratic mean $Q(a,b)$, contra-harmonic mean $C(a,b)$, second contra-harmonic mean $\overline{C}(a,b)$, centroidal mean $\widetilde{C}(a,b)$, Toader mean $T(a,b)$ [33-36], pth power mean $M_p(a,b)$ [37-43], and Schwab-Borchardt mean $SB(a,b)$ [44-48] of a and b are given by

$$\mathcal{K}(r)=\int_0^{\pi/2}\left(1-r^2\sin^2 t\right)^{-1/2}dt,\quad \mathcal{E}(r)=\int_0^{\pi/2}\sqrt{1-r^2\sin^2(t)}dt,$$

$$G(a,b)=\sqrt{ab},\quad A(a,b)=\frac{a+b}{2},\quad Q(a,b)=\sqrt{\frac{a^2+b^2}{2}},$$

$$C(a,b)=\frac{a^2+b^2}{a+b},\quad \overline{C}(a,b)=\frac{a^3+b^3}{a^2+b^2},\quad \widetilde{C}(a,b)=\frac{2(a^2+ab+b^2)}{3(a+b)},$$

$$T(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2(t)+b^2\sin^2(t)}dt,$$
$$=\begin{cases}2a\mathcal{E}\left(\sqrt{1-(b/a)^2}\right)/\pi, & a>b,\\ 2b\mathcal{E}\left(\sqrt{1-(a/b)^2}\right)/\pi, & a<b,\end{cases}\tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary: 26E60; Secondary: 33E05.

Key words and phrases. Toader mean, Neuman mean, geometric mean, arithmetic mean, quadratic mean, contra-harmonic mean.

*The research was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11301127, 11701176, 11626101, 11601485), the Science and Technology Research Program of Zhejiang Educational Committee (Grant No. Y201635325) and the Natural Science Foundation of Huzhou City (Grant No. 2018YZ07).

**Corresponding author: Yu-Ming Chu, Email: chuyuming2005@126.com.

1

$$M_p(a,b)=\begin{cases}\left(\frac{a^p+b^p}{2}\right)^{1/p}, & p\neq 0,\\ \sqrt{ab}, & p=0\end{cases}$$

and

$$SB(a,b)=\begin{cases}\frac{\sqrt{b^2-a^2}}{\arccos(a/b)}, & a<b,\\ \frac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & a>b,\end{cases}$$

respectively, where $\cosh^{-1}(x)=\log(x+\sqrt{x^2-1})$ is the inverse hyperbolic cosine functions.

Recently, the bivariate means have attracted the attention of many researchers [49-82]. Neuman [83] introduced the Neuman mean

$$N(a,b)=\frac{1}{2}\left[a+\frac{b^2}{SB(a,b)}\right],$$

provided the explicit formulae for $N_{AG}(a,b)(a,b)$, $N_{GA}(a,b)$, $N_{AQ}(a,b)$ and $N_{QA}(a,b)$ as follows

$$N_{AG}(a,b)=:N[A(a,b),G(a,b)]=\frac{1}{2}A(a,b)\left[1+(1-v^2)\frac{\tanh^{-1}(v)}{v}\right],$$

$$N_{GA}(a,b)=:N[G(a,b),A(a,b)]=\frac{1}{2}A(a,b)\left[\sqrt{1-v^2}+\frac{\arcsin(v)}{v}\right], \tag{1.2}$$

$$N_{AQ}(a,b)=:N[A(a,b),Q(a,b)]=\frac{1}{2}A(a,b)\left[1+(1+v^2)\frac{\arctan(v)}{v}\right],$$

$$N_{QA}(a,b)=:N[Q(a,b),A(a,b)]=\frac{1}{2}A(a,b)\left[\sqrt{1+v^2}+\frac{\sinh^{-1}(v)}{v}\right], \tag{1.3}$$

where $v=(a-b)/(a+b)$, $\tanh^{-1}(x)=\log[(1+x)/(1-x)]/2$ and $\sinh^{-1}(x)=\log(x+\sqrt{x^2+1})$ are the inverse hyperbolic tangent and sine functions, respectively.

It is well known that the power mean $M_p(a,b)$ is continuous and strictly increasing with respect to $p\in\mathbb{R}$ for fixed $a,b>0$ with $a\neq b$ and the inequalities

$$\begin{aligned}G(a,b)=M_0(a,b)<A(a,b)=M_1(a,b)<\widetilde{C}(a,b)\\ <Q(a,b)=M_2(a,b)<C(a,b)<\overline{C}(a,b)\end{aligned} \tag{1.4}$$

hold for all $a,b>0$ with $a\neq b$.

Barnard, Pearce and Richards [84], and Alzer and Qiu [85] proved that the double inequality

$$M_{3/2}(a,b)<T(a,b)<M_{\log 2/\log(\pi/2)}(a,b)$$

holds all $a,b>0$ with $a\neq b$.

In [86], the authors stated that the double inequality

$$\alpha Q(a,b)+(1-\alpha)A(a,b)<T(a,b)<\beta Q(a,b)+(1-\beta)A(a,b) \tag{1.5}$$

is valid for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq 1/2$ and $\beta\geq(4-\pi)/[(\sqrt{2}-1)\pi]=0.6596\cdots$.

Neuman [83] presented the inequalities

$$\begin{aligned}G(a,b)<N_{AG}(a,b)<N_{GA}(a,b)<A(a,b)\\ <N_{QA}(a,b)<N_{AQ}(a,b)<Q(a,b),\end{aligned} \tag{1.6}$$

$$\alpha_1A(a,b)+(1-\alpha_1)G(a,b)<N_{GA}(a,b)<\beta_1A(a,b)+(1-\beta_1)G(a,b),$$

$$\alpha_2Q(a,b)+(1-\alpha_2)A(a,b)<N_{AQ}(a,b)<\beta_2Q(a,b)+(1-\beta_2)A(a,b),$$

$$\alpha_3A(a,b)+(1-\alpha_3)G(a,b)<N_{AG}(a,b)<\beta_3A(a,b)+(1-\beta_3)G(a,b),$$

$$\alpha_4Q(a,b)+(1-\alpha_4)A(a,b)<N_{QA}(a,b)<\beta_4Q(a,b)+(1-\beta_4)A(a,b)$$

for all $a,b>0$ with $a\neq b$ if $\alpha_1\leq 2/3$, $\beta_1\geq\pi/4$, $\alpha_2\leq 2/3$, $\beta_2\geq(\pi-2)/[4(\sqrt{2}-1)]=0.6890\cdots$, $\alpha_3\leq 1/3$, $\beta_3\geq 1/2$, $\alpha_4\leq 1/3$, $\beta_4\geq(\log(1+\sqrt{2})+\sqrt{2}-2)/[2(\sqrt{2}-1)]=0.3568\cdots$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Li, Qian and Chu [87] proved that the double inequalities

$$\alpha N_{AQ}(a,b)+(1-\alpha)A(a,b)<T(a,b)<\beta N_{AQ}(a,b)+(1-\beta)A(a,b),$$

$$Q[\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a]<T(a,b)<Q[\mu a+(1-\mu)b,\mu b+(1-\mu)a]$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq 3/4$ and $\beta\geq 4(4-\pi)/[\pi(\pi-2)]=0.9753\cdots$, $\lambda\leq 1/2+\sqrt{2}/4=0.8535\cdots$ and $\mu\geq 1/2+\sqrt{16/\pi^2-1}/2=0.8940\cdots$ if $\lambda,\mu\in(1/2,1)$.

Qian, Song, Zhang and Chu [88] proved that the two-sided inequalities

$$\lambda_1\overline{C}(a,b)+(1-\lambda_1)A(a,b)<T(a,b)<\mu_1\overline{C}(a,b)+(1-\mu_1)A(a,b)$$

$$\overline{C}[\lambda_2 a+(1-\lambda_2)b,\lambda_2 b+(1-\lambda_2)a]<T(a,b)<\overline{C}[\mu_2 a+(1-\mu_2)b,\mu_2 b+(1-\mu_2)a]$$

are valid for all $a,b>0$ with $a\neq b$ if and only if $\lambda_1\leq 1/8$, $\mu_1\geq 4/\pi-1=0.2732\cdots$, $\lambda_2\leq 1/2+\sqrt{2}/8=0.6767\cdots$ and $\mu_2\geq 1/2+\sqrt{(4-\pi)/(3\pi-4)}/2=0.6988\cdots$ if $\lambda_2,\mu_2\in(1/2,1)$.

In [89], Song, Qian and Chu found that the inequalities

$$\alpha_1 A(a,b)+(1-\alpha_1)\widetilde{C}(a,b)<N_{QA}(a,b)<\beta_1 A(a,b)+(1-\beta_1)\widetilde{C}(a,b), \tag{1.7}$$

$$A^{\alpha_2}(a,b)\widetilde{C}^{1-\alpha_2}(a,b)<N_{QA}(a,b)<A^{\beta_2}(a,b)\widetilde{C}^{1-\beta_2}(a,b),$$

$$\widetilde{C}[\alpha_3 a+(1-\alpha_3)b,\alpha_3 b+(1-\alpha_3)a]<N_{QA}(a,b)<\widetilde{C}[\beta_3 a+(1-\beta_3)b,\beta_3 b+(1-\beta_3)a]$$

take place if and only if $\alpha_1\geq 4-3[\sqrt{2}+\log(1+\sqrt{2})]/2=0.5566\cdots$, $\beta_1\leq 1/2$, $\alpha_2\geq 1-[\log(\sqrt{2}+\log(1+\sqrt{2}))-\log 2]/(2\log 2-\log 3)=0.5208\cdots$, $\beta_2\leq 1/2$, $\beta_3\geq 1/2+\sqrt{2}/4=0.8535\cdots$ and $\alpha_3\leq 1/2+\sqrt{6[\sqrt{2}+\log(1+\sqrt{2})]-12}/4=0.8329\cdots$ if $\alpha_3,\beta_3\in(1/2,1)$.

From (1.4)-(1.7) we clearly see that the inequalities

$$N_{GA}(a,b)<N_{QA}(a,b)<\frac{1}{2}A(a,b)+\frac{1}{2}\widetilde{C}(a,b) \tag{1.8}$$
$$<\frac{1}{2}A(a,b)+\frac{1}{2}Q(a,b)<T(a,b)<Q(a,b)<C(a,b)$$

hold for all $a,b>0$ with $a\neq b$.

Motivated by inequality (1.8), in the article we deal with the optimality of the parameters α_1, α_2, α_3, α_4, β_1, β_2, β_3 and β_4 such that the double inequalities

$$\alpha_1 Q(a,b)+(1-\alpha_1)N_{GA}(a,b)<T(a,b)<\beta_1 Q(a,b)+(1-\beta_1)N_{GA}(a,b),$$
$$\alpha_2 Q(a,b)+(1-\alpha_2)N_{QA}(a,b)<T(a,b)<\beta_2 Q(a,b)+(1-\beta_2)N_{QA}(a,b),$$
$$\alpha_3 C(a,b)+(1-\alpha_3)N_{GA}(a,b)<T(a,b)<\beta_3 C(a,b)+(1-\beta_3)N_{GA}(a,b),$$
$$\alpha_4 C(a,b)+(1-\alpha_4)N_{QA}(a,b)<T(a,b)<\beta_4 C(a,b)+(1-\beta_4)N_{QA}(a,b)$$

hold for all $a,b>0$ with $a\neq b$.

2. Lemmas

In order to prove our main results, we need several formulas and lemmas which we present in this section.

The following formulas for $\mathcal{K}(r)$ and $\mathcal{E}(r)$ can be found in the literature [90]:

$$\frac{d\mathcal{K}(r)}{dr}=\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r(1-r^2)},\quad \frac{d\mathcal{E}(r)}{dr}=\frac{\mathcal{E}(r)-\mathcal{K}(r)}{r},$$

$$\frac{d\left[\mathcal{K}(r)-\mathcal{E}(r)\right]}{dr}=\frac{r\mathcal{E}(r)}{1-r^2},\quad \mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right)=\frac{2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{1+r},$$

$$\mathcal{K}(0^+)=\mathcal{E}(0^+)=\frac{\pi}{2},\quad \mathcal{K}(1^-)=\infty,\quad \mathcal{E}(1^-)=1.$$

Lemma 2.1. *(See [90, Theorem 1.25]) Let* $-\infty < a < b < \infty$*,* $f, g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b)*, and* $g'(x) \neq 0$ *on* (a,b)*. If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b)*, then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2.2. *The following statements are true:*

(1) The function $r \mapsto \left[\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4, 1)$*;*

(2) The function $r \mapsto \mathcal{K}(r)$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/2, \infty)$*;*

(3) The function $r \mapsto [\mathcal{K}(r)-\mathcal{E}(r)]/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4, +\infty)$*;*

(4) The function $r \mapsto \phi(r) = \left[3\mathcal{E}(r)-2(1-r^2)\mathcal{K}(r)\right]/\sqrt{1+r^2}$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/2, 3\sqrt{2}/2)$*.*

Proof. Parts (1)-(3) can be found in [8, Theorem 3.21(1), (2) and Exercise 3.43(11)]. For part (4), it is not difficult to verify that

$$\phi(0^+) = \frac{\pi}{2}, \qquad \phi(1^+) = \frac{3\sqrt{2}}{2}, \tag{2.1}$$

$$\begin{aligned}\phi'(r) &= \frac{\mathcal{E}(r)-2r^2\mathcal{E}(r)-\mathcal{K}(r)+3r^2\mathcal{K}(r)}{r(1+r^2)^{3/2}}\\ &= \frac{r}{(1+r^2)^{3/2}}\left[\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r^2}\right]+\frac{2r^3}{(1+r^2)^{3/2}}\left[\frac{\mathcal{K}(r)-\mathcal{E}(r)}{r^2}\right].\end{aligned} \tag{2.2}$$

It follows from (2.2) together with Lemma 2.2(1) and (3) that

$$\phi'(r) > 0 \tag{2.3}$$

for $r \in (0,1)$.

Therefore, part (4) follows from (2.1) and (2.3). ☐

Lemma 2.3. *The function*

$$\varphi(r) = \frac{2r^2+1-\frac{2}{\pi}\sqrt{1+r^2}\left[3\mathcal{E}(r)-2(1-r^2)\mathcal{K}(r)\right]}{r^2}$$

is strictly decreasing from $(0,1)$ *onto* $(3-6\sqrt{2}/\pi, 3/4)$*.*

Proof. Let $\varphi_1(r) = 2r^2+1-2\sqrt{1+r^2}\left[3\mathcal{E}(r)-2(1-r^2)\mathcal{K}(r)\right]/\pi$, $\varphi_2(r) = r^2$. Then simple computations lead to

$$\varphi_1(0^+) = \varphi_2(0^+) = 0, \quad \varphi(r) = \frac{\varphi_1(r)}{\varphi_2(r)}, \tag{2.4}$$

$$\varphi(1^-) = 3-\frac{6\sqrt{2}}{\pi}, \tag{2.5}$$

$$\frac{\varphi_1'(r)}{\varphi_2'(r)} = 2-\frac{1}{\pi}\left\{\frac{3\mathcal{E}(r)-2(1-r^2)\mathcal{K}(r)}{\sqrt{1+r^2}}+\sqrt{1+r^2}\left[\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r^2}+\mathcal{K}(r)\right]\right\}. \tag{2.6}$$

It is not difficult to verify that the function $r \mapsto \sqrt{1+r^2}$ is strictly increasing on $(0,1)$. Then it follows from Lemma 2.2(1), (2) and (4) together with (2.6) that $\varphi_1'(r)/\varphi_2'(r)$ is strictly decreasing on $(0,1)$ and

$$\varphi(0^+) = \lim_{r\to 0^+}\frac{\varphi_1'(r)}{\varphi_2'(r)} = \frac{3}{4}. \tag{2.7}$$

Therefore, Lemma 2.3 follows from Lemma 2.1, (2.4), (2.5) and (2.7) together with the monotonicity of $\varphi_1'(r)/\varphi_2'(r)$. ☐

Lemma 2.4. *The function*

$$\psi(r)=\frac{3r^2+1-\frac{2}{\pi}\left[3\mathcal{E}(r)-2(1-r^2)\mathcal{K}(r)\right]}{r^2}$$

is strictly decreasing from $(0,1)$ *onto* $(4-6/\pi, 9/4)$.

Proof. Let $\psi_1(r)=3r^2+1-2\left[3\mathcal{E}(r)-2(1-r^2)\mathcal{K}(r)\right]/\pi$, $\psi_2(r)=r^2$. Then simple computations lead to

$$\psi_1(0^+)=\psi_2(0^+)=0,\quad \psi(r)=\frac{\psi_1(r)}{\psi_2(r)}, \tag{2.8}$$

$$\psi(1^-)=4-\frac{6}{\pi}, \tag{2.9}$$

$$\frac{\psi_1'(r)}{\psi_2'(r)}=3-\frac{1}{\pi}\left[\frac{\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)}{r^2}+\mathcal{K}(r)\right]. \tag{2.10}$$

From Lemma 2.2(1), (2) and (2.10) we know that $\psi_1'(r)/\psi_2'(r)$ is strictly decreasing on $(0,1)$ and

$$\psi(0^+)=\lim_{r\to 0^+}\frac{\psi_1'(r)}{\psi_2'(r)}=\frac{9}{4}. \tag{2.11}$$

Therefore, Lemma 2.4 follows from Lemma 2.1, (2.8), (2.9) and (2.11) together with the monotonicity of $\psi_1'(r)/\psi_2'(r)$. □

3. Main Results

Theorem 3.1. *The double inequality*

$$\alpha_1 Q(a,b)+(1-\alpha_1)N_{GA}(a,b)<T(a,b)<\beta_1 Q(a,b)+(1-\beta_1)N_{GA}(a,b)$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_1\le 5/8$ *and* $\beta_1\ge(16-\pi^2)/[\pi(4\sqrt{2}-\pi)]=0.7758\cdots$.

Proof. Since $Q(a,b)$, $N_{GA}(a,b)$ and $T(a,b)$ are symmetric and homogenous of degree one. Without loss of generality, we assume that $a>b$. Let $r=(a-b)/(a+b)\in(0,1)$. Then from (1.1) and (1.2) one has

$$T(a,b)=\frac{2}{\pi}A(a,b)\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right], \tag{3.1}$$

$$N_{GA}(a,b)=\frac{1}{2}A(a,b)\left[\sqrt{1-r^2}+\frac{\arcsin(r)}{r}\right]. \tag{3.2}$$

It follows from (3.1) and (3.2) together with $Q(a,b)=A(a,b)\sqrt{1+r^2}$ that

$$\frac{T(a,b)-N_{GA}(a,b)}{Q(a,b)-N_{GA}(a,b)}=\frac{\frac{2}{\pi}\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]-\frac{1}{2}\left[\sqrt{1-r^2}+\frac{\arcsin(r)}{r}\right]}{\sqrt{1+r^2}-\frac{1}{2}\left[\sqrt{1-r^2}+\frac{\arcsin(r)}{r}\right]}$$

$$=1-\frac{2r\sqrt{1+r^2}-\frac{4}{\pi}r\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]}{2r\sqrt{1+r^2}-r\sqrt{1-r^2}-\arcsin(r)}:=1-F(r). \tag{3.3}$$

Let $f_1(r)=2r\sqrt{1+r^2}-4r\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]/\pi$ and $g_1(r)=2r\sqrt{1+r^2}-r\sqrt{1-r^2}-\arcsin(r)$. Then simple computations lead to

$$f_1(0^+)=g_1(0^+)=0,\quad F(r)=\frac{f_1(r)}{g_1(r)}, \tag{3.4}$$

$$\frac{f_1'(r)}{g_1'(r)}=\frac{2r^2+1-\frac{2}{\pi}\sqrt{1+r^2}\left[3\mathcal{E}(r)-2(1-r^2)\mathcal{K}(r)\right]}{2r^2-\sqrt{1-r^4}+1}$$

$$= \frac{\varphi(r)}{(2r^2 - \sqrt{1-r^4}+1)/r^2}, \tag{3.5}$$

where $\varphi(r)$ is defined as in Lemma 2.3.

It is easy to verify that the function $r \mapsto (2r^2 - \sqrt{1-r^4}+1)/r^2$ is positive and strictly increasing on (0, 1), then (3.5) and Lemma 2.3 lead to the conclusion that $f_1'(r)/g_1'(r)$ is strictly decreasing on (0,1). Hence from Lemma 2.1 and (3.4) we know that $F(r)$ is strictly decreasing on (0,1). Moreover,

$$\lim_{r\to 0^+} \frac{2r\sqrt{1+r^2} - \frac{4}{\pi}r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{2r\sqrt{1+r^2} - r\sqrt{1-r^2} - \arcsin(r)} = \frac{3}{8}, \tag{3.6}$$

$$\lim_{r\to 1^-} \frac{2r\sqrt{1+r^2} - \frac{4}{\pi}r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{2r\sqrt{1+r^2} - r\sqrt{1-r^2} - \arcsin(r)} = \frac{4(\sqrt{2}\pi - 4)}{\pi(4\sqrt{2} - \pi)}. \tag{3.7}$$

Therefore, Theorem 3.1 follows from (3.3), (3.6) and (3.7) together with the monotonicity of $F(r)$. □

Theorem 3.2. *The double inequality*

$$\alpha_2 Q(a,b) + (1-\alpha_2)N_{QA}(a,b) < T(a,b) < \beta_2 Q(a,b) + (1-\beta_2)N_{QA}(a,b)$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq 1/4$ and $\beta_2 \geq 1 - 2(\sqrt{2}\pi - 4)/\left[(\sqrt{2} - \log(1+\sqrt{2}))\pi\right] = 0.4708\cdots$.

Proof. Since $Q(a,b)$, $N_{QA}(a,b)$ and $T(a,b)$ are symmetric and homogenous of degree one. Without loss of generality, we assume that $a > b$. Let $r = (a-b)/(a+b) \in (0,1)$. Then from (1.4) we have

$$N_{QA}(a,b) = \frac{1}{2}A(a,b)\left[\sqrt{1+r^2} + \frac{\sinh^{-1}(r)}{r}\right]. \tag{3.8}$$

It follows from (3.1) and (3.8) together with $Q(a,b) = A(a,b)\sqrt{1+r^2}$ that

$$\frac{T(a,b) - N_{QA}(a,b)}{Q(a,b) - N_{QA}(a,b)} = \frac{\frac{2}{\pi}\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right] - \frac{1}{2}\left[\sqrt{1+r^2} + \frac{\sinh^{-1}(r)}{r}\right]}{\sqrt{1+r^2} - \frac{1}{2}\left[\sqrt{1+r^2} + \frac{\sinh^{-1}(r)}{r}\right]}$$

$$= 1 - \frac{2r\sqrt{1+r^2} - \frac{4}{\pi}r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{r\sqrt{1+r^2} - \sinh^{-1}(r)} := 1 - G(r). \tag{3.9}$$

Let $f_1(r) = 2r\sqrt{1+r^2} - 4r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]/\pi$ and $g_2(r) = r\sqrt{1+r^2} - arcsinh(r)$. Then simple computations lead to

$$f_1(0^+) = g_2(0^+) = 0, \quad G(r) = \frac{f_1(r)}{g_2(r)}, \tag{3.10}$$

$$\frac{f_1'(r)}{g_2'(r)} = \frac{2r^2 + 1 - \frac{2}{\pi}\sqrt{1+r^2}\left[3\mathcal{E}(r) - 2(1-r^2)\mathcal{K}(r)\right]}{r^2} = \varphi(r), \tag{3.11}$$

where $\varphi(r)$ is defined as in Lemma 2.3.

It follows from Lemma 2.3 and (3.11) that $f_1'(r)/g_2'(r)$ is strictly decreasing on (0,1). Then Lemma 2.1 and (3.10) lead to the conclusion that $G(r)$ is strictly decreasing on (0,1). Moreover,

$$\lim_{r\to 0^+} \frac{2r\sqrt{1+r^2} - \frac{4}{\pi}r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{r\sqrt{1+r^2} - \sinh^{-1}(r)} = \frac{3}{4}, \tag{3.12}$$

$$\lim_{r\to 1^-} \frac{2r\sqrt{1+r^2} - \frac{4}{\pi}r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{r\sqrt{1+r^2} - \sinh^{-1}(r)} = \frac{2(\sqrt{2}\pi - 4)}{\left[\sqrt{2} - \log(1+\sqrt{2})\right]\pi}. \tag{3.13}$$

Therefore, Theorem 3.2 follows from (3.9), (3.12) and (3.13) together with the monotonicity of $G(r)$. □

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Theorem 3.3. *The double inequality*

$$\alpha_3 C(a,b)+(1-\alpha_3)N_{GA}(a,b)<T(a,b)<\beta_3 C(a,b)+(1-\beta_3)N_{GA}(a,b),$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_3\leq 5/14$ *and* $\beta_3\geq(16-\pi^2)/[\pi(8-\pi)]=0.4016\cdots$.

Proof. Without loss of generality, we assume that $a>b$. Let $r=(a-b)/(a+b)\in(0,1)$. Then it follows from (3.1), (3.2) and $C(a,b)=A(a,b)(1+r^2)$ that

$$\frac{T(a,b)-N_{GA}(a,b)}{C(a,b)-N_{GA}(a,b)}=\frac{\frac{2}{\pi}\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]-\frac{1}{2}\left[\sqrt{1-r^2}+\frac{\arcsin(r)}{r}\right]}{1+r^2-\frac{1}{2}\left[\sqrt{1-r^2}+\frac{\arcsin(r)}{r}\right]}$$

$$=1-\frac{2r(1+r^2)-\frac{4}{\pi}r\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]}{2r(1+r^2)-r\sqrt{1-r^2}-\arcsin(r)}:=1-H(r). \tag{3.14}$$

Let $f_2(r)=2r(1+r^2)-4r\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]/\pi$ and $g_3(r)=2r(1+r^2)-r\sqrt{1-r^2}-\arcsin(r)$. Then simple computations lead to

$$f_2(0^+)=g_3(0^+)=0,\quad H(r)=\frac{f_2(r)}{g_3(r)}, \tag{3.15}$$

$$\frac{f_2'(r)}{g_3'(r)}=\frac{3r^2+1-\frac{2}{\pi}\left[3\mathcal{E}(r)-2(1-r^2)\mathcal{K}(r)\right]}{3r^2-\sqrt{1-r^2}+1}$$

$$=\frac{\psi(r)}{(3r^2-\sqrt{1-r^2}+1)/r^2}, \tag{3.16}$$

where $\psi(r)$ is defined as in Lemma 2.4.

It is easy to verify that the function $r\mapsto(3r^2-\sqrt{1-r^2}+1)/r^2$ is positive and strictly increasing on (0, 1). Then from Lemma 2.4 and (3.16) we know that $f_2'(r)/g_3'(r)$ is strictly decreasing on (0,1). Hence Lemma 2.1 and (3.15) lead to the conclusion that $H(r)$ is strictly decreasing on (0,1). Moreover,

$$\lim_{r\to 0^+}\frac{2r(1+r^2)-\frac{4}{\pi}r\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]}{2r(1+r^2)-r\sqrt{1-r^2}-\arcsin(r)}=\frac{9}{14}, \tag{3.17}$$

$$\lim_{r\to 1^-}\frac{2r(1+r^2)-\frac{4}{\pi}r\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]}{2r(1+r^2)-r\sqrt{1-r^2}-\arcsin(r)}=\frac{8(\pi-2)}{\pi(8-\pi)}. \tag{3.18}$$

Therefore, Theorem 3.3 follows from (3.14), (3.17) and (3.18) together with the monotonicity of $H(r)$. □

Theorem 3.4. *The double inequality*

$$\alpha_4 C(a,b)+(1-\alpha_4)N_{QA}(a,b)<T(a,b)<\beta_4 C(a,b)+(1-\beta_4)N_{QA}(a,b)$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_4\leq 1/10$ *and* $\beta_4\geq 1-4(\pi-2)/\left[(4-\sqrt{2}-\log(1+\sqrt{2}))\pi\right]=0.1472$.

Proof. Without loss of generality, we assume that $a>b$. Let $r=(a-b)/(a+b)\in(0,1)$. Then it follows from (3.1), (3.8) and $C(a,b)=A(a,b)(1+r^2)$ that

$$\frac{T(a,b)-N_{QA}(a,b)}{C(a,b)-N_{QA}(a,b)}=\frac{\frac{2}{\pi}\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]-\frac{1}{2}\left[\sqrt{1+r^2}+\frac{\sinh^{-1}(r)}{r}\right]}{1+r^2-\frac{1}{2}\left[\sqrt{1+r^2}+\frac{\sinh^{-1}(r)}{r}\right]}$$

$$=1-\frac{2r(1+r^2)-\frac{4}{\pi}r\left[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)\right]}{2r(1+r^2)-r\sqrt{1+r^2}-\sinh^{-1}(r)}:=1-J(r). \tag{3.19}$$

Let $f_2(r) = 2r(1+r^2) - 4r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]/\pi$ and $g_4(r) = 2r(1+r^2) - r\sqrt{1+r^2} - \sinh^{-1}(r)$. Then simple computations lead to

$$f_2(0^+) = g_4(0^+) = 0, \quad J(r) = \frac{f_2(r)}{g_4(r)}, \tag{3.20}$$

$$\begin{aligned}\frac{f_2'(r)}{g_4'(r)} &= \frac{3r^2 + 1 - \frac{2}{\pi}\left[3\mathcal{E}(r) - 2(1-r^2)\mathcal{K}(r)\right]}{3r^2 - \sqrt{1+r^2} + 1} \\ &= \frac{\psi(r)}{(3r^2 - \sqrt{1+r^2} + 1)/r^2},\end{aligned} \tag{3.21}$$

where $\psi(r)$ is defined as in Lemma 2.4.

It is easy to verify that the function $r \mapsto (3r^2 - \sqrt{1+r^2} + 1)/r^2$ is positive and strictly increasing on (0,1). Then from Lemma 2.4 and (3.21) we know that $f_2'(r)/g_4'(r)$ is strictly decreasing on (0,1). Hence Lemma 2.1 and (3.20) lead to the conclusion that $J(r)$ is strictly decreasing on (0,1). Moreover,

$$\lim_{r\to 0^+} \frac{2r(1+r^2) - \frac{4}{\pi}r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{2r(1+r^2) - r\sqrt{1+r^2} - \sinh^{-1}(r)} = \frac{9}{10}, \tag{3.22}$$

$$\lim_{r\to 1^-} \frac{2r(1+r^2) - \frac{4}{\pi}r\left[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)\right]}{2r(1+r^2) - r\sqrt{1+r^2} - \sinh^{-1}(r)} = \frac{4(\pi-2)}{\left[4 - \sqrt{2} - \log(1+\sqrt{2})\right]\pi}. \tag{3.23}$$

Therefore, Theorem 3.4 follows from (3.19), (3.22) and (3.23) together with the monotonicity of $J(r)$. □

Let $r_0 = \log(1+\sqrt{2})$, $r^* = r^2/(1+\sqrt{1-r^2})^2$. Then (1.1) and Theorems 3.1-3.4 lead to Corollary 3.5 immediately.

Corollary 3.5. *The double inequalities*

$$\frac{\pi}{64}\left[10\sqrt{2}\sqrt{2-r^2} + 3(1+\sqrt{1-r^2})\left(\sqrt{1-r^{*2}} + \frac{\arcsin(r^*)}{r^*}\right)\right] < \mathcal{E}(r)$$
$$< \frac{\sqrt{2}(16-\pi^2)}{4(4\sqrt{2}-\pi)}\sqrt{2-r^2} + \frac{\sqrt{2}\pi - 4}{2(4\sqrt{2}-\pi)}(1+\sqrt{1-r^2})\left(\sqrt{1-r^{*2}} + \frac{\arcsin(r^*)}{r^*}\right),$$
$$\frac{\pi}{32}\left[2\sqrt{2}\sqrt{2-r^2} + 3(1+\sqrt{1-r^2})\left(\sqrt{1+r^{*2}} + \frac{\sinh^{-1}(r^*)}{r^*}\right)\right] < \mathcal{E}(r)$$
$$< \frac{\sqrt{2}(8-\pi(\sqrt{2}+r_0))}{4(\sqrt{2}-r_0)}\sqrt{2-r^2} + \frac{\sqrt{2}\pi - 4}{4(\sqrt{2}-r_0)}(1+\sqrt{1-r^2})\left(\sqrt{1+r^{*2}} + \frac{\sinh^{-1}(r^*)}{r^*}\right),$$
$$\frac{\pi}{112}\left[\frac{20(2-r^2)}{1+\sqrt{1-r^2}} + 9(1+\sqrt{1-r^2})\left(\sqrt{1-r^{*2}} + \frac{\arcsin(r^*)}{r^*}\right)\right] < \mathcal{E}(r)$$
$$< \frac{16-\pi^2}{2(8-\pi)}\frac{2-r^2}{1+\sqrt{1-r^2}} + \frac{\pi-2}{8-\pi}(1+\sqrt{1-r^2})\left(\sqrt{1-r^{*2}} + \frac{\arcsin(r^*)}{r^*}\right),$$
$$\frac{\pi}{80}\left[\frac{4(2-r^2)}{1+\sqrt{1-r^2}} + 9(1+\sqrt{1-r^2})\left(\sqrt{1+r^{*2}} + \frac{\sinh^{-1}(r^*)}{r^*}\right)\right] < \mathcal{E}(r)$$
$$< \frac{8-\pi(\sqrt{2}+r_0)}{2(4-\sqrt{2}-r_0)}\frac{2-r^2}{1+\sqrt{1-r^2}} + \frac{\pi-2}{2(4-\sqrt{2}-r_0)}(1+\sqrt{1-r^2})\left(\sqrt{1+r^{*2}} + \frac{\sinh^{-1}(r^*)}{r^*}\right).$$

hold for all $r \in (0,1)$.

4. Results and discussion

In the article, we present the best possible parameters α_1, α_2, α_3, α_4, β_1, β_2, β_3 and β_4 such that the double inequalities

$$\alpha_1 Q(a,b)+(1-\alpha_1)N_{GA}(a,b)<T(a,b)<\beta_1 Q(a,b)+(1-\beta_1)N_{GA}(a,b),$$
$$\alpha_2 Q(a,b)+(1-\alpha_2)N_{QA}(a,b)<T(a,b)<\beta_2 Q(a,b)+(1-\beta_2)N_{QA}(a,b),$$
$$\alpha_3 C(a,b)+(1-\alpha_3)N_{GA}(a,b)<T(a,b)<\beta_3 C(a,b)+(1-\beta_3)N_{GA}(a,b),$$
$$\alpha_4 C(a,b)+(1-\alpha_4)N_{QA}(a,b)<T(a,b)<\beta_4 C(a,b)+(1-\beta_4)N_{QA}(a,b)$$

hold for all $a,b>0$ with $a\neq b$. Our results are the improvements and refinements of the previously results.

5. Conclusion

We present several sharp bounds for the Toader mean in terms of the Neuman mean, quadratic mean and contraharmonic mean, and give new bounds for the complete elliptic integral of the second kind $\mathcal{E}(r)$. Our approach may have further applications in the theory of bivariate means and special functions.

References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Dover, New York, 1965.

[2] J. M. Borwein and P. B. Borwein, Pi and AGM, John Wiley & Sons, New York, 1987.

[3] X.-Y. Ma, S.-L. Qiu, G.-H. Zhong and Y.-M. Chu, Some inequalities for the generalized linear distortion function, Appl. Math. J. Chinese Univ., 2012, **27B**(1), 87–93.

[4] Z.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, Monotonicity rule for the quotient of two functions and its applications, J. Inequal. Appl., 2017, **2017**, Article 106, 13 pages.

[5] Y.-M. Chu and W.-M. Qian, Optimal inequalities between harmonic, geometric, logarithmic, and arithmetic-geometric means, J. Appl. Math., 2011, **2011**, Article ID 618929, 9 pages.

[6] M.-K. Wang, Y.-M. Chu, S.-L. Qiu and Y.-P. Jiang, Bounds for the perimeter of an ellipse, J. Approx. Theory, 2012, **164**(7), 928–937.

[7] Y.-M. Chu, M.-K. Wang and Y.-F. Qiu, On Alzer and Qiu's conjecture for complete elliptic integral and inverse hyperbolic tangent function, Abstr. Appl. Anal., 2011, **2011**, Article ID 697547, 7 pages.

[8] G.-D. Wang, X.-H. Zhang and Y.-P. Jiang, Concavity with respect to Hölder means involving the generalized Grötzsch function, J. Math. Anal. Appl., 2011, **379**(1), 200–204.

[9] M.-K. Wang, Y.-M. Chu, Y.-F. Qiu, S.-L. Qiu, An optimal power mean inequality for the complete elliptic integrals, Appl. Math. Lett., 2011, **24**(6), 887–890.

[10] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, Hölder mean inequalities for the complete elliptic integrals, Integral Transforms Spec. Funct., 2012, **23**(7), 521–527.

[11] Y.-M. Chu, M.-K. Wang, Y.-P. Jiang and S.-L. Qiu, Concavity of the complete elliptic integrals of the second kind with respect to Hölder means, J. Math. Anal. Appl., 2012, **395**(2), 637–642.

[12] Y.-M. Chu, M.-K. Wang, S.-L. Qiu and Y.-P. Jiang, Bounds for complete integrals of the second kind with applications, Comput. Math. Appl., 2012, **63**(7), 1177–1184.

[13] M.-K. Wang, S.-L. Qiu, Y.-M. Chu and Y.-P. Jiang, Generalized Hersch-Pfluger distortion function and complete elliptic integrals, J. Math. Anal. Appl., 2012, **385**(1), 221–229.

[14] M.-K. Wang, Y.-M. Chu, S.-L. Qiu and Y.-P. Jiang, Convexity of the complete elliptic integrals of the first kind with respect to Hölder means, J. Math. Anal. Appl., 2012, **388**(2), 1141–1146.

[15] Y.-M. Chu, S.-L. Qiu and M.-K. Wang, Sharp inequalities involving the power mean and complete elliptic integral of the first kind, Rocky Mountain J. Math., 2013, **43**(3), 1489–1496.

[16] M.-K. Wang and Y.-M. Chu, Asymptotical bounds for complete elliptic integrals of the second kind, J. Math. Anal. Appl., 2013, **402**(1), 119–126.

[17] M.-K. Wang, Y.-M. Chu and S.-L. Qiu, Some monotonicity properties of generalized elliptic integrals with applications, Math. Inequal. Appl., 2013, **16**(3), 671–677.
[18] G.-D. Wang, X.-H. Zhang and Y.-M. Chu, A power mean inequality involving the complete elliptic integrals, Rocky Mountain J. Math., 2014, **44**(5), 1661–1667.
[19] M.-K. Wang, Y.-M. Chu and Y.-Q. Song, Ramanujan's cubic transformation and generalized modular equation, Sci. China. Math., 2015, **58**(11), 2387–2404.
[20] M.-K. Wang, Y.-M. Chu and S.-L. Qiu, Sharp bounds for generalized elliptic integrals of the first kind, J. Math. Anal. Appl., 2015, **429**(2), 744–757.
[21] M.-K. Wang, Y.-M. Chu and Y.-P. Jiang, Ramanujan's cubic transformation inequalities for zero-balanced hypergeometric functions, Rocky Mountain J. Math., 2016, **46**(2), 679–691.
[22] M.-K. Wang, Y.-M. Chu and Y.-Q. Song, Asymptotical formulas for Gaussian and generalized hypergeometric functions, Appl. Math. Comput. 2016, **276**, 44–60.
[23] W.-M. Qian and Y.-M. Chu, Sharp bounds for a special quasi-arithmetic mean in terms of arithmetic and geometric means with two parameters, J. Inequal. Appl., 2017, **2017**, Article 274, 10 pages.
[24] M.-K. Wang and Y.-M. Chu, Refinements of transformation inequalities for zero-balanced hypergeometric functions, Acta Math. Sci., 2017, **37B**(3), 607–622.
[25] Z.-H. Yang and Y.-M. Chu, A monotonicity property involving the generalized elliptic integral of the first kind, Math. Inequal. Appl., 2017, **20**(3), 729–735.
[26] M.-K. Wang, Y.-M. Li and Y.-M. Chu, Inequalities and infinite product formula for Ramanujan generalized modular equation function, Ramanujan J., 2018, **46**(1), 189–200.
[27] Z.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the arithmetic-geometric mean and complete elliptic integral of the first kind, J. Math. Anal. Appl., 2018, **462**(1), 1714–1726.
[28] M.-K. Wang and Y.-M. Chu, Landen inequalities for a class of hypergeometric functions with applications, Math. Inequal. Appl., 2018, **21**(2), 521–537.
[29] M.-K. Wang, S.-L. Qiu and Y.-M. Chu, Infinite series formula for Hübner upper bound function with applications to Hersch-Pfluger distortion, Math. Inequal. Appl., 2018, **21**(3), 629–648.
[30] T.-R. Huang, S.-Y. Tan, X.-Y. Ma and Y.-M. Chu, Monotonicity properties and bounds for the complete p-elliptic integrals, J. Inequal. Appl., 2018, **2018**, Article ID 239, 11 pages.
[31] T.-H. Zhao, M.-K. Wang, W. Zhang and Y.-M. Chu, Quadratic transformation inequalities for Gaussian hypergeoemtric function, J. Inequal. Appl., 2018, **2018**, Article 251, 15 pages.
[32] Z.-H. Yang, W.-M. Qian and Y.-M. Chu, Monotonicity properties and bounds involving the complete elliptic integrals of the first kind, Math. Inequal. Appl., 2018, **21**(4), 1185–1199.
[33] Gh. Toader, Some mean values related to the arithmetic-geometric mean, J. Math. Anal. Appl., 1998, **218**(2), 358–368.
[34] Y.-M. Chu, M.-K. Wang, S.-L. Qiu and Y.-F. Qiu, Sharp generalized Seiffert mean bounds for Toader mean, Abstr. Appl. Anal., 2011, **2011**, Article ID 605259, 8 pages.
[35] Y.-M. Chu and M.-K. Wang, Inequalities between arithmetic-geometric, Gini, and Toader means, Abstr. Appl. Appl., 2012, **2012**, Article ID 830585, 11 pages.
[36] Y.-M. Chu and M.-K. Wang, Optimal Lehmer mean bounds for the Toader mean, Results Math., 2012, **61**(3-4), 223–229.
[37] W.-F. Xia, Y.-M. Chu and G.-D. Wang, The optimal upper and lower power mean bounds for a convex combination of the arithmetic and logarithmic means, Abstr. Appl. Anal., 2010, **2010**, Article ID 604804, 9 pages.
[38] Y.-M. Chu and W.-F. Xia, Two optimal double inequalities between power mean and logarithmic mean, Comput. Math. Appl., 2010, **60**(1), 83–89.
[39] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang, Sharp power mean bounds for the combination of Seiffert and geometric means, Abstr. Appl. Anal., 2010, **2010**, Article ID 108920, 12 pages.
[40] Y.-M. Chu, S.-S. Wang and C. Zong, Optimal lower power mean bound for the convex combination of harmonic and logarithmic means, Abstr. Appl. Anal., 2011, **2011**, Article ID 520648, 9 pages.
[41] G.-D. Wang, X.-H. Zhang and Y.-M. Chu, A power mean inequality for the Grötzch ring function, Math. Inequal. Appl., 2011, **14**(4), 833–837.
[42] Y.-M. Li, M.-K. Wang and Y.-M. Chu, Sharp power mean bounds for Seiffert mean, Appl. Math. J. Chinese Univ., 2014, **29B**(1), 101–107.

[43] W.-F. Xia, W. Janous and Y.-M. Chu, The optimal convex combination bounds of arithmetic and harmonic mean in terms of power mean, J. Math. Inequal., 2012, **6**(2), 241–248.
[44] E. Neuman and J. Sándor, On the Schwab-Borchardt mean, Math. Pannon., 2003, **14**(2), 253–266.
[45] Z.-Y. He, Y.-M. Chu and M.-K. Wang, Optimal bounds for Neuman means in terms of harmonic and contraharmonic means, J. Appl. Math., 2013, **2013**, Article ID 807623, 4 pages.
[46] Y.-Y. Yang, W.-M. Qian, and Y.-M. Chu, Refinements of bounds for Neuman means with applications, J. Nonlinear Sci. Appl., 2016, **9**(4), 1529–1540.
[47] Y.-M. Chu and B.-Y. Long, Sharp inequalities between means, Math. Inequal. Appl., 2011, **14**(3), 647–655.
[48] Y.-M. Chu, M.-K. Wang and W.-M. Gong, Two sharp double inequalities for Seiffert mean, J. Inequal. Appl., 2011, **2011**, Article 44, 7 pages.
[49] B.-Y. Long and Y.-M. Chu, Optimal inequalities for generalized logarithmic, arithmetic, and geometric means, J. Inequal. Appl., 2010, **2010**, Article ID 806825, 10 pages.
[50] M.-K. Wang, Y.-M. Chu and Y.-F. Qiu, Some comparison inequalities for generalized Muirhead and identric means, J. Inequal. Appl., 2010, **2010**, Article ID 295620, 10 pages.
[51] Y.-M. Chu and B.-Y. Long, Best possible inequalities between generalized logarithmic mean and classical means, Abstr. Appl. Anal., 2010, **2010**, Article ID 303286, 13 pages.
[52] X.-M. Zhang and Y.-M. Chu, Convexity of the integral arithmetic mean of a convex function, Rocky Mountain J. Math., 2010, **40**(3), 1061–1068.
[53] Y.-M. Chu, Y.-F. Qiu, M.-K. Wang and G.-D. Wang, The optimal convex combination bounds of arithmetic and harmonic means for the Seiffert's mean, J. Inequal. Appl., 2010, **2010**, Article ID 436457, 7 pages.
[54] M.-K. Wang, Y.-F. Qiu and Y.-M. Chu, Sharp bounds for Seiffert means in terms of Lehmer means, J. Math. Inequal., 2010, **4**(4), 581–586.
[55] W.-F. Xia, Y.-M. Chu and G.-D. Wang, Necessary and sufficient conditions for the Schur harmonic convexity or concavity of the extended mean values, Rev. Un. Mat. Argentina, 2011, **52**(1), 121–132.
[56] W.-F. Xia and Y.-M. Chu, The Schur convexity of Gini mean values in the sense of harmonic mean, Acta Math. Sci., 2011, **31B**(3), 1103–1112.
[57] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, A sharp double inequality between harmonic and identric means, Abstr. Appl. Anal., 2011, **2011**, Article ID 657935, 7 pages.
[58] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On approximating the error function, Math. Inequal. Appl., 2018, **21**(2), 469–479.
[59] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, An optimal double inequality between Seiffert and geometric means, J. Appl. Math., 2011, **2011**, Article ID 261237, 6 pages.
[60] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, A best possible double inequality between Seiffert and harmonic means, J. Inequal. Appl., 2011, **2011**, Article 94, 7 pages.
[61] Y.-F. Qiu, M.-K. Wang, Y.-M. Chu and G.-D. Wang, Two sharp inequalities for Lehmer mean, identric mean and logarithmic mean, J. Math. Inequal., 2011, **5**(3), 301–306.
[62] Y.-M. Chu, C. Zong and G.-D. Wang, Optimal convex combiantion bounds of Seiffert and geometric means for the arithmetic mean, J. Math. Inequal., 2011, **5**(3), 429–434.
[63] M.-K. Wang, Z.-K. Wang and Y.-M. Chu, An optimal double inequality between geometric and identric means, Appl. Math. Lett., 2012, **25**(3), 471–475.
[64] Y.-M. Chu and S.-W. Hou, Sharp bounds for Seiffert mean in terms of contraharmonic mean, Abstr. Appl. Anal., 2012, **2012**, Article ID 425175, 6 pages.
[65] S.-L. Qiu, Y.-F. Qiu, M.-K. Wang and Y.-M. Chu, Hölder mean inequalities for the generalized Grötzsch ring and Hersch-Pfluger distortion functions, Math. Inequal. Appl., 2012, **15**(1), 237–245.
[66] Y.-M. Chu, M.-K. Wang and G.-D. Wang, The optimal generalized logarithmic mean bounds for Seiffert's mean, Acta Math. Sci., 2012, **32B**(4), 1619–1626.
[67] Y.-M. Chu, M.-K. Wang and Z.-K. Wang, Best possible inequalities among harmonic, geometric, logarithmic and Seiffert means, Math. Inequal. Appl., 2012, **15**(2), 415–422.
[68] W.-M. Gong, Y.-Q. Song, M.-K. Wang and Y.-M. Chu, A sharp double inequality betwee Seiffert, arithmetic, and geometric means, Abstr. Appl. Anal., 2012, **2012**, Article ID 648834, 7 pages.

[69] H.-N. Hu, G.-Y. Tu and Y.-M. Chu, Optimal bounds for Seiffert mean in terms of one-parameter means, J. Appl. Math., 2012, **2012**, Article ID 917120, 7 pages.
[70] Y.-M. Li, B.-Y. Long and Y.-M. Chu, Sharp bounds for the Neuman-Sándor mean in terms of generalized logarithmic mean, J. Math. Inequal., 2012, **6**(4), 567–577.
[71] Y.-M. Chu and B.-Y. Long, Bounds of the Neuman-Sándor mean using power and identric means, Abstr. Appl. Anal., 2013, **2013**, Article ID 832591, 6 pages.
[72] Y.-M. Chu, S.-W. Hou and W.-F. Xia, Optimal convex combination bounds of centroidal and harmonic means for logarithmic and inentric means, Bull. Iranian Math. Soc., 2013, **39**(2), 259–269.
[73] Y.-M. Chu, M.-K. Wang and X.-Y. Ma, Sharp bounds for Toader mean in terms of contraharmonic mean with applications, J. Math. Inequal., 2013, **7**(2), 161–166.
[74] W.-F. Xia and Y.-M. Chu, Optimal inequaliites between Neuman-Sándor, centroidal and harmonic means, J. Math. Inequal., 2013, **7**(4), 593–600.
[75] Y.-M. chu, Y.-M. Li, W.-F. Xia and X.-H. Zhang, Best possible inequalities for the harmonic mean of error function, J. Inequal. Appl., 2014, **2014**, Article 525, 9 pages.
[76] W.-M. Qian and Y.-M. Chu, Best possible bounds for Yang mean using generalized logarithmic mean, Math. Probl. Eng., 2016, **2016**, Article ID 8901258, 7 pages.
[77] W.-M. Qian, X.-H. Zhang and Y.-M. Chu, Sharp bounds for the Toader-Qi mean in terms of harmonic and geometric means, J. Math. Inequal., 2017, **11**(1), 121–127.
[78] H.-Z. Xu, Y.-M. Chu and W.-M. Qian, Sharp bounds for the Sándor-Yang means in terms of arithmetic and contra-harmonic means, J. Inequal. Appl., 2018, **2018**, Article 127, 13 pages.
[79] Zh.-H. Yang, W. Zhang and Y.-M. Chu, Sharp Gautschi inequality for parameter $0 < p < 1$ with applications, Math. Inequal. Appl., 2017, **20**(4), 1107–1120.
[80] Zh.-H. Yang, Y.-M. Chu and W. Zhang, Accurate approximations for the complete elliptic of the second kind, J. Math. Anal. Appl., 2016, **438**(2), 875–888.
[81] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, Monotonicity rule for the quotient of two function and its applications, J. Inequal. Appl., 2017, **2017**, Article 106, 13 pages.
[82] Zh.-H. Yang, W.-M. Qian, Y.-M. Chu and W. Zhang, On rational bounds for the gamma function, J. Inequal. Appl., 2017, **2017**, Article 210, 17 pages.
[83] E. Neuman, On a new bivariate mean, Aequat. Math., 2014, **88**(3), 277–289.
[84] R. W. Barnard, K. Pearce and K. C. Richards, An inequality involving the generalized hypergeometric function and the arc length of an ellipse, SIAM J. Math. Anal., 2000, **31**(3), 693–699.
[85] H. Alzer and S.-L. Qiu, Monotonicity theorems and inequalities for the complete elliptic integrals, J. Comput. Appl. Math., 2004, **172**(2), 289–312.
[86] Y.-M. Chu, M.-K. Wang and S.-L. Qiu, Optimal combinations bounds of root-square and arithmetic means for Toader mean, Proc. Indian Acad. Sci. Math. Sci., 2012, **122**(1), 41–51.
[87] J.-F. Li, W.-M. Qian and Y.-M. Chu, Sharp bounds for Toader mean in terms of arithmetic, quadratic, and Neuman means, J. Inequal. Appl., 2015, **2015**, Article ID 277, 9 pages.
[88] W.-M. Qian, Y.-Q. Song, X.-H. Zhang and Y.-M. Chu, Sharp bounds for Toader mean in terms of arithmetic and second contraharmonic means, J. Funct. Spaces, 2015, **2015**, Article ID 452823, 5 pages.
[89] Y.-Q. Song, W.-M. Qian and Y.-M. Chu, Optimal bounds for Neuman mean using arithmetic and centroidal means, J. Funct. Spaces, 2016, **2016**, Article ID 5131907, 7 pages.
[90] G. D. Anderson, M. K. Vamanamurthy and M. K. Vuorinen, Conformal Invariants, Inequalities, and Quasiconformal Maps, John Wiley & Sons, New York, 1997.

Wei-Mao Qian, College of Science, Hunan City University, Yiyang 413000, Hunan, China.
E-mail address: qwm661977@126.com

Zai-Yin He, College of Mathematics and Econometrics, Hunan University, Changsha 410082, Hunan, China
E-mail address: hzy@zjhu.edu.com

Yu-Ming Chu (Corresponding author), Department of Mathematics, Huzhou University, Huzhou 313000, Zhejiang, China
E-mail address: chuyuming2005@126.com

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

ON STRONGLY STARLIKENESS OF STRONGLY CONVEX FUNCTIONS

ADEL A. ATTIYA, NAK EUN CHO, AND M. F. YASSEN

Abstract. In this paper we introduce an argument property which gives an interesting relation between the classes of strongly convex and strongly starlike functions of order α and type β in the open unit disk. Also, the sufficient condition of starlikeness under certain restrictions is obtained.

1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. The function $f(z)$ is called strongly starlike of order β and type α and strongly convex of order β and type α, respectively if it satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta \tag{1.2}$$

and

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta, \tag{1.3}$$

where $\alpha \in [0, 1)$ and $\beta \in (0, 1]$. We denote by $S^*(\alpha, \beta)$ and $C(\alpha, \beta)$ the classes of functions satisfy the conditions (1.2) and (1.3) respectively. We note that both $S^*(\alpha, 1) = S^*(\alpha)$ and $C(\alpha, 1) = C(\alpha)$, are the well known classes of starlike functions of order α and convex functions of order α.

MacGregor [2] Wilken and Feng [5] obtained the following result:

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Analytic functions; Strongly convex functions; Strongly starlike functions.

$$f(z) \in C(\alpha) \;\Rightarrow\; f(z) \in \; S^*(\beta) \qquad (0 \leq \alpha < 1),$$

where

$$\beta := \beta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2-2\alpha}(1-2^{2\alpha-1}))}\;, & \alpha \neq \frac{1}{2} \\ \frac{1}{2\log 2}, & \alpha = \frac{1}{2}. \end{cases} \tag{1.4}$$

Also, Nunokawa *et al.*[4] investigated a certain relation between $S^*(\alpha,\beta)$ and $C(\alpha,\beta)$. In the present paper, we obtain a relationship between strongly convex and strongly starlike functions by using the result given by Nunokawa [3].

In our investigation, we need the following lemma:

Lemma 1.1. [3] *Let $P(z)$ be analytic in $\mathbb{U}$, $P(0) = 1$, $P(z) \neq 0$ in $\mathbb{U}$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg(P(z_0))| = \frac{\pi}{2}\delta,$$

where $0 < \delta$. Then we have

$$\frac{z_0 P'(z_0)}{P(z_0)} = ik\delta,$$

where

$$k \geq \; \frac{1}{2}\left(a + \frac{1}{a}\right) \;\; when \;\; \arg(P(z_0)) = \frac{\pi}{2}\delta$$

and

$$k \leq -\; \frac{1}{2}\left(a + \frac{1}{a}\right) \;\; when \;\; \arg(P(z_0)) = -\frac{\pi}{2}\delta,$$

where $(P(z_0))^{1/\delta} = \pm ia$ and $a > 0$.

2. Main Result

Theorem 2.1. *Let $f(z)$ be analytic function defined by (1.1) and also, let*

$$f(z) \in C(\alpha, \gamma) \quad (z \in \mathbb{U}), \tag{2.5}$$

where $0 \leq \alpha < 1$ and $0 < \gamma < 1$.
Then

$$f(z) \in S(\beta, \delta) \quad (z \in \mathbb{U}), \tag{2.6}$$

where

$$\gamma = \frac{2}{\pi}\ \arctan\left(\frac{\delta(1-\beta)a_0^{\delta-1}\left(a_0^2+1\right)}{2\left(\beta+(1-\beta)a_0^{\delta}\right)\left((\beta-\alpha)+(1-\beta)a_0^{\delta}\right)}\right), \tag{2.7}$$

β is defined by (1.4), $0<\delta<1$ and a_0 is the positive root of the equation:

$$\begin{aligned}(\beta-\alpha)\beta\left((1+\delta)\,x^2-(1-\delta)\right)+x^{\delta}\,(1-\beta)\,(2\beta-\alpha)\left(x^2-1\right)\\ +x^{2\delta}\,(1-\beta)^2\left((1-\delta)\,x^2-(1+\delta)\right)=0,\end{aligned} \tag{2.8}$$

which satisfies

$$a_0^{\delta} \geq \left(\frac{\beta}{1-\beta}\left(\sqrt{\csc^2\left(\frac{\pi}{2}\delta\right)+\left(\frac{\beta-\alpha}{\beta}\right)}-\csc\left(\frac{\pi}{2}\delta\right)\right)\right)^{1/\delta} \tag{2.9}$$

Proof. Let

$$p(z)=\frac{z\ f'(z)}{f(z)},\ p(0)=1 \quad \text{and} \quad p(z)\neq\beta \quad (z\in\mathbb{U}).$$

Then we have

$$1+\frac{z\ f''(z)}{f'(z)}=p(z)+\frac{zp'(z)}{p(z)}.$$

If there exists $z_0\in\mathbb{U}$ such that

$$|\arg\left(P(z)\right)|=|\arg\left(p(z)-\beta\right)|<\frac{\pi}{2}\delta$$

for $|z|<|z_0|$ and

$$|\arg\left(P(z_0)\right)|=|\arg\left(p(z_0)-\beta\right)|=\frac{\pi}{2}\delta,$$

where

$$P(z)=\frac{p(z)-\beta}{1-\beta}\ .$$

Since $P(0)=1$ and by using Lemma 1.1, we have

$$\frac{z_0\ P'(z_0)}{P(z_0)}=\frac{z_0p'(z_0)}{p(z_0)-\beta}=i\delta k.$$

The first case, if

$$\arg\left(P(z_0)\right)=\arg\left(p(z_0)-\beta\right)=\frac{\pi}{2}\delta,$$

then we have

$$\begin{aligned}
&\arg\left(1+\frac{z\ f''(z)}{f'(z)}-\alpha\right)\\
&=\arg\left((p(z_0)-\beta)\left(1+\frac{z_0p'(z_0)/p(z_0)}{p(z_0)-\beta}+\frac{\beta-\alpha}{p(z_0)-\beta}\right)\right)\\
&=\frac{\pi}{2}\delta+\arg\left(1+\frac{i\delta k}{\beta+(1-\beta)(ia)^\delta}+\frac{\beta-\alpha}{(1-\beta)(ia)^\delta}\right)\\
&=\frac{\pi}{2}\delta+\arg\left(1+\frac{i\delta k}{\beta+(1-\beta)a^\delta e^{i\frac{\pi}{2}\delta}}+\frac{\beta-\alpha}{(1-\beta)a^\delta e^{i\frac{\pi}{2}\delta}}\right)\\
&=\arg\left(e^{i\frac{\pi}{2}\delta}+\frac{i\delta k}{\beta e^{-i\frac{\pi}{2}\delta}+(1-\beta)a^\delta}+\frac{(\beta-\alpha)}{(1-\beta)a^\delta}\right)\\
&\geq\arctan\left(\frac{\frac{\delta k(1-\beta)a^\delta+\delta k\beta\cos\left(\frac{\pi}{2}\delta\right)}{\left(\beta+(1-\beta)a^\delta\right)^2}+\sin\left(\frac{\pi}{2}\delta\right)}{\frac{\beta-\alpha}{(1-\beta)a^\delta}+\cos\left(\frac{\pi}{2}\delta\right)-\frac{\beta\delta k\sin\left(\frac{\pi}{2}\delta\right)}{\left(\beta+(1-\beta)a^\delta\right)^2}}\right).
\end{aligned}$$

Since the function $h(k)$ defined by

$$h(k)=\arctan\left(\frac{\frac{\delta k(1-\beta)a^\delta+\delta k\beta\ \cos\left(\frac{\pi}{2}\delta\right)}{\left(\beta+(1-\beta)a^\delta\right)^2}+\sin\left(\frac{\pi}{2}\delta\right)}{\frac{\beta-\alpha}{(1-\beta)a^\delta}+\cos\left(\frac{\pi}{2}\delta\right)-\frac{\beta\delta k\sin\left(\frac{\pi}{2}\delta\right)}{\left(\beta+(1-\beta)a^\delta\right)^2}}\right)$$

is an increasing function of k $(k\geq 1)$, we have

$$\begin{aligned}
&\arg\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right)\\
&\geq\arctan\left(\frac{\frac{\left(\delta\ (1-\beta)a^\delta+\delta\beta\cos\left(\frac{\pi}{2}\delta\right)\right)(a+1/a)}{2\left(\beta+(1-\beta)a^\delta\right)^2}+\sin\left(\frac{\pi}{2}\delta\right)}{\frac{\beta-\alpha}{(1-\beta)a^\delta}+\cos\left(\frac{\pi}{2}\delta\right)-\frac{\beta\delta\sin\left(\frac{\pi}{2}\delta\right)(a+1/a)}{2\left(\beta+(1-\beta)a^\delta\right)^2}}\right).
\end{aligned}$$

Also, the function $f(\theta)$ defined by

$$f(\theta)=\arctan\left(\frac{\frac{\delta(1-\beta)a^\delta(a+1/a)}{2\left(\beta+(1-\beta)a^\delta\right)^2}+\frac{\delta\beta(a+1/a)}{2\left(\beta+(1-\beta)a^\delta\right)^2}\cos\theta+\sin\theta}{\frac{\beta-\alpha}{(1-\beta)a^\delta}+\cos\theta-\frac{\beta\delta(a+1/a)}{2\left(\beta+(1-\beta)a^\delta\right)^2}\sin\theta}\right)$$

is an increasing and continuous function of θ $(0 < \theta < \frac{\pi}{2})$ when a^δ satisfies (2.9). Therefore, we have

$$\arg\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right) \geq \arctan\left(\frac{\delta(1-\beta)\,(a+1/a)\,a^\delta}{2\,(\beta+(1-\beta)a^\delta)\,((\beta-\alpha)+(1-\beta)a^\delta)}\right). \tag{2.10}$$

On the other hand, since the function $g(x)$ defined by

$$g(x)=\frac{\delta(1-\beta)\left(x+\frac{1}{x}\right)x^\delta}{2\,(\beta+(1-\beta)x^\delta)\,((\beta-\alpha)+(1-\beta)x^\delta)} \qquad (x>0), \tag{2.11}$$

takes its minimum value when x is defined by (2.8), we see that this contradicts the hypothesis of Theorem 2.1.

The second case, if

$$\arg\left(P(z_0)\right)=\arg\left(p(z_0)-\beta\right)=-\frac{\pi}{2}\delta,$$

then we have

$$\begin{aligned}
&\arg\left(1+\frac{z\,f''(z)}{f'(z)}-\alpha\right)\\
&=\arg\left((p(z_0)-\beta)\left(1+\frac{z_0p'(z_0)/p(z_0)}{p(z_0)-\beta}+\frac{\beta-\alpha}{p(z_0)-\beta}\right)\right)\\
&=-\frac{\pi}{2}\delta+\arg\left(1+\frac{i\delta k}{\beta+(1-\beta)a^\delta e^{-i\frac{\pi}{2}\delta}}+\frac{\beta-\alpha}{(1-\beta)a^\delta e^{-i\frac{\pi}{2}\delta}}\right)\\
&=\arg\left(e^{-i\frac{\pi}{2}\delta}+\frac{i\;\delta k}{\beta\;e^{i\frac{\pi}{2}\delta}+(1-\beta)a^\delta}+\frac{(\beta-\alpha)}{(1-\beta)a^\delta}\right)\\
&=\arctan\left(\frac{\frac{\delta k(1-\beta)a^\delta+\delta k\beta\cos\left(\frac{\pi}{2}\delta\right)}{\left(\beta+(1-\beta)a^\delta\right)^2}-\sin\left(\frac{\pi}{2}\delta\right)}{\frac{\beta-\alpha}{(1-\beta)a^\delta}+\cos\left(\frac{\pi}{2}\delta\right)+\frac{\beta\delta k\sin\left(\frac{\pi}{2}\delta\right)}{\left(\beta+(1-\beta)a^\delta\right)^2}}\right).
\end{aligned}$$

Since the function $h(k)$ defined by

$$h(k)=\arctan\left(\frac{\delta k(1-\beta)a^\delta+\delta k\beta\cos\left(\frac{\pi}{2}\delta\right)-\sin\left(\frac{\pi}{2}\delta\right)}{\frac{\beta-\alpha}{(1-\beta)a^\delta}+\cos\left(\frac{\pi}{2}\delta\right)+\beta\delta k\sin\left(\frac{\pi}{2}\delta\right)}\right)$$

is a decreasing function of k $(k\leq -1)$, we have

$$\arg\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right) \leq \arctan\left(\frac{-\frac{\left(\delta(1-\beta)a^{\delta}+\delta\beta\cos\left(\frac{\pi}{2}\delta\right)\right)(a+1/a)}{2\left(\beta+(1-\beta)a^{\delta}\right)^{2}}-\sin\left(\frac{\pi}{2}\delta\right)}{\frac{\beta-\alpha}{(1-\beta)a^{\delta}}+\cos\left(\frac{\pi}{2}\delta\right)-\frac{\beta\delta\sin\left(\frac{\pi}{2}\delta\right)(a+1/a)}{2\left(\beta+(1-\beta)a^{\delta}\right)^{2}}}\right).$$

Also, the function $f(\theta)$ defined by

$$f(\theta)=-\arctan\left(\frac{\frac{\delta(1-\beta)a^{\delta}(a+1/a)}{2\left(\beta+(1-\beta)a^{\delta}\right)^{2}}+\frac{\delta\beta(a+1/a)}{2\left(\beta+(1-\beta)a^{\delta}\right)^{2}}\cos\theta+\sin\theta}{\frac{\beta-\alpha}{(1-\beta)a^{\delta}}+\cos\theta-\frac{\beta\delta(a+1/a)}{2\left(\beta+(1-\beta)a^{\delta}\right)^{2}}\sin\theta}\right)$$

is a decreasing and continuous function of θ $(0<\theta<\frac{\pi}{2})$, when a^{δ} satisfies (2.9). Therefore, we have

$$\arg\left(1+\frac{z\ f''(z)}{f'(z)}-\alpha\right) \leq -\arctan\left(\frac{\delta(1-\beta)\left(a+\frac{1}{a}\right)a^{\delta}}{2\left(\beta+(1-\beta)a^{\delta}\right)\left((\beta-\alpha)+(1-\beta)a^{\delta}\right)}\right).$$

Also, by using the function $g(x)$ defind by (2.11) which contradicts hypothesis of Theorem 2.1. Therefore, it completes the proof of the theorem. □

Putting $f(z)$ instead of $zf'(z)$ in Theorem 2.1, we have the following corollary

Corollary 2.1. *Let $f(z)$ be analytic function defined by* (1.1) *and also, let*

$$f(z)\in S^{*}(\alpha,\gamma) \quad (z\in\mathbb{U}), \tag{2.12}$$

where $0\leq\alpha<1$ and $0<\gamma<1$. Then

$$\left|\arg\left(\frac{f(z)}{A(z)}-\beta\right)\right|<\frac{\pi}{2}\delta \quad (z\in\mathbb{U}), \tag{2.13}$$

where $A(z)=\int_0^z(f(t)/t)dt$ is Alexander operator defined by Alexander [1],

$$\gamma=\frac{2}{\pi}\ \arctan\left(\frac{\delta(1-\beta)a_0^{\delta-1}\left(a_0^2+1\right)}{2\left(\beta+(1-\beta)a_0^{\delta}\right)\left((\beta-\alpha)+(1-\beta)a_0^{\delta}\right)}\right), \tag{2.14}$$

β is defined by (1.4), $0 < \delta < 1$ *and* a_0 *is the positive root of the equation:*

$$(2.15)\quad (\beta-\alpha)\beta\left((1+\delta)\,x^2-(1-\delta)\right)+x^{\delta}\,(1-\beta)\,(2\beta-\alpha)\left(x^2-1\right) + x^{2\delta}\,(1-\beta)^2\left((1-\delta)\,x^2-(1+\delta)\right)=0.$$

which satisfies

$$(2.16)\quad a_0^{\delta} \geq \left(\frac{\beta}{1-\beta}\left(\sqrt{\csc^2\left(\frac{\pi}{2}\delta\right)+\left(\frac{\beta-\alpha}{\beta}\right)}-\csc\left(\frac{\pi}{2}\delta\right)\right)\right)^{1/\delta}.$$

Corollary 2.2. *Let $f(z)$ be analytic function defined by* (1.1) *and also, let*

$$(2.17)\qquad f(z)\in C(\alpha,\gamma)\quad (z\in\mathbb{U}),$$

where $0\leq\alpha<1$ and $0<\gamma<1$. Then

$$(2.18)\qquad f(z)\in S(\beta,\delta)\quad (z\in\mathbb{U}),$$

where

$$(2.19)\qquad \gamma=\frac{2}{\pi}\ \arctan\left(\frac{\delta\sqrt{\beta(\beta-\alpha)}}{\left(\beta+\sqrt{\beta(\beta-\alpha)}\right)\left((\beta-\alpha)+\sqrt{\beta(\beta-\alpha)}\right)}\right),$$

and β is defined by (1.4).

Proof. Let $f(z)\in C(\alpha,\gamma)$. Since the inequality (2.10) is satisfied when a^{δ} satisfies (2.9), we have

$$\frac{\delta(1-\beta)\,(a+1/a)\,a^{\delta}}{2\,(\beta+(1-\beta)a^{\delta})\,((\beta-\alpha)+(1-\beta)a^{\delta})} \geq \frac{\delta(1-\beta)a^{\delta}}{(\beta+(1-\beta)a^{\delta})\,((\beta-\alpha)+(1-\beta)a^{\delta})}.$$

Then the function $k(x)$ defined by

$$k(x)=\frac{\delta(1-\beta)x}{(\beta+(1-\beta)x)\,((\beta-\alpha)+(1-\beta)x)}\qquad (x>0)$$

takes its minimum value when $x=\frac{\sqrt{\beta(\beta-\alpha)}}{1-\beta}$.

On the other hand , we have

$$\frac{\sqrt{\beta(\beta-\alpha)}}{1-\beta}\geq\left(\frac{\beta}{1-\beta}\left(\sqrt{\csc^2\left(\frac{\pi}{2}\delta\right)+\left(\frac{\beta-\alpha}{\beta}\right)}-\csc\left(\frac{\pi}{2}\delta\right)\right)\right).$$

Hence we have $f(z)\in S(\beta,\delta)$. □

Acknowledgement. The first and the second authors of this research were supported by the Deanship of Scientific Research, Hail University, Hail, under research no. 0150323 and by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2016R1D1A1A09916450).

References

[1] J.W. Alexander, Functions which map the interior of the unit circle upon simple region, Ann. Math. 17(1915), 12-22.

[2] T.H. MacGregor, A subordination for convex functions of order α J. London Math. Soc. (2)9(1974/75), 530–536.

[3] M. Nunokawa, On the order of strongly starlikeness of strongly convex functions, Proc. Japan Acad. Ser. A Math. Sci. 69(1993), 234-237.

[4] M. Nunokawa, S. Owa and H. Shiraishi, On a remark of strongly convex functions of order β and convex of order α, RIMS Kôkyûroku, Vol. 2010, No. 1717, 100-106.

[5] D.R. Wilken and J.A. Feng, A remark on convex and starlike functions. J. London Math. Soc. 21(1980), no. 2, 287-290.

Department of Mathematics, Faculty of Science, University of Hail, Hail, Saudi Arabia,
and,
Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura, 35516, Egypt
E-mail address: aattiy@mans.edu.eg

Department of Applied Mathematics, Pukyong National University, Busan, Korea
E-mail address: necho@pknu.ac.kr

Department of Mathematics, Faculty of Science, Damietta University, New Damietta, 34517, Egypt
Current address: Department of Mathematics, Faculty of Sciences and Humanities Aflaj, Prince Sattam bin Abdulaziz University, Kingdom of Saudi Arabia
E-mail address: mansouralieg@yahoo.com

Invariance analysis of a four-dimensional system of fourth-order difference equations with variable coefficients

Mensah Folly-Gbetoula*

School of Mathematics, University of the Witwatersrand, 2050, Johannesburg, South Africa.

Abstract

A class of a four-dimensional system of difference equations is considered. A Lie symmetry analysis is performed and symmetries are derived. We use the differential invariant approach to obtain exact solutions. The link between the similarity variables and these symmetries is clearly given. Furthermore, we show the existence of periodic solutions for some specific coefficients. This work considerably extends some findings by El-Dessoky and Hobiny [M. M. El-Dessoky and A. Hobiny, J. Computational Analysis and Applications, **26:8** (2019), 1428–1439].

Keywords: System of difference equation; invariance analysis; group invariant solutions; periodicity
MSC: 39A11, 39A05

1 Introduction

The group theoretical approach for finding exact solutions to differential equations is now well reported [2, 14] and its application to difference equations has sparked interest recently [6–8, 10–13]. This approach, commonly known as Lie symmetry analysis, permits one to lower the order of the difference equations via a convenient choice of canonical coordinates obtained using a group of transformations admitted by the equation. Its application to higher dimensional system of difference equations is somewhat new and the calculation one deals with when finding symmetries in the latter can become cumbersome. Hydon in [10] extends the idea of Maeda [16] by developing a systematic algorithm permitting one to obtain the Lie algebra of a difference equation. Several authors have studied difference equations from different approaches and some interesting results can be found in [3–5, 17]

In this paper, inspired by the work in [1] where the authors study the behavior and existence of solutions of

$$\begin{aligned} x_{n+1} &= \frac{x_{n-3}}{\pm 1 \pm x_{n-3}y_{n-2}z_{n-1}t_n},\ y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm x_n y_{n-3}z_{n-2}t_{n-1}} \\ z_{n+1} &= \frac{z_{n-3}}{\pm 1 \pm x_{n-1}y_n z_{n-3}t_{n-2}},\ t_{n+1} = \frac{t_{n-3}}{\pm 1 \pm x_{n-2}y_{n-1}z_n t_{n-3}}, \end{aligned} \tag{1}$$

we utilize Hydon's idea in a slightly modified manner to investigate the solutions to

$$\begin{aligned} x_{n+1} &= \frac{x_{n-3}}{a_n + b_n x_{n-3}y_{n-2}z_{n-1}t_n},\ y_{n+1} = \frac{y_{n-3}}{c_n + d_n x_n y_{n-3}z_{n-2}t_{n-1}} \\ z_{n+1} &= \frac{z_{n-3}}{e_n + f_n x_{n-1}y_n z_{n-3}t_{n-2}},\ t_{n+1} = \frac{t_{n-3}}{g_n + h_n x_{n-2}y_{n-1}z_n t_{n-3}}, \end{aligned} \tag{2}$$

where $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, $(c_n)_{n\in\mathbb{N}_0}$, $(d_n)_{n\in\mathbb{N}_0}$, $(e_n)_{n\in\mathbb{N}_0}$, $(f_n)_{n\in\mathbb{N}_0}$, $(g_n)_{n\in\mathbb{N}_0}$ and $(h_n)_{n\in\mathbb{N}_0}$ are non-zero sequences of real numbers. The solutions of (2) are derived after a series of steps. Firstly, we obtain the Lie algebra of (2). We make use of point symmetries and additional assumptions on the characteristics to allow us derive analytic expressions for the symmetry generators. Secondly, we lower the order via the invariants and finally, find the solutions. We have showed that results in [1] are special cases of our findings.

*Mensah.Folly-Gbetoula@wits.ac.za

1.1 Preliminaries

In this section, we commence with some background necessary for understanding symmetry analysis. Note that throughout this paper, we utilize definitions and notation in [10, 14]. The notion of symmetry is strongly related to the notion of group transformations. Basically, it is a group of transformations that map a solution of a given equation onto another solution. Suppose G is a group of transformations acting on a manifold $\mathcal{M}$. Certain subsets $\mathcal{H}$ of this group, called $\mathcal{H}$-invariant, transform solutions onto themselves. Often times, for system of difference equations, the difference invariants of $\mathcal{H}$ are the new variables of the much simpler difference equations equivalent to the original system of equations.

Let S^i be the forward shift operator that maps n to $n+i$. We shall assume that a system of fourth order ordinary difference equations is of the form

$$S^p(u^k) = \Omega_k(n, [u^k]), \quad k = 1, 2, 3, 4, \tag{3}$$

where $[u^i]$ denotes the dependent variable u^i and its shifts. The invertible mapping $(n, u^k) \mapsto (n, \tilde{u}^k = u^k + \varepsilon Q_k(n, [u^k]) + O(\varepsilon^2)),\ k = 1, 2, 3, 4,$ is a symmetry group of transformations if and only if it satisfies the following linearized symmetry condition

$$S^p(Q_k) - \mathcal{X}(\Omega_k) = 0, \quad k = 1, 2, 3, 4, \tag{4}$$

where $\mathcal{X}$ is the $(p-1)$st prolongation of the symmetry generator

$$X = \sum_{k=1}^{4} Q_k \frac{\partial}{\partial u^k}, \tag{5}$$

i. e.,

$$\mathcal{X} = X^{[p-1]} = \sum_{j=0}^{p-1} \sum_{k=1}^{4} S^j(Q_k) \frac{\partial}{\partial S_j(u^k)}. \tag{6}$$

We shall refer to $Q_k = Q_k(n, u_n)$ as characteristics and for simplicity we shall consider point transformations only, that is, $Q_k = Q_k(n, u^k)$.

Definition 1.1 *[14] Let G be a connected group of transformations acting on a manifold M. A smooth real-valued function $\zeta : M \to \mathbb{R}$ is an invariant function for G if and only if*

$$X(\zeta) = 0 \qquad \textit{for all} \qquad x \in M,$$

Without any lucky guess, the reduction of order can readily be done via the canonical coordinates [9]

$$s^k = \int \frac{du^k}{Q_k(n, u^k)}, \quad k = 1, 2, 3, 4. \tag{7}$$

Eventually, the constraining restrictions on the constants in the characteristics, Q_k, $k = 1, 2, 3, 4$, hint on a perfect choice of invariants.

2 Main results

To start, we consider the corresponding forward system

$$\begin{aligned} x_{n+4} = \Omega_1 = \frac{x_n}{A_n + B_n x_n y_{n+1} z_{n+2} t_{n+3}},\ y_{n+4} = \Omega_2 = \frac{y_n}{C_n + D_n x_{n+3} y_n z_{n+1} t_{n+2}} \\ z_{n+4} = \Omega_3 = \frac{z_n}{E_n + F_n x_{n+2} y_{n+3} z_n t_{n+1}},\ t_{n+4} = \Omega_4 = \frac{t_n}{G_n + H_n x_{n+1} y_{n+2} z_{n+3} t_n}, \end{aligned} \tag{8}$$

where $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$, $(C_n)_{n\in\mathbb{N}_0}$, $(D_n)_{n\in\mathbb{N}_0}$, $(E_n)_{n\in\mathbb{N}_0}$, $(F_n)_{n\in\mathbb{N}_0}$, $(G_n)_{n\in\mathbb{N}_0}$ and $(H_n)_{n\in\mathbb{N}_0}$ are non-zero sequences of real numbers, equivalent to (2).

2.1 Symmetries

To construct the characteristics of the system of fourth order difference equations (8), we must impose linearized symmetry criterion (4). This amounts to

$$S^4Q_1 + \frac{B_n x_n^2 (t_{n+3}z_{n+2}(S^1Q_2) + t_{n+3}y_{n+1}(S^2Q_3) + y_{n+1}z_{n+2}(S^3Q_4)) - A_nQ_1}{(A_n + B_n x_n y_{n+1} z_{n+2} t_{n+3})^2} = 0, \tag{9a}$$

$$S^4Q_2 + \frac{D_n y_n^2 (t_{n+2}x_{n+3}(S^1Q_3) + x_{n+3}z_{n+1}(S^2Q_4) + t_{n+2}z_{n+1}(S^3Q_1)) - C_nQ_2}{(C_n + D_n x_{n+3} y_n z_{n+1} t_{n+2})^2} = 0, \tag{9b}$$

$$S^4Q_3 + \frac{F_n z_n^2 (t_{n+1}x_{n+2}(S^3Q_2) + x_{n+2}y_{n+3}(S^1Q_4) + t_{n+1}y_{n+3}(S^2Q_1)) - E_nQ_3}{(E_n + F_n x_{n+2} y_{n+3} z_n t_{n+1})^2} = 0, \tag{9c}$$

$$S^4Q_4 + \frac{H_n t_n^2 (x_{n+1}z_{n+3}(S^2Q_2) + x_{n+1}y_{n+2}(S^3Q_3) + y_{n+2}z_{n+3}(S^1Q_1)) - G_nQ_4}{(G_n + H_n x_{n+1} y_{n+2} z_{n+3} t_n)^2} = 0. \tag{9d}$$

We act the operators $\partial/\partial x_n - [(\partial\Omega_1/\partial x_n)/(\partial\Omega_1/\partial y_{n+1})]\partial/\partial y_{n+1}$, $\partial/\partial y_n - [(\partial\Omega_2/\partial y_n)(\partial\Omega_2/\partial z_{n+1})]\partial/\partial z_{n+1}$, $\partial/\partial z_n - [(\partial\Omega_3/\partial z_n)(\partial\Omega_3/\partial y_{n+3})]\partial/\partial y_{n+3}$ and $\partial/\partial t_n - [(\partial\Omega_4/\partial t_n)(\partial\Omega_4/\partial y_{n+2})]\partial/\partial y_{n+2}$ on equations in (9), respectively, to get

$$(S^1Q_2)' - Q^{1'} + (1/z_{n+2})(S^2Q_3) + (1/t_{n+3})(S^3Q^4) + (2/x_n)Q_1 = 0 \tag{10a}$$

$$-Q^{2'} + (S^1Q_3)' + (2/y_n)Q_2 + (1/t_{n+2})(S^2Q_4) + (1/x_{n+3})(S^3Q_1) = 0 \tag{10b}$$

$$(S^3Q^2)' - Q^{3'} + (2/z_n)Q^3 + (1/t_{n+1})(S^1Q_4) + (1/x_{n+2})(S^2Q^1) = 0 \tag{10c}$$

$$(S^2Q^2)' - {Q_4}' + (1/z_{n+3})(S^3Q_3) + (2/t_n)Q_4 + (1/x_{n+1})(S^1Q^1) = 0 \tag{10d}$$

after simplification. Note that $'$ denotes the derivative with respect to the continuous variable.
Next, we differentiate equations in (10) with respect to x_n, y_n, z_n and t_n, respectively. The latter leads to the differential equations

$$\begin{aligned}&-Q^{1''} + (2/x_n)Q^{1'} - (2/x_n^2)Q^1 = 0,\ -Q^{2''} + (2/y_n)Q^{2'} - (2/y_n^2)Q^2 = 0,\\ &-Q^{3''} + (2/z_n)Q^{3'} - (2/z_n^2)Q^3 = 0, -Q^{4''} + (2/t_n)Q^{4'} - (2/t_n^2)Q^4 = 0\end{aligned} \tag{11}$$

whose solutions are given by

$$\begin{aligned}&Q_1(n,x_n) = \alpha_1(n){x_n}^2 + \beta_1(n)x_n, \quad Q_2(n,y_n) = \alpha_2(n){y_n}^2 + \beta_2(n)y_n,\\ &Q_3(n,z_n) = \alpha_3(n){z_n}^2 + \beta_3(n)z_n, \quad Q_4(n,t_n) = \alpha_4(n){t_n}^2 + \beta_4(n)t_n,\end{aligned} \tag{12}$$

for some functions α_i and β_i, respectively.
We replace (12) and their shits in (9). Due to the fact that the α_i's and β_i's depend on the independent variable only, we equate all products of shifts of dependent variables x_n, y_n, z_n and t_n in the resulting equations to zero; this yields the 'final constraints' below

$$\beta_1(n) + \beta_2(n+1) + \beta_3(n+2) + \beta_4(n+3) = 0, \alpha_1(n) = \alpha_2(n) = \alpha_3(n) = \alpha_4(n) = 0, \tag{13}$$

with $\beta_1(n) = \beta_1(n+4), \beta_2(n) = \beta_2(n+4), \beta_3(n) = \beta_3(n+4), \beta_4(n) = \beta_4(n+4)$. The reader can easily verify that the functions satisfying the above constraints are of the forms:

$$\begin{aligned}&\alpha_j(n) = 0,\ j = 1,2,3,4;\ \beta_1(n) = c_1 + c_2(-i)^n + c_3(i)^n + c_4(-1)^n;\ \beta_2(n) = c_5 + c_6(-i)^n + c_7(i)^n + c_8(-1)^n;\\ &\beta_3(n) = c_9 + c_{10}(-i)^n + c_{11}(i)^n + c_{12}(-1)^n;\ \beta_4(n) = (ic_2 + c_6 - ic_{10})(-i)^n + (c_7 - ic_3 + ic_{11})(i)^n + (c_4 - c_8\\ &+ c_{12})(-1)^n - c_1 - c_5 - c_9,\end{aligned} \tag{14}$$

where the c_i's, $i = 1, \ldots, 12$, are arbitrary constants. Consequently, thanks to (5), (12) and (14), we obtain twelve symmetry generators:

$$\begin{aligned}
&X_1 = x_n\partial x_n - t_n\partial t_n, X_2 = (-i)^n(x_n\partial x_n + it_n\partial t_n), X_3 = i^n(x_n\partial x_n - it_n\partial t_n), X_4 = (-1)^n(x_n\partial x_n + t_n\partial t_n),\\
&X_5 = y_n\partial y_n - t_n\partial t_n, X_6 = (-i)^n(y_n\partial y_n + t_n\partial t_n), X_7 = i^n(y_n\partial y_n + t_n\partial t_n), X_8 = (-1)^n(y_n\partial y_n - t_n\partial t_n),\\
&X_9 = z_n\partial z_n - t_n\partial t_n, X_{10} = (-i)^n(z_n\partial z_n - it_n\partial t_n), X_{11} = i^n(z_n\partial z_n + it_n\partial t_n), X_{12} = (-1)^n(z_n\partial z_n + t_n\partial t_n).
\end{aligned} \tag{15}$$

Note that for simplicity, we adopt the notation $\partial x = \partial/\partial x$.

2.2 Reduction of order via symmetries and formulas for solutions

Using any linear combinations of the symmetries in (15) that involves all four independent variables x_n, y_n, z_n and t_n, say $X = X_1 + X_2 + X_3 = x_n\partial x_n + y_n\partial y_n + z_n\partial z_n - 3t_n\partial t_n$, we derive the corresponding canonical coordinates

$$s_1(n) = \int \frac{dx_n}{x_n},\ s_2(n) = \int \frac{dy_n}{y_n},\ s_3(n) = \int \frac{dz_n}{z_n}, s_4(n) = \int \frac{dt_n}{-3t_n}. \tag{16}$$

Inspired by the form of the equations in the final constraints (13), we construct the invariants:

$$\begin{aligned}
\tilde{X}_n =& \beta_1(n)s_1(n) + \beta_2(n+1)s_2(n+1) + \beta_3(n+2)s_3(n+2) + \beta_4(n+3)s_4(n+3) = \ln|x_n y_{n+1} z_{n+2} t_{n+3}|\\
\tilde{Y}_n =& \beta_1(n+3)s_1(n+3) + \beta_2(n)s_2(n) + \beta_3(n+1)s_3(n+1) + \beta_4(n+2)s_4(n+2) = \ln|x_{n+3} y_n z_{n+1} t_{n+2}|\\
\tilde{Z}_n =& \beta_1(n+2)s_1(n+2) + \beta_2(n+3)s_2(n+3) + \beta_3(n)s_3(n) + \beta_4(n+1)s_4(n+1) = \ln|x_{n+2} y_{n+3} z_n t_{n+1}|\\
\tilde{T}_n =& \beta_1(n+1)s_1(n+1) + \beta_2(n+2)s_2(n+2) + \beta_3(n+3)s_3(n+3) + \beta_4(n)s_4(n) = \ln|x_{n+1} y_{n+2} z_{n+3} t_n|,
\end{aligned}$$

obtained by replacing $\beta_i(n+j)$ by $s_i(n+j)\beta_i(n+j)$ in the left hand sides of equations in (13).
Using Definition 1.1, the reader can easily confirm that $\tilde{X}_n$, $\tilde{Y}_n$, $\tilde{Z}_n$ and $\tilde{T}_n$ are invariant functions. For simplicity, we introduce the variables

$$X_n = \exp(-\tilde{X}_n),\ Y_n = \exp(-\tilde{Y}_n),\ Z_n = \exp(-\tilde{Z}_n),\ T_n = \exp(-\tilde{T}_n). \tag{17}$$

Thus

$$X_{n+1} = H_n + G_nT_n,\ Y_{n+1} = B_n + A_nX_n,\ Z_{n+1} = D_n + C_nY_n,\ T_{n+1} = F_n + E_nZ_n \tag{18a}$$

and so

$$x_{n+4} = \frac{X_n}{Y_{n+1}}x_n,\ y_{n+4} = \frac{Y_n}{Z_{n+1}}y_n,\ z_{n+4} = \frac{Z_n}{T_{n+1}}z_n,\ t_{n+4} = \frac{T_n}{X_{n+1}}t_n. \tag{18b}$$

Straightforward iterations (using equation (18a)) yield

$$X_{n+4} = \Lambda_n^x + (\Theta_n^x)X_n,\ Y_{n+4} = \Delta_n^y + (\Theta_n^y)Y_n,\ Z_{n+4} = \Delta_n^z + (\Theta_n^z)Z_n,\ T_{n+4} = \Delta_n^t + (\Theta_n^t)T_n$$

that is

$$U_{4n+j} = U_j\left(\prod_{k_1=0}^{n-1}\Theta^u_{4k_1+j}\right) + \sum_{l=0}^{n-1}\left(\Lambda^u_{4l+j}\prod_{k_2=l+1}^{n-1}\Theta^u_{4k_2+j}\right), \tag{19a}$$

for $j = 0, 1, 2, 3$ and $(U, u) \in \{(X, x), (Y, y), (Z, z), (T, t)\}$, where

$$\begin{aligned}
&\Lambda_n^x = H_{n+3} + G_{n+3}F_{n+2} + G_{n+3}E_{n+2}D_{n+1} + G_{n+3}E_{n+2}C_{n+1}B_n, \Theta_n^x = G_{n+3}E_{n+2}C_{n+1}A_n;\\
&\Lambda_n^y = B_{n+3} + A_{n+3}H_{n+2} + A_{n+3}G_{n+2}F_{n+1} + A_{n+3}G_{n+2}E_{n+1}D_n, \Theta_n^y = A_{n+3}G_{n+2}E_{n+1}C_n;\\
&\Lambda_n^z = D_{n+3} + C_{n+3}B_{n+2} + C_{n+3}A_{n+2}H_{n+1} + C_{n+3}A_{n+2}G_{n+1}F_n, \Theta_n^z = C_{n+3}A_{n+2}G_{n+1}E_n;\\
&\Lambda_n^t = F_{n+3} + E_{n+3}D_{n+2} + E_{n+3}C_{n+2}B_{n+1} + E_{n+3}C_{n+2}A_{n+1}H_n, \Theta_n^t = E_{n+3}C_{n+2}A_{n+1}G_n;
\end{aligned} \tag{19b}$$

Also, straightforward iterations (using equation (18b)) yield

$$x_{4n+j} = x_j \prod_{k=0}^{n-1} \frac{X_{4k+j}}{Y_{4k+1+j}}, y_{4n+j} = y_j \prod_{k=0}^{n-1} \frac{Y_{4k+j}}{Z_{4k+1+j}}, z_{4n+j} = z_j \prod_{k=0}^{n-1} \frac{Z_{4k+j}}{T_{4k+1+j}}, t_{4n+j} = t_j \prod_{k=0}^{n-1} \frac{T_{4k+j}}{X_{4k+1+j}}, \quad (19c)$$

$j = 0, 1, 2, 3$. Combining equations in (19), we obtain the following solutions $\{x_n\}$ of the system of equations (8):

$$\begin{aligned} x_{4n+j} =& x_j \prod_{s=0}^{n-1} \left[\frac{X_j \left(\prod_{k_1=0}^{s-1} \Theta^x_{4k_1+j} \right) + \sum_{l=0}^{s-1} \left(\Lambda^x_{4l+j} \prod_{k_2=l+1}^{s-1} \Theta^x_{4k_2+j} \right)}{Y_{j+1} \left(\prod_{k_1=0}^{s-1} \Theta^y_{4k_1+j+1} \right) + \sum_{l=0}^{s-1} \left(\Lambda^y_{4l+j+1} \prod_{k_2=l+1}^{s-1} \Theta^y_{4k_2+j+1} \right)} \right], \quad j = 0, 1, 2, \\ x_{4n+3} =& x_3 \prod_{s=0}^{n-1} \left[\frac{X_3 \left(\prod_{k_1=0}^{s-1} \Theta^x_{4k_1+3} \right) + \sum_{l=0}^{s-1} \left(\Lambda^x_{4l+3} \prod_{k_2=l+1}^{s-1} \Theta^x_{4k_2+3} \right)}{Y_0 \left(\prod_{k_1=0}^{s} \Theta^y_{4k_1} \right) + \sum_{l=0}^{s} \left(\Lambda^y_{4l} \prod_{k_2=l+1}^{s} \Theta^y_{4k_2} \right)} \right], \end{aligned} \quad (20)$$

where Θ^u_n and Λ^u_n, $u \in \{x, y, z, t\}$ are defined in (19b); and $X_0 = 1/(x_0y_1z_2t_3),\ X_1 = H_0+G_0/(t_0x_1y_2z_3),\ X_2 = F_0G_1+H_1+(E_0G_1)/(t_1x_2y_3z_0)$ $X_3 = D_0E_1G_2+F_1G_2+H_2+(C_0E_1G_2)/(t_2x_3y_0z_1),\ Y_0 = 1/(t_2x_3y_0z_1),\ Y_1 = B_0 + A_0/(t_3x_0y_1z_2),\ Y_2 = A_1H_0 + B_1 + (A_1G_0)/(t_0x_1y_2z_3),\ Y_3 = A_2F_0G_1 + B_1 + (A_2E_0G_1)/(t_1x_2y_3z_0)$.

Recall that we forward shifted equation (2) thrice to obtain (8) whose solutions x_n is giving in (20). Now, we go backward thrice and replace the capital letters in the right hand sides of equations in (19b) with lower cases letters to get the solutions x_n corresponding to (8). In other words, solutions $\{x_n\}$ of the system of equations (2) is giving by

$$\begin{aligned} x_{4n-3} =& x_{-3} \prod_{s=0}^{n-1} \frac{\left(\prod_{i=0}^{s-1} \theta^x_{4i} \right) + x_{-3}y_{-2}z_{-1}t_0 \sum_{l=0}^{s-1} \left(\lambda^x_{4l} \prod_{i=l+1}^{s-1} \theta^x_{4i} \right)}{(a_0 + b_0x_{-3}y_{-2}z_{-1}t_0) \left(\prod_{i=0}^{s-1} \theta^y_{4i+1} \right) + x_{-3}y_{-2}z_{-1}t_0 \sum_{l=0}^{s-1} \left(\lambda^y_{4l+1} \prod_{i=l+1}^{s-1} \theta^y_{4i+1} \right)} \\ x_{4n-2} =& x_{-2} \prod_{s=0}^{n-1} \frac{(g_0 + h_0t_{-3}x_{-2}y_{-1}z_0) \left(\prod_{i=0}^{s-1} \theta^x_{4i+1} \right) + t_{-3}x_{-2}y_{-1}z_0 \sum_{l=0}^{s-1} \left(\lambda^x_{4l+1} \prod_{i=l+1}^{s-1} \theta^x_{4i+1} \right)}{((a_1h_0 + b_1)t_{-3}x_{-2}y_{-1}z_0 + a_1g_0) \left(\prod_{i=0}^{s-1} \theta^y_{4i+2} \right) + t_{-3}x_{-2}y_{-1}z_0 \sum_{l=0}^{s-1} \left(\lambda^y_{4l+2} \prod_{i=l+1}^{s-1} \theta^y_{4i+2} \right)} \\ x_{4n-1} =& x_{-1} \prod_{s=0}^{n-1} \frac{((f_0g_1 + h_1)t_{-2}x_{-1}y_0z_{-3} + e_0g_1) \left(\prod_{i=0}^{s-1} \theta^x_{4i+2} \right) + t_{-2}x_{-1}y_0z_{-3} \sum_{l=0}^{s-1} \left(\lambda^x_{4l+2} \prod_{i=l+1}^{s-1} \theta^x_{4i+2} \right)}{((a_0f_0g_1 + a_2h_1 + b_2)t_{-2}x_{-1}y_0z_{-3} + a_2e_0g_1) \left(\prod_{i=0}^{s-1} \theta^y_{4i+3} \right) + t_{-2}x_{-1}y_0z_{-3} \sum_{l=0}^{s-1} \left(\lambda^y_{4l+3} \prod_{i=l+1}^{s-1} \theta^y_{4i+3} \right)} \\ x_{4n} =& x_0 \prod_{s=0}^{n-1} \frac{((d_0e_1g_2 + f_1g_2 + h_2)t_{-1}x_0y_{-3}z_{-2} + c_0e_1g_2) \left(\prod_{i=0}^{s-1} \theta^x_{4i+3} \right) + t_{-1}x_0y_{-3}z_{-2} \sum_{l=0}^{s-1} \left(\lambda^x_{4l+3} \prod_{i=l+1}^{s-1} \theta^x_{4i+3} \right)}{\left(\prod_{i=0}^{s} \theta^y_{4i} \right) + t_{-1}x_0y_{-3}z_{-2} \sum_{l=0}^{s} \left(\lambda^y_{4l} \prod_{i=l+1}^{s} \theta^y_{4i} \right)}. \end{aligned}$$

Similar computations yield

$$y_{4n-3} = y_{-3}\prod_{s=0}^{n-1}\frac{\left(\prod_{i=0}^{s-1}\theta_{4i}^{y}\right)+t_{-1}x_0y_{-3}z_{-2}\sum_{l=0}^{s-1}\left(\lambda_{4l}^{y}\prod_{i=l+1}^{s-1}\theta_{4i}^{y}\right)}{(c_0+d_0t_{-1}x_0y_{-3}z_{-2})\left(\prod_{i=0}^{s-1}\theta_{4i+1}^{z}\right)+t_{-1}x_0y_{-3}z_{-2}\sum_{l=0}^{s-1}\left(\lambda_{4l+1}^{z}\prod_{i=l+1}^{s-1}\theta_{4i+1}^{z}\right)}$$

$$y_{4n-2} = y_{-2}\prod_{s=0}^{n-1}\frac{(a_0+b_0t_0x_{-3}y_{-2}z_{-1})\left(\prod_{i=0}^{s-1}\theta_{4i+1}^{y}\right)+t_0x_{-3}y_{-2}z_{-1}\sum_{l=0}^{s-1}\left(\lambda_{4l+1}^{y}\prod_{i=l+1}^{s-1}\theta_{4i+1}^{y}\right)}{((b_0c_1+d_1)t_0x_{-3}y_{-2}z_{-1}+a_0c_1)\left(\prod_{i=0}^{s-1}\theta_{4i+2}^{z}\right)+t_0x_{-3}y_{-2}z_{-1}\sum_{l=0}^{s-1}\left(\lambda_{4l+2}^{z}\prod_{i=l+1}^{s-1}\theta_{4i+2}^{z}\right)}$$

$$y_{4n-1} = y_{-1}\prod_{s=0}^{n-1}\frac{((a_1h_0+b_1)t_{-3}x_{-2}y_{-1}z_0+a_1g_0)\left(\prod_{i=0}^{s-1}\theta_{4i+2}^{y}\right)+t_{-3}x_{-2}y_{-1}z_0\sum_{l=0}^{s-1}\left(\lambda_{4l+2}^{y}\prod_{i=l+1}^{s-1}\theta_{4i+2}^{y}\right)}{((a_1c_2h_0+b_1c_2+d_2)t_{-3}x_{-2}y_{-1}z_0+a_1c_2g_0)\left(\prod_{i=0}^{s-1}\theta_{4i+3}^{z}\right)+t_{-3}x_{-2}y_{-1}z_0\sum_{l=0}^{s-1}\left(\lambda_{4l+3}^{z}\prod_{i=l+1}^{s-1}\theta_{4i+3}^{z}\right)}$$

$$y_{4n} = y_0\prod_{s=0}^{n-1}\frac{((a_2f_0g_1+a_2h_1+b_2)t_{-2}x_{-1}y_0z_{-3}+a_2e_0g_1)\left(\prod_{i=0}^{s-1}\theta_{4i+3}^{y}\right)+t_{-2}x_{-1}y_0z_{-3}\sum_{l=0}^{s-1}\left(\lambda_{4l+3}^{y}\prod_{i=l+1}^{s-1}\theta_{4i+3}^{y}\right)}{\left(\prod_{i=0}^{s}\theta_{4i}^{z}\right)+t_{-2}x_{-1}y_0z_{-3}\sum_{l=0}^{s}\left(\lambda_{4l}^{z}\prod_{i=l+1}^{s}\theta_{4i}^{z}\right)}$$

$$z_{4n-3} = z_{-3}\prod_{s=0}^{n-1}\frac{\left(\prod_{i=0}^{s-1}\theta_{4i}^{z}\right)+t_{-2}x_{-1}y_0z_{-3}\sum_{l=0}^{s-1}\left(\lambda_{4l}^{z}\prod_{i=l+1}^{s-1}\theta_{4i}^{z}\right)}{(e_0+f_0t_{-2}x_{-1}y_0z_{-3})\left(\prod_{i=0}^{s-1}\theta_{4i+1}^{t}\right)+t_{-2}x_{-1}y_0z_{-3}\sum_{l=0}^{s-1}\left(\lambda_{4l+1}^{t}\prod_{i=l+1}^{s-1}\theta_{4i+1}^{t}\right)}$$

$$z_{4n-2} = z_{-2}\prod_{s=0}^{n-1}\frac{(c_0+d_0t_{-1}x_0y_{-3}z_{-2})\left(\prod_{i=0}^{s-1}\theta_{4i+1}^{z}\right)+t_{-1}x_0y_{-3}z_{-2}\sum_{l=0}^{s-1}\left(\lambda_{4l+1}^{z}\prod_{i=l+1}^{s-1}\theta_{4i+1}^{z}\right)}{((d_0e_1+f_1)t_{-1}x_0y_{-3}z_{-2}+c_0e_1)\left(\prod_{i=0}^{s-1}\theta_{4i+2}^{t}\right)+t_{-1}x_0y_{-3}z_{-2}\sum_{l=0}^{s-1}\left(\lambda_{4l+2}^{t}\prod_{i=l+1}^{s-1}\theta_{4i+2}^{t}\right)}$$

$$z_{4n-1} = z_{-1}\prod_{s=0}^{n-1}\frac{((b_0c_1+d_1)t_0x_{-3}y_{-2}z_{-1}+a_0c_1)\left(\prod_{i=0}^{s-1}\theta_{4i+2}^{z}\right)+t_0x_{-3}y_{-2}z_{-1}\sum_{l=0}^{s-1}\left(\lambda_{4l+2}^{z}\prod_{i=l+1}^{s-1}\theta_{4i+2}^{z}\right)}{((b_0c_1e_2+d_1e_2+f_2)t_0x_{-3}y_{-2}z_{-1}+a_0c_1e_2)\left(\prod_{i=0}^{s-1}\theta_{4i+3}^{t}\right)+t_0x_{-3}y_{-2}z_{-1}\sum_{l=0}^{s-1}\left(\lambda_{4l+3}^{t}\prod_{i=l+1}^{s-1}\theta_{4i+3}^{t}\right)}$$

$$z_{4n} = z_0\prod_{s=0}^{n-1}\frac{((a_1c_2h_0+b_1c_2+d_2)t_{-3}x_{-2}y_{-1}z_0+a_1c_2g_0)\left(\prod_{i=0}^{s-1}\theta_{4i+3}^{z}\right)+t_{-3}x_{-2}y_{-1}z_0\sum_{l=0}^{s-1}\left(\lambda_{4l+3}^{z}\prod_{i=l+1}^{s-1}\theta_{4i+3}^{z}\right)}{\left(\prod_{i=0}^{s}\theta_{4i}^{t}\right)+t_{-3}x_{-2}y_{-1}z_0\sum_{l=0}^{s}\left(\lambda_{4l}^{t}\prod_{i=l+1}^{s}\theta_{4i}^{t}\right)}$$

$$t_{4n-3} = t_{-3}\prod_{s=0}^{n-1}\frac{\left(\prod_{i=0}^{s-1}\theta_{4i}^{t}\right)+t_{-3}x_{-2}y_{-1}z_0\sum_{l=0}^{s-1}\left(\lambda_{4l}^{t}\prod_{i=l+1}^{s-1}\theta_{4i}^{t}\right)}{(g_0+h_0t_{-3}x_{-2}y_{-1}z_0)\left(\prod_{i=0}^{s-1}\theta_{4i+1}^{x}\right)+t_{-3}x_{-2}y_{-1}z_0\sum_{l=0}^{s-1}\left(\lambda_{4l+1}^{x}\prod_{i=l+1}^{s-1}\theta_{4i+1}^{x}\right)}$$

6

$$t_{4n-2} = t_{-2}\prod_{s=0}^{n-1}\frac{(e_0+f_0t_{-2}x_{-1}y_0z_{-3})\left(\prod_{i=0}^{s-1}\theta^t_{4i+1}\right)+t_{-2}x_{-1}y_0z_{-3}\sum_{l=0}^{s-1}\left(\lambda^t_{4l+1}\prod_{i=l+1}^{s-1}\theta^t_{4i+1}\right)}{((f_0g_1+h_1)t_{-2}x_{-1}y_0z_{-3}+e_0g_1)\left(\prod_{i=0}^{s-1}\theta^x_{4i+2}\right)+t_{-2}x_{-1}y_0z_{-3}\sum_{l=0}^{s-1}\left(\lambda^x_{4l+2}\prod_{i=l+1}^{s-1}\theta^x_{4i+2}\right)}$$

$$t_{4n-1} = t_{-1}\prod_{s=0}^{n-1}\frac{((d_0e_1+f_1)t_{-1}x_0y_{-3}z_{-2}+c_0e_1)\left(\prod_{i=0}^{s-1}\theta^t_{4i+2}\right)+t_{-1}x_0y_{-3}z_{-2}\sum_{l=0}^{s-1}\left(\lambda^t_{4l+2}\prod_{i=l+1}^{s-1}\theta^t_{4i+2}\right)}{((d_0e_1g_2+f_1g_2+h_2)t_{-1}x_0y_{-3}z_{-2}+c_0e_1g_2)\left(\prod_{i=0}^{s-1}\theta^x_{4i+3}\right)+t_{-1}x_0y_{-3}z_{-2}\sum_{l=0}^{s-1}\left(\lambda^x_{4l+3}\prod_{i=l+1}^{s-1}\theta^x_{4i+3}\right)}$$

$$t_{4n} = t_0\prod_{s=0}^{n-1}\frac{((b_0c_1e_2+d_1e_2+f_2)t_0x_{-3}y_{-2}z_{-1}+a_0c_1e_2)\left(\prod_{i=0}^{s-1}\theta^t_{4i+3}\right)+t_0x_{-3}y_{-2}z_{-1}\sum_{l=0}^{s-1}\left(\lambda^t_{4l+3}\prod_{i=l+1}^{s-1}\theta^t_{4i+3}\right)}{\left(\prod_{i=0}^{s}\theta^x_{4i}\right)+t_0x_{-3}y_{-2}z_{-1}\sum_{l=0}^{s}\left(\lambda^x_{4l}\prod_{i=l+1}^{s}\theta^x_{4i}\right)}. \tag{21a}$$

Note that

$$\begin{aligned}
\lambda^x_n &= h_{n+3}+g_{n+3}f_{n+2}+g_{n+3}e_{n+2}d_{n+1}+g_{n+3}e_{n+2}c_{n+1}b_n, \theta^x_n = g_{n+3}e_{n+2}c_{n+1}a_n;\\
\lambda^y_n &= b_{n+3}+a_{n+3}h_{n+2}+a_{n+3}g_{n+2}f_{n+1}+a_{n+3}g_{n+2}e_{n+1}d_n, \theta^y_n = a_{n+3}g_{n+2}e_{n+1}c_n;\\
\lambda^z_n &= d_{n+3}+c_{n+3}b_{n+2}+c_{n+3}a_{n+2}h_{n+1}+c_{n+3}a_{n+2}g_{n+1}f_n, \theta^z_n = c_{n+3}a_{n+2}g_{n+1}e_n;\\
\lambda^t_n &= f_{n+3}+e_{n+3}d_{n+2}+e_{n+3}c_{n+2}b_{n+1}+e_{n+3}c_{n+2}a_{n+1}h_n, \theta^t_n = e_{n+3}c_{n+2}a_{n+1}g_n.
\end{aligned} \tag{21b}$$

2.3 Case where $a_n,\ b_n,\ c_n,\ d_n,\ e_n,\ f_n,\ g_n$ and h_n are periodic of period four

Suppose $\{a_n\}=\{a_0,a_1,a_2,a_3,a_0,\dots\}$, $\{b_n\}=\{b_0,b_1,b_2,b_3,b_0,\dots\}$, $\{c_n\}=\{c_0,c_1,c_2,c_3,c_0,\dots\}$, $\{d_n\}=\{d_0,d_1,d_2,d_3,d_0,\dots\}$, $\{e_n\}=\{e_0,e_1,e_2,e_3,e_0,\dots\}$, $\{f_n\}=\{f_0,f_1,f_2,f_3,f_0,\dots\}$ and $\{g_n\}=\{g_0,g_1,g_2,g_3,\ g_0,\dots\}$. Equations in (21) simplify to

$$x_{4n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{(\theta^x_0)^s+x_{-3}y_{-2}z_{-1}t_0(\lambda^x_0)\sum_{l=0}^{s-1}(\theta^x_0)^l}{(a_0+b_0x_{-3}y_{-2}z_{-1}t_0)\,(\theta^y_1)^s+x_{-3}y_{-2}z_{-1}t_0(\lambda^y_1)\sum_{l=0}^{s-1}(\theta^y_1)^l}$$

$$x_{4n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{(g_0+h_0t_{-3}x_{-2}y_{-1}z_0)(\theta^x_1)^s+t_{-3}x_{-2}y_{-1}z_0(\lambda^x_1)\sum_{l=0}^{s-1}(\theta^x_1)^l}{((a_1h_0+b_1)t_{-3}x_{-2}y_{-1}z_0+a_1g_0)\,[\theta^y_2]^s+t_{-3}x_{-2}y_{-1}z_0(\lambda^y_2)\sum_{l=0}^{s-1}(\theta^y_2)^l}$$

$$x_{4n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{((f_0g_1+h_1)t_{-2}x_{-1}y_0z_{-3}+e_0g_1)\,(\theta^x_2)^s+t_{-2}x_{-1}y_0z_{-3}\lambda^x_2\sum_{l=0}^{s-1}(\theta^x_2)^l}{((a_0f_0g_1+a_2h_1+b_2)t_{-2}x_{-1}y_0z_{-3}+a_2e_0g_1)\,(\theta^y_3)^s+t_{-2}x_{-1}y_0z_{-3}(\lambda^y_3)\sum_{l=0}^{s-1}(\theta^y_3)^l}$$

$$x_{4n} = x_0\prod_{s=0}^{n-1}\frac{((d_0e_1g_2+f_1g_2+h_2)t_{-1}x_0y_{-3}z_{-2}+c_0e_1g_2)\,(\theta^x_3)^s+t_{-1}x_0y_{-3}z_{-2}(\lambda^x_3)\sum_{l=0}^{s-1}(\theta^x_3)^l}{(\theta^y_0)^{s+1}+t_{-1}x_0y_{-3}z_{-2}(\lambda^y_0)\sum_{l=0}^{s}(\theta^y_0)^l}$$

$$y_{4n-3}=y_{-3}\prod_{s=0}^{n-1}\frac{(\theta_0^y)^s+t_{-1}x_0y_{-3}z_{-2}(\lambda_0^y)\sum_{l=0}^{s-1}(\theta_0^y)^l}{(c_0+d_0t_{-1}x_0y_{-3}z_{-2})(\theta_1^z)^s+t_{-1}x_0y_{-3}z_{-2}(\lambda_1^z)\sum_{l=0}^{s-1}(\theta_1^z)^l}$$

$$y_{4n-2}=y_{-2}\prod_{s=0}^{n-1}\frac{(a_0+b_0t_0x_{-3}y_{-2}z_{-1})(\theta_1^y)^s+t_0x_{-3}y_{-2}z_{-1}(\lambda_1^y)\sum_{l=0}^{s-1}(\theta_1^y)^l}{((b_0c_1+d_1)t_0x_{-3}y_{-2}z_{-1}+a_0c_1)(\theta_2^z)^s+t_0x_{-3}y_{-2}z_{-1}(\lambda_2^z)\sum_{l=0}^{s-1}(\theta_2^z)^l}$$

$$y_{4n-1}=y_{-1}\prod_{s=0}^{n-1}\frac{((a_1h_0+b_1)t_{-3}x_{-2}y_{-1}z_0+a_1g_0)(\theta_2^y)^s+t_{-3}x_{-2}y_{-1}z_0(\lambda_2^y)\sum_{l=0}^{s-1}(\theta_2^y)^l}{((a_1c_2h_0+b_1c_2+d_2)t_{-3}x_{-2}y_{-1}z_0+a_1c_2g_0)(\theta_3^z)^s+t_{-3}x_{-2}y_{-1}z_0(\lambda_3^z)\sum_{l=0}^{s-1}(\theta_3^z)^l}$$

$$y_{4n}=y_0\prod_{s=0}^{n-1}\frac{((a_2f_0g_1+a_2h_1+b_2)t_{-2}x_{-1}y_0z_{-3}+a_2e_0g_1)(\theta_3^y)^s+t_{-2}x_{-1}y_0z_{-3}(\lambda_3^y)\sum_{l=0}^{s-1}(\theta_3^y)^l}{(\theta_0^z)^{s+1}+t_{-2}x_{-1}y_0z_{-3}(\lambda_0^z)\sum_{l=0}^{s}(\theta_0^z)^l}$$

$$z_{4n-3}=z_{-3}\prod_{s=0}^{n-1}\frac{(\theta_0^z)^s+t_{-2}x_{-1}y_0z_{-3}(\lambda_0^z)\sum_{l=0}^{s-1}(\theta_0^z)^l}{(e_0+f_0t_{-2}x_{-1}y_0z_{-3})(\theta_1^t)^s+t_{-2}x_{-1}y_0z_{-3}(\lambda_1^t)\sum_{l=0}^{s-1}(\theta_1^t)^l}$$

$$z_{4n-2}=z_{-2}\prod_{s=0}^{n-1}\frac{(c_0+d_0t_{-1}x_0y_{-3}z_{-2})(\theta_1^z)^s+t_{-1}x_0y_{-3}z_{-2}(\lambda_1^z)\sum_{l=0}^{s-1}(\theta_1^z)^l}{((d_0e_1+f_1)t_{-1}x_0y_{-3}z_{-2}+c_0e_1)(\theta_2^t)^s+t_{-1}x_0y_{-3}z_{-2}(\lambda_2^t)\sum_{l=0}^{s-1}(\theta_2^t)^l}$$

$$z_{4n-1}=z_{-1}\prod_{s=0}^{n-1}\frac{((b_0c_1+d_1)t_0x_{-3}y_{-2}z_{-1}+a_0c_1)(\theta_2^z)^s+t_0x_{-3}y_{-2}z_{-1}(\lambda_2^z)\sum_{l=0}^{s-1}(\theta_2^z)^l}{((b_0c_1e_2+d_1e_2+f_2)t_0x_{-3}y_{-2}z_{-1}+a_0c_1e_2)(\theta_3^t)^s+t_0x_{-3}y_{-2}z_{-1}(\lambda_3^t)\sum_{l=0}^{s-1}(\theta_3^t)^l}$$

$$z_{4n}=z_0\prod_{s=0}^{n-1}\frac{((a_1c_2h_0+b_1c_2+d_2)t_{-3}x_{-2}y_{-1}z_0+a_1c_2g_0)(\theta_3^z)^s+t_{-3}x_{-2}y_{-1}z_0(\lambda_3^z)\sum_{l=0}^{s-1}(\theta_3^z)^l}{(\theta_0^t)^{s+1}+t_{-3}x_{-2}y_{-1}z_0(\lambda_0^t)\sum_{l=0}^{s}(\theta_0^t)^l}$$

$$t_{4n-3}=t_{-3}\prod_{s=0}^{n-1}\frac{(\theta_0^t)^s+t_{-3}x_{-2}y_{-1}z_0(\lambda_0^t)\sum_{l=0}^{s-1}(\theta_0^t)^l}{(g_0+h_0t_{-3}x_{-2}y_{-1}z_0)(\theta_1^x)^s+t_{-3}x_{-2}y_{-1}z_0(\lambda_1^x)\sum_{l=0}^{s-1}(\theta_1^x)^l}$$

$$t_{4n-2}=t_{-2}\prod_{s=0}^{n-1}\frac{(e_0+f_0t_{-2}x_{-1}y_0z_{-3})(\theta_1^t)^s+t_{-2}x_{-1}y_0z_{-3}(\lambda_1^t)\sum_{l=0}^{s-1}(\theta_1^t)^l}{((f_0g_1+h_1)t_{-2}x_{-1}y_0z_{-3}+e_0g_1)(\theta_2^x)^s+t_{-2}x_{-1}y_0z_{-3}(\lambda_2^x)\sum_{l=0}^{s-1}(\theta_2^x)^l}$$

$$t_{4n-1}=t_{-1}\prod_{s=0}^{n-1}\frac{((d_0e_1+f_1)t_{-1}x_0y_{-3}z_{-2}+c_0e_1)(\theta_2^t)^s+t_{-1}x_0y_{-3}z_{-2}(\lambda_2^t)\sum_{l=0}^{s-1}(\theta_2^t)^l}{((d_0e_1g_2+f_1g_2+h_2)t_{-1}x_0y_{-3}z_{-2}+c_0e_1g_2)(\theta_3^x)^s+t_{-1}x_0y_{-3}z_{-2}(\lambda_3^x)\sum_{l=0}^{s-1}(\theta_3^x)^l}$$

$$t_{4n} = t_0 \prod_{s=0}^{n-1} \frac{((b_0c_1e_2 + d_1e_2 + f_2)t_0x_{-3}y_{-2}z_{-1} + a_0c_1e_2)\left(\theta_3^t\right)^s + t_0x_{-3}y_{-2}z_{-1}(\lambda_3^t)\sum_{l=0}^{s-1}\left(\theta_3^t\right)^l}{\left(\theta_0^x\right)^{s+1} + t_0x_{-3}y_{-2}z_{-1}(\lambda_0^x)\sum_{l=0}^{s}\left(\theta_0^x\right)^l}, \tag{22}$$

where $\theta_0^u,\ \lambda_0^u,\ u = x, y, z, t$ are defined in (21b).

2.4 Case where $a_n,\ b_n,\ c_n,\ d_n,\ e_n,\ f_n,\ g_n$ and h_n are constant

Suppose that $a_n = a,\ b_n = b, c_n = c, d_n = d,\ e_n = e,\ f_n = f$ and $g_n = g$. Equations in (22) simplify to

$$x_{4n-3} = x_{-3} \prod_{s=0}^{n-1} \left[\frac{(aceg)^s + x_{-3}y_{-2}z_{-1}t_0(h + gf + ged + gecb)\sum_{l=0}^{s-1}(aceg)^l}{(a + bx_{-3}y_{-2}z_{-1}t_0)(aceg)^s + x_{-3}y_{-2}z_{-1}t_0(b + ah + agf + aged)\sum_{l=0}^{s-1}(aceg)^l}\right]$$

$$x_{4n-2} = x_{-2} \prod_{s=0}^{n-1} \frac{(g + htx_{-2}y_{-1}z_0)(aceg)^s + t_{-3}x_{-2}y_{-1}z_0(h + gf + ged + gecb)\sum_{l=0}^{s-1}(aceg)^l}{((ah + b)t_{-3}x_{-2}y_{-1}z_0 + ag)(aceg)^s + t_{-3}x_{-2}y_{-1}z_0(b + ah + agf + aged)\sum_{l=0}^{s-1}(aceg)^l}$$

$$x_{4n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{((fg + h)t_{-2}x_{-1}y_0z_{-3} + eg)(aceg)^s + t_{-2}x_{-1}y_0z_{-3}(h + gf + ged + gecb)\sum_{l=0}^{s-1}(aceg)^l}{((afg + a_2h + b)t_{-2}x_{-1}y_0z_{-3} + aeg)(aceg)^s + t_{-2}x_{-1}y_0z_{-3}(b + ah + agf + aged)\sum_{l=0}^{s-1}(aceg)^l}$$

$$x_{4n} = x_0 \prod_{s=0}^{n-1} \frac{((deg + fg + h)t_{-1}x_0y_{-3}z_{-2} + ceg)(aceg)^s + t_{-1}x_0y_{-3}z_{-2}(h + gf + ged + gecb)\sum_{l=0}^{s-1}(aceg)^l}{(aceg)^{s+1} + t_{-1}x_0y_{-3}z_{-2}(b + ah + agf + aged)\sum_{l=0}^{s}(aceg)^l}$$

$$y_{4n-3} = y_{-3} \prod_{s=0}^{n-1} \frac{(aceg)^s + t_{-1}x_0y_{-3}z_{-2}(b + ah + agf + aged)\sum_{l=0}^{s-1}(aceg)^l}{(c + dt_{-1}x_0y_{-3}z_{-2})(aceg)^s + t_{-1}x_0y_{-3}z_{-2}(d + cb + cah + cagf)\sum_{l=0}^{s-1}(aceg)^l}$$

$$y_{4n-2} = y_{-2} \prod_{s=0}^{n-1} \frac{(a + bt_0x_{-3}y_{-2}z_{-1})(aceg)^s + t_0x_{-3}y_{-2}z_{-1}(b + ah + agf + aged)\sum_{l=0}^{s-1}(aceg)^l}{((bc + d)t_0x_{-3}y_{-2}z_{-1} + ac)(aceg)^s + t_0x_{-3}y_{-2}z_{-1}(d + cb + cah + cagf)\sum_{l=0}^{s-1}(aceg)^l}$$

$$y_{4n-1} = y_{-1} \prod_{s=0}^{n-1} \frac{((ah + b)t_{-3}x_{-2}y_{-1}z_0 + ag)(aceg)^s + t_{-3}x_{-2}y_{-1}z_0(b + ah + agf + aged)\sum_{l=0}^{s-1}(aceg)^l}{((ach + bc + d)t_{-3}x_{-2}y_{-1}z_0 + acg)(aceg)^s + t_{-3}x_{-2}y_{-1}z_0(d + cb + cah + cagf)\sum_{l=0}^{s-1}(aceg)^l}$$

$$y_{4n} = y_0 \prod_{s=0}^{n-1} \frac{((afg + ah + b)t_{-2}x_{-1}y_0z_{-3} + aeg)(aceg)^s + t_{-2}x_{-1}y_0z_{-3}(b + ah + agf + aged)\sum_{l=0}^{s-1}(aceg)^l}{(aceg)^{s+1} + t_{-2}x_{-1}y_0z_{-3}(d + cb + cah + cagf)\sum_{l=0}^{s}(aceg)^l}$$

$$z_{4n-3} = z_{-3} \prod_{s=0}^{n-1} \frac{(aceg)^s + t_{-2}x_{-1}y_0z_{-3}(d + cb + cah + cagf)\sum_{l=0}^{s-1}(aceg)^l}{(e + ft_{-2}x_{-1}y_0z_{-3})(aceg)^s + t_{-2}x_{-1}y_0z_{-3}(f + ed + ecb + ecah)\sum_{l=0}^{s-1}(aceg)^l}$$

9

$$
\begin{aligned}
z_{4n-2} =& z_{-2}\prod_{s=0}^{n-1}\frac{\left(c+dt_{-1}x_0y_{-3}z_{-2}\right)(aceg)^s+t_{-1}x_0y_{-3}z_{-2}(d+cb+cah+cagf)\sum\limits_{l=0}^{s-1}(aceg)^l}{\left((de+f)t_{-1}x_0y_{-3}z_{-2}+ce\right)(aceg)^s+t_{-1}x_0y_{-3}z_{-2}(f+ed+ecb+ecah)\sum\limits_{l=0}^{s-1}(aceg)^l}\\
z_{4n-1} =& z_{-1}\prod_{s=0}^{n-1}\frac{\left((bc+d)t_0x_{-3}y_{-2}z_{-1}+ac\right)(aceg)^s+t_0x_{-3}y_{-2}z_{-1}(d+cb+cah+cagf)\sum\limits_{l=0}^{s-1}(aceg)^l}{\left((bce+de+f)t_0x_{-3}y_{-2}z_{-1}+ace\right)(aceg)^s+t_0x_{-3}y_{-2}z_{-1}(f+ed+ecb+ecah)\sum\limits_{l=0}^{s-1}(aceg)^l}\\
z_{4n} =& z_0\prod_{s=0}^{n-1}\frac{\left((ach+bc+d)t_{-3}x_{-2}y_{-1}z_0+acg\right)(aceg)^s+t_{-3}x_{-2}y_{-1}z_0(d+cb+cah+cagf)\sum\limits_{l=0}^{s-1}(aceg)^l}{(aceg)^{s+1}+t_{-3}x_{-2}y_{-1}z_0(f+ed+ecb+ecah)\sum\limits_{l=0}^{s}(aceg)^l}\\
t_{4n-3} =& t_{-3}\prod_{s=0}^{n-1}\frac{(aceg)^s+t_{-3}x_{-2}y_{-1}z_0(f+ed+ecb+ecah)\sum\limits_{l=0}^{s-1}(aceg)^l}{\left(g+ht_{-3}x_{-2}y_{-1}z_0\right)(aceg)^s+t_{-3}x_{-2}y_{-1}z_0(h+gf+ged+gecb)\sum\limits_{l=0}^{s-1}(aceg)^l}\\
t_{4n-2} =& t_{-2}\prod_{s=0}^{n-1}\frac{\left(e+ft_{-2}x_{-1}y_0z_{-3}\right)(aceg)^s+t_{-2}x_{-1}y_0z_{-3}(f+ed+ecb+ecah)\sum\limits_{l=0}^{s-1}(aceg)^l}{\left((fg+h)t_{-2}x_{-1}y_0z_{-3}+eg\right)(aceg)^s+t_{-2}x_{-1}y_0z_{-3}(h+gf+ged+gecb)\sum\limits_{l=0}^{s-1}(aceg)^l}\\
t_{4n-1} =& t_{-1}\prod_{s=0}^{n-1}\frac{\left((de+f)t_{-1}x_0y_{-3}z_{-2}+ce\right)(aceg)^s+t_{-1}x_0y_{-3}z_{-2}(f+ed+ecb+ecah)\sum\limits_{l=0}^{s-1}(aceg)^l}{\left((deg+fg+h)t_{-1}x_0y_{-3}z_{-2}+ceg\right)(aceg)^s+t_{-1}x_0y_{-3}z_{-2}(h+gf+ged+gecb)\sum\limits_{l=0}^{s-1}(aceg)^l}\\
t_{4n} =& t_0\prod_{s=0}^{n-1}\frac{\left((bce+de+f)t_0x_{-3}y_{-2}z_{-1}+ace\right)(aceg)^s+t_0x_{-3}y_{-2}z_{-1}(f+ed+ecb+ecah)\sum\limits_{l=0}^{s-1}(aceg)^l}{(aceg)^{s+1}+t_0x_{-3}y_{-2}z_{-1}(h+gf+ged+gecb)\sum\limits_{l=0}^{s}(aceg)^l}.
\end{aligned}
\tag{23}
$$

2.4.1 Case where $a=1,\ b=1,\ c=1,\ d=1,\ e=1,\ f=1,\ g=1$ and $h=1$

Here, $\theta^x=\theta^y=\theta^z=\theta^t=1$ and $\lambda^x=\lambda^y=\lambda^z=\lambda^t=4$. Thus, equations in (23) simplify to

$$
\begin{aligned}
x_{4n-3} =& x_{-3}\prod_{s=0}^{n-1}\left[\frac{1+4sx_{-3}y_{-2}z_{-1}t_0}{1+(4s+1)x_{-3}y_{-2}z_{-1}t_0}\right], x_{4n-2}=x_{-2}\prod_{s=0}^{n-1}\left[\frac{1+(4s+1)t_{-3}x_{-2}y_{-1}z_0}{1+(4s+2)t_{-3}x_{-2}y_{-1}z_0}\right],\\
x_{4n-1} =& x_{-1}\prod_{s=0}^{n-1}\left[\frac{1+(4s+2)t_{-2}x_{-1}y_0z_{-3}}{1+(4s+3)t_{-2}x_{-1}y_0z_{-3}}\right], x_{4n}=x_0\prod_{s=0}^{n-1}\left[\frac{1+(4s+3)t_{-1}x_0y_{-3}z_{-2}}{1+(4s+4)t_{-1}x_0y_{-3}z_{-2}}\right],\\
y_{4n-3} =& y_{-3}\prod_{s=0}^{n-1}\left[\frac{1+4st_{-1}x_0y_{-3}z_{-2}}{1+(4s+1)t_{-1}x_0y_{-3}z_{-2}}\right], y_{4n-2}=y_{-2}\prod_{s=0}^{n-1}\left[\frac{1++(4s+1)t_0x_{-3}y_{-2}z_{-1}}{1+(4s+2)t_0x_{-3}y_{-2}z_{-1}}\right],\\
y_{4n-1} =& y_{-1}\prod_{s=0}^{n-1}\left[\frac{1+(4s+2)t_{-3}x_{-2}y_{-1}z_0}{1+(4s+3)t_{-3}x_{-2}y_{-1}z_0}\right], y_{4n}=y_0\prod_{s=0}^{n-1}\left[\frac{1+(4s+3)t_{-2}x_{-1}y_0z_{-3}}{1+(4s+4)t_{-2}x_{-1}y_0z_{-3}}\right],
\end{aligned}
$$

10

$$
\begin{aligned}
z_{4n-3} =& z_{-3}\prod_{s=0}^{n-1}\left[\frac{1+4st_{-2}x_{-1}y_0z_{-3}}{1+(4s+1)t_{-2}x_{-1}y_0z_{-3}}\right], z_{4n-2}=z_{-2}\prod_{s=0}^{n-1}\left[\frac{1+(4s+1)t_{-1}x_0y_{-3}z_{-2}}{1+(4s+2)t_{-1}x_0y_{-3}z_{-2}}\right],\\
z_{4n-1} =& z_{-1}\prod_{s=0}^{n-1}\left[\frac{1+(4s+2)t_0x_{-3}y_{-2}z_{-1}}{1+(4s+3)t_0x_{-3}y_{-2}z_{-1}}\right], z_{4n}=z_0\prod_{s=0}^{n-1}\left[\frac{1+(4s+3)t_{-3}x_{-2}y_{-1}z_0}{1+(4s+4)t_{-3}x_{-2}y_{-1}z_0}\right],\\
t_{4n-3} =& t_{-3}\prod_{s=0}^{n-1}\left[\frac{1+4st_{-3}x_{-2}y_{-1}z_0}{1+(4s+1)t_{-3}x_{-2}y_{-1}z_0}\right], t_{4n-2}=t_{-2}\prod_{s=0}^{n-1}\left[\frac{1+(4s+1)t_{-2}x_{-1}y_0z_{-3}}{1+(4s+2)t_{-2}x_{-1}y_0z_{-3}},\right]\\
t_{4n-1} =& t_{-1}\prod_{s=0}^{n-1}\left[\frac{1+(4s+2)t_{-1}x_0y_{-3}z_{-2}}{1+(4s+3)t_{-1}x_0y_{-3}z_{-2}}\right], t_{4n}=t_0\prod_{s=0}^{n-1}\left[\frac{1+(4s+3)t_0x_{-3}y_{-2}z_{-1}}{1+(4s+4)t_0x_{-3}y_{-2}z_{-1}}\right].
\end{aligned}
\tag{24}
$$

2.5 Case where $a = c = h = -1$ and $b = d = e = f = g = 1$

Here, $\theta^x = \theta^y = \theta^z = \theta^t = 1$ and $\lambda^x = \lambda^y = \lambda^z = \lambda^t = 0$. Thus, equations in (23) simplify to Theorem 2.2 in [1].

2.6 Case where $a = c = e = g = -1$ and $b = d = f = h = 1$

Here, $\theta^x = \theta^y = \theta^z = \theta^t = 1$ and $\lambda^x = \lambda^y = \lambda^z = \lambda^t = 0$. Thus, equations in (23) simplify to Theorem 2.3 in [1].

2.7 Case where $a = b = c = d = e = f = g = 1$ and $h = -1$

Here, $\theta^x = \theta^y = \theta^z = \theta^t = 1$ and $\lambda^x = \lambda^y = \lambda^z = \lambda^t = 0$. Thus, equations in (23) simplify to Theorem 3.1 in [1].

3 Existence of four periodic solutions

If

$$x_{-3}y_{-2}z_{-1}t_0 = x_{-2}y_{-1}z_0t_{-3} = x_{-1}y_0z_{-3}t_{-2} = x_0y_{-3}z_{-2}t_{-1} = \frac{1-a}{b} = \frac{1-c}{d} = \frac{1-e}{f} = \frac{1-g}{h},$$

then

$$\theta^x = \theta^y = \theta^z = \theta^t = geca$$

and

$$\lambda^x = \lambda^y = \lambda^z = \lambda^t = \frac{b}{1-a}(1-geca).$$

Thus, equations in (23) simplify to

$$
\begin{aligned}
&x_{4n-3} = x_{-3}, x_{4n-2} = x_{-2}, x_{4n-1} = x_{-1}, x_{4n} = x_0,\\
&y_{4n-3} = y_{-3}, y_{4n-2} = y_{-2}, y_{4n-1} = y_{-1},\ y_{4n} = y_0,\\
&z_{4n-3} = z_{-3}, z_{4n-2} = z_{-2}, z_{4n-1} = z_{-1}, z_{4n} = z_0,\\
&t_{4n-3} = t_{-3}, t_{4n-2} = t_{-2}, t_{4n-1} = t_{-1}, t_{4n} = t_0
\end{aligned}
$$

and therefore all solutions of (8) are periodic with period four.
Below are the figures of some numerical examples that illustrate two cases of systems where solutions are periodic with period four.

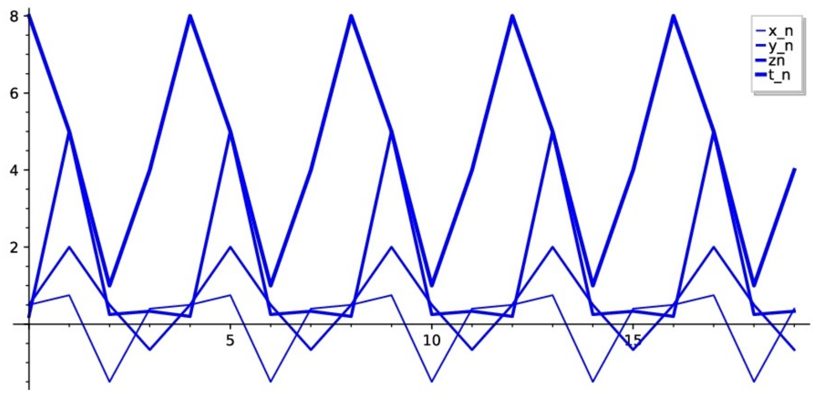


Figure 1: Periodic solutions of (8) when $a = 2$, $b = -1$, $c = 3$, $d = -2$, $e = 4$, $f = -3$, $g = 5$, $h = -4$ with initial conditions $x_0 = 0.5$, $x_1 = 0.75$, $x_2 = -3/2$, $x_3 = 0.4$, $y_0 = 0.5$, $y_1 = 2$, $y_2 = 0.5$, $y_3 = -2/3$, $z_0 = 1/5$, $z_1 = 5$, $z_2 = 0.25$, $z_3 = 1/3$, $t_0 = 8$, $t_1 = 5$, $t_2 = 1$, $t_3 = 4$.

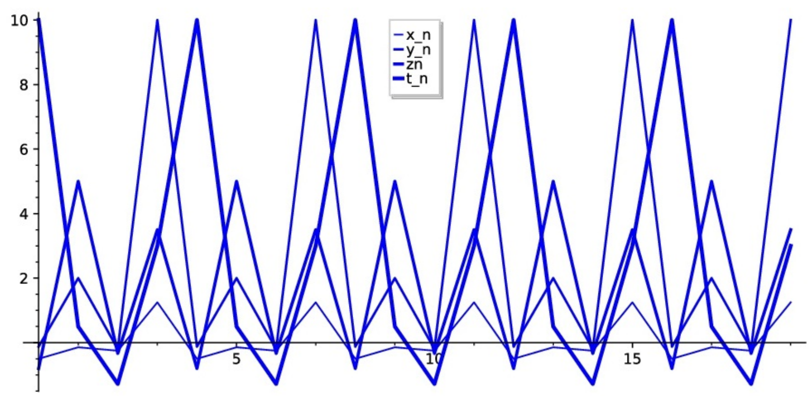


Figure 2: Periodic solutions of (8) when $a = 0.5$, $b = 0.5$, $c = 0.75$, $d = 0.25$, $e = 6$, $f = -5$, $g = -1$, $h = 2$ with initial conditions $x_0 = -0.5$, $x_1 = -1/7$, $x_2 = -1/4$, $x_3 = 1.25$, $y_0 = -0.125$, $y_1 = 2$, $y_2 = -1/5$, $y_3 = 10$, $z_0 = -0.8$, $z_1 = 5$, $z_2 = -1/3$, $z_3 = 3.5$, $t_0 = 10$, $t_1 = 0.5$, $t_2 = -1.28$, $t_3 = 3$.

References

[1] M. M. El-Dessoky and A. Hobiny, On the existence and behavior of the solutions for some difference equations systems, *J. Computational Analysis and Applications* **26:8** (2019), 1428–1439.

[2] G. Bluman and S. Anco, Symmetry and Integration Methods for Differential Equations, Springer, New York (2002).

[3] M. M. El-Dessoky, Solution for Rational Systems of Difference Equations of Order Three, *Mathematics* **4:53** (2016).

[4] E. M. Elsayed, Solutions of rational difference systems of order two, *Mathematical and Computer Modelling*, **55** (2012), 378–384.

[5] E. M. Elsayed and T.F. Ibrahim, Periodicity and solutions for some systems of nonlinear rational difference equations, *Hacettepe Journal of Mathematics and Statistics*, **44:6** (2015), 1361–1390.

[6] M. Folly-Gbetoula, Symmetry, reductions and exact solutions of the difference equation $u_{n+2} = au_n/(1+bu_nu_{n+1})$, *J.Diff. Eq. and Appl.*, **23:6** (2017), 1017–1024.

[7] M. Folly-Gbetoula and A.H. Kara, Symmetries, conservation laws, and 'integrability' of difference equations, *Advances in Difference Equations*, **2014:224** (2014).

[8] M. Folly-Gbetoula and D. Nyirenda, On some sixth-order rational recursive sequences, *Journal of computational analysis and applications*, **27:6** (2019), 1057–1069.

[9] N. Joshi and P. Vassiliou, The existence of Lie Symmetries for First-Order Analytic Discrete Dynamical Systems, *Journal of Mathematical Analysis and Applications*, **195** (1995), 872–887.

[10] P. E. Hydon, Difference Equations by Differential Equation Methods, *Cambridge University Press*, (2014).

[11] P. E. Hydon, Symmetries and first integrals of ordinary difference equations, Proc. Roy. Soc. Lond. A, **456** (2000), 2835–2855.

[12] N. Mnguni and M. Folly-Gbetoula, Invariance analysis of a third-order difference equation with variable coefficients, *Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms* **25** (2018), 63–73.

[13] D. Nyirenda and M. Folly-Gbetoula, Invariance analysis and exact solutions of some sixth-order difference equations, *J. Nonlinear Sci. Appl.* **10** (2017), 6262–6273.

[14] P. J. Olver, Applications of Lie Groups to Differential Equations, Second Edition, Springer, New York, (1993).

[15] G. R. W. Quispel and R. Sahadevan, Lie symmetries and the integration of difference equations, Physics Letters A, **184** (1993), 64-70.

[16] S. Maeda, The similarity method for difference equations, IMA *J. Appl. Math.* **38** (1987), 129–134.

[17] S. Stevic, On a system of difference equation, *Applied Mathematics and Computation*, **218** (2011), 3372–3378.

Dynamics of an anti-competitive system of difference equations

J. Ma* A. Q. Khan†

Abstract

In this paper, we study the dynamical properties of an anti-competitive system of second-order rational difference equations. The proposed work is considerably extended and improve some exiting results in the literature.

Keywords and phrases: difference equations; boundedness and persistence; asymptotic behavior
2010 AMS: 39A10, 40A05

1 Introduction

In [1], Hamza *et al*. have investigated the global behavior of the difference equation: $x_{n+1} = \frac{Ax_{n-1}}{B+Cx_n^2}$, $n = 0, 1, \cdots$, where A, B, C and initial conditions x_0, x_{-1} are positive real numbers. Motivated by the above studies, our aim in this paper is to investigate the dynamical properties of the following anti-competitive system of second-order rational difference equations:

$$x_{n+1} = \frac{\alpha + \beta y_{n-1}}{\gamma + \delta x_n^2},\ y_{n+1} = \frac{\alpha_1 + \beta_1 x_{n-1}}{\gamma_1 + \delta_1 y_n^2},\ n = 0, 1, \cdots, \tag{1}$$

where $\alpha,\ \beta,\ \gamma,\ \delta,\ \alpha_1,\ \beta_1,\ \gamma_1,\ \delta_1$ and the initial conditions $x_0,\ x_{-1},\ y_0,\ y_{-1}$ are positive real numbers. The rest of the paper is dedicated to investigate the boundedness and persistence, existence of unbounded solutions, existence and uniqueness of positive equilibrium point, local and global stability about the unique positive equilibrium point of the system (1).

2 Main results

2.1 Boundedness and persistence

Theorem 1. *If $\beta\beta_1 < \gamma\gamma_1$ then every solution $\{(x_n, y_n)/x_n, y_n > 0\}$ of the system (1) is bounded and persists.*

Proof. If $\{(x_n, y_n)/x_n, y_n > 0\}$ is a solution of the system (1) then

$$x_{n+1} \leq \frac{\alpha}{\gamma} + \frac{\beta}{\gamma} y_{n-1},\ y_{n+1} \leq \frac{\alpha_1}{\gamma_1} + \frac{\beta_1}{\gamma_1} x_{n-1},\ n = 0, 1, \cdots. \tag{2}$$

From (2), one get

$$x_{n+1} \leq \frac{\alpha}{\gamma} + \frac{\alpha_1\beta}{\gamma\gamma_1} + \frac{\beta\beta_1}{\gamma\gamma_1} x_{n-3},\ y_{n+1} \leq \frac{\alpha_1}{\gamma_1} + \frac{\alpha\beta_1}{\gamma\gamma_1} + \frac{\beta\beta_1}{\gamma\gamma_1} y_{n-3},\ n = 0, 1, \cdots. \tag{3}$$

Consider

$$\Phi_{n+1} = \frac{\alpha}{\gamma} + \frac{\alpha_1\beta}{\gamma\gamma_1} + \frac{\beta\beta_1}{\gamma\gamma_1} \Phi_{n-3},\ \xi_{n+1} = \frac{\alpha_1}{\gamma_1} + \frac{\alpha\beta_1}{\gamma\gamma_1} + \frac{\beta\beta_1}{\gamma\gamma_1} \xi_{n-3},\ n = 0, 1, \cdots. \tag{4}$$

The solution $\{(\Phi_n, \xi_n)\}$ of (4) is

$$\begin{aligned} \Phi_n &= r_1 \left(\sqrt[4]{\frac{\beta\beta_1}{\gamma\gamma_1}}\right)^n + r_2 \left(-\sqrt[4]{\frac{\beta\beta_1}{\gamma\gamma_1}}\right)^n + r_3 \left(\iota\sqrt[4]{\frac{\beta\beta_1}{\gamma\gamma_1}}\right)^n + r_4 \left(-\iota\sqrt[4]{\frac{\beta\beta_1}{\gamma\gamma_1}}\right)^n + \frac{\alpha\gamma_1 + \beta\alpha_1}{\gamma\gamma_1 - \beta\beta_1}, \\ \xi_n &= s_1 \left(\sqrt[4]{\frac{\beta\beta_1}{\gamma\gamma_1}}\right)^n + s_2 \left(-\sqrt[4]{\frac{\beta\beta_1}{\gamma\gamma_1}}\right)^n + s_3 \left(\iota\sqrt[4]{\frac{\beta\beta_1}{\gamma\gamma_1}}\right)^n + s_4 \left(-\iota\sqrt[4]{\frac{\beta\beta_1}{\gamma\gamma_1}}\right)^n + \frac{\alpha_1\gamma + \alpha\beta_1}{\gamma\gamma_1 - \beta\beta_1}, \end{aligned} \tag{5}$$

*College of Science, University of Shanghai for Science and Technology, Shanghai, 200093, P. R. China, e-mail: majiying100@126.com
†Department of Mathematics, University of Azad Jammu & Kashmir, Muzaffarabad 13100, Pakistan, e-mail: abdulqadeerkhan1@gmail.com

where $r_1,\ r_2,\ r_3,\ r_4,\ s_1,\ s_2,\ s_3,\ s_4$ depend upon the initial values $\Phi_{-3},\ \Phi_{-2},\ \Phi_{-1},\ \Phi_0,\ \xi_{-3},\ \xi_{-2},\ \xi_{-1},\ \xi_0$. Assuming that $\beta\beta_1 < \gamma\gamma_1$ then (5) implies that Φ_n and ξ_n are bounded. Now consider the solution $\{(\Phi_n, \xi_n)\}$ of (5) such that

$$\begin{aligned} &\Phi_{-3} = x_{-3},\ \Phi_{-2} = x_{-2},\ \Phi_{-1} = x_{-1},\ \Phi_0 = x_0, \\ &\xi_{-3} = y_{-3},\ \xi_{-2} = y_{-2},\ \xi_{-1} = y_{-1},\ \xi_0 = y_0. \end{aligned} \tag{6}$$

From (3), (5) and (6) one get

$$x_n \le \frac{\alpha\gamma_1 + \beta\alpha_1}{\gamma\gamma_1 - \beta\beta_1} + \epsilon = U_1 + \epsilon,\ y_n \le \frac{\alpha_1\gamma + \alpha\beta_1}{\gamma\gamma_1 - \beta\beta_1} + \epsilon = U_2 + \epsilon, \tag{7}$$

where for large n, ϵ is a sufficiently small number. In addition from (1) and (7), we get

$$x_n \ge \frac{\alpha}{\gamma + \delta x_n^2} \ge \frac{\alpha(\gamma\gamma_1 - \beta\beta_1)^2}{\gamma(\gamma\gamma_1 - \beta\beta_1)^2 + \delta(\alpha\gamma_1 + \beta\alpha_1)^2} = L_1. \tag{8}$$

$$y_n \ge \frac{\alpha_1}{\gamma_1 + \delta_1 y_n^2} \ge \frac{\alpha_1(\gamma\gamma_1 - \beta\beta_1)^2}{\gamma_1(\gamma\gamma_1 - \beta\beta_1)^2 + \delta_1(\alpha_1\gamma + \beta_1\alpha)^2} = L_2. \tag{9}$$

Finally, from (7), (8) and (9) one get

$$L_1 \le x_n \le U_1,\ L_2 \le y_n \le U_2,\ n = 0, 1, \cdots. \tag{10}$$

□

2.2 Existence of unbounded solution

Theorem 2. *For solution $\{(x_n, y_n)/x_n, y_n > 0\}$ of the system (1), the following statements hold:*

(i) If $\beta\beta_1 > (\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)$ then $x_n \to \infty$ as $n \to \infty$.

(ii) If $\beta\beta_1 > (\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)$ then $y_n \to \infty$ as $n \to \infty$.

Proof. (i) If $\{(x_n, y_n)/x_n, y_n > 0\}$ is a solution of the system (1) then

$$x_{n+1} = \frac{\alpha + \beta y_{n-1}}{\gamma + \delta x_n^2} \ge \frac{\alpha + \beta y_{n-1}}{\gamma + \delta U_1^2} = \frac{\alpha}{\gamma + \delta U_1^2} + \frac{\beta}{\gamma + \delta U_1^2} y_{n-1}. \tag{11}$$

$$y_{n+1} = \frac{\alpha_1 + \beta_1 x_{n-1}}{\gamma_1 + \delta_1 y_n^2} \ge \frac{\alpha_1 + \beta_1 x_{n-1}}{\gamma_1 + \delta_1 U_2^2} = \frac{\alpha_1}{\gamma_1 + \delta_1 U_2^2} + \frac{\beta_1}{\gamma_1 + \delta_1 U_2^2} x_{n-1}. \tag{12}$$

From (12)

$$y_{n-1} \ge \frac{\alpha_1}{\gamma_1 + \delta_1 U_2^2} + \frac{\beta_1}{\gamma_1 + \delta_1 U_2^2} x_{n-3}. \tag{13}$$

Using (13) in (11), one get

$$x_{n+1} \ge \frac{\alpha}{\gamma + \delta U_1^2} + \frac{\beta\alpha_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)} + \frac{\beta\beta_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)} x_{n-3}. \tag{14}$$

Consider

$$\tau_{n+1} = \frac{\alpha}{\gamma + \delta U_1^2} + \frac{\beta\alpha_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)} + \frac{\beta\beta_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)} \tau_{n-3}. \tag{15}$$

The solution of (15) is

$$\begin{aligned} \tau_n \ = \ & c_1 \left(\sqrt[4]{\frac{\beta\beta_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)}} \right)^n + c_2 \left(-\sqrt[4]{\frac{\beta\beta_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)}} \right)^n + c_3 \left(\iota \sqrt[4]{\frac{\beta\beta_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)}} \right)^n + \\ & c_4 \left(-\iota \sqrt[4]{\frac{\beta\beta_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2)}} \right)^n + \frac{\alpha(\gamma_1 + \delta_1 U_2^2) + \beta\alpha_1}{(\gamma + \delta U_1^2)(\gamma_1 + \delta_1 U_2^2) - \beta\beta_1}, \end{aligned}$$

where $c_1,\ c_2,\ c_3,\ c_4$ depends on $\tau_{-3},\ \tau_{-2},\ \tau_{-1},\ \tau_0$. Now if $\beta\beta_1 > (\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)$ then $\{\tau_n\}$ is divergent. Hence by comparison $x_n \to \infty$ as $n \to \infty$.

(ii) Similarly from (11), we have

$$x_{n-1} \geq \frac{\alpha}{\gamma+\delta U_1^2} + \frac{\beta}{\gamma+\delta U_1^2} y_{n-3}. \tag{16}$$

Using (16) in (12), we get

$$y_{n+1} \geq \frac{\alpha_1}{\gamma_1+\delta_1 U_2^2} + \frac{\beta_1\alpha}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)} + \frac{\beta\beta_1}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)} y_{n-3}. \tag{17}$$

Consider

$$\mu_{n+1} = \frac{\alpha_1}{\gamma_1+\delta_1 U_2^2} + \frac{\beta_1\alpha}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)} + \frac{\beta\beta_1}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)} \mu_{n-3}. \tag{18}$$

The solution of (18) is given by

$$\begin{aligned}\mu_n &= c_5\left(\sqrt[4]{\frac{\beta\beta_1}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)}}\right)^n + c_6\left(-\sqrt[4]{\frac{\beta\beta_1}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)}}\right)^n + c_7\left(\iota\sqrt[4]{\frac{\beta\beta_1}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)}}\right)^n + \\ &\quad c_8\left(-\iota\sqrt[4]{\frac{\beta\beta_1}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)}}\right)^n + \frac{\alpha_1(\gamma+\delta U_1^2)+\beta_1\alpha}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)-\beta\beta_1},\end{aligned}$$

where $c_5,\ c_6,\ c_7,\ c_8$ depends on $\mu_{-3},\ \mu_{-2},\ \mu_{-1},\ \mu_0$. If $\beta\beta_1 > (\gamma+\delta U_1^2)(\gamma_1+\delta_1 U_2^2)$ then $\{\mu_n\}$ is divergent. Hence by comparison $y_n \to \infty$ as $n \to \infty$. □

2.3 Existence and uniqueness of positive equilibrium point

Theorem 3. *If*

$$\alpha_1 + \beta_1 L_1 < \left(\gamma_1 + \delta_1\left(\frac{(\gamma+\delta L_1^2)L_1-\alpha}{\beta}\right)^2\right)\frac{(\gamma+\delta L_1^2)L_1-\alpha}{\beta}, \tag{19}$$

$$\alpha_1 + \beta_1 U_1 > \left(\gamma_1 + \delta_1\left(\frac{(\gamma+\delta U_1^2)L_1-\alpha}{\beta}\right)^2\right)\frac{(\gamma+\delta U_1^2)U_1-\alpha}{\beta}, \tag{20}$$

and

$$\frac{\left(\gamma+3\delta {U_1}^2\right)\left(\gamma_1\beta^2+3\delta_1\left(\left(\gamma+\delta {U_1}^2\right)U_1-\alpha\right)^2\right)}{\beta^3\beta_1} < 1, \tag{21}$$

then the system (1) has a unique positive equilibrium point $\Omega = (\bar{x}, \bar{y}) \in [L_1, U_1] \times [L_2, U_2]$.

Proof. Consider

$$x = \frac{\alpha+\beta y}{\gamma+\delta x^2}, \quad y = \frac{\alpha_1+\beta_1 x}{\gamma_1+\delta_1 y^2}. \tag{22}$$

From (22), we have

$$y = \frac{(\gamma+\delta x^2)x-\alpha}{\beta}, \quad x = \frac{(\gamma_1+\delta_1 y^2)y-\alpha_1}{\beta_1}.$$

Taking

$$F(x) = \frac{(\gamma_1+\delta_1(h(x))^2)h(x)-\alpha_1}{\beta_1} - x, \tag{23}$$

where

$$h(x) = \frac{(\gamma+\delta x^2)x-\alpha}{\beta}, \tag{24}$$

and $x \in [L_1, U_1]$. Now

$$F(L_1) = \frac{(\gamma_1+\delta_1(h(L_1))^2)h(L_1)-\alpha_1}{\beta_1} - L_1 = \frac{\left(\gamma_1+\delta_1\left(\frac{(\gamma+\delta L_1^2)L_1-\alpha}{\beta}\right)^2\right)\frac{(\gamma+\delta L_1^2)L_1-\alpha}{\beta} - \alpha_1}{\beta_1} - L_1. \tag{25}$$

Assume that (19) hold then (25) implies that $F(L_1) > 0$. Also,

$$F(U_1) = \frac{(\gamma_1 + \delta_1 (h(U_1))^2) h(U_1) - \alpha_1}{\beta_1} - U_1 = \frac{\left(\gamma_1 + \delta_1 \left(\frac{(\gamma+\delta U_1^2)U_1-\alpha}{\beta}\right)^2\right) \frac{(\gamma+\delta U_1^2)U_1-\alpha}{\beta} - \alpha_1}{\beta_1} - U_1. \tag{26}$$

Assuming (20) hold then from (26) one get $F(U_1) < 0$. Hence, $F(x)$ has at least one positive solution in $x \in [L_1, U_1]$. Furthermore,

$$F'(x) = h'(x) \frac{\gamma_1 + 3\delta_1 (h(x))^2}{\beta_1} - 1, \tag{27}$$

where

$$h'(x) = \frac{\gamma + 3\delta x^2}{\beta}. \tag{28}$$

Let $\bar{x}$ be a solution of equation $F(x) = 0$, then from (23), (24) and (28) one get

$$\bar{x} = \frac{(\gamma_1 + \delta_1 (h(\bar{x}))^2) h(\bar{x}) - \alpha_1}{\beta_1}, \tag{29}$$

$$h(\bar{x}) = \frac{(\gamma + \delta \bar{x}^2)\bar{x} - \alpha}{\beta}, \tag{30}$$

$$h'(\bar{x}) = \frac{\gamma + 3\delta \bar{x}^2}{\beta}. \tag{31}$$

In view of (30) and (31), equation (27) takes the following form

$$\begin{aligned} F'(\bar{x}) &= \frac{(\gamma + 3\delta \bar{x}^2)\left(\gamma_1 \beta^2 + 3\delta_1 \left((\gamma + \delta \bar{x}^2)\bar{x} - \alpha\right)^2\right)}{\beta^3 \beta_1} - 1, \\ &\leq \frac{(\gamma + 3\delta {U_1}^2)\left(\gamma_1 \beta^2 + 3\delta_1 \left((\gamma + \delta {U_1}^2) U_1 - \alpha\right)^2\right)}{\beta^3 \beta_1} - 1. \end{aligned} \tag{32}$$

Assume that (21) hold then from (32) one get $F'(\bar{x}) < 0$. □

2.4 Local stability

Theorem 4. *For equilibrium Ω of the system (1), the following statements hold:*

(i) Ω of the system (1) is locally asymptotically stable if

$$\frac{2\delta U_1^2}{\gamma + \delta L_1^2}\left(1 + \frac{2\delta_1 U_2^2}{\gamma_1 + \delta_1 L_2^2}\right) + \frac{1}{\gamma_1 + \delta_1 L_2^2}\left(2\delta_1 U_2^2 + \frac{\beta\beta_1}{\gamma + \delta L_1^2}\right) < 1. \tag{33}$$

(ii) Ω of the system (1) is unstable if

$$\frac{2\delta L_1^2}{\gamma + \delta U_1^2}\left(1 + \frac{2\delta_1 L_2^2}{\gamma_1 + \delta_1 U_2^2}\right) + \frac{1}{\gamma_1 + \delta_1 U_2^2}\left(2\delta_1 L_2^2 + \frac{\beta\beta_1}{\gamma + \delta U_1^2}\right) > 1. \tag{34}$$

Proof. If $(\bar{x}, \bar{y})$ is an equilibrium point of the system (1) then

$$\bar{x} = \frac{\alpha + \beta \bar{y}}{\gamma + \delta \bar{x}^2}, \ \bar{y} = \frac{\alpha_1 + \beta_1 \bar{x}}{\gamma_1 + \delta_1 \bar{y}^2}. \tag{35}$$

Consider the following transformation in order to construct the corresponding linearized form of the system (1):

$$(x_{n+1}, x_n, y_{n+1}, y_n) \mapsto (f, f_1, g, g_1), \tag{36}$$

where

$$f=\frac{\alpha+\beta y_{n-1}}{\gamma+\delta x_n^2},\ f_1=x_n,\ g=\frac{\alpha_1+\beta_1x_{n-1}}{\gamma_1+\delta_1y_n^2},\ g_1=y_n. \tag{37}$$

The Jacobian matrix $J|_{(\bar{x},\bar{y})}$ about $(\bar{x},\bar{y})$ under the transformation (36) is given by

$$J|_{(\bar{x},\bar{y})}=\begin{pmatrix} a & 0 & 0 & b\\ 1 & 0 & 0 & 0\\ 0 & a_1 & b_1 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}, \tag{38}$$

where

$$a=-\frac{2\delta\bar{x}^2}{\gamma+\delta\bar{x}^2},\ b=\frac{\beta}{\gamma+\delta\bar{x}^2},\ a_1=\frac{\beta_1}{\gamma_1+\delta_1\bar{y}^2},\ b_1=-\frac{2\delta_1\bar{y}^2}{\gamma_1+\delta_1\bar{y}^2}. \tag{39}$$

The characteristic equation of $J|_{(\bar{x},\bar{y})}$ about $(\bar{x},\bar{y})$ is given by

$$\lambda^4-(a+b_1)\lambda^3+ab_1\lambda^2-a_1b=0. \tag{40}$$

Now,

$$\begin{aligned}|a|+|b_1|+|ab_1|+|a_1b| &= \frac{2\delta\bar{x}^2}{\gamma+\delta\bar{x}^2}+\frac{2\delta_1\bar{y}^2}{\gamma_1+\delta_1\bar{y}^2}+\frac{4\delta\delta_1\ \bar{x}^2\bar{y}^2}{(\gamma+\delta\bar{x}^2)(\gamma_1+\delta_1\bar{y}^2)}+\frac{\beta\beta_1}{(\gamma+\delta\bar{x}^2)(\gamma_1+\delta_1\bar{y}^2)},\\ &\le \frac{2\delta U_1^2}{\gamma+\delta L_1^2}+\frac{2\delta_1U_2^2}{\gamma_1+\delta_1L_2^2}+\frac{4\delta\delta_1U_1^2U_2^2}{(\gamma+\delta L_1^2)(\gamma_1+\delta_1L_2^2)}+\frac{\beta\beta_1}{(\gamma+\delta L_1^2)(\gamma_1+\delta_1L_2^2)},\\ &= \frac{2\delta U_1^2}{\gamma+\delta L_1^2}\left(1+\frac{2\delta_1U_2^2}{\gamma_1+\delta_1L_2^2}\right)+\frac{1}{\gamma_1+\delta_1L_2^2}\left(2\delta_1U_2^2+\frac{\beta\beta_1}{\gamma+\delta L_1^2}\right).\end{aligned} \tag{41}$$

Assuming that (33) hold then from (41) one gets $|a|+|b_1|+|ab_1|+|a_1b|<1$. Hence from Remark 1.3.1 of [2], Ω of (1) is locally asymptotically stable.

Proof (ii). Using same manipulations as for the proof of (i) and assume that (34) hold then

$$\begin{aligned}|a|+|b_1|+|ab_1|+|a_1b| &= \frac{2\delta\bar{x}^2}{\gamma+\delta\bar{x}^2}+\frac{2\delta_1\bar{y}^2}{\gamma_1+\delta_1\bar{y}^2}+\frac{4\delta\delta_1\ \bar{x}^2\bar{y}^2}{(\gamma+\delta\bar{x}^2)(\gamma_1+\delta_1\bar{y}^2)}+\frac{\beta\beta_1}{(\gamma+\delta\bar{x}^2)(\gamma_1+\delta_1\bar{y}^2)},\\ &\ge \frac{2\delta L_1^2}{\gamma+\delta U_1^2}+\frac{2\delta_1L_2^2}{\gamma_1+\delta_1U_2^2}+\frac{4\delta\delta_1L_1^2L_2^2}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1U_2^2)}+\frac{\beta\beta_1}{(\gamma+\delta U_1^2)(\gamma_1+\delta_1U_2^2)},\\ &= \frac{2\delta L_1^2}{\gamma+\delta U_1^2}\left(1+\frac{2\delta_1L_2^2}{\gamma_1+\delta_1U_2^2}\right)+\frac{1}{\gamma_1+\delta_1U_2^2}\left(2\delta_1L_2^2+\frac{\beta\beta_1}{\gamma+\delta U_1^2}\right)>1.\end{aligned} \tag{42}$$

Hence Ω of system (1) is unstable. □

2.5 Global character

Now we will study the global dynamics of (1) about Ω by utilizing Theorem 1.16 of [3].

Theorem 5. *Ω of the system (1) is a global attractor.*

Proof. If $f(x,y)=\frac{\alpha+\beta y}{\gamma+\delta x^2}$ and $g(x,y)=\frac{\alpha_1+\beta_1x}{\gamma_1+\delta_1y^2}$ then it is easy to examine that $f(x,y)$ is non-increasing (resp. non-decreasing) in x (resp. y) $\forall\ (x,y)\in\left[\frac{\alpha(\gamma\gamma_1-\beta\beta_1)^2}{\gamma(\gamma\gamma_1-\beta\beta_1)^2+\delta(\alpha\gamma_1+\beta\alpha_1)^2},\frac{\alpha\gamma_1+\beta\alpha_1}{\gamma\gamma_1-\beta\beta_1}\right]\times\left[\frac{\alpha_1(\gamma\gamma_1-\beta\beta_1)^2}{\gamma_1(\gamma\gamma_1-\beta\beta_1)^2+\delta_1(\alpha_1\gamma+\beta_1\alpha)^2},\frac{\alpha_1\gamma+\alpha\beta_1}{\gamma\gamma_1-\beta\beta_1}\right]$. Also $g(x,y)$ is non-decreasing (resp. non-increasing) in x (resp. y) $\forall\ (x,y)\in\left[\frac{\alpha(\gamma\gamma_1-\beta\beta_1)^2}{\gamma(\gamma\gamma_1-\beta\beta_1)^2+\delta(\alpha\gamma_1+\beta\alpha_1)^2},\frac{\alpha\gamma_1+\beta\alpha_1}{\gamma\gamma_1-\beta\beta_1}\right]\times\left[\frac{\alpha_1(\gamma\gamma_1-\beta\beta_1)^2}{\gamma_1(\gamma\gamma_1-\beta\beta_1)^2+\delta_1(\alpha_1\gamma+\beta_1\alpha)^2},\frac{\alpha_1\gamma+\alpha\beta_1}{\gamma\gamma_1-\beta\beta_1}\right]$. Let (m_1,M_1,m_2,M_2) be a solution of the system

$$m_1=\frac{\alpha+\beta m_2}{\gamma+\delta M_1^2},\ M_1=\frac{\alpha+\beta M_2}{\gamma+\delta m_1^2}. \tag{43}$$

and

$$m_2=\frac{\alpha_1+\beta_1m_1}{\gamma_1+\delta_1M_2^2},\ M_2=\frac{\alpha_1+\beta_1M_1}{\gamma_1+\delta_1m_2^2}. \tag{44}$$

From (43) and (44), we get

$$\frac{m_1}{M_1} = \frac{(\alpha + \beta m_2)(\gamma + \delta m_1^2)}{(\gamma + \delta M_1^2)(\alpha + \beta M_2)}. \tag{45}$$

$$\frac{m_2}{M_2} = \frac{(\alpha_1 + \beta_1 m_1)(\gamma_1 + \delta_1 m_2^2)}{(\gamma_1 + \delta_1 M_2^2)(\alpha_1 + \beta_1 M_1)}. \tag{46}$$

Setting

$$\frac{m_1}{M_1} = a_1 \leq 1, \ \frac{m_2}{M_2} = a_2 \leq 1. \tag{47}$$

In view of (47), equations (45) and (46) then implies that

$$\begin{aligned} \beta\gamma(a_1 - a_2)M_2 &= \alpha\delta(a_1 - 1)a_1 M_1^2 + \beta\delta(a_1 a_2 - 1)a_1 M_1^2 M_2 - \alpha\gamma(a_1 - 1), \\ \beta_1\gamma_1(a_2 - a_1)M_1 &= \alpha_1\delta_1(a_2 - 1)a_2 M_2^2 + \beta_1\delta_1(a_1 a_2 - 1)a_2 M_1 M_2^2 - \alpha_1\gamma_1(a_2 - 1). \end{aligned} \tag{48}$$

So right-hand sides of (48) are less then or equal to zero, and thus

$$a_1 - a_2 \leq 0, \ a_2 - a_1 \leq 0.$$

This implies that

$$a_1 \leq a_2 \leq a_1,$$

which hold if and only if $a_1 = a_2$. In view of (48) it follows that $a_1 = a_2 = 1$ and thus $m_1 = M_1, \ m_2 = M_2$. Hence, by Theorem 1.16 of [3], Ω of the system (1) is a global attractor. $\square$

3 Conclusion

This work is related to the dynamical properties of an anti-competitive system of rational difference equations. We proved that if $\beta\beta_1 < \gamma\gamma_1$ then every solution $\{(x_n, y_n)/x_n, y_n > 0\}$ of the system (1) is bounded and persists. We proved that if $\alpha_1+\beta_1 L_1 < \left(\gamma_1 + \delta_1\left(\frac{(\gamma+\delta L_1^2)L_1-\alpha}{\beta}\right)^2\right)\frac{(\gamma+\delta L_1^2)L_1-\alpha}{\beta}$, $\alpha_1+\beta_1 U_1 > \left(\gamma_1 + \delta_1\left(\frac{(\gamma+\delta U_1^2)L_1-\alpha}{\beta}\right)^2\right)\frac{(\gamma+\delta U_1^2)U_1-\alpha}{\beta}$ and $\frac{(\gamma+3\delta {U_1}^2)\left(\gamma_1\beta^2+3\delta_1\left(\left(\gamma+\delta {U_1}^2\right)U_1-\alpha\right)^2\right)}{\beta^3\beta_1} < 1$ then system (1) has a unique positive equilibrium point $\Omega = (\bar{x}, \bar{y}) \in [L_1, U_1] \times [L_2, U_2]$. Furthermore method of Linearization is used to study the local stability about the unique positive equilibrium point Ω. Linear stability analysis shows that Ω is locally asymptotically stable if $\frac{2\delta U_1^2}{\gamma+\delta L_1^2}\left(1 + \frac{2\delta_1 U_2^2}{\gamma_1+\delta_1 L_2^2}\right) + \frac{1}{\gamma_1+\delta_1 L_2^2}\left(2\delta_1 U_2^2 + \frac{\beta\beta_1}{\gamma+\delta L_1^2}\right) < 1$ and unstable if $\frac{2\delta L_1^2}{\gamma+\delta U_1^2}\left(1 + \frac{2\delta_1 L_2^2}{\gamma_1+\delta_1 U_2^2}\right) + \frac{1}{\gamma_1+\delta_1 U_2^2}\left(2\delta_1 L_2^2 + \frac{\beta\beta_1}{\gamma+\delta U_1^2}\right) > 1$. Finally global dynamics about Ω is also investigated.

Acknowledgements

J. Ma's research is supported by the National Natural Science Foundations of China [grant number 11501364] while A. Q. Khan's research is supported by the Higher Education Commission (HEC) of Pakistan.

References

[1] A. E. Hamza, R. Khalaf-Allah, Dynamics of second-order rational difference equation, *BAMS*, 23(1)(2008):206-214.

[2] V. L. Kocic, G. Ladas, Global behavior of nonlinear difference equations of higher order with applications, Kluwer Academic Publishers, Dordrecht, (1993).

[3] E. A. Grove, G. Ladas, Periodicities in monlinear difference equations, Chapman and Hall/CRC Press, Boca Raton, (2004).

AN ITERATIVE SCHEME FOR SOLVING SPLIT SYSTEM OF MINIMIZATION PROBLEMS

ANTENEH GETACHEW GEBRIE AND RABIAN WANGKEEREE

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand

Abstract. In this paper, we propose iterative algorithm for solving split system of minimization problems. We prove strong convergence of the sequences generated by the proposed algorithms. The iterative schemes are proposed in such a way that the selection of the step-sizes does not need any prior information about the operator norm. We further give some example to numerically verify the efficiency and implementation of our method.

Keywords: Minimization problem, strong convergence, Moreau-Yosida approximate, Hilbert space.
AMS Subject Classification: 49J53, 49J52, 47J05, 90C25, 65K10.

1. Introduction

Let H_1 and H_2 be real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Given nonempty closed convex subsets C_i $(i = 1, \dots, N)$ and Q_i $(i = 1, \dots, M)$ of H_1 and H_2, respectively. The multiple-set split feasibility problem (MSSFP) which was introduced by Censor et al. [10] is formulated as finding a point

$$\bar{x} \in \bigcap_{i=1}^{N} C_i \text{ such that } A\bar{x} \in \bigcap_{j=1}^{M} Q_j. \tag{1.1}$$

In particular, if $N = M = 1$, then the MSSFP (1.1) is reduced to find a point

$$\bar{x} \in C \text{ such that } A\bar{x} \in Q. \tag{1.2}$$

where C and Q are nonempty closed convex subsets of H_1 and H_2, respectively. The problem (1.2) is known as the split feasibility problem (SFP) which was first introduced by Censor and Elfving [9] for modeling inverse problems in finite-dimensional Hilbert spaces. Many authors studied the SFP, see for example in [5, 9, 13, 14, 17, 24], and MSSFP, see for example in [10, 15, 19, 34, 35], provided the solution exists. The SFP and MSSFP arises in many fields in the real world, such as image reconstruction, modeling inverse problems, radiation therapy treatment planning and signal processing, and medical care; for details see [6, 7, 8] and the references therein.

Throughout this paper, unless otherwise stated, we assume that H_1 and H_2 are real Hilbert spaces, $A : H_1 \to H_2$ is nonzero bounded linear operator, I denotes the identity operator on a Hilbert space and $\mathbb{R}$ denotes set of real numbers.

Let us consider the following problem: find $x \in H_1$ with the property that

$$\min_{x \in H_1} \{f(x) + g_\lambda(Ax)\}, \tag{1.3}$$

where $f : H_1 \to \mathbb{R} \cup \{+\infty\}$, $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ are two proper, convex, lower-semicontinuous functions and g_λ is Moreau-Yosida approximate [26] of the function g of parameter λ given by $g_\lambda(y) = \min_{u \in H_2}\{g(u) + \frac{1}{2\lambda}\|y - u\|^2\}$. In [21], Moudafi and Thakur introduced a weakly convergent algorithm solving the (1.3) in case $\arg\min f \cap A^{-1}(\arg\min g) \neq \emptyset$. Note that if we take $f = \delta_C$ [defined as $\delta_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise], the indicator function of nonempty, closed and convex subset C of H_1 and $g = \delta_Q$, the indicator

*Corresponding author: R. Wangkeeree.
Email address: antgetm@gmail.com (A.G Gebrie) and rabianw@nu.ac.th (R. Wangkeeree).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

function of nonempty, closed and convex subset Q of H_2, then problem problem (1.3) is reduced to the following minimization problem:

$$\min_{x\in C}\left\{\frac{1}{2\lambda}\|(I-P_Q)(Ax)\|^2\right\} \tag{1.4}$$

which, when $C\cap A^{-1}(Q)\neq\emptyset$, is equivalent to the split feasibility problem (SEP). It should also be noticed that (1.3) is equivalent to the problem of finding a point $\bar{x}\in H_1$ with the property

$$\bar{x}\in\arg\min f \text{ such that } A\bar{x}\in\arg\min g. \tag{1.5}$$

Moudafi and Thakur [21] used the idea of Lopez et al. [17] to introduce a new way of selecting the step sizes given by

$$\theta_{\lambda\mu}(x)=\sqrt{\|A^*(I-\text{prox}_{\lambda g})Ax\|^2+\|(I-\text{prox}_{\lambda\mu f})x\|^2}$$

with $h_\lambda(x)=\frac{1}{2}\|(I-\text{prox}_{\lambda g})Ax\|^2$ and $l_{\lambda\mu}(x)=\frac{1}{2}\|(I-\text{prox}_{\lambda\mu f})x\|^2$ where $\text{prox}_{\lambda f}(x)=\arg\min\limits_{u\in H_1}\{f(u)+\frac{1}{2\lambda}\|u-y\|^2\}$ stands for the proximal mapping of f. They proposed the following split proximal algorithm, which generates, from an initial point $x_1\in H_1$ assume that x_n has been constructed and $\theta_\lambda(x_n)\neq 0$, then compute x_{n+1} via the rule

$$x_{n+1}=\text{prox}_{\lambda\mu_n f}\big(x_n-\mu_n A^*(I-\text{prox}_{\lambda g})Ax_n\big) \tag{1.6}$$

where stepsize $\mu_n=\rho_n\frac{h_\lambda(x_n)+l_{\lambda\mu_n}(x_n)}{\theta^2_{\lambda\mu_n}(x_n)}$ with $0<\rho_n<4$ and if $\theta_{\lambda\mu_n}(x_n)=0$, then $x_{n+1}=x_n$ is a solution of (1.5) and the iterative process stops; otherwise, we set $n:=n+1$ and go to (1.6). Based on Moudafi and Thakur [21] many iterative algorithms are proposed for solving split minimization problem (1.5), see eg, Shehu and Iyiola in [28, 29, 30, 31], Shehu and Ogbuisi in [27], Shehu et al. in [32], Abbas et al. in [1].

Very recently, Shehu and Iyiola [29] proposed algorithm for solving (1.5) as follows:

$$\begin{cases} u, x_1\in H_1,\\ z_n=(1-\alpha_n)x_n+\alpha_n u,\\ y_n=z_n-\rho_n\frac{h(z_n)+l(z_n)}{\theta^2(z_n)}\big((I-\text{prox}_{\lambda f})z_n+A^*(I-\text{prox}_{\lambda g})Az_n\big),\\ x_{n+1}=(1-\beta_n)z_n+\beta_n y_n,\end{cases} \tag{1.7}$$

where $l(x)=\frac{1}{2}\|(I-\text{prox}_{\lambda f})x\|^2$, $h(x)=\frac{1}{2}\|(I-\text{prox}_{\lambda g})Ax\|^2$ and $\theta(x)=\|(I-\text{prox}_{\lambda f})x+A^*(I-\text{prox}_{\lambda g})Ax\|$. It was shown that the sequence $\{x_n\}$ generated by iterative algorithm (1.7) converges strongly to the solution of problem (1.5) under the following conditions:

- **(a) :** $0<\alpha_n<1$, $\lim\limits_{n\to\infty}\alpha_n=0$ and $\sum\limits_{n=1}^{\infty}\alpha_n=\infty$.
- **(b) :** $0<\beta\leq\beta_n\leq\delta<1$,
- **(c) :** $0<\rho_n<4$, $\liminf\limits_{n\to\infty}\rho_n(4-\rho_n)>0$.

To prove the strong convergence of iterative algorithm (1.7) the authors used simpler alternative proof without recourse to 'two cases method' of proof studied by other authors [1, 27, 30, 31, 32] and is also different from the approaches used in the proofs of [21, 28].

Motivated and inspired by results in [10, 21, 29], in this paper, we introduce and study the following *split system of minimization problem* (SSMP): finding a point $\bar{x}\in H_1$ with the property

$$\bar{x}\in\bigcap_{i=1}^{N}(\arg\min f_i) \text{ such that } A\bar{x}\in\bigcap_{j=1}^{M}(\arg\min g_j) \tag{1.8}$$

where $f_i:H_1\to\mathbb{R}\cup\{+\infty\}$ and $g_j:H_2\to\mathbb{R}\cup\{+\infty\}$ are proper, lower semicontinuous convex functions, $\arg\min f_i=\{\bar{x}\in H_1: f_i(\bar{x})\leq f_i(x),\ \forall x\in H_1\}$, $\arg\min g_j=\{\bar{y}\in H_2: g_j(\bar{y})\leq g_j(y),\ \forall y\in H_2\}$ and $i\in\Phi=\{1,\dots,N\}$, $j\in\Psi=\{1,\dots,M\}$. The solution set Γ of problem (1.8) is denoted by

$$\Gamma=\Big\{\bar{x}\in H_1:\bar{x}\in\bigcap_{i=1}^{N}(\arg\min f_i) \text{ and } A\bar{x}\in\bigcap_{j=1}^{M}(\arg\min g_j)\Big\}.$$

Minimizers of any proper, lower semicontinuous function are exactly fixed points of its proximal mappings and proximal mappings are nonexpansive mapping (whose set of fixed points is closed and convex), we have that the set of minimizers of any proper, lower semicontinuous function is closed and convex. Therefore, since A bounded linear operator the solution set Γ of problem (1.8) is closed convex set. We assume Γ is nonempty.

We propose an iterative scheme using extended form of selecting step sizes used to solve (1.5) to the context of solving split system of minimization problem (1.8). The iterative scheme is developed by computation of proximal of f_i at z_n and g_j at Az_n in a parallel setting under simple assumptions on step sizes. Moreover, the technique of the proof takes some steps of [29, 33] so that it takes few steps to complete the proof. Note that if $f_i = f$ for all $i \in \Phi$ and $g_j = g$ for all $j \in \Psi$, then problem (1.8) reduces to the problem of split minimization problem (1.5) considered in [1, 21, 27, 28, 29, 30, 31, 32].

This paper is organized in the following way. In Section 2, we collect some basic and useful lemmas for further study. In Section 3, we propose and analyze the convergence result of our algorithm. In Section 4, we give a numerical example to discuss performance of the proposed algorithm.

2. Preliminary

In order to prove our main results, we recall some basic definitions and lemmas, which will be needed in the sequel. The symbols " $\rightharpoonup$ " and " $\to$ " denote weak and strong convergence, respectively.

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The metric projection on C is a mapping $P_C : H \to C$ defined by

$$P_C(x) = \arg\min\{\|y - x\| : y \in C\}, \quad x \in H.$$

Lemma 2.1. *Let C be a closed convex subset of H. Given $x \in H$ and a point $z \in C$, then $z = P_C(x)$ if and only if*

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C.$$

Let $T : H \to H$. Then,

(I): T is L-Lipschitz if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in H.$$

If $L \in (0, 1)$, then we call T a contraction. If $L = 1$, then T is called a nonexpansive mapping.

(II): T is firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - ||(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H,$$

which is equivalent to

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \quad \forall x, y \in H.$$

If T is firmly nonexpansive, $I - T$ is also firmly nonexpansive.

(III): strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2$$

for all $x, y \in H$.

(IV): inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2$$

for all $x, y \in H$.

Note that the proximal mapping of f is nonexpansive and firmly nonexpansive mapping. The minimizers of any proper, lower semicontinuous function are exactly fixed points of its proximal mappings. Many properties of proximal operator can be found in [12] and the references therein.

Lemma 2.2. *Let H be a real Hilbert space. Then,*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle, \quad \forall x, y \in H$$

The following facts will be used several times in the paper.

Lemma 2.3. [2] *Let H be a real Hilbert space. Then,*

$$\|(1-\alpha)x+\alpha y\|^2 = (1-\alpha)\|x\|^2 + \alpha\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2,$$

$\forall \alpha \in \mathbb{R}$, $\forall x, y \in H$.

Let H be a real Hilbert space, $\{x_1, x_2, \ldots, x_d\} \subset H$ and $\{\lambda_1, \lambda_2, \ldots, \lambda_d\} \subset [0,1]$ with $\sum\limits_{i=1}^{d} \lambda_i = 1$. Then, from [2, 37] one can see that

$$\Big\|\sum_{i=1}^{d} \lambda_i x_i\Big\|^2 \le \sum_{i=1}^{d} \lambda_i \|x_i\|^2,$$

i.e., convexity of $\|.\|^2$.

Lemma 2.4. [18] *Let $\{a_n\}$ be the sequence of nonnegative numbers such that*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n\delta_n,$$

where $\{\delta_n\}$ is a sequence of real numbers bounded from above and $0 \le \alpha_n \le 1$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. Then it holds that

$$\limsup_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \delta_n.$$

3. Main result

First we introduce the following settings which is an extension of settings introduced by Moudafi and Thakur [21]. Let $\lambda > 0$. For $x \in H_1$,

(i): for each $i \in \Phi$, define

$$l_i(x) = \frac{1}{2}\|(I - \mathrm{prox}_{\lambda f_i})x\|^2 \text{ and } \nabla l_i(x) = (I - \mathrm{prox}_{\lambda f_i})x,$$

(ii): $l(x)$ and $\nabla l(x)$ are defined as $l(x) = l_{i_x}(x)$ and so $\nabla l(x) = \nabla l_{i_x}(x)$ where i_x is in Φ such that

$$i_x \in \arg\max\{\|(I - \mathrm{prox}_{\lambda f_i})x\| : i \in \Phi\},$$

(iii): for each $j \in \Psi$, define

$$h_j(x) = \frac{1}{2}\|(I - \mathrm{prox}_{\lambda g_j})Ax\|^2 \text{ and } \nabla h_j(x) = A^*(I - \mathrm{prox}_{\lambda g_j})Ax,$$

(iv): for each $j \in \Psi$, define

$$\theta_j(x) = \max\{\|\nabla h_j(x)\|, \|\nabla l(x)\|\}.$$

It is easy to see that, for $x \in H_1$

$$\|\nabla l_i(x)\| \le \|\nabla l_{i_x}(x)\| = \|\nabla l(x)\|, \ \ \forall i \in \Phi$$

and

$$l_i(x) = \frac{1}{2}\|\nabla l_i(x)\|^2, \ \ \forall i \in \Phi.$$

In this section, we propose algorithm for solving SSMP (1.8) and we analyse the convergence of the iteration sequence generated by the algorithm by assuming that the solution set Γ is nonempty. In order to design the algorithm, we consider the parameter sequences satisfying the following conditions.

Condition 1

(C1) : $0 < \alpha_n < 1$, $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$.

(C2) : $0 < \beta \le \beta_n \le \delta < 1$,

(C3) : $0 < \xi \leq \xi_n^j \leq 1$ such that $\sum_{j=1}^{M} \xi_n^j = 1$ for each $n \geq 1$.

(C4) : $0 < \delta \leq \delta_n^i \leq 1$ such that $\sum_{i=1}^{N} \delta_n^i = 1$ for each $n \geq 1$.

(C5) : $0 < \rho_n < 2\delta$, $\liminf_{n\to\infty} \rho_n(2\delta - \rho_n) > 0$.

Throughout this paper, unless otherwise stated, Condition 1 refers to conditions (C1)-(C5) above. Using the definitions of ∇l_i, l_i, l, ∇l, h_j, ∇h_j and θ_j given in **(i)-(iv)**, we are now in a position to introduce our algorithm.

Algorithm 1

Initialization: Choose $u,\ x_1 \in H_1$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\rho_n\}$, $\{\delta_n^i\}$ and $\{\xi_n^j\}$ be real sequences satisfying Condition 1.

Step 1: Evaluate $z_n = (1 - \alpha_n)x_n + \alpha_n u$.

Step 2: For each $j \in \Psi$ compute $\theta_j(z_n)$, $h_j(z_n)$ and $l(z_n)$.

Let $\Psi_n = \{j \in \Psi : \theta_j(z_n) \neq 0\}$.

If $\Psi_n = \emptyset$, then z_n is a solution of (1.8) and the iterative process stops, otherwise, go to Step 3.

Step 3: For each $j \in \Psi$ evaluate $\mu_n^j = \rho_n \eta_n^j$ where

$$\eta_n^j = \begin{cases} 0, & \text{if } j \notin \Psi_n \\ \frac{h_j(z_n)+l(z_n)}{\theta_j^2(z_n)}, & \text{if } j \in \Psi_n. \end{cases}$$

Step 4: Evaluate

$$w_n = z_n - \Big(\sum_{j\in\Psi} \xi_n^j \mu_n^j\Big) \sum_{i\in\Phi} \delta_n^i \nabla l_i(z_n)$$

and

$$t_n = z_n - \sum_{j\in\Psi} \xi_n^j \mu_n^j \nabla h_j(z_n).$$

Step 5: Evaluate

$$y_n = \frac{w_n + t_n}{2}.$$

Step 6: Evaluate $x_{n+1} = (1 - \beta_n)z_n + \beta_n y_n$.

Step 7: Set $n := n + 1$ and go to Step 1.

Lemma 3.1. *If $\Psi_n = \emptyset$, then z_n is the solution of (1.8).*

Proof. Suppose $\Psi_n = \emptyset$ at some iteration n.
Then, from $\Psi_n = \{j \in \Psi : \theta_j(z_n) \neq 0\} = \emptyset$, we have

$$\begin{aligned} &\max\{\|\nabla h_j(z_n)\|, \|\nabla l(z_n)\|\} = 0, \forall j \in \Psi \\ &\Leftrightarrow \|\nabla h_j(z_n)\| = 0 = \|\nabla l(z_n)\|, \forall j \in \Psi, \\ &\Leftrightarrow \|\nabla h_j(z_n)\| = 0 = \|\nabla l_i(z_n)\|, \forall i \in \Phi, \forall j \in \Psi, \\ &\Leftrightarrow A^*(I - \text{prox}_{\lambda g_j})Az_n = 0 = (I - \text{prox}_{\lambda f_i})z_n, \forall i \in \Phi, \forall j \in \Psi, \end{aligned}$$

and this implies that $z_n \in \Gamma$. □

Remark 3.2. Note that we can also use $\theta_j(x) = \sqrt{\|\nabla h_j(x)\|^2 + \|\nabla l(x)\|^2}$ instead of $\theta_j(x) = \max\{\|\nabla h_j(x)\|, \|\nabla l(x)\|\}$ and the proof for convergence will be the same. It is clear to see that

$$\max\{\|\nabla h_j(x)\|, \|\nabla l(x)\|\} \leq \sqrt{\|\nabla h_j(x)\|^2 + \|\nabla l(x)\|^2}.$$

If Algorithm 1 does not stop, then we have the following strong convergence theorem for approximation of solution of problem (1.8).

Theorem 3.3. *The sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to $\bar{x} \in \Gamma$ where $\bar{x} = P_\Gamma u$.*

Proof. Let $\bar{x} = P_\Gamma u$. Since $\text{prox}_{\lambda f_i}$ and $\text{prox}_{\lambda g_j}$ are firmly nonexpansive, $I - \text{prox}_{\lambda f_i}$ and $I - \text{prox}_{\lambda g_j}$ are also firmly nonexpansive, and since $\bar{x}$ verifies (1.8) (since minimizers of any function are exactly fixed-points of its proximal mapping), we have for all $z \in H_1$

$$\begin{aligned}\langle \nabla l_i(z), z - \bar{x}\rangle &= \langle (I - \text{prox}_{\lambda f_i})z, z - \bar{x}\rangle \\ &\geq \|(I - \text{prox}_{\lambda f_i})z\|^2 = 2l_i(z)\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\langle \nabla h_j(z), z - \bar{x}\rangle &= \langle A^*(I - \text{prox}_{\lambda g_j})Az, z - \bar{x}\rangle \\ &= \langle (I - \text{prox}_{\lambda g_j})Az, Az - A\bar{x}\rangle \\ &\geq \|(I - \text{prox}_{\lambda g_j})Az\|^2 = 2h_j(z), \quad \forall j \in \Psi.\end{aligned} \tag{3.2}$$

Note that, for all $z \in H_1$, $\|\nabla l(z)\| \leq \theta_j(z)$, $\|\nabla h_j(z)\| \leq \theta_j(z)$, $\forall j \in \Psi$,

$$\sum_{i\in\Phi} \delta_n^i \|\nabla l_i(z)\|^2 \leq \|\nabla l(z)\|^2 \text{ and } \sum_{i\in\Phi} \delta_n^i l_i(z) \geq \zeta l(z).$$

Using convexity of $\|.\|^2$ together with (3.1), we have

$$\begin{aligned}\|w_n - \bar{x}\|^2 &= \|z_n - \Big(\sum_{j\in\Psi} \xi_n^j \mu_n^j\Big) \sum_{i\in\Phi} \delta_n^i \nabla l_i(z_n) - \bar{x}\|^2 \\ &= \|z_n - \bar{x}\|^2 + \Big\|\Big(\sum_{j\in\Psi} \xi_n^j \mu_n^j\Big) \sum_{i\in\Phi} \delta_n^i \nabla l_i(z_n)\Big\|^2 \\ &\qquad - 2\Big\langle\Big(\sum_{j\in\Psi} \xi_n^j \mu_n^j\Big) \sum_{i\in\Phi} \delta_n^i \nabla l_i(z_n), z_n - \bar{x}\Big\rangle \\ &\leq \|z_n - \bar{x}\|^2 + \Big(\sum_{j\in\Psi} \xi_n^j \mu_n^j\Big)^2 \sum_{i\in\Phi} \delta_n^i \big\|\nabla l_i(z_n)\big\|^2 \\ &\qquad - 2\Big(\sum_{j\in\Psi} \xi_n^j \mu_n^j\Big) \sum_{i\in\Phi} \delta_n^i \langle \nabla l_i(z_n), z_n - \bar{x}\rangle \\ &\leq \|z_n - \bar{x}\|^2 + \Big(\sum_{j\in\Psi} \xi_n^j (\mu_n^j)^2\Big) \sum_{i\in\Phi} \delta_n^i \big\|\nabla l_i(z_n)\big\|^2 \\ &\qquad - 2\Big(\sum_{j\in\Psi} \xi_n^j \mu_n^j\Big) \sum_{i\in\Phi} \delta_n^i \langle \nabla l_i(z_n), z_n - \bar{x}\rangle \\ &\leq \|z_n - \bar{x}\|^2 + \Big(\sum_{j\in\Psi} \xi_n^j (\mu_n^j)^2\Big) \sum_{i\in\Phi} \delta_n^i \big\|\nabla l_i(z_n)\big\|^2 \\ &\qquad - 4\Big(\sum_{j\in\Psi} \xi_n^j \mu_n^j\Big) \sum_{i\in\Phi} \delta_n^i l_i(z_n).\end{aligned} \tag{3.3}$$

Similarly, using convexity of $\|.\|^2$ together with (3.2), we have

$$\begin{aligned}&\|t_n - \bar{x}\|^2 = \|z_n - \sum_{j\in\Psi} \xi_n^j \mu_n^j \nabla h_j(z_n) - \bar{x}\|^2 \\ &= \|z_n - \bar{x}\|^2 + \|\sum_{j\in\Psi} \xi_n^j \mu_n^j \nabla h_j(z_n)\|^2 - 2\langle \sum_{j\in\Psi} \xi_n^j \mu_n^j \nabla h_j(z_n), z_n - \bar{x}\rangle \\ &\leq \|z_n - \bar{x}\|^2 + \sum_{j\in\Psi} \xi_n^j (\mu_n^j)^2 \|\nabla h_j(z_n)\|^2 - 2\sum_{j\in\Psi} \xi_n^j \mu_n^j \langle \nabla h_j(z_n), z_n - \bar{x}\rangle \\ &\leq \|z_n - \bar{x}\|^2 + \sum_{j\in\Psi} \xi_n^j (\mu_n^j)^2 \|\nabla h_j(z_n)\|^2 - 4\sum_{j\in\Psi} \xi_n^j \mu_n^j h_j(z_n).\end{aligned} \tag{3.4}$$

Now,

$$\begin{aligned}
&\Big(\sum_{j\in\Psi}\xi_n^j(\mu_n^j)^2\Big)\sum_{i\in\Phi}\delta_n^i\big\|\nabla l_i(z_n)\big\|^2-4\Big(\sum_{j\in\Psi}\xi_n^j\mu_n^j\Big)\sum_{i\in\Phi}\delta_n^i l_i(z_n)\\
&\le\Big(\sum_{j\in\Psi}\xi_n^j(\mu_n^j)^2\Big)\|\nabla l(z_n)\|^2-4\Big(\sum_{j\in\Psi}\xi_n^j\mu_n^j\Big)\delta_n^{i_{z_n}}l(z_n)\\
&\le\Big(\sum_{j\in\Psi}\xi_n^j(\mu_n^j)^2\Big)\|\nabla l(z_n)\|^2-4\zeta\Big(\sum_{j\in\Psi}\xi_n^j\mu_n^j\Big)l(z_n)\\
&=\Big(\sum_{j\in\Psi}\xi_n^j(\rho_n\eta_n^j)^2\Big)\|\nabla l(z_n)\|^2-4\zeta\Big(\sum_{j\in\Psi}\xi_n^j\rho_n\eta_n^j\Big)l(z_n)\\
&=\sum_{j\in\Psi_n}\xi_n^j\Big(\rho_n\tfrac{h_j(z_n)+l(z_n)}{\theta_j^2(z_n)}\Big)^2\|\nabla l(z_n)\|^2-4\zeta\sum_{j\in\Psi_n}\xi_n^j\rho_n\tfrac{h_j(z_n)+l(z_n)}{\theta_j^2(z_n)}l(z_n)\\
&\le\rho_n^2\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^4(z_n)}\theta_j^2(z_n)-4\zeta\rho_n\sum_{j\in\Psi_n}\xi_n^j\tfrac{h_j(z_n)+l(z_n)}{\theta_j^2(z_n)}l(z_n)\\
&=\rho_n^2\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}-4\zeta\rho_n\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}\tfrac{l(z_n)}{h_j(z_n)+l(z_n)}\\
&=\rho_n\sum_{j\in\Psi_n}\xi_n^j\Big(\rho_n-\tfrac{4\zeta l(z_n)}{h_j(z_n)+l(z_n)}\Big)\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)},
\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}
&\sum_{j\in\Psi}\xi_n^j(\mu_n^j)^2\|\nabla h_j(z_n)\|^2-4\sum_{j\in\Psi}\xi_n^j\mu_n^j h_j(z_n)\\
&=\sum_{j\in\Psi}\xi_n^j(\rho_n\eta_n^j)^2\|\nabla h_j(z_n)\|^2-4\sum_{j\in\Psi}\xi_n^j\rho_n\eta_n^j h_j(z_n)\\
&=\sum_{j\in\Psi_n}\xi_n^j\Big(\rho_n\tfrac{h_j(z_n)+l(z_n)}{\theta_j^2(z_n)}\Big)^2\|\nabla h_j(z_n)\|^2-4\sum_{j\in\Psi_n}\xi_n^j\rho_n\tfrac{h_j(z_n)+(z_n)}{\theta_j^2(z_n)}h_j(z_n)\\
&\le\rho_n^2\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^4(z_n)}\theta_j^2(z_n)-4\rho_n\sum_{j\in\Psi_n}\xi_n^j\tfrac{h_j(z_n)+l(z_n)}{\theta_j^2(z_n)}h_j(z_n)\\
&=\rho_n^2\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(x_n)}-4\rho_n\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}\tfrac{h_j(z_n)}{h_j(z_n)+l(z_n)}\\
&\le\rho_n^2\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}-4\zeta\rho_n\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}\tfrac{h_j(z_n)}{h_j(z_n)+l(z_n)}\\
&=\rho_n\sum_{j\in\Psi_n}\xi_n^j\Big(\rho_n-\tfrac{4\zeta h_j(z_n)}{h_j(z_n)+l(z_n)}\Big)\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}.
\end{aligned}\tag{3.6}$$

From convexity of $\|.\|^2$ and (3.3)-(3.6), we have

$$\begin{aligned}
&\|y_n-\bar{x}\|^2=\|\tfrac{1}{2}(w_n+t_n)-\bar{x}\|^2\le\tfrac{1}{2}\|w_n-\bar{x}\|^2+\tfrac{1}{2}\|t_n-\bar{x}\|^2\\
&\le\|z_n-\bar{x}\|^2+\tfrac{\rho_n}{2}\sum_{j\in\Psi_n}\xi_n^j\Big(\rho_n-\tfrac{4\zeta l(z_n)}{h_j(z_n)+l(z_n)}\Big)\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}\\
&\qquad+\tfrac{\rho_n}{2}\sum_{j\in\Psi_n}\xi_n^j\Big(\rho_n-\tfrac{4\zeta h_j(z_n)}{h_j(z_n)+l(z_n)}\Big)\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}\\
&=\|z_n-\bar{x}\|^2+\rho_n(\rho_n-2\zeta)\sum_{j\in\Psi_n}\xi_n^j\tfrac{(h_j(z_n)+l(z_n))^2}{\theta_j^2(z_n)}.
\end{aligned}\tag{3.7}$$

From (3.7) and (C5), we have

$$\|y_n-\bar{x}\|\le\|z_n-\bar{x}\|.\tag{3.8}$$

Using (3.8) and the definition of x_{n+1}, we get

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2&=\|(1-\beta_n)z_n+\beta_n y_n-\bar{x}\|^2\\
&=\|(1-\beta_n)(z_n-\bar{x})+\beta_n(y_n-\bar{x})\|^2\\
&=(1-\beta_n)\|z_n-\bar{x}\|^2+\beta_n\|y_n-\bar{x}\|^2-\beta_n(1-\beta_n)\|z_n-y_n\|^2\\
&\le\|z_n-\bar{x}\|^2-\beta_n(1-\beta_n)\|z_n-y_n\|^2.
\end{aligned}\tag{3.9}$$

From (3.9) and the definition of z_n, we get

$$\begin{aligned}\|x_{n+1}-\bar{x}\| \leq \|z_n-\bar{x}\| &= (1-\alpha_n)\|x_n-\bar{x}\|+\alpha_n\|u-\bar{x}\| \\ &\leq \max\{\|x_n-\bar{x}\|,\|u-\bar{x}\|\} \\ &\quad\vdots \\ &\leq \max\{\|x_n-\bar{x}\|,\|u-\bar{x}\|\}\end{aligned} \tag{3.10}$$

which shows that $\{x_n\}$ is bounded. Consequently, $\{y_n\}$, $\{Ay_n\}$ and $\{z_n\}$ are all bounded. Now,

$$\tfrac{1}{\beta_n}(x_{n+1}-z_n)=\tfrac{1}{\beta_n}\big((1-\beta_n)z_n+\beta_n y_n-z_n\big)=y_n-z_n \tag{3.11}$$

and

$$\|y_n-z_n\|^2=\tfrac{1}{\beta_n^2}\|x_{n+1}-z_n\|^2=\tfrac{\alpha_n}{\beta_n}\Big(\tfrac{\|x_{n+1}-z_n\|^2}{\alpha_n\beta_n}\Big). \tag{3.12}$$

Using (3.9) and (3.11), we have

$$\|x_{n+1}-\bar{x}\|^2\leq \|z_n-\bar{x}\|^2-\tfrac{1-\beta_n}{\beta_n}\|x_{n+1}-z_n\|^2. \tag{3.13}$$

From the definition of z_n, we have

$$\begin{aligned}\|z_n-\bar{x}\|^2 &= \|(1-\alpha_n)x_n+\alpha_n u-\bar{x}\|^2 \\ &= (1-\alpha_n)^2\|x_n-\bar{x}\|^2+\alpha_n^2\|u-\bar{x}\|^2+2\alpha_n(1-\alpha_n)\langle x_n-\bar{x},u-\bar{x}\rangle \\ &= (1-\alpha_n)\|x_n-\bar{x}\|^2+\alpha_n^2\|u-\bar{x}\|^2+2\alpha_n(1-\alpha_n)\langle x_n-\bar{x},u-\bar{x}\rangle\end{aligned} \tag{3.14}$$

Thus, (3.13) and (3.14) gives

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2 &\leq (1-\alpha_n)\|x_n-\bar{x}\|^2+\alpha_n^2\|u-\bar{x}\|^2 \\ &\quad +2\alpha_n(1-\alpha_n)\langle x_n-\bar{x},u-\bar{x}\rangle-\tfrac{1-\beta_n}{\beta_n}\|x_{n+1}-z_n\|^2.\end{aligned} \tag{3.15}$$

That is,

$$\|x_{n+1}-\bar{x}\|^2\leq (1-\alpha_n)\|x_n-\bar{x}\|^2-\alpha_n\Gamma_n \tag{3.16}$$

where

$$\Gamma_n=-\alpha_n\|u-\bar{x}\|^2+2(1-\alpha_n)\langle\bar{x}-x_n,u-\bar{x}\rangle+\frac{1-\beta_n}{\alpha_n\beta_n}\|x_{n+1}-z_n\|^2.$$

We know that $\{x_n\}$ is bounded and so it is bounded below. Hence, Γ_n is bounded below. Furthermore, using Lemma 2.4 and (C1), we have

$$\limsup_{n\to\infty}\|x_n-\bar{x}\|\leq\limsup_{n\to\infty}(-\Gamma_n)=-\liminf_{n\to\infty}\Gamma_n. \tag{3.17}$$

Therefore, $\liminf_{n\to\infty}\Gamma_n$ is a finite real number and by (C1), we have

$$\liminf_{n\to\infty}\Gamma_n=\liminf_{n\to\infty}\big(2\langle\bar{x}-x_n,u-\bar{x}\rangle+\tfrac{1-\beta_n}{\alpha_n\beta_n}\|x_{n+1}-z_n\|^2\big).$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup p$ in H_1 and

$$\liminf_{n\to\infty}\Gamma_n=\liminf_{k\to\infty}\big(2\langle\bar{x}-x_{n_k},u-\bar{x}\rangle+\tfrac{1-\beta_{n_k}}{\alpha_{n_k}\beta_{n_k}}\|x_{n_k+1}-z_{n_k}\|^2\big). \tag{3.18}$$

Since $\{x_n\}$ is bounded and $\liminf_{n\to\infty}\Gamma_n$ is finite, we have that $\frac{1-\beta_{n_k}}{\alpha_{n_k}\beta_{n_k}}\|x_{n_k+1}-z_{n_k}\|^2$ is bounded. Also, by (C2), we have $\frac{1-\beta_n}{\alpha_n\beta_n}\geq\frac{1-\delta}{\alpha_n\beta_n}>0$ and so we have that $\frac{1}{\alpha_{n_k}\beta_{n_k}}\|x_{n_k+1}-z_{n_k}\|^2$ is bounded. Observe from (C1) and (C2), we have

$$0<\frac{\alpha_{n_k}}{\beta_{n_k}}\leq\frac{\alpha_{n_k}}{\beta}\to 0,\quad k\to\infty.$$

Therefore, we obtain from (3.12) and $\frac{\alpha_{n_k}}{\beta_{n_k}}\to 0,\ \ k\to\infty$ that

$$\|y_{n_k}-z_{n_k}\|\to 0,\quad k\to\infty. \tag{3.19}$$

From the definition of x_{n+1}, we have

$$\|x_{n_k+1}-z_{n_k}\|=\beta_{n_k}\|y_{n_k}-z_{n_k}\|\to 0,\quad k\to\infty$$

and

$$\|z_{n_k} - x_{n_k}\| = \alpha_{n_k}\|u - x_{n_k}\| \to 0, \quad k \to \infty. \tag{3.20}$$

Hence,

$$\|x_{n_k+1} - x_{n_k}\| \le \|x_{n_k+1} - z_{n_k}\| + \|z_{n_k} - x_{n_k}\| \to 0, \quad k \to \infty.$$

Now, using (3.7), we obtain

$$\begin{aligned} \rho_{n_k}(2\zeta - \rho_{n_k}) \sum_{j\in\Psi_{n_k}} \xi^j_{n_k} \tfrac{(h_j(z_{n_k})+l(z_{n_k}))^2}{\theta_j^2(z_{n_k})} &\le \|z_{n_k} - \bar{x}\|^2 - \|y_{n_k} - \bar{x}\|^2 \\ \le (\|z_{n_k} - \bar{x}\| - \|y_{n_k} - \bar{x}\|)(\|z_{n_k} - \bar{x}\| + \|y_{n_k} - \bar{x}\|) \\ = \|z_{n_k} - y_{n_k}\|(\|z_{n_k} - \bar{x}\| + \|y_{n_k} - \bar{x}\|). \end{aligned} \tag{3.21}$$

Therefore, (3.19), (3.21) and (C5) gives

$$\rho_{n_k}(2\zeta - \rho_{n_k}) \sum_{j\in\Psi_{n_k}} \xi^j_n \frac{(h_j(z_{n_k}) + l(z_{n_k}))^2}{\theta_j^2(z_{n_k})} \to 0, \quad k \to \infty. \tag{3.22}$$

Again using (C5) together with (3.22) yields

$$\sum_{j\in\Psi_{n_k}} \xi^j_n \frac{(h_j(z_{n_k}) + l(z_{n_k}))^2}{\theta_j^2(z_{n_k})} \to 0, \quad k \to \infty. \tag{3.23}$$

Hence, in view of (3.23) and restriction condition imposed on ξ^j_n, we have

$$\frac{(h_j(z_{n_k}) + l(z_{n_k}))^2}{\theta_j^2(z_{n_k})} \to 0, \quad k \to \infty \tag{3.24}$$

for all $j \in \Psi_{n_k}$.
For each $i \in \Phi$ and for each $j \in \Psi$, $\nabla h_j(.)$ and $\nabla l_i(.)$ are Lipschitz continuous with constant $\|A\|^2$ and 1, respectively. Since the sequence $\{z_n\}$ is bounded and

$$\|\nabla h_j(z_n)\| = \|\nabla h_j(z_n)\| = \|\nabla h_j(z_n) - \nabla h_j(\bar{x})\| \le \|A\|^2\|z_n - \bar{x}\|, \forall j \in \Psi,$$

$$\|\nabla l_i(z_n)\| = \|\nabla l_i(z_n)\| = \|\nabla l_i(z_n) - \nabla l_i(\bar{x})\| \le \|z_n - \bar{x}\|, \forall i \in \Phi,$$

we have the sequences $\{\|\nabla l_i(z_n)\|\}_{n=1}^{+\infty}$ and $\{\|\nabla h_j(z_n)\|\}_{n=1}^{+\infty}$ are bounded. Hence, the boundedness of $\{\|\nabla l_i(z_n)\|\}_{n=1}^{+\infty}$ for all $i \in \Phi$ gives $\{\|\nabla l(z_n)\|\}_{n=1}^{+\infty}$ is bounded. Thus, we have $\{\theta_j^2(z_n)\}_{n=1}^{+\infty}$ is bounded and hence $\{\theta_j^2(z_{n_k})\}_{k=1}^{+\infty}$ is bounded. Consequently, using (3.24), we have for each $j \in \Psi_{n_k}$

$$\lim_{k\to+\infty}(h_j(z_{n_k}) + l(z_{n_k})) = 0 \Leftrightarrow \lim_{k\to+\infty} h_j(z_{n_k}) = \lim_{k\to+\infty} l(z_{n_k}) = 0.$$

Since $\theta_j(z_{n_k}) = 0$ for each $j \notin \Psi_{n_k}$ and this results $h_j(z_{n_k}) = 0 = l(z_{n_k})$ for each $j \notin \Psi_{n_k}$. Hence, using $\lim\limits_{n\to+\infty} h_j(z_{n_k}) = \lim\limits_{k\to+\infty} l(z_n) = 0$ for each $j \in \Psi_{n_k}$ and $h_j(z_{n_k}) = 0 = l(z_{n_k})$ for each $j \notin \Psi_{n_k}$, we have

$$\lim_{k\to+\infty} h_j(z_{n_k}) = \lim_{k\to+\infty} l(z_{n_k}) = 0, \quad \forall j \in \Psi.$$

From the definition of $l(z_{n_k})$, we can have $l_i(z_{n_k}) \le l(z_{n_k}), \quad \forall i \in \Phi$. Therefore,

$$\lim_{k\to+\infty} h_j(z_{n_k}) = \lim_{k\to+\infty} l_i(z_{n_k}) = 0, \quad \forall i \in \Phi, \forall j \in \Psi.$$

Since $x_{n_k} \to p$ and using (3.20), we have $z_{n_k} \to p$.
The lower-semicontinuity of $h_j(.)$ implies that

$$0 \le h_j(p) \le \liminf_{k\to\infty} h_j(z_{n_k}) = \lim_{k\to\infty} h_j(z_{n_k}) = 0, \quad \forall j \in \Psi.$$

That is, $h_j(p) = \frac{1}{2}\|(I - \text{prox}_{\lambda g_j})Ap\|^2 = 0$ for all $j \in \Psi$, i.e., Ap is a fixed point of the proximal mapping of each g_j or equivalently, $0 \in \partial g_j(Ap)$ for all $j \in \Psi$. In other words, Ap is a minimizer of each g_j for all $j \in \Psi$. Likewise, the lower-semicontinuity of $l_i(.)$ implies that

$$0 \le l_i(p) \le \liminf_{k\to\infty} l_i(z_{n_k}) = \lim_{k\to\infty} l_i(z_{n_k}) = 0, \quad \forall i \in \Phi.$$

That is, $l_i(p) = \frac{1}{2}\|(I - \text{prox}_{\lambda f_i})p\|^2 = 0$ for all $i \in \Phi$, i.e., p is a fixed point of the proximal mapping of each f_i or equivalently, $0 \in \partial f_i(p)$ for all $i \in \Phi$. In other words, p is a minimizer of each f_i for all $i \in \Phi$. Thus, $p \in \Gamma$.

Now, we obtain from (3.18), Lemma 2.1 and $\bar{x} = P_\Gamma u$ that

$$\begin{aligned}\liminf_{n\to\infty} \Gamma_n &= \liminf_{k\to\infty} \left(2\langle \bar{x} - x_{n_k}, u - \bar{x}\rangle + \tfrac{1-\beta_{n_k}}{\alpha_{n_k}\beta_{n_k}}\|x_{n_k+1} - z_{n_k}\|^2\right)\\ &\geq 2\liminf_{k\to\infty}\langle \bar{x} - x_{n_k}, u - \bar{x}\rangle\\ &\geq 2\langle \bar{x} - p, u - \bar{x}\rangle \geq 0.\end{aligned}$$

Then we have from (3.17) that

$$\limsup_{n\to\infty} \|x_n - \bar{x}\|^2 \leq \limsup_{n\to\infty}(-\Gamma_n) = -\liminf_{n\to\infty} \Gamma_n \leq 0.$$

Therefore, $\|x_n - \bar{x}\| \to 0$ and this implies that $\{x_n\}$ converges strongly to $\bar{x}$. This completes the proof. □

It is worth mentioning that our approach also works for approximation of solution of split minimization problem (1.5). Let Ω_1 denote the solution set of (1.5), i.e.,

$$\Omega_1 = \{\bar{x} \in H_1 : \bar{x} \in \arg\min f \text{ and } A\bar{x} \in \arg\min g\}.$$

For $x \in H_1$, set $l(x) = \frac{1}{2}\|(I - \text{prox}_{\lambda f})x\|^2$, $\nabla l(x) = (I - \text{prox}_{\lambda f})x$, $h(x) = \frac{1}{2}\|(I - \text{prox}_{\lambda g})Ax\|^2$, $\nabla h(x) = A^*(I - \text{prox}_{\lambda g})Ax$ and $\theta(x) = \max\{\|\nabla h(x)\|, \|\nabla l(x)\|\}$. Thus, the following Corollary is an immediate consequence of Theorem 3.3.

Corollary 3.4. *If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\rho_n\}$ are real sequences satisfying the following conditions:*

- **(a) :** $0 < \alpha_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.
- **(b) :** $0 < \beta \leq \beta_n \leq \delta < 1$,
- **(c) :** $0 < \rho_n < 2\delta$, $\liminf_{n\to\infty} \rho_n(2\delta - \rho_n) > 0$.

then the sequence $\{x_n\}$ generated by iterative algorithm

$$\begin{cases} u, x_1 \in H_1,\\ z_n = (1-\alpha_n)x_n + \alpha_n u,\\ \mu_n = \begin{cases} \rho_n 0, & \text{if } \theta(z_n) = 0\\ \rho_n \frac{h(z_n)+l(z_n)}{\theta^2(z_n)}, & \text{if } \theta(z_n) \neq 0. \end{cases}\\ y_n = z_n - \frac{1}{2}\mu_n\big(\nabla l(z_n) + \nabla h(z_n)\big),\\ x_{n+1} = (1-\beta_n)z_n + \beta_n y_n, \end{cases}, \tag{3.25}$$

converges strongly to $\bar{x} \in \Omega_1$ where $\bar{x} = P_{\Omega_1} u$.

Proof. Setting $f_i = f$ for all $i \in \Phi$ and $g_j = g$ for all $j \in \Psi$ in Theorem 3.3, we obtain the desired result. □

Remark 3.5. Iterative algorithm (3.25) seems to share a similar structure with the proposed algorithm in [29]. However, the selection of the step-sizes and their restriction slightly different.

The feasibility problem (convex feasibility problem), equilibrium problem and inclusion problem can be converted to the fixed point problem of firmly nonexpansive mapping. We can apply our algorithm to solve split system of feasibility problems (MSSFPs), split system of equilibrium problems and split system of inclusion problems.

1. *Multiple-set split feasibility problem* (1.1) by replacing $\text{prox}_{\lambda f_i}$ by projection mapping P_{C_i} and $\text{prox}_{\lambda g_j}$ by projection mapping P_{Q_j} in the Algorithm 1, for all $i' \in \Phi'$, $i \in \Phi = \{1, 2, \ldots, N\}$ and $j \in \Psi = \{1, 2, \ldots, M\}$.

2. *Split system of equilibrium problem:* Let $f_i : H_1 \times H_1 \to \mathbb{R}$ and $g_j : H_2 \times H_2 \to \mathbb{R}$ be bifunctions where $i \in \Phi = \{1, \dots, N\}$, $j \in \Psi = \{1, \dots, M\}$. Split system of equilibrium problem of a problem of find $\bar{x} \in H_1$ such that
$$\begin{cases} f_i(\bar{x}, x) \geq 0, & \forall x \in H_1, \forall i \in \Phi, \\ g_j(A\bar{x}, u) \geq 0, & \forall u \in H_2, , \forall j \in \Psi. \end{cases} \tag{3.26}$$
Our iterative algorithm solves (3.26) by replacing proximal mappings by the resolvent operators associated to monotone equilibrium bifunctions, see [11, 3, 22].
3. *Split null point problem*: Let $T_i : H_1 \to 2^{H_1}$, $U_j : H_2 \to 2^{H_2}$ be maximal monotone mappings for all $i \in \Phi = \{1, \dots, N\}$ and $j \in \Psi = \{1, \dots, M\}$. The split system of inclusion problem is to find $\bar{x} \in H_1$ such that
$$\begin{cases} 0 \in T_i(\bar{x}), & \forall i \in \Phi, \\ 0 \in U_j(A\bar{x}), & \forall j \in \Psi. \end{cases} \tag{3.27}$$
Our iterative algorithm solves (3.27) by replacing proximal mappings by the resolvent operators associated to the maximal monotone operators, see, [4, 25, 16, 20, 23, 36].

Our algorithm works for several split type problems and avoids the computational cost of finding operator norm.

4. Numerical results

Now in this section we will consider SSMP (1.8) involving quadratic optimization problems. The algorithm has been coded in Matlab R2017a running on MacBook 1.1 GHz Intel Core m3 8 GB 1867 MHz LPDDR3. Let $H_1 = \mathbb{R}^p$ and $H_2 = \mathbb{R}^q$. Consider
$$f_i(x) = \frac{1}{2} x^T B_i x + x^T D_i, \quad i \in \Phi = \{1, \dots, N\},$$
$$g_1(u) = \|u\|_q \text{ and } g_2(u) = \sum_{k=1}^{q} h(u_k)$$
where for each $i \in \Phi$, B_i is invertible symmetric positive semidefinite $p \times p$ matrix and each D_i are vectors in $\mathbb{R}^p$, $u = (u_1, u_2, \dots, u_q) \in \mathbb{R}^q$, $\|.\|_q$ is the Euclidean norm in $\mathbb{R}^q$ and
$$h(u_k) = \max\{|u_k| - 1, 0\}$$
for $k = 1, 2, \dots, q$.
Now for $\lambda = 1$, the proximal operators are given by
$$\operatorname{prox}_{\lambda f_i}(x) = (I + B_i)^{-1}(x - D_i), \quad i \in \Phi,$$
$$\operatorname{prox}_{\lambda g_1}(u) = \begin{cases} \left(1 - \frac{1}{\|u\|_q}\right)u, & \|u\|_q \geq 1 \\ 0, & \text{otherwise} \end{cases} \tag{4.1}$$
and
$$\operatorname{prox}_{\lambda g_2}(u) = (\operatorname{prox}_{\lambda h}(u_1), \operatorname{prox}_{\lambda h}(u_2), \dots, \operatorname{prox}_{\lambda h}(u_q))$$
where
$$\operatorname{prox}_{\lambda h}(u_k) = \begin{cases} u_k, & \text{if } |u_k| < 1 \\ \operatorname{sign}(u_k), & \text{if } 1 \leq |u_k| \leq 2 \\ \operatorname{sign}(u_k - 1), & \text{if } |u_k| > 2. \end{cases}$$
The proximal operator (4.1) is called the block soft thresholding obtained in de-noising model.
We set $D_i = 0$ (zero vector in $\mathbb{R}^p$) for all $i \in \Phi$. Let $N = 3$, $p = q$, A is identity $p \times p$ matrix and B_1, B_2 and B_3 are randomly generated invertible symmetric positive semidefinite $p \times p$ matrices. Hence, with this setting, it is clear to see that $\Gamma = \{0\}$. In all the experiments we took $\delta_n^i = \frac{i}{6}$ and $\xi_n^j = \frac{j}{3}$ for $i \in \Phi = \{1, 2, 3\}$, $j \in \Psi = \{1, 2\}$, $\rho_n = \frac{1}{10}$ as $0 < \rho_n < 2\zeta$ for $\zeta = \frac{1}{6}$. Table 1, 2 and 3 describe the average execution time in second (CPU-t(s)) and the number of iterations (Iter(n)) of our algorithm for this example. The stopping criteria in the tables 1, 2 and 3 is defined as $\frac{\|x_{n+1} - x_n\|}{\|x_2 - x_1\|} \leq \text{TOL}$.

TABLE 1. For $p = q = 4$, $\alpha_n = \frac{1}{\sqrt{n+1}}$, $\beta_n = 0.9$, $u = (1, 1, 1, 1)$, $x_1 = 10u$.

Iter(n)	TOL	CPU-t(s)	x_n				$\|x_{n+1} - x_n\|$
			x_n^1	x_n^2	x_n^3	x_n^4	
1			10	10	10	10	9.0304
2			5.4841	5.4854	5.4842	5.4852	3.0046
3			3.9817	3.9834	3.9820	3.9826	1.5000
4			3.2317	3.2341	3.2314	3.2326	0.8991
5			2.7825	2.7839	2.7820	2.7831	0.5988
6			2.4828	2.4844	2.4827	2.4838	0.4277
7			2.2691	2.2702	2.2690	2.2699	0.3204
$\vdots$			$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
23			1.3796	1.3800	1.3796	1.3799	0.0323
24	10^{-3}	0.0352	1.3634	1.3638	1.3634	1.3637	0.0297

TABLE 2. For $p = q = 100$, $\alpha_n = \frac{1}{n+1}$, $\beta_n = 0.5$ and randomly generated starting points u and x_1 in $\mathbb{R}^{100}$.

Iter(n)	TOL	CPU-t(s)	$\|x_{n+1} - x_n\|$
1			6383.1845
2			1519.9088
3			554.2387
4			247.0358
5			124.2771
$\vdots$			$\vdots$
15			1.5845
16	10^{-4}	0.2923	0.5736

TABLE 3. For $p = q = 200$, $\alpha_n = \frac{1}{10(n+1)})$, $\beta_n = 0.1$ and randomly generated starting points u and x_1 in $\mathbb{R}^{200}$.

Iter(n)	TOL	CPU-t(s)	$\|x_{n+1} - x_n\|$
1			14554.8769
2			3475.8500
3			1270.6095
4			567.6027
5			286.1360
6	10^{-2}	0.0093	156.6170

From the tables 1-3 we can see that our proposed algorithm is efficient and easy to implement.

REFERENCES

1. M. Abbas, M. AlShahrani, Q. Ansari, O.S. Iyiola and Y. Shehu, Iterative methods for solving proximal split minimization problems, *Numer. Algorithms* pp. 1–23 (2018)
2. H.H. Bauschke, P.L. Combettes, L. Patrick and others, Convex analysis and monotone operator theory in Hilbert spaces, vol. 408. Springer (2011)
3. E. Blum, From optimization and variational inequalities to equilibrium problems, *Math. Student* 63, 123–145 (1994)
4. H. Brezis, Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, vol. 5. Elsevier (1973)

5. C. Byrne, Iterative oblique projection onto convex sets and the split feasibility problem. *Inverse probl.* 18(2), 441 (2002)
6. C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse probl.* 20(1), 103 (2003)
7. Y. Censor, T. Bortfeld, B. Martin and A. Trofimov, The split feasibility model leading to a unified approach for inversion problems in intensity-modulated radiation therapy, *Technical Report 20 April: Department of Mathematics, University of Haifa, Israel* (2005)
8. Y. Censor, T. Bortfeld, B. Martin and A. Trofimov, A unified approach for inversion problems in intensity-modulated radiation therapy, *Phy. Med. Biol.* 51(10), 2353 (2006)
9. Y. Censor and T. Elfving, A multiprojection algorithm using bregman projections in a product space, *Numer. Algorithms* 8(2), 221–239 (1994)
10. Y. Censor, T. Elfving, N. Kopf and T. Bortfeld, The multiple-sets split feasibility problem and its applications for inverse problems, *Inverse probl.* 21(6), 2071 (2005)
11. P.L. Combettes, S.A. Hirstoaga, and others, Equilibrium programming in hilbert spaces, *J. Nonlinear Convex Anal.* 6(1), 117–136 (2005)
12. P.L. Combettes and J.C. Pesquet, Proximal splitting methods in signal processing, *In: Fixed-point algorithms for inverse problems in science and engineering*, pp. 185–212. Springer (2011)
13. Y. Dang and Y. Gao, The strong convergence of a three-step algorithm for the split feasibility problem, *Optim. Lett.* 7(6), 1325–1339 (2013)
14. Y. Dang, Y. Gao and L. Li, Inertial projection algorithms for convex feasibility problem, *J. Syst. Eng. Electron* 23(5), 734–740 (2012)
15. A. Latif, J. Vahidi, and M. Eslamian, Strong convergence for generalized multiple-set split feasibility problem, Filomat 30(2), 459–467 (2016)
16. B. Lemaire, Which fixed point does the iteration method select? *In: Recent Advances in Optimization*, pp. 154–167. Springer (1997)
17. G. López, V. Martín-Márquez, F. Wang and H.K. Xu, Solving the split feasibility problem without prior knowledge of matrix norms, *Inverse Probl.* 28(8), 085004 (2012)
18. P.E. Maingé and Ş. Măruşter, Convergence in norm of modified krasnoselski–mann iterations for fixed points of demicontractive mappings, *Appl. Math. Comput.* 217(24), 9864–9874 (2011)
19. E. Masad and S. Reich, A note on the multiple-set split convex feasibility problem in Hilbert space, *J. Nonlinear Convex Anal.* 8(3), 367 (2007)
20. A. Moudafi, Split monotone variational inclusions, *J. Comput. Appl. Math.* 150(2), 275–283 (2011)
21. A. Moudafi, B. Thakur, Solving proximal split feasibility problems without prior knowledge of operator norms, *Optim. Lett.* 8(7), 2099–2110 (2014)
22. A. Moudafi and M. Théra, Proximal and dynamical approaches to equilibrium problems, *In: Ill-Posed Variational Problems and Regularization Techniques* pp. 187–201. Springer (1999)
23. S. Plubtieng and W. Sriprad, A viscosity approximation method for finding common solutions of variational inclusions, equilibrium problems, and fixed point problems in hilbert spaces, *Fixed Point Theory Appl.* 2009(1), 567147 (2009)
24. B. Qu, and N. Xiu, A note on the CQ algorithm for the split feasibility problem, *Inverse Probl.* 21(5), 1655 (2005)
25. R. Rockafellar, On the maximality of sums of nonlinear monotone operators, *Trans. Amer. Math. Soc.* 149(1), 75–88 (1970)
26. R.T. Rockafellar and R.J.B. Wets, Variational analysis, vol. 317. Springer Science & Business Media (2009)
27. Y. Shehu, G. Cai and O.S. Iyiola, Iterative approximation of solutions for proximal split feasibility problems, *Fixed Point Theory Appl.* 2015(1), 123 (2015)
28. Y. Shehu and O.S. Iyiola, Convergence analysis for the proximal split feasibility problem using an inertial extrapolation term method, *J. Fixed Point Theory Appl.* 19(4), 2483– 2510 (2017)
29. Y. Shehu and O.S. Iyiola, Strong convergence result for proximal split feasibility problem in Hilbert spaces, *Optimization* 66(12), 2275–2290 (2017)
30. Y. Shehu and O.S. Iyiola, Accelerated hybrid viscosity and steepest-descent method for proximal split feasibility problems, *Optimization* 67(4), 475–492 (2018)
31. Y. Shehu and O.S. Iyiola, Nonlinear iteration method for proximal split feasibility problems, *Math. Meth. Appl. Sci.* (2018)
32. Y. Shehu and F.U.Ogbuisi, Convergence analysis for proximal split feasibility problems and fixed point problems, *J. Appl. Math. Comp.* 48(1-2), 221–239 (2015)
33. Y. Wang, F. Wang and H.K. Xu, Error sensitivity for strongly convergent modifications of the proximal point algorithm, *J. Opt. Theory Appl.* 168(3), 901–916 (2016)
34. M. Wen, J. Peng and Y. Tang, A cyclic and simultaneous iterative method for solving the multiple-sets split feasibility problem, *J. Opt. Theory Appl.* 166(3), 844–860 (2015)
35. H.K. Xu, A variable krasnosel'skii–mann algorithm and the multiple-set split feasibility problem, *Inverse probl.* 22(6), 2021 (2006)
36. Y. Yao, Y.J. Cho and Y.C. Liou, Iterative algorithms for variational inclusions, mixed equilibrium and fixed point problems with application to optimization problems, *Cent. Eur. J. Math.* 9(3), 640–656 (2011)
37. H. Zegeye and N. Shahzad, Convergence of mann's type iteration method for generalized asymptotically nonexpansive mappings, *Comput. Math. Appl.* 62(11), 4007–4014 (2011)

Complex Korovkin Theory

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Let K be a compact convex subspace of $\mathbb{C}$ and $C(K,\mathbb{C})$ the space of continuous functions from K into $\mathbb{C}$. We consider bounded linear functionals from $C(K,\mathbb{C})$ into $\mathbb{C}$ and bounded linear operators from $C(K,\mathbb{C})$ into itself. We assume that these are bounded by companion real positive linear entities, respectively. We study quantitatively the rate of convergence of the approximation of these linearities to the corresponding unit elements. Our results are inequalities of Korovkin type involving the complex modulus of continuity and basic test functions.

2010 Mathematics Subject Classification : 41A17, 41A25, 41A36.
Keywords and phrases: positive linear functional, positive linear operator, complex functions, Korovkin theory, complex modulus of continuity.

1 Introduction

The study of the convergence of positive linear operators became more intensive and attractive when P. Korovkin (1953) proved his famous theorem (see [7], p. 14).

Korovkin's First Theorem. Let $[a,b]$ be a compact interval in $\mathbb{R}$ and $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators L_n mapping $C([a,b])$ into itself. Assume that $(L_n f)$ converges uniformly to f for the three test functions $f = 1, x, x^2$. Then $(L_n f)$ converges uniformly to f on $[a,b]$ for all functions of $f \in C([a,b])$.

So a lot of authors since then have worked on the theoretical aspects of the above convergence. But R. A. Mamedov (1959) (see [8]) was the first to put Korovkin's theorem in a quantitative scheme.

Mamedov's Theorem. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators in the space $C([a,b])$, for which $L_n 1 = 1$, $L_n(t,x) = x + \alpha_n(x)$, $L_n(t^2,x) = x^2 + \beta_n(x)$. Then it holds

$$\|L_n(f,x) - f(x)\|_\infty \leq 3\omega_1\left(f, \sqrt{d_n}\right),$$

1

where ω_1 is the first modulus of continuity and $d_n = \|\beta_n(x) - 2x\alpha_n(x)\|_\infty$.

An improvement of the last result was the following.

Shisha and Mond's Theorem. (1968, see [10]). Let $[a,b] \subset \mathbb{R}$ be a compact interval. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear operators acting on $C([a,b])$. For $n = 1,2,...$, suppose $L_n(1)$ is bounded. Let $f \in C([a,b])$. Then for $n = 1,2,...$, it holds

$$\|L_n f - f\|_\infty \leq \|f\|_\infty \cdot \|L_n 1 - 1\|_\infty + \|L_n(1) + 1\|_\infty \cdot \omega_1(f, \mu_n),$$

where

$$\mu_n := \left\|\left(L_n\left((t-x)^2\right)\right)(x)\right\|_\infty^{\frac{1}{2}}.$$

Shisha-Mond inequality generated and inspired a lot of research done by many authors worldwide on the rate of convergence of a sequence of positive linear operators to the unit operator, always producing similar inequalities however in many different directions, e.g., see the important work of H. Censka of 1983 in [6], etc.

The author (see [1]) in his 1993 research monograph, produces in many directions best upper bounds for $|(L_n f)(x_0) - f(x_0)|$, $x_0 \in Q \subseteq \mathbb{R}^n$, $n \geq 1$, compact and convex, which lead for the first time to sharp/attained inequalities of Shisha-Mond type. The method of proving is probabilistic from the theory of moments. His pointwise approach is closely related to the study of the weak convergence with rates of a sequence of finite positive measures to the unit measure at a specific point.

The author in [3], pp. 383-412 continued this work in an abstract setting: Let X be a normed vector space, Y be a Banach lattice; $M \subset X$ is a compact and convex subset. Consider the space of continuous functions from M into Y, denoted by $C(M,Y)$; also consider the space of bounded functions $B(M,Y)$. He studied the rate of the uniform convergence of lattice homomorphisms $T : C(M,Y) \to C(M,Y)$ or $T : C(M,Y) \to B(M,Y)$ to the unit operator I. See also [2].

Also the author in [4], pp. 175-188 continued the last abstract work for bounded linear operators that are bounded by companion real positive linear operators. Here the invoved functions are from $[a,b] \subset \mathbb{R}$ into $(X, \|\cdot\|)$ a Banach space.

All the above have inspired and motivated the work of this article. Our results are of Shisha-Mond type, i.e., of Korovkin type.

Namely here let K be a convex and compact subset of $\mathbb{C}$ and l be a linear functional from $C(K,\mathbb{C})$ into $\mathbb{C}$, and let $\widetilde{l}$ be a positive linear functional from $C(K,\mathbb{R})$ into $\mathbb{R}$, such that $|l(f)| \leq \widetilde{l}(|f|)$, $\forall\, f \in C(K,\mathbb{C})$.

Clearly then l is a bounded linear functional. Initially we create a quantitative Korovkin type theory over the last described setting, then we transfer these results to related bounded linear operators with similar properties.

2 Background

We need

Theorem 1 *Let $K \subseteq (\mathbb{C}, |\cdot|)$ and f a function from K into $\mathbb{C}$. Consider the first complex modulus of continuity*

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in K \\ |x-y|<\delta}} |f(x) - f(y)|, \quad \delta > 0. \tag{1}$$

We have:

(1)' If K is open convex or compact convex, then $\omega_1(f,\delta) < \infty$, $\forall\ \delta > 0$, where $f \in UC(K,\mathbb{C})$ (uniformly continuous functions).

(2)' If K is open convex or compact convex, then $\omega_1(f,\delta)$ is continuous on $\mathbb{R}_+$ in δ, for $f \in UC(K,\mathbb{C})$.

(3)' If K is convex, then

$$\omega_1(f, t_1 + t_2) \leq \omega_1(f, t_1) + \omega_1(f, t_2), \quad t_1, t_2 > 0, \tag{2}$$

that is the subadditivity property is true. Also it holds

$$\omega_1(f, n\delta) \leq n\omega_1(f,\delta) \tag{3}$$

and

$$\omega_1(f, \lambda\delta) \leq \lceil\lambda\rceil\, \omega_1(f,\delta) \leq (\lambda+1)\,\omega_1(f,\delta), \tag{4}$$

where $n \in \mathbb{N}$, $\lambda > 0$, $\delta > 0$, $\lceil\cdot\rceil$ is the ceiling of the number.

(4)' Clearly in general $\omega_1(f,\delta) \geq 0$ and is increasing in $\delta > 0$ and $\omega_1(f,0) = 0$.

(5)' If K is open or compact, then $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, iff $f \in UC(K,\mathbb{C})$.

(6)' It holds

$$\omega_1(f+g,\delta) \leq \omega_1(f,\delta) + \omega_1(g,\delta), \tag{5}$$

for $\delta > 0$, any $f, g : K \to \mathbb{C}$, $K \subset \mathbb{C}$ is arbitrary.

Proof. (1)' Here K is open convex. Let here $f \in UC(K,\mathbb{C})$, iff $\forall\ \varepsilon > 0$, $\exists\ \delta > 0 : |x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Let $\varepsilon_0 > 0$ then $\exists\ \delta_0 > 0$: $|x-y| \leq \delta_0$ with $|f(x) - f(y)| < \varepsilon_0$, hence $\omega_1(f,\delta_0) \leq \varepsilon_0 < \infty$.

Let $\delta > 0$ arbitrary and $x, y \in K : |x-y| \leq \delta$. Choose $n \in \mathbb{N} : n\delta_0 > \delta$, and set $x_i = x + \frac{i}{n}(y-x)$, $0 \leq i \leq n$. Notice that all $x_i \in K$. Then

$$|f(x) - f(y)| = \left|\sum_{i=0}^{n-1} (f(x_i) - f(x_{i+1}))\right| \leq$$

$$|f(x) - f(x_1)| + |f(x_1) - f(x_2)| + |f(x_2) - f(x_3)| + ... + |f(x_{n-1}) - f(y)| \leq$$

$$n\omega_1(f,\delta_0) \leq n\varepsilon_0 < \infty,$$

since $|x_i - x_{i+1}| = \frac{1}{n}|x-y| \leq \frac{1}{n}\delta < \delta_0$.

3

Thus $\omega_1(f,\delta) \leq n\varepsilon_0 < \infty$, proving the claim. If K is compact convex, then claim is obvious.

(2)' Let $x, y \in K$ and let $|x-y| \leq t_1+t_2$, then there exists a point $z \in \overline{xy}$, $z \in K : |x-z| \leq t_1$ and $|y-z| \leq t_2$, where $t_1, t_2 > 0$.

Notice that

$$|f(x)-f(y)| \leq |f(x)-f(z)| + |f(z)-f(y)| \leq \omega_1(f,t_1) + \omega_1(f,t_2).$$

Hence

$$\omega_1(f,t_1+t_2) \leq \omega_1(f,t_1) + \omega_1(f,t_2),$$

proving (3)'. Then by the obvious property (4)' we get

$$0 \leq \omega_1(f,t_1+t_2) - \omega_1(f,t_1) \leq \omega_1(f,t_2),$$

and

$$|\omega_1(f,t_1+t_2) - \omega_1(f,t_1)| \leq \omega_1(f,t_2).$$

Let $f \in UC(K,\mathbb{C})$, then $\lim_{t_2 \downarrow 0} \omega_1(f,t_2) = 0$, by property (5)'. Hence $\omega_1(f,\cdot)$ is continuous on $\mathbb{R}_+$.

(5)' ($\Rightarrow$) Let $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$. Then $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ with $\omega_1(f,\delta) \leq \varepsilon$. I.e. $\forall\, x, y \in K : |x-y| \leq \delta$ we get $|f(x)-f(y)| \leq \varepsilon$. That is $f \in UC(K,\mathbb{C})$.

($\Leftarrow$) Let $f \in UC(K,\mathbb{C})$. Then $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$: whenever $|x-y| \leq \delta$, $x, y \in K$, it implies $|f(x)-f(y)| \leq \varepsilon$. I.e. $\forall\, \varepsilon > 0$, $\exists\, \delta > 0 : \omega_1(f,\delta) \leq \varepsilon$. That is $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$.

(6)' Notice that

$$|(f(x)+g(x)) - (f(y)+g(y))| \leq |f(x)-f(y)| + |g(x)-g(y)|.$$

That is property (6)' now is clear. ■

We need

Theorem 2 *([1], p. 208) Let $(V_1, \|\cdot\|)$, $(V_2, \|\cdot\|)$ be real normed vector spaces and $Q \subseteq V_1$ which is star- shaped relative to the fixed point x_0. Consider $f : Q \to V_2$ with the properties:*

$$f(x_0) = 0, \text{ and } \|s-t\| \leq h \text{ implies } \|f(s)-f(t)\| \leq w; \quad w, h > 0. \tag{6}$$

Then, there exists a maximal such function Φ, namely

$$\Phi(t) := \left\lceil \frac{\|t-x_0\|}{h} \right\rceil \cdot w \cdot \overrightarrow{i}, \tag{7}$$

where $\overrightarrow{i}$ is any unit vector in V_2.

That is

$$\|f(t)\| \leq \|\Phi(t)\|, \text{ all } t \in Q. \tag{8}$$

4

Corollary 3 *Let $K \subseteq (\mathbb{C}, |\cdot|)$ be a compact convex subset, and $f \in C(K, \mathbb{C})$. Then*

$$|f(x) - f(x_0)| \leq \omega_1(f, \delta) \left\lceil \frac{|x - x_0|}{\delta} \right\rceil, \quad \delta > 0, \tag{9}$$

$\forall\ x, x_0 \in K$.

We make

Remark 4 *Let $K \subseteq (\mathbb{C}, |\cdot|)$ be a compact subset and $g \in C(K, \mathbb{R})$.*

A linear functional I from $C(K, \mathbb{R})$ into $\mathbb{R}$ is positive, iff $I(g_1) \geq I(g_2)$, whenever $g_1 \geq g_2$, where $g_1, g_2 \in C(K, \mathbb{R})$.

Let us assume that I is a positive linear functional. Then by Riesz representation theorem, [9], p. 304, there exists a unique Borel measure μ on K such that

$$I(g) = \int_K g(t)\, d\mu(t), \tag{10}$$

$\forall\ g \in C(K, \mathbb{R})$.

We make

Remark 5 *Here initially we follow [5].*

Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a, b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$.

We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\, dz = \int_{\gamma_{u,w}} f(z)\, dz := \int_a^b f(z(t))\, z'(t)\, dt. \tag{11}$$

By triangle inequality we have

$$\left| \int_\gamma f(z)\, dz \right| = \left| \int_a^b f(z(t))\, z'(t)\, dt \right| \leq \int_a^b |f(z(t))|\, |z'(t)|\, dt := \int_\gamma |f(z)|\, |dz|. \tag{12}$$

Inequalities (12) provide a typical example on linear functionals: clearly $\int_\gamma f(z)\, dz$ induces a linear functional from $C(\gamma, \mathbb{C})$ into $\mathbb{C}$, and $\int_\gamma |f(z)|\, |dz|$ involves a positive linear functional from $C(\gamma, \mathbb{R})$ into $\mathbb{R}$.

Thus, be given K a convex and compact subset of $\mathbb{C}$ and l be a linear functional from $C(K, \mathbb{C})$ into $\mathbb{C}$, it is not strange to assume that there exists a positive linear functional $\widetilde{l}$ from $C(K, \mathbb{R})$ into $\mathbb{R}$, such that

$$|l(f)| \leq \widetilde{l}(|f|), \quad \forall\ f \in C(K, \mathbb{C}). \tag{13}$$

Furthermore, we may assume that $\widetilde{l}(1(\cdot)) = 1$, where $1(t) = 1$, $\forall\ t \in K$, $l(c(\cdot)) = c, \forall c \in \mathbb{C}$ where $c(t) = c$, $\forall\ t \in K\cdot$

We call $\widetilde{l}$ the companion functional to l.

Here $\mathbb{C}$ is a vector space over the field of reals. The functional l is linear over $\mathbb{R}$ and the functional $\widetilde{l}$ is linear over $\mathbb{R}$.

Next we study approximation properties of $\left(l_n, \widetilde{l}_n\right)$ pairs, $n \in \mathbb{N}$.

5

3 Main Results - I

First about linear functionals:

We present the following quantitative approximation result of Korovkin type.

Theorem 6 *Here K is a convex and compact subset of $\mathbb{C}$ and l_n is a sequence of linear functionals from $C(K,\mathbb{C})$ into $\mathbb{C}$, $n \in \mathbb{N}$. There is a sequence of companion positive linear functionals $\widetilde{l}_n$ from $C(K,\mathbb{R})$ into $\mathbb{R}$, such that*

$$|l_n(f)| \leq \widetilde{l}_n(|f|), \ \ \forall\ f \in C(K,\mathbb{C}), \ \forall\ n \in \mathbb{N}. \tag{14}$$

Additionally, we assume that $\widetilde{l}_n(1(\cdot)) = 1$ and $l_n(c(\cdot)) = c, \forall c \in \mathbb{C}\ \forall\ n \in \mathbb{N}$.

Then

$$|l_n(f) - f(x_0)| \leq 2\omega_1\left(f, \widetilde{l}_n(|\cdot - x_0|)\right), \ \ \forall\ n \in \mathbb{N}, \ \forall\ x_0 \in K, \tag{15}$$

$\forall\ f \in C(K,\mathbb{C})$.

Proof. We notice that

$$|l_n(f) - f(x_0)| = |l_n(f) - l_n(f(x_0)(\cdot))| =$$

$$|l_n(f(\cdot) - f(x_0)(\cdot))| \overset{(14)}{\leq} \widetilde{l}_n(|f(\cdot) - f(x_0)(\cdot)|) \overset{(\text{by } \delta>0, \ (9))}{\leq}$$

$$\widetilde{l}_n\left(\omega_1(f,\delta)\left\lceil \frac{|\cdot - x_0|}{\delta} \right\rceil\right) \leq \omega_1(f,\delta)\,\widetilde{l}_n\left(1(\cdot) + \frac{|\cdot - x_0|}{\delta}\right) =$$

$$\omega_1(f,\delta)\left[\widetilde{l}_n(1(\cdot)) + \frac{1}{\delta}\widetilde{l}_n(|\cdot - x_0|)\right] =$$

$$\omega_1(f,\delta)\left[1 + \frac{1}{\delta}\widetilde{l}_n(|\cdot - x_0|)\right] = 2\omega_1\left(f, \widetilde{l}_n(|\cdot - x_0|)\right), \tag{16}$$

by choosing

$$\delta := \widetilde{l}_n(|\cdot - x_0|),$$

if $\widetilde{l}_n(|\cdot - x_0|) > 0$, that is proving (15).

Next, we consider the case of $\widetilde{l}_n(|\cdot - x_0|) = 0$. By Riesz representation theorem, see (10) there exists a probability measure μ such that

$$\widetilde{l}_n(g) = \int_K g(t)\,d\mu(t), \ \ \forall\ g \in C(K,\mathbb{R}). \tag{17}$$

That is, here it holds

$$\int_K |t - x_0|\,d\mu(t) = 0,$$

which implies $|t - x_0| = 0$, a.e, hence $t - x_0 = 0$, a.e, and $t = x_0$, a.e. Consequently $\mu(\{t \in K : t \neq x_0\}) = 0$. Hence $\mu = \delta_{x_0}$, the Dirac measure with support only $\{x_0\}$.

Therefore in that case $\widetilde{l}_n(g) = g(x_0)$, $\forall\ g \in C(K,\mathbb{R})$. Thus, it holds $\omega_1\left(f, \widetilde{l}_n(|\cdot - x_0|)\right) = \omega_1(f,0) = 0$, and $\widetilde{l}_n(|f(\cdot) - f(x_0)(\cdot)|) = |f(x_0) - f(x_0)| = 0$, giving $|l_n(f) - f(x_0)| = 0$. That is (15) is again true. ■

6

Remark 7 *We have that*

$$\widetilde{l}_n\left(|\cdot - x_0|\right) = \int_K |t - x_0|\, d\mu\left(t\right)$$

(by Schwarz's inequality)

$$\leq \left(\int_K 1 d\mu\left(t\right)\right)^{\frac{1}{2}} \left(\int_K |t - x_0|^2\, d\mu\left(t\right)\right)^{\frac{1}{2}} =$$

$$\left(\widetilde{l}_n\left(1\right)\right)^{\frac{1}{2}} \left(\int_K |t - x_0|^2\, d\mu\left(t\right)\right)^{\frac{1}{2}} = \left(\widetilde{l}_n\left(|\cdot - x_0|^2\right)\right)^{\frac{1}{2}}. \tag{18}$$

We give

Corollary 8 *All as in Theorem 6. Then*

$$\left|l_n\left(f\right) - f\left(x_0\right)\right| \leq 2\omega_1\left(f, \left(\widetilde{l}_n\left(|\cdot - x_0|^2\right)\right)^{\frac{1}{2}}\right), \quad \forall\ n \in \mathbb{N},\ \forall\ x_0 \in K. \tag{19}$$

Conclusion 9 *All as in Theorem 6. By (15) and/or (19), as* $\widetilde{l}_n\left(|\cdot - x_0|\right) \to 0$*, or* $\widetilde{l}_n\left(|\cdot - x_0|^2\right) \to 0$*, as* $n \to +\infty$*, we obtain that* $l_n\left(f\right) \to f\left(x_0\right)$ *with rates,* $\forall$ $x_0 \in K$*.*

Next comes a more general quantitative approximation result of Korovkin type.

Theorem 10 *Here* K *is a convex and compact subset of* $\mathbb{C}$ *and* l_n *is a sequence of linear functionals from* $C\left(K, \mathbb{C}\right)$ *into* $\mathbb{C}$*,* $n \in \mathbb{N}$*. There is a sequence of companion positive linear functionals* $\widetilde{l}_n$ *from* $C\left(K, \mathbb{R}\right)$ *into* $\mathbb{R}$*, such that*

$$\left|l_n\left(f\right)\right| \leq \widetilde{l}_n\left(|f|\right), \ \ \forall\ f \in C\left(K, \mathbb{C}\right), \ \forall\ n \in \mathbb{N}. \tag{20}$$

Additionally, we assume that

$$l_n\left(cg\right) = c\widetilde{l}_n\left(g\right), \quad \forall\ g \in C\left(K, \mathbb{R}\right), \ \forall\ c \in \mathbb{C}. \tag{21}$$

Then, for any $f \in C\left(K, \mathbb{C}\right)$*, we have*

$$\left|l_n\left(f\right) - f\left(x_0\right)\right| \leq \left|f\left(x_0\right)\right| \left|\widetilde{l}_n\left(1\left(\cdot\right)\right) - 1\right| + \left(\widetilde{l}_n\left(1\left(\cdot\right)\right) + 1\right)\omega_1\left(f, \widetilde{l}_n\left(|\cdot - x_0|\right)\right), \tag{22}$$

$\forall$ $x_0 \in K$*,* $\forall$ $n \in \mathbb{N}$*.*

(Notice if $\widetilde{l}_n\left(1\left(\cdot\right)\right) = 1$*, then (22) collapses to (15). So Theorem 10 generalizes Theorem 6).*

By (22), as $\widetilde{l}_n\left(1\left(\cdot\right)\right) \to 1$ *and* $\widetilde{l}_n\left(|\cdot - x_0|\right) \to 0$*, then* $l_n\left(f\right) \to f\left(x_0\right)$*, as* $n \to +\infty$*, with rates, and as here* $\widetilde{l}_n\left(1\left(\cdot\right)\right)$ *is bounded.*

Proof. We observe that

$$\begin{aligned} &\left|l_n(f)-f(x_0)\right| = \left|l_n(f)-l_n(f(x_0)(\cdot))+l_n(f(x_0)(\cdot))-f(x_0)\right| \leq \\ &\left|l_n(f)-l_n(f(x_0)(\cdot))\right| + \left|f(x_0)\widetilde{l}_n(1(\cdot))-f(x_0)\right| = \\ &\left|l_n(f(\cdot)-f(x_0)(\cdot))\right| + \left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot))-1\right| \leq \qquad (23) \\ &\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot))-1\right| + \widetilde{l}_n(\left|f(\cdot)-f(x_0)(\cdot)\right|) \leq \\ &\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot))-1\right| + \widetilde{l}_n\left(\omega_1(f,\delta)\left\lceil\frac{|\cdot-x_0|}{\delta}\right\rceil\right) \leq \\ &\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot))-1\right| + \widetilde{l}_n(\omega_1(f,\delta))\left(1(\cdot)+\frac{|\cdot-x_0|}{\delta}\right) = \\ &\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot))-1\right| + \omega_1(f,\delta)\left[\widetilde{l}_n(1(\cdot))+\frac{1}{\delta}\widetilde{l}_n(|\cdot-x_0|)\right] = \\ &\left|f(x_0)\right|\left|\widetilde{l}_n(1(\cdot))-1\right| + \left(\widetilde{l}_n(1(\cdot))+1\right)\omega_1\left(f,\widetilde{l}_n(|\cdot-x_0|)\right), \end{aligned}$$

by choosing

$$\delta := \widetilde{l}_n(|\cdot-x_0|), \tag{24}$$

if $\widetilde{l}_n(|\cdot-x_0|)>0$.

Next we consider the case of

$$\widetilde{l}_n(|\cdot-x_0|)=0. \tag{25}$$

By Riesz representation theorem there exists a positive finite measure μ such that

$$\widetilde{l}_n(g)=\int_K g(t)\,d\mu(t),\ \ \forall\ g\in C(K,\mathbb{R}). \tag{26}$$

That is

$$\int_K |t-x_0|\,d\mu(t)=0, \tag{27}$$

which implies $|t-x_0|=0$, a.e., hence $t-x_0=0$, a.e, and $t=x_0$, a.e. on K. Consequently $\mu(\{t\in K: t\neq x_0\})=0$. That is $\mu=\delta_{x_0}M$ (where $0<M:=\mu(K)=\widetilde{l}_n(1(\cdot))$). Hence, in that case $\widetilde{l}_n(g)=g(x_0)M$. Consequently it holds $\omega_1\left(f,\widetilde{l}_n(|\cdot-x_0|)\right)=0$, and the right hand side of (22) equals $|f(x_0)||M-1|$. Also, it is $\widetilde{l}_n(|f(\cdot)-f(x_0)(\cdot)|)=|f(x_0)-f(x_0)|M=0$. Hence from the first part of this proof we get $|l_n(f)-l_n(f(x_0)(\cdot))|=0$, and $l_n(f)=l_n(f(x_0)(\cdot))=f(x_0)\widetilde{l}_n(1(\cdot))=Mf(x_0)$.

Consequently the left hand side of (22) becomes

$$|l_n(f)-f(x_0)|=|Mf(x_0)-f(x_0)|=|f(x_0)||M-1|.$$

So that (22) becomes an equality, and both sides equal $|f(x_0)||M-1|$ in the extreme case of $\widetilde{l}_n(|\cdot-x_0|)=0$. Thus inequality (22) is proved completely in all cases. ■

We make

8

Remark 11 *By Schwartz's inequality we get*

$$\widetilde{l}_n\left(|\cdot - x_0|\right) \leq \left(\widetilde{l}_n\left(|\cdot - x_0|^2\right)\right)^{\frac{1}{2}}\left(\widetilde{l}_n\left(1\left(\cdot\right)\right)\right)^{\frac{1}{2}}. \tag{28}$$

We give

Corollary 12 *All as in Theorem 10. Then*

$$|l_n(f) - f(x_0)| \leq |f(x_0)|\left|\widetilde{l}_n(1(\cdot)) - 1\right| +$$

$$\left(\widetilde{l}_n(1(\cdot)) + 1\right)\omega_1\left(f, \left(\widetilde{l}_n(1(\cdot))\right)^{\frac{1}{2}}\left(\widetilde{l}_n\left(|\cdot - x_0|^2\right)\right)^{\frac{1}{2}}\right), \tag{29}$$

$\forall\ x_0 \in K$, $\forall\ n \in \mathbb{N}$.

Next we give another version of our Korovkin type result.

Theorem 13 *Here all are as in Theorem 10. Then, for any $f \in C(K, \mathbb{C})$, we have*

$$|l_n(f) - f(x_0)| \leq |f(x_0)|\left|\widetilde{l}_n(1(\cdot)) - 1\right| + \left(\widetilde{l}_n(1(\cdot)) + 1\right)\omega_1\left(f, \left(\widetilde{l}_n\left(|\cdot - x_0|^2\right)\right)^{\frac{1}{2}}\right), \tag{30}$$

$\forall\ x_0 \in K$, $\forall\ n \in \mathbb{N}$.

By (30), as $\widetilde{l}_n(1(\cdot)) \to 1$ and $\widetilde{l}_n\left(|\cdot - x_0|^2\right) \to 0$, then $l_n(f) \to f(x_0)$, as $n \to +\infty$, with rates, and as here $\widetilde{l}_n(1(\cdot))$ is bounded.

Proof. Let $t, x_0 \in K$ and $\delta > 0$. If $|t - x_0| > \delta$, then

$$|f(t) - f(x_0)| \leq \omega_1(f, |t - x_0|) = \omega_1\left(f, |t - x_0|\,\delta^{-1}\delta\right) \leq \tag{31}$$

$$\left(1 + \frac{|t - x_0|}{\delta}\right)\omega_1(f, \delta) \leq \left(1 + \frac{|t - x_0|^2}{\delta^2}\right)\omega_1(f, \delta).$$

The estimate

$$|f(t) - f(x_0)| \leq \left(1 + \frac{|t - x_0|^2}{\delta^2}\right)\omega_1(f, \delta) \tag{32}$$

also holds trivially when $|t - x_0| \leq \delta$.

So (32) is true always, $\forall\ t \in K$, for any $x_0 \in K$.

We can rewrite

$$|f(\cdot) - f(x_0)| \leq \left(1 + \frac{|\cdot - x_0|^2}{\delta^2}\right)\omega_1(f, \delta). \tag{33}$$

As in the proof of Theorem 10 we have

$$|l_n(f) - f(x_0)| \leq ... \leq |f(x_0)|\left|\widetilde{l}_n(1(\cdot)) - 1\right| +$$

$$\widetilde{l}_n\left(\omega_1(f,\delta)\left(1(\cdot)+\frac{|\cdot-x_0|^2}{\delta^2}\right)\right)=$$
$$|f(x_0)|\left|\widetilde{l}_n(1(\cdot))-1\right|+\omega_1(f,\delta)\left[\widetilde{l}_n(1(\cdot))+\frac{1}{\delta^2}\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right]= \tag{34}$$
$$|f(x_0)|\left|\widetilde{l}_n(1(\cdot))-1\right|+\omega_1\left(f,\left(\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right)^{\frac{1}{2}}\right)\left(\widetilde{l}_n(1(\cdot))+1\right),$$

by choosing

$$\delta:=\left(\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right)^{\frac{1}{2}}, \tag{35}$$

if $\widetilde{l}_n\left(|\cdot-x_0|^2\right)>0$.

Next we consider the case of

$$\widetilde{l}_n\left(|\cdot-x_0|^2\right)=0. \tag{36}$$

By Riesz representation theorem there exists a positive finite measure μ such that

$$\widetilde{l}_n(g)=\int_K g(t)\,d\mu(t),\ \ \forall\ g\in C(K,\mathbb{R}). \tag{37}$$

That is

$$\int_K |t-x_0|^2\,d\mu(t)=0,$$

which implies $|t-x_0|^2=0$, a.e., hence $t-x_0=0$, a.e, and $t=x_0$, a.e. on K. Consequently $\mu(\{t\in K: t\neq x_0\})=0$. That is $\mu=\delta_{x_0}M$ (where $0<M:=\mu(K)=\widetilde{l}_n(1(\cdot))$). Hence, in that case $\widetilde{l}_n(g)=g(x_0)M$. Consequently it holds $\omega_1\left(f,\left(\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right)^{\frac{1}{2}}\right)=0$, and the right hand side of (30) equals $|f(x_0)||M-1|$.

Also, it is $\widetilde{l}_n(|f(\cdot)-f(x_0)(\cdot)|)=|f(x_0)-f(x_0)|M=0$. Hence from the first part of this proof we get: $|l_n(f)-l_n(f(x_0)(\cdot))|=0$, and $l_n(f)=l_n(f(x_0)(\cdot))=f(x_0)\widetilde{l}_n(1(\cdot))=Mf(x_0)$.

Consequently the left hand side of (30) becomes

$$|l_n(f)-f(x_0)|=|f(x_0)||M-1|.$$

So that (30) is true again. The proof of the theorem is now complete. ∎

Corollary 14 *Here all are as in Theorem 10. Then*

$$|l_n(f)-f(x_0)|\leq|f(x_0)|\left|\widetilde{l}_n(1(\cdot))-1\right|+\left(\widetilde{l}_n(1(\cdot))+1\right)\cdot$$
$$\min\left\{\omega_1\left(f,\left(\widetilde{l}_n(1(\cdot))\right)^{\frac{1}{2}}\left(\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right)^{\frac{1}{2}}\right),\omega_1\left(f,\left(\widetilde{l}_n\left(|\cdot-x_0|^2\right)\right)^{\frac{1}{2}}\right)\right\}, \tag{38}$$

$\forall\ x_0\in K$, $\forall\ n\in\mathbb{N}$.

10

Proof. By (29) and (30). ■

So (29) is better that (30) only if $\widetilde{l}_n(1(\cdot)) < 1$.

We need

Theorem 15 *Let $K \subseteq \mathbb{C}$ convex, $x_0 \in K^0$ (interior of K) and $f : K \to \mathbb{R}$ such that $|f(t) - f(x_0)|$ is convex in $t \in K$. Furthermore let $\delta > 0$ so that the closed disk $D(x_0, \delta) \subset K$. Then*

$$|f(t) - f(x_0)| \leq \frac{\omega_1(f, \delta)}{\delta} |t - x_0|, \quad \forall\, t \in K. \tag{39}$$

Proof. Let $g(t) := |f(t) - f(x_0)|$, $t \in K$, which is convex in $t \in K$ and $g(x_0) = 0$.

Then by Lemma 8.1.1, p. 243 of [1], we obtain

$$g(t) \leq \frac{\omega_1(g, \delta)}{\delta} |t - x_0|, \quad \forall\, t \in K. \tag{40}$$

We notice the following

$$|f(t_1) - f(x_0)| = |f(t_1) - f(t_2) + f(t_2) - f(x_0)| \leq$$

$$|f(t_1) - f(t_2)| + |f(t_2) - f(x_0)|,$$

hence

$$|f(t_1) - f(x_0)| - |f(t_2) - f(x_0)| \leq |f(t_1) - f(t_2)|. \tag{41}$$

Similarly, it holds

$$|f(t_2) - f(x_0)| - |f(t_1) - f(x_0)| \leq |f(t_1) - f(t_2)|. \tag{42}$$

Therefore for any $t_1, t_2 \in K : |t_1 - t_2| \leq \delta$ we get

$$\big|\, |f(t_1) - f(x_0)| - |f(t_2) - f(x_0)| \,\big| \leq |f(t_1) - f(t_2)| \leq \omega_1(f, \delta). \tag{43}$$

That is

$$\omega_1(g, \delta) \leq \omega_1(f, \delta). \tag{44}$$

The last and (40) imply

$$|f(t) - f(x_0)| \leq \frac{\omega_1(f, \delta)}{\delta} |t - x_0|, \quad \forall\, t \in K, \tag{45}$$

proving (39). ■

We continue with a convex Korovkin type result:

Theorem 16 *All as in Theorem 10. Let $x_0 \in K^0$ and assume that $|f(t) - f(x_0)|$ is convex in $t \in K$. Let $\delta > 0$, such that the closed disk $D(x_0, \delta) \subset K$. Then*

$$|l_n(f) - f(x_0)| \leq |f(x_0)| \left|\widetilde{l}_n(1(\cdot)) - 1\right| + \omega_1\left(f, \widetilde{l}_n(|\cdot - x_0|)\right), \quad \forall\, n \in \mathbb{N}. \tag{46}$$

11

Proof. As in the proof Theorem 10 we have

$$\left|l_n(f) - f(x_0)\right| \leq \ldots \leq |f(x_0)| \left|\widetilde{l}_n(1(\cdot)) - 1\right| + \widetilde{l}_n(|f(\cdot) - f(x_0)(\cdot)|) \overset{(39)}{\leq} \tag{47}$$

$$|f(x_0)| \left|\widetilde{l}_n(1(\cdot)) - 1\right| + \frac{\omega_1(f,\delta)}{\delta}\widetilde{l}_n(|\cdot - x_0|) =$$

$$|f(x_0)| \left|\widetilde{l}_n(1(\cdot)) - 1\right| + \omega_1\left(f, \widetilde{l}_n(|\cdot - x_0|)\right),$$

by choosing

$$\delta := \widetilde{l}_n(|\cdot - x_0|) > 0,$$

if the last is positive. The case of $\widetilde{l}_n(|\cdot - x_0|) = 0$ is treated similarly as in the proof of Theorem 10. The theorem is proved. ■

Theorem 17 *All as in Theorem 16. Inequality (46) is sharp, in fact it is attained by* $f^*(t) = \overrightarrow{j}|t - x_0|$*, where* $\overrightarrow{j}$ *is a unit vector of* $(\mathbb{C}, |\cdot|)$*;* $t, x_0 \in K$*.*

Proof. Indeed, f^* here fulfills the assumptions of the theorem. We further notice that $f^*(x_0) = 0$, and $|f^*(t) - f^*(x_0)| = |t - x_0|$ is convex in $t \in K$. The left hand side of (46) is

$$|l_n(f^*) - f^*(x_0)| = |l_n(f^*)| = \left|l_n\left(\overrightarrow{j}|\cdot - x_0|\right)\right| \overset{(21)}{=}$$

$$\left|\overrightarrow{j}\widetilde{l}_n(|\cdot - x_0|)\right| = \left|\widetilde{l}_n(|\cdot - x_0|)\right|. \tag{48}$$

The right hand side of (46) is

$$\omega_1\left(f^*, \widetilde{l}_n(|\cdot - x_0|)\right) = \omega_1\left(\overrightarrow{j}|\cdot - x_0|, \widetilde{l}_n(|\cdot - x_0|)\right) =$$

$$\sup_{\substack{t_1,t_2 \in K \\ |t_1 - t_2| \leq \widetilde{l}_n(|\cdot - x_0|)}} \left|\overrightarrow{j}|t_1 - x_0| - \overrightarrow{j}|t_2 - x_0|\right| =$$

$$\sup_{\substack{t_1,t_2 \in K \\ |t_1 - t_2| \leq \widetilde{l}_n(|\cdot - x_0|)}} \left||t_1 - x_0| - |t_2 - x_0|\right| \leq \tag{49}$$

$$\sup_{\substack{t_1,t_2 \in K \\ |t_1 - t_2| \leq \widetilde{l}_n(|\cdot - x_0|)}} |t_1 - t_2| = \widetilde{l}_n(|\cdot - x_0|).$$

Hence we have found that

$$\omega_1\left(f^*, \widetilde{l}_n(|\cdot - x_0|)\right) \leq \widetilde{l}_n(|\cdot - x_0|). \tag{50}$$

Clearly (46) is attained.

The theorem is proved. ■

12

4 Main Results - II

Next we give results on linear operators:

Let K be a compact convex subset of $\mathbb{C}$. Consider $L : C(K, \mathbb{C}) \to C(K, \mathbb{C})$ a linear operator and $\widetilde{L} : C(K, \mathbb{R}) \to C(K, \mathbb{R})$ a positve linear operator (i.e. for $f_1.f_2 \in C(K, \mathbb{R})$ with $f_1 \geq f_2$ we get $\widetilde{L}(f_1) \geq \widetilde{L}(f_2)$) both over the field of $\mathbb{R}$.

We assume that

$$|L(f)| \leq \widetilde{L}(|f|), \ \forall \ f \in C(K, \mathbb{C}),$$

(i.e. $|L(f)(z)| \leq \widetilde{L}(|f|)(z)$, $\forall \ z \in K$).

We call $\widetilde{L}$ the companion operator of L.

Let $x_0 \in K$. Clearly, then $L(\cdot)(x_0)$ is a linear functional from $C(K, \mathbb{C})$ into $\mathbb{C}$, and $\widetilde{L}(\cdot)(x_0)$ is a positive linear functional from $C(K, \mathbb{R})$ into $\mathbb{R}$. Notice $L(f)(z) \in \mathbb{C}$ and $\widetilde{L}(|f|)(z) \in \mathbb{R}$, $\forall \ f \in C(K, \mathbb{C})$ (thus $|f| \in C(K, \mathbb{R})$). Here $L(f) \in C(K, \mathbb{C})$, and $\widetilde{L}(|f|) \in C(K, \mathbb{R})$, $\forall \ f \in C(K, \mathbb{C})$.

Notice that $C(K, \mathbb{C}) = UC(K, \mathbb{C})$, also $C(K, \mathbb{R}) = UC(K, \mathbb{R})$ (uniformly continuous functions).

By [3], p. 388, we have that $\widetilde{L}(|\cdot - x_0|^r)(x_0)$, $r > 0$, is a continuous function in $x_0 \in K$.

After this preparation we transfer the main results from section 3 to linear operators.

We have the following approximation results with rates of Korovkin type.

Theorem 18 *Here K is a convex and compact subset of $\mathbb{C}$ and L_n is a sequence of linear operators from $C(K, \mathbb{C})$ into itself, $n \in \mathbb{N}$. There is a sequence of companion positive linear operators $\widetilde{L}_n$ from $C(K, \mathbb{R})$ into itself, such that*

$$|L_n(f)| \leq \widetilde{L}_n(|f|), \ \forall \ f \in C(K, \mathbb{C}), \ \forall \ n \in \mathbb{N} \tag{51}$$

(i.e. $|L_n(f)(x_0)| \leq \left(\widetilde{L}_n(|f|)\right)(x_0)$, $\forall \ x_0 \in K$).

Additionally, we assume that

$$L_n(cg) = c\widetilde{L}_n(g), \ \ \forall \ g \in C(K, \mathbb{R}), \ \forall \ c \in \mathbb{C} \tag{52}$$

(i.e. $(L_n(cg))(x_0) = c\left(\widetilde{L}_n(g)\right)(x_0)$, $\forall \ x_0 \in K$).

Then, for any $f \in C(K, \mathbb{C})$, we have

$$|(L_n(f))(x_0) - f(x_0)| \leq |f(x_0)| \left|\widetilde{L}_n(1(\cdot))(x_0) - 1\right| +$$

$$\left(\widetilde{L}_n(1(\cdot))(x_0) + 1\right)\omega_1\left(f, \widetilde{L}_n(|\cdot - x_0|)(x_0)\right), \tag{53}$$

$\forall \ x_0 \in K$, $\forall \ n \in \mathbb{N}$.

Proof. By Theorem 10. ■

Corollary 19 *All as in Theorem 18. Then*

$$\left\| L_n(f) - f \right\|_{\infty,K} \leq \|f\|_{\infty,K} \left\| \widetilde{L}_n(1(\cdot)) - 1 \right\|_{\infty,K} +$$

$$\left\| \widetilde{L}_n(1(\cdot)) + 1 \right\|_{\infty,K} \omega_1 \left(f, \left\| \widetilde{L}_n(|\cdot - x_0|)(x_0) \right\|_{\infty,K} \right), \tag{54}$$

$\forall\ n \in \mathbb{N}$.

If $\widetilde{L}_n(1(\cdot)) = 1$, $\forall\ n \in \mathbb{N}$, *then*

$$\left\| L_n(f) - f \right\|_{\infty,K} \leq 2\omega_1 \left(f, \left\| \widetilde{L}_n(|\cdot - x_0|)(x_0) \right\|_{\infty,K} \right), \tag{55}$$

$\forall\ n \in \mathbb{N}$.

As $\widetilde{L}_n(1(\cdot)) \xrightarrow{u} 1$, $\left\| \widetilde{L}_n(|\cdot - x_0|)(x_0) \right\|_{\infty,K} \xrightarrow{u} 0$, *then (by (54))* $L_n(f) \xrightarrow{u} f$, *as* $n \to +\infty$, *where* u *means uniformly. Notice* $\widetilde{L}_n(1(\cdot))$ *is bounded, and all the suprema in (54) are finite.*

We continue with

Theorem 20 *Here all as in Theorem 18. Then, for any* $f \in C(K, \mathbb{C})$, *we have*

$$|(L_n(f))(x_0) - f(x_0)| \leq |f(x_0)| \left| \widetilde{L}_n(1(\cdot))(x_0) - 1 \right| +$$

$$\left(\widetilde{L}_n(1(\cdot))(x_0) + 1 \right) \omega_1 \left(f, \left(\widetilde{L}_n\left(|\cdot - x_0|^2 \right)(x_0) \right)^{\frac{1}{2}} \right), \tag{56}$$

$\forall\ x_0 \in K$, $\forall\ n \in \mathbb{N}$.

Proof. By Theorem 13. ∎

Corollary 21 *All as in Theorem 18. Then, for any* $f \in C(K, \mathbb{C})$, *we have*

$$\left\| L_n(f) - f \right\|_{\infty,K} \leq \|f\|_{\infty,K} \left\| \widetilde{L}_n(1(\cdot)) - 1 \right\|_{\infty,K} +$$

$$\left\| \widetilde{L}_n(1(\cdot)) + 1 \right\|_{\infty,K} \omega_1 \left(f, \left\| \widetilde{L}_n\left(|\cdot - x_0|^2 \right)(x_0) \right\|_{\infty,K}^{\frac{1}{2}} \right), \tag{57}$$

$\forall\ n \in \mathbb{N}$.

If $\widetilde{L}_n(1(\cdot)) = 1$, *then*

$$\left\| L_n(f) - f \right\|_{\infty,K} \leq 2\omega_1 \left(f, \left\| \widetilde{L}_n\left(|\cdot - x_0|^2 \right)(x_0) \right\|_{\infty,K}^{\frac{1}{2}} \right), \tag{58}$$

$\forall\ n \in \mathbb{N}$.

As $\widetilde{L}_n(1(\cdot)) \xrightarrow{u} 1$, $\left\| \widetilde{L}_n\left(|\cdot - x_0|^2 \right)(x_0) \right\|_{\infty,K} \xrightarrow{u} 0$, *then (by (57))* $L_n(f) \xrightarrow{u} f$, *as* $n \to +\infty$.

14

We continue with a convex Korovkin type result:

Theorem 22 *All as in Theorem 18. Let a fixed* $x_0^* \in K^0$ *and assume that* $|f(t) - f(x_0^*)|$ *is convex in* $t \in K$. *Let* $\delta > 0$,*such that the closed disk* $D(x_0^*, \delta) \subset K$. *Then*

$$|(L_n(f))(x_0^*) - f(x_0^*)| \leq |f(x_0^*)| \left| \widetilde{L}_n(1(\cdot))(x_0^*) - 1 \right|$$

$$+\omega_1\left(f, \widetilde{L}_n(|\cdot - x_0^*|)(x_0^*)\right), \quad \forall\, n \in \mathbb{N}. \tag{59}$$

As $\widetilde{L}_n(1(\cdot))(x_0^*) \to 1$, *and* $\widetilde{L}_n(|\cdot - x_0^*|)(x_0^*) \to 0$, *we get that* $(L_n(f))(x_0^*) \to f(x_0^*)$, *as* $n \to +\infty$, *a pointwise convergence.*

Proof. By Theorem 16. ■

Note: Theorem 22 goes throw if (51), (52) are valid only for the particular x_0^*.

We finish with

Proposition 23 *All as in Theorem 22. Inequality (59) is sharp, in fact it is attained by* $\overline{f}(t) = \overrightarrow{j}\,|t - x_0^*|$, *where* $\overrightarrow{j}$ *is a unit vector of* $\mathbb{C}$; $x_0^*, t \in K$.

Proof. By Theorem 17. ■

Note: Let K be a convex compact subset of a real normed vector space $(V, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ is a Banach space. We can consider bounded linear functionals and bounded operators on $C(K, X)$. This paper's methodology can be applied to this more general setting and produce a similar Korovkin theory in full strength.

References

[1] G.A. Anastassiou, *Moments in Probability and Approximation Theory*, Pitman Research Notes in Math., Vol. 287, Longman Sci. & Tech., Harlow, U.K., 1993.

[2] G.A. Anastassiou, *Lattice homomorphism- Korovkin type inequalities for vector valued functions*, Hokkaido Math. J., 26 (1997), 337-364.

[3] G.A. Anastassiou, *Quantitative Approximations*, Chapman &Hall / CRC, Boca Raton, New York, 2001.

[4] G.A. Anastassiou, *Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations*, Springer, Heidelberg, New York, 2018.

[5] S.S. Dragomir, *An integral representation of the remainder in Taylor's expansion formula for analytic function on general domains*, RGMIA Res. Rep. Call. 22 (2019), Art. 2, 14 pp., rgmia.org/v22.php.

[6] H. Gonska, *On approximation of continuously differentiable functions by positive linear operators*, Bull. Austral. Math. Soc., 27 (1983), 73-81.

[7] P.P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan Publ. Corp., Delhi, India, 1960.

[8] R.G. Mamedov, *On the order of the approximation of functions by linear positive operators*, Dokl. Akad. Nauk USSR, 128 (1959), 674-676.

[9] H.L. Royden, *Real Analysis*, 2nd edition, Macmillan, New York, 1968.

[10] O. Shisha and B. Mond, *The degree of convergence of sequences of linear positive operators*, Nat. Acad. of Sci., 60 (1968), 1196-1200.

ADDITIVE ρ-FUNCTIONAL INEQUALITIES IN NON-ARCHIMEDEAN BANACH SPACES

INHO HWANG

Abstract. In this paper, we solve the additive ρ-functional inequalities

$$\|f(x+y)+f(x-y)-2f(x)\| \le \left\|\rho\left(2f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)\right)\right\|, \tag{0.1}$$

where ρ is a fixed non-Archimedean number with $|\rho|<1$, and

$$\left\|2f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)\right\| \le \|\rho(f(x+y)+f(x-y)-2f(x))\|, \tag{0.2}$$

where ρ is a fixed non-Archimedean number with $|\rho|<|2|$.

Furthermore, we prove the Hyers-Ulam stability of the additive ρ-functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces.

1. Introduction and preliminaries

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s| \le |r|+|s|, \qquad \forall r,s\in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \le \max\{|r|,|s|\}, \qquad \forall r,s\in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n|\le 1$ for all $n\in\mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0|=0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1. ([6]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X\to[0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$;

(ii) $\|rx\| = |r|\|x\| \qquad (r\in K, x\in X)$;

(iii) the strong triangle inequality

$$\|x+y\| \le \max\{\|x\|,\|y\|\}, \qquad \forall x,y\in X$$

2010 *Mathematics Subject Classification.* Primary 46S10, 39B62, 39B52, 47S10, 12J25.

Key words and phrases. Hyers-Ulam stability; non-Archimedean normed space; additive ρ-functional inequality.

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space.*

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \leq \varepsilon$$

for all $n, m \geq N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \leq \varepsilon$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space.*

The stability problem of functional equations originated from a question of Ulam [13] concerning the stability of group homomorphisms. The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation.* In particular, every solution of the Cauchy equation is said to be an *additive mapping.* Hyers [5] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [10] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [4] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The functional equation $f(x+y) + f(x-y) = 2f(x)$ is called the *Jensen type additive functional equation.*

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping.* The stability of quadratic functional equation was proved by Skof [12] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [2, 7, 8, **?**, 11]).

In this paper, we solve the additive ρ-functional inequalities (0.1) and (0.2) and prove the Hyers-Ulam stability of the additive ρ-functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces.

Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$.

2. Additive ρ-functional inequality (0.1) in non-Archimedean normed spaces

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < 1$.

In this section, we solve the additive ρ-functional inequality (0.1) in non-Archimedean normed spaces.

Lemma 2.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\|f(x+y) + f(x-y) - 2f(x)\| \leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\right)\right\| \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $y = x$ in (2.1), we get $\|f(2x) - 2f(x)\| \le 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x+y) + f(x-y) - 2f(x)\| &\le \left\|\rho\left(2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\right)\right\| \\ &= |\rho|\|f(x+y) + f(x-y) - 2f(x)\| \end{aligned}$$

and so $f(x+y) + f(x-y) = 2f(x)$ for all $x, y \in X$. It is easy to show that f is additive. □

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (2.1) in non-Archimedean Banach spaces.

Theorem 2.2. *Let $r < 1$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} \|f(x+y) + f(x-y) - 2f(x)\| &\le \left\|\rho\left(2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\right)\right\| \\ &+ \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{2.3}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{|2|^r}\|x\|^r \tag{2.4}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.3), we get

$$\|f(2x) - 2f(x)\| \le 2\theta\|x\|^r \tag{2.5}$$

for all $x \in X$. So $\|f(x) - 2f\left(\frac{x}{2}\right)\| \le \frac{2}{|2|^r}\theta\|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned} &\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \qquad (2.6) \\ &\le \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &= \max\left\{|2|^l\left\|f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |2|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\le \max\left\{\frac{|2|^l}{|2|^{rl+r}}, \cdots, \frac{|2|^{m-1}}{|2|^{r(m-1)+r}}\right\}2\theta\|x\|^r = \frac{2\theta}{|2|^{(r-1)l+r}}\|x\|^r \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

It follows from (2.3) that

$$\begin{aligned}
&\|A(x+y)+A(x-y)-2A(x)\| = \lim_{n\to\infty}|2|^n\left\|f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)\right\| \\
&\le \lim_{n\to\infty}|2|^n|\rho|\left\|2f\left(\frac{x+y}{2^{n+1}}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)\right\|+\lim_{n\to\infty}\frac{|2|^n\theta}{|2|^{nr}}(\|x\|^r+\|y\|^r) \\
&= |\rho|\left\|2A\left(\frac{x+y}{2}\right)+A(x-y)-2A(x)\right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\|A(x+y)+A(x-y)-2A(x)\| \le \left\|\rho\left(2A\left(\frac{x+y}{2}\right)+A(x-y)-2A(x)\right)\right\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive .

Now, let $T : X \to Y$ be another additive mapping satisfying (2.4). Then we have

$$\begin{aligned}
\|A(x)-T(x)\| &= \left\|2^qA\left(\frac{x}{2^q}\right)-2^qT\left(\frac{x}{2^q}\right)\right\| \\
&\le \max\left\{\left\|2^qA\left(\frac{x}{2^q}\right)-2^qf\left(\frac{x}{2^q}\right)\right\|,\left\|2^qT\left(\frac{x}{2^q}\right)-2^qf\left(\frac{x}{2^q}\right)\right\|\right\} \le \frac{2\theta}{|2|^{(r-1)q+r}}\|x\|^r,
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $A : X \to Y$ is a unique additive mapping satisfying (2.4). □

Theorem 2.3. *Let $r > 1$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x)-A(x)\| \le \frac{2\theta}{|2|}\|x\|^r$$

for all $x \in X$.

Proof. It follows from (2.5) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\| \le \frac{2}{|2|}\theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{2^l}f\left(2^lx\right)-\frac{1}{2^m}f\left(2^mx\right)\right\| \\
&\le \max\left\{\left\|\frac{1}{2^l}f\left(2^lx\right)-\frac{1}{2^{l+1}}f\left(2^{l+1}x\right)\right\|,\cdots,\left\|\frac{1}{2^{m-1}}f\left(2^{m-1}x\right)-\frac{1}{2^m}f\left(2^mx\right)\right\|\right\} \\
&= \max\left\{\frac{1}{|2|^l}\left\|f\left(2^lx\right)-\frac{1}{2}f\left(2^{l+1}x\right)\right\|,\cdots,\frac{1}{|2|^{m-1}}\left\|f\left(2^{m-1}x\right)-\frac{1}{2}f\left(2^mx\right)\right\|\right\} \\
&\le \max\left\{\frac{|2|^{lr}}{|2|^{l+1}},\cdots,\frac{|2|^{r(m-1)}}{|2|^{(m-1)+1}}\right\}2\theta\|x\|^r = \frac{2\theta}{|2|^{(1-r)l+1}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

3. Additive ρ-functional inequality (0.2)

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < |2|$.

In this section, we solve the additive ρ-functional inequality (0.2) in non-Archimedean normed spaces.

Lemma 3.1. *If a mapping $f : X \to Y$ satisfies*

$$\left\|2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\right\| \le \|\rho(f(x+y) + f(x-y) - 2f(x))\| \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|f(0)\| \le 0$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get $\|2f\left(\frac{x}{2}\right) - f(x)\| \le 0$ and so

$$2f\left(\frac{x}{2}\right) = f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} \|f(x+y) + f(x-y) - 2f(x)\| &= \left\|2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\right\| \\ &\le |\rho|\|f(x+y) + f(x-y) - 2f(x)\| \end{aligned}$$

and so $f(x+y) + f(x-y) = 2f(x)$ for all $x, y \in X$. It is easy to show that f is additive. □

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (3.1) in non-Archimedean Banach spaces.

Theorem 3.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} \left\|2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\right\| &\le \|\rho(f(x+y) + f(x-y) - 2f(x))\| \\ &+ \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \theta\|x\|^r \tag{3.4}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.3), we get $f(0) = 0$.

Letting $y = 0$ in (3.3), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \le \theta\|x\|^r \tag{3.5}$$

for all $x \in X$. So

$$\begin{aligned}
&\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \qquad (3.6)\\
&\quad \le \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\quad = \max\left\{|2|^l \left\|f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |2|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\quad \le \max\left\{\frac{|2|^l}{|2|^{rl}}, \cdots, \frac{|2|^{m-1}}{|2|^{r(m-1)}}\right\}\theta\|x\|^r = \frac{\theta}{|2|^{(r-1)l}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.3. *Let $r > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (3.3)*. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{|2|^r\theta}{|2|}\|x\|^r \qquad (3.7)$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{|2|^r\theta}{|2|}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| \qquad (3.8)\\
&\quad \le \max\left\{\left\|\frac{1}{2^l}f\left(2^l x\right) - \frac{1}{2^{l+1}}f\left(2^{l+1}x\right)\right\|, \cdots, \left\|\frac{1}{2^{m-1}}f\left(2^{m-1}x\right) - \frac{1}{2^m}f\left(2^m x\right)\right\|\right\}\\
&\quad = \max\left\{\frac{1}{|2|^l}\left\|f\left(2^l x\right) - \frac{1}{2}f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|2|^{m-1}}\left\|f\left(2^{m-1}x\right) - \frac{1}{2}f\left(2^m x\right)\right\|\right\}\\
&\quad \le \max\left\{\frac{|2|^{rl}}{|2|^{l+1}}, \cdots, \frac{|2|^{r(m-1)}}{|2|^{(m-1)+1}}\right\}|2|^r\theta\|x\|^r = \frac{|2|^r\theta}{|2|^{(1-r)l+1}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.7).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.2. □

Competing interests

The author declares that he has no competing interests.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[2] A. Chahbi and N. Bounader, *On the generalized stability of d'Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.

[3] P. W. Cholewa, *Remarks on the stability of functional equations,* Aequationes Math. **27** (1984), 76–86.

[4] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–43.

[5] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

[6] M.S. Moslehian and Gh. Sadeghi, *A Mazur-Ulam theorem in non-Archimedean normed spaces*, Nonlinear Anal.–TMA **69** (2008), 3405–3408.

[7] C. Park, *C^*-ternary biderivations and C^*-ternary bihomomorphisms,* Math. **6** (2018), Art. 30.

[8] C. Park, *Bi-additive s-functional inequalities and quasi-$*$-multipliers on Banach algebras,* Math. **6** (2018), Art. 31.

[9] C. Park, A. Najati and S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.

[10] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[11] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.

[12] F. Skof, *Propriet locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.

[13] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.

Inho Hwang
Department of Mathematics, Incheon National University, Incheon 22012, Korea
Email address: ho818@inu.ac.kr

Square root and 3rd root functional equations in C^*-algebras

Choonkil Park ,Sun Young Jang* and, Jieun Ahn

Abstract. In this paper, we introduce a square root functional equation and a 3rd root functional equation. We prove the Hyers-Ulam stability of the square root functional equation and of the 3rd root functional equation in C^*-algebras.

1. Introduction and preliminaries

The stability problem of functional equations was originated from a question of Ulam [7] concerning the stability of group homomorphisms. Hyers [5] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [6] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [3] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Rassias' approach.

Definition 1.1. [2] Let A be a C^*-algebra and $x \in A$ a self-adjoint element, i.e., $x^* = x$. Then x is said to be *positive* if it is of the form yy^* for some $y \in A$.

The set of positive elements of A is denoted by A^+.

Note that A^+ is a closed convex cone (see [2]).

It is well-known that for a positive element x and a positive integer n there exists a unique positive element $y \in A^+$ such that $x = y^n$. We denote y by $x^{\frac{1}{n}}$ (see [4]).

In this paper, we introduce a *square root functional equation*

$$S\left(x + y + x^{\frac{1}{4}} y^{\frac{1}{2}} x^{\frac{1}{4}} + y^{\frac{1}{4}} x^{\frac{1}{2}} y^{\frac{1}{4}}\right) = S(x) + S(y) \tag{1.1}$$

and a *3rd root functional equation*

$$T\left(x + y + 3x^{\frac{1}{3}} y^{\frac{1}{3}} x^{\frac{1}{3}} + 3y^{\frac{1}{3}} x^{\frac{1}{3}} y^{\frac{1}{3}}\right) = T(x) + T(y) \tag{1.2}$$

for all $x, y \in A^+$. Each solution of the square root functional equation is called a *square root mapping* and each solution of the 3rd root functional equation is called a *3rd root mapping*.

Note that the functions $S(x) = \sqrt{x} = x^{\frac{1}{2}}$ and $T(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ in the set of non-negative real numbers are solutions of the functional equations (1.1) and (1.2), respectively.

In this paper, we prove the Hyers-Ulam stability of the functional equations (1.1) and (1.2) in C^*-algebras.

Throughout this paper, let A^+ and B^+ be the sets of positive elements in C^*-algebras A and B, respectively.

---

[0]**2010 Mathematics Subject Classification**: 46L05, 39B52.

[0]**Keywords**: Hyers-Ulam stability, C^*-algebra, square root functional equation, 3rd root functional equation.

*Corresponding author: Sun Young Jang (email: jsym@ulsan.ac.kr).

2. Stability of the square root functional equation

In this section, we investigate the square root functional equation in C^*-algebras.

Lemma 2.1. *Let $S : A^+ \to B^+$ be a square root mapping satisfying* (1.1). *Then S satisfies*

$$S(4^n x) = 2^n S(x) \tag{2.1}$$

for all $x \in A^+$ and all $n \in \mathbb{Z}$.

Proof. Putting $x = y = 0$ in (1.1), we obtain $S(0) = 0$. Letting $y = 0$ in (1.1), we obtain

$$S(4^0 x) = S(x) = 2^0 S(x)$$

for all $x \in A^+$.

First of all, we use the induction on n to prove the equality (2.1) for all positive integers n.

Replacing y by x in (1.1), we get

$$S(4x) = 2S(x) \tag{2.2}$$

for all $x \in A^+$. So the equality (2.1) holds for $n = 1$.

Assume that

$$S(4^k x) = 2^k S(x) \tag{2.3}$$

holds for a positive integer k. Replacing x by $4x$ in (2.3) and using (2.2), we obtain

$$S(4^{k+1}x) = S(4^k \cdot 4x) = 2^k S(4x) = 2^{k+1} S(x)$$

for all $x \in A^+$. So the equality (2.1) holds for $n = k + 1$. Thus

$$S(4^n x) = 2^n S(x) \tag{2.4}$$

for all $x \in A^+$ and all positive integers n.

Next, replacing x by $4^{-n}x$ in (2.4), we obtain

$$S(x) = S(4^n \cdot 4^{-n} x) = 2^n S(4^{-n} x)$$

for all $x \in A^+$ and all positive integers n. So

$$S(4^n x) = 2^n S(x)$$

for all $x \in A^+$ and all negative integers n.

Therefore,

$$S(4^n x) = 2^n S(x)$$

for all $x \in A^+$ and all $n \in \mathbb{Z}$. □

We prove the Hyers-Ulam stability of the square root functional equation in C^*-algebras.

Theorem 2.2. *Let $f : A^+ \to B^+$ be a mapping for which there exists a function $\varphi : A^+ \times A^+ \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{4^j}, \frac{y}{4^j}\right) \quad < \quad \infty, \tag{2.5}$$

$$\left\| f\left(x + y + x^{\frac{1}{4}} y^{\frac{1}{2}} x^{\frac{1}{4}} + y^{\frac{1}{4}} x^{\frac{1}{2}} y^{\frac{1}{4}}\right) - f(x) - f(y) \right\| \quad \leq \quad \varphi(x, y) \tag{2.6}$$

for all $x, y \in A^+$. Then there exists a unique square root mapping $S : A^+ \to A^+$ satisfying (1.1) *and*

$$\|f(x) - S(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, y) \tag{2.7}$$

for all $x \in A^+$.

Proof. Letting $y = x$ in (2.6), we get

$$\|f(4x) - 2f(x)\| \leq \varphi(x, x) \tag{2.8}$$

for all $x \in A^+$. It follows from (2.8) that

$$\left\|f(x) - 2f\left(\frac{x}{4}\right)\right\| \leq \varphi\left(\frac{x}{4}, \frac{x}{4}\right)$$

for all $x \in A^+$. Hence

$$\left\|2^l f\left(\frac{x}{4^l}\right) - 2^m f\left(\frac{x}{4^m}\right)\right\| \leq \frac{1}{2}\sum_{j=l+1}^{m} 2^j \varphi\left(\frac{x}{4^j}, \frac{x}{4^j}\right) \tag{2.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A^+$. It follows from (2.5) and (2.9) that the sequence $\{2^k f(\frac{x}{4^k})\}$ is Cauchy for all $x \in A^+$. Since B^+ is complete, the sequence $\{2^k f(\frac{x}{4^k})\}$ converges. So one can define the mapping $S : A^+ \to B^+$ by

$$S(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{4^k}\right)$$

for all $x \in A^+$.

By (2.8) and (2.9),

$$\begin{aligned}
&\left\|S\left(x + y + x^{\frac{1}{4}}y^{\frac{1}{2}}x^{\frac{1}{4}} + y^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{1}{4}}\right) - S(x) - S(y)\right\| \\
&= \lim_{k\to\infty} 2^k \left\|f\left(\frac{x + y + x^{\frac{1}{4}}y^{\frac{1}{2}}x^{\frac{1}{4}} + y^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{1}{4}}}{4^k}\right) - f\left(\frac{x}{4^k}\right) - f\left(\frac{y}{4^k}\right)\right\| \\
&\leq \lim_{k\to\infty} 2^k \varphi\left(\frac{x}{4^k}, \frac{y}{4^k}\right) = 0
\end{aligned}$$

for all $x, y \in A^+$. So

$$S\left(x + y + x^{\frac{1}{4}}y^{\frac{1}{2}}x^{\frac{1}{4}} + y^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{1}{4}}\right) - S(x) - S(y) = 0.$$

Hence the mapping $S : A^+ \to B^+$ is a square root mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.7). So there exists a square root mapping $S : A^+ \to B^+$ satisfying (1.1) and (2.7).

Now, let $S' : A^+ \to B^+$ be another square root mapping satisfying (1.1) and (2.7). Then we have

$$\begin{aligned}
\|S(x) - S'(x)\| &= 2^q \left\|S\left(\frac{x}{4^q}\right) - S'\left(\frac{x}{4^q}\right)\right\| \\
&\leq 2^q \left\|S\left(\frac{x}{4^q}\right) - f\left(\frac{x}{4^q}\right)\right\| + 2^q \left\|S'\left(\frac{x}{4^q}\right) - f\left(\frac{x}{4^q}\right)\right\| \\
&\leq \frac{2 \cdot 2^q}{2} \widetilde{\varphi}\left(\frac{x}{4^q}, \frac{x}{4^q}\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in A^+$. So we can conclude that $S(x) = S'(x)$ for all $x \in A^+$. This proves the uniqueness of S. □

Corollary 2.3. *Let $p > \frac{1}{2}$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping such that*

$$\left\|f\left(x + y + x^{\frac{1}{4}}y^{\frac{1}{2}}x^{\frac{1}{4}} + y^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{1}{4}}\right) - f(x) - f(y)\right\| \leq \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}} \tag{2.10}$$

for all $x, y \in A^+$. Then there exists a unique square root mapping $S : A^+ \to B^+$ satisfying (1.1) *and*

$$\|f(x) - S(x)\| \leq \frac{2\theta_1 + \theta_2}{4^p - 2}||x||^p$$

for all $x \in A^+$.

Proof. Define $\varphi(x,y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$, and apply Theorem 2.2. Then we get the desired result. □

Theorem 2.4. *Let* $f : A^+ \to B^+$ *be a mapping for which there exists a function* $\varphi : A^+ \times A^+ \to [0,\infty)$ *satisfying* (2.6) *such that*

$$\widetilde{\varphi}(x,y) := \sum_{j=0}^{\infty} 2^{-j}\varphi(4^j x, 4^j y) < \infty$$

for all $x, y \in A^+$. *Then there exists a unique square root mapping* $S : A^+ \to B^+$ *satisfying* (1.1) *and*

$$\|f(x) - S(x)\| \le \frac{1}{2}\widetilde{\varphi}(x,x)$$

for all $x \in A^+$.

Proof. It follows from (2.8) that

$$\left\| f(x) - \frac{1}{2} f(4x) \right\| \le \frac{1}{2}\varphi(x,x)$$

for all $x \in A^+$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let* $0 < p < \frac{1}{2}$ *and* θ_1, θ_2 *be non-negative real numbers, and let* $f : A^+ \to B^+$ *be a mapping satisfying* (2.10). *Then there exists a unique square root mapping* $S : A^+ \to B^+$ *satisfying* (1.1) *and*

$$\|f(x) - S(x)\| \le \frac{2\theta_1 + \theta_2}{2 - 4^p}||x||^p$$

for all $x \in A^+$.

Proof. Define $\varphi(x,y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$, and apply Theorem 2.4. Then we get the desired result. □

3. Stability of the 3rd root functional equation

In this section, we investigate the 3rd root functional equation in C^*-algebras.

Lemma 3.1. *Let* $T : A^+ \to B^+$ *be a 3rd root mapping satisfying* (1.2). *Then* T *satisfies*

$$T(8^n x) = 2^n T(x)$$

for all $x \in A^+$ *and all* $n \in \mathbb{Z}$.

Proof. The proof is similar to the proof of Lemma 2.1. □

We prove the Hyers-Ulam stability of the 3rd root functional equation in C^*-algebras.

Theorem 3.2. *Let* $f : A^+ \to B^+$ *be a mapping for which there exists a function* $\varphi : A^+ \times A^+ \to [0,\infty)$ *such that*

$$\widetilde{\varphi}(x,y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{8^j}, \frac{y}{8^j}\right) \quad < \quad \infty,$$

$$\left\| f\left(x + y + 3x^{\frac{1}{3}}y^{\frac{1}{3}}x^{\frac{1}{3}} + 3y^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}\right) - f(x) - f(y) \right\| \quad \le \quad \varphi(x,y) \tag{3.1}$$

for all $x, y \in A^+$. Then there exists a unique 3rd root mapping $T : A^+ \to A^+$ satisfying (1.2) *and*

$$\|f(x) - T(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, y)$$

for all $x \in A^+$.

Proof. Letting $y = x$ in (3.1), we get

$$\|f(8x) - 2f(x)\| \leq \varphi(x, x) \tag{3.2}$$

for all $x \in A^+$.

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Corollary 3.3. *Let $p > \frac{1}{3}$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping such that*

$$\left\| f\left(x + y + 3x^{\frac{1}{3}}y^{\frac{1}{3}}x^{\frac{1}{3}} + 3y^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}\right) - f(x) - f(y) \right\| \leq \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}} \tag{3.3}$$

for all $x, y \in A^+$. Then there exists a unique 3rd root mapping $T : A^+ \to B^+$ satisfying (1.2) *and*

$$\|f(x) - T(x)\| \leq \frac{2\theta_1 + \theta_2}{8^p - 2}||x||^p$$

for all $x \in A^+$.

Proof. Define $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$, and apply Theorem 3.2. Then we get the desired result. $\square$

Theorem 3.4. *Let $f : A^+ \to B^+$ be a mapping for which there exists a function $\varphi : A^+ \times A^+ \to [0, \infty)$ satisfying* (3.1) *such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 2^{-j}\varphi(8^j x, 8^j y) < \infty$$

for all $x, y \in A^+$. Then there exists a unique 3rd root mapping $T : A^+ \to B^+$ satisfying (1.2) *and*

$$\|f(x) - T(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, x)$$

for all $x \in A^+$.

Proof. It follows from (3.2) that

$$\left\| f(x) - \frac{1}{2}f(8x) \right\| \leq \frac{1}{2}\varphi(x, x)$$

for all $x \in A^+$.

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Corollary 3.5. *Let $0 < p < \frac{1}{3}$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping satisfying* (3.3). *Then there exists a unique 3rd root mapping $T : A^+ \to B^+$ satisfying* (1.2) *and*

$$\|f(x) - T(x)\| \leq \frac{2\theta_1 + \theta_2}{2 - 8^p}||x||^p$$

for all $x \in A^+$.

Proof. Define $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$, and apply Theorem 3.4. Then we get the desired result. $\square$

4. Square root and 3rd root functional equations in C^*-algebras

We have defined a *square root functional equation*

$$S\left(x+y+x^{\frac{1}{4}}y^{\frac{1}{2}}x^{\frac{1}{4}}+y^{\frac{1}{4}}x^{\frac{1}{2}}y^{\frac{1}{4}}\right)=S(x)+S(y)$$

and a *3rd root functional equation*

$$T\left(x+y+3x^{\frac{1}{3}}y^{\frac{1}{3}}x^{\frac{1}{3}}+3y^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}\right)=T(x)+T(y)$$

for all $x, y \in A^+$. Each solution of the square root functional equation is called a *square root mapping* and each solution of the 3rd root functional equation is called a *3rd root mapping*.

It was shown that each square root mapping $S : A^+ \to B^+$ satisfies $S(4^n x) = 2^n S(x)$ for all $x \in A^+$ and all $n \in \mathbb{Z}$ and that each 3rd root mapping $T : A^+ \to B^+$ satisfies $T(8^n x) = 2^n T(x)$ for all $x \in A^+$ and all $n \in \mathbb{Z}$. Moreover, we prove that there exists a square root mapping near a given approximate square root mapping and that there exists a 3rd root mapping near a given approximate 3rd root mapping by using the Hyer-Ulam-Rassias approach.

Acknowledgments

The authors would like to thank the useful comments of referees.

J.S. Jang is supoorted by NRF 201807042748.

Competing interests

The authors declare that they have no competing interests.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[2] J. Dixmier, *C^*-Algebras*, North-Holland Publ. Com., Amsterdam, New York and Oxford, 1977.

[3] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[4] K.R. Goodearl, *Notes on Real and Complex C^*-Algebras*, Shiva Math. Series *IV*, Shiva Publ. Limited, Cheshire, England, 1982.

[5] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[6] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Am. Math. Soc. **72** (1978), 297–300.

[7] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.

Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
Email address: baak@hanyang.ac.kr

Sun Young Jang
Department of Mathematics, University of Ulsan, Ulsan 44610, Korea
Email address: jsym@ulsan.ac.kr

Jieun Ahn
Department of Mathematics, University of Ulsan, Ulsan 44610, Republic of Korea
Email address: je302@naver.com

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Approximation by Multivariate Sublinear and Max-product Operators, Revisited

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we study quantitatively the approximation of multivariate function by general multivariate positive sublinear operators with applications to multivariate Max-product operators. These are of Bernstein type, of Favard-Szász-Mirakjan type, of Baskakov type, of sampling type, of Lagrange interpolation type and of Hermite-Fejér interpolation type. Our results are both: under the presence of smoothness and without any smoothness assumption on the function to be approximated.

2010 AMS Mathematics Subject Classification: 41A17, 41A25, 41A36, 41A63.

Keywords and Phrases: multivariate positive sublinear operators, multivariate Max-product operators, multivariate modulus of continuity.

1 Background

Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$ and let $x_0 := (x_{01}, ..., x_{0k}) \in Q$ be fixed. Let $f \in C^n(Q)$ and suppose that each nth partial derivative $f_\alpha = \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, ..., \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., k$, and $|\alpha| := \sum_{i=1}^k \alpha_i = n$, has relative to Q and the l_1-norm $\|\cdot\|$, a modulus of continuity $\omega_1(f_\alpha, h) \leq w$, where h and w are fixed positive numbers. Here

$$\omega_1(f_\alpha, h) := \sup_{\substack{x,y \in Q \\ \|x-y\|_{l_1} \leq h}} |f_\alpha(x) - f_\alpha(y)|. \tag{1}$$

The jth derivative of $g_z(t) = f(x_0 + t(z - x_0))$, $(z = (z_1, ..., z_k) \in Q)$ is given by

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^j f \right] (x_{01} + t(z_1 - x_{01}), ..., x_{0k} + t(z_k - x_{0k})). \tag{2}$$

Consequently it holds

$$f(z_1, ..., z_k) = g_z(1) = \sum_{j=0}^{n} \frac{g_z^{(j)}(0)}{j!} + R_n(z, 0), \tag{3}$$

where

$$R_n(z, 0) := \int_0^1 \left(\int_0^{t_1} ... \left(\int_0^{t_{n-1}} \left(g_z^{(n)}(t_n) - g_z^{(n)}(0) \right) dt_n \right) ... \right) dt_1. \tag{4}$$

We apply Lemma 7.1.1, [1], pp. 208-209, to $(f_\alpha(x_0 + t(z - x_0)) - f_\alpha(x_0))$ as a function of z, when $\omega_1(f_\alpha, h) \leq w$.

$$|f_\alpha(x_0 + t(z - x_0)) - f_\alpha(x_0)| \leq w \left\lceil \frac{t \|z - x_0\|}{h} \right\rceil, \tag{5}$$

all $t \geq 0$, where $\lceil \cdot \rceil$ is the ceiling function.

For $\|z - x_0\| \neq 0$, it follows from (2)

$$|R_n(z, 0)| \leq$$

$$\int_0^1 \int_0^{t_1} ... \int_0^{t_{n-1}} \left(\sum_{|\alpha|=n} \frac{n!}{\alpha_1! ... \alpha_k!} |z_1 - x_{01}|^{\alpha_1} ... |z_k - x_{0k}|^{\alpha_k} w \left\lceil \frac{t_n \|z - x_0\|}{h} \right\rceil \right) dt_n ... dt_1 \tag{6}$$

$$= \sum_{|\alpha|=n} \frac{n!}{\alpha_1! ... \alpha_k!} \frac{\prod_{i=1}^{k} |z_i - x_{0i}|^{\alpha_i}}{\|z - x_0\|^n} w \Phi_n(\|z - x_0\|) = w \Phi(\|z - x_0\|),$$

since $\|z - x_0\| = \sum_{i=1}^{k} |z_i - x_{0i}|$. Above we denote (for $h > 0$ fixed):

$$\Phi_n(x) := \int_0^{|x|} \left\lceil \frac{t}{h} \right\rceil \frac{(|x| - t)^{n-1}}{(n-1)!} dt, \quad (x \in \mathbb{R}), \tag{7}$$

equivalently

$$\Phi_n(x) = \int_0^{|x|} \int_0^{x_1} ... \left(\int_0^{x_{n-1}} \left\lceil \frac{x_n}{h} \right\rceil dx_n \right) ... dx_1, \tag{8}$$

see [1], p. 210-211.

Therefore we have

$$|R_n(z,0)| \leq w\Phi_n(\|z-x_0\|), \quad \text{for all } z \in Q. \tag{9}$$

Also we have $g_z(0) = f(x_0)$.

One obtains ([1], p. 210)

$$\Phi_n(x) = \frac{1}{n!}\left(\sum_{j=0}^{\infty}(|x| - jh)_+^n\right), \tag{10}$$

which is a polynomial spline function.

Furthermore we get ([1], pp. 210-211)

$$\Phi_n(x) \leq \Phi_{*n}(x) := \left(\frac{|x|^{n+1}}{(n+1)!h} + \frac{|x|^n}{2n!} + \frac{h|x|^{n-1}}{8(n-1)!}\right), \tag{11}$$

with equality only at $x = 0$.

Moreover, Φ_n is convex on $\mathbb{R}$ and strictly increasing on $\mathbb{R}_+$, $n \geq 1$.

In case of $Q := \{x \in \mathbb{R}^* : \|x\| \leq 1\}$, where $\|\cdot\|$ is the l_1-norm in $\mathbb{R}^k$ we have

$$0 \leq \|z - x_0\| \leq \|z\| + \|x_0\| \leq 1 + \|x_0\|, \ \forall\, z \in Q,$$

hence $\Phi_n(\|z-x_0\|) \leq \Phi_n(1+\|x_0\|)$, and by convexity of Φ_n we get

$$\frac{\Phi_n(\|z-x_0\|)}{\|z-x_0\|} \leq \frac{\Phi_n(1+\|x_0\|)}{(1+\|x_0\|)}, \tag{12}$$

$\forall\, z \in Q : \|z - x_0\| \neq 0$,

and hence

$$\Phi_n(\|z-x_0\|) \leq \|z-x_0\| \frac{\Phi_n(1+\|x_0\|)}{(1+\|x_0\|)}, \ \ \forall\, z \in Q. \tag{13}$$

Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$, $x_0 \in Q$ fixed, $f \in C^n(Q)$. Then for $j = 1, ..., n$, we have

$$g_z^{(j)}(0) = \sum_{\substack{\alpha:=(\alpha_1,...,\alpha_k),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,...,k,\ |\alpha|:=\sum_{i=1}^k \alpha_i = j}} \left(\frac{j!}{\prod_{i=1}^k \alpha_i!}\right)\left(\prod_{i=1}^k (z_i - x_{0i})^{\alpha_i}\right) f_\alpha(x_0). \tag{14}$$

If $f_\alpha(x_0) = 0$, for all $\alpha : |\alpha| = 1, ..., n$, then $g_z^{(j)}(0) = 0$, $j = 1, ..., n$, and by (3):

$$f(z) - f(x_0) = R_n(z, 0), \tag{15}$$

that is

$$|f(z) - f(x_0)| \leq w\Phi_n(\|z-x_0\|), \ \forall\, z \in Q, \tag{16}$$

3

where $x_0 \in Q$ is fixed.

Using (11) we derive

$$\|f(z) - f(x_0)\| \leq w \left(\frac{\|z - x_0\|^{n+1}}{(n+1)!h} + \frac{\|z - x_0\|^n}{2n!} + h \frac{\|z - x_0\|^{n-1}}{8(n-1)!} \right), \quad \forall\, z \in Q. \tag{17}$$

We have proved the following fundamental result:

Theorem 1 *Let $(Q, \|\cdot\|)$, where $\|\cdot\|$ is the l_1-norm, be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$ and let $x_0 \in Q$ be fixed. Let $f \in C^n(Q)$, $n \in \mathbb{N}$, $h > 0$. We assume that $f_\alpha(x_0) = 0$, for all $\alpha : |\alpha| = 1, ..., n$. Then*

$$\|f(z) - f(x_0)\| \leq \left(\max_{\alpha:|\alpha=n|} \omega_1(f_\alpha, h) \right) \cdot$$

$$\left(\frac{\|z - x_0\|^{n+1}}{(n+1)!h} + \frac{\|z - x_0\|^n}{2n!} + h \frac{\|z - x_0\|^{n-1}}{8(n-1)!} \right), \quad \forall\, z \in Q. \tag{18}$$

In conclusion we have

Theorem 2 *Let $(Q, \|\cdot\|)$, where $\|\cdot\|$ is the l_1-norm, be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$ and let $x \in Q$ ($x = (x_1, ..., x_k)$) be fixed. Let $f \in C^n(Q)$, $n \in \mathbb{N}$, $h > 0$. We assume that $f_\alpha(x) = 0$, for all $\alpha : |\alpha| = 1, ..., n$. Then*

$$\|f(t) - f(x)\| \leq \left(\max_{\alpha:|\alpha|=n} \omega_1(f_\alpha, h) \right) \cdot \tag{19}$$

$$\left(\frac{\|t - x\|^{n+1}}{(n+1)!h} + \frac{\|t - x\|^n}{2n!} + h \frac{\|t - x\|^{n-1}}{8(n-1)!} \right) \leq$$

$$\left(\max_{\alpha:|\alpha|=n} \omega_1(f_\alpha, h) \right) \left(\frac{k^n \left(\sum_{i=1}^k |t_i - x_i|^{n+1} \right)}{(n+1)!h} + \frac{k^{n-1} \left(\sum_{i=1}^k |t_i - x_i|^n \right)}{2n!} \right.$$

$$\left. + \frac{hk^{n-2}}{8(n-1)!} \left(\sum_{i=1}^k |t_i - x_i|^{n-1} \right) \right), \; \forall\, t \in Q, \tag{20}$$

where $t = (t_1, ..., t_k)$.

Proof. By Theorem 1 and a convexity argument. ■

We need

Definition 3 *Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$. Here we denote*

$$C_+(Q) = \{f : Q \to \mathbb{R}_+ \text{ and continuous}\}.$$

4

Let $L_N : C_+(Q) \rightarrow C_+(Q)$, $N \in \mathbb{N}$, be a sequence of operators satisfying the following properties:

(i) (positive homogeneous)

$$L_N(\alpha f) = \alpha L_N(f), \ \forall \ \alpha \geq 0, \ f \in C_+(Q); \tag{21}$$

(ii) (monotonicity)

if $f, g \in C_+(Q)$ satisfy $f \leq g$, then

$$L_N(f) \leq L_N(g), \ \forall \ N \in \mathbb{N}, \tag{22}$$

and

(iii) (subadditivity)

$$L_N(f+g) \leq L_N(f) + L_N(g), \ \forall \ f, g \in C_+(Q). \tag{23}$$

We call L_N positive sublinear operators.

Remark 4 *(to Definition 3) Let $f, g \in C_+(Q)$. We see that $f = f - g + g \leq |f-g| + g$. Then $L_N(f) \leq L_N(|f-g|) + L_N(g)$, and $L_N(f) - L_N(g) \leq L_N(|f-g|)$.*

Similarly $g = g - f + f \leq |g-f| + f$, hence $L_N(g) \leq L_N(|f-g|) + L_N(f)$, and $L_N(g) - L_N(f) \leq L_N(|f-g|)$.

Consequently it holds

$$|L_N(f)(x) - L_N(g)(x)| \leq L_N(|f-g|)(x), \ \forall \ x \in Q. \tag{24}$$

In this article we treat $L_N : L_N(1) = 1$.

We observe that

$$|L_N(f)(x) - f(x)| = |L_N(f)(x) - L_N(f(x))(x)| \overset{(24)}{\leq}$$

$$L_N(|f(\cdot) - f(x)|)(x), \ \forall \ x \in Q. \tag{25}$$

We give

Theorem 5 *Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$ and let $x \in Q$ be fixed. Let $f \in C^n(Q, \mathbb{R}_+)$, $n \in \mathbb{N}$, $h > 0$. We assume that $f_\alpha(x) = 0$, for all $\alpha : |\alpha| = 1, ..., n$. Let $\{L_N\}_{N \in \mathbb{N}}$ positive sublinear operators mapping $C_+(Q)$ into itself, such that $L_N(1) = 1$. Then*

$$|L_N(f)(x) - f(x)| \leq \left(\max_{\alpha:|\alpha|=n} \omega_1(f_\alpha, h)\right) \cdot$$

$$\left(\frac{k^n}{(n+1)!h}\left(\sum_{i=1}^k L_N\left(|t_i - x_i|^{n+1}\right)(x)\right) + \frac{k^{n-1}}{2n!}\left(\sum_{i=1}^k L_N\left(|t_i - x_i|^n\right)(x)\right)\right.$$

$$\left. + \frac{hk^{n-2}}{8(n-1)!}\left(\sum_{i=1}^k L_N\left(|t_i - x_i|^{n-1}\right)(x)\right)\right), \ \forall \ N \in \mathbb{N}. \tag{26}$$

5

Proof. By Theorem 2, see Definition 3, and by (25). ■

We need

The Maximum Multiplicative Principle 6 *Here $\vee$ stands for maximum. Let $\alpha_i > 0$, $i = 1, ..., n$; $\beta_j > 0$, $j = 1, ..., m$. Then*

$$\vee_{i=1}^{n} \vee_{j=1}^{m} \alpha_i \beta_j = \left(\vee_{i=1}^{n} \alpha_i\right) \left(\vee_{j=1}^{m} \beta_j\right). \tag{27}$$

Proof. Obvious. ■

We make

Remark 7 *In [4], p. 10, the authors introduced the basic Max-product Bernstein operators*

$$B_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} p_{N,k}(x) f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} p_{N,k}(x)}, \quad N \in \mathbb{N}, \tag{28}$$

where $p_{N,k}(x) = \binom{N}{k} x^k (1-x)^{N-k}$, $x \in [0,1]$, and $f : [0,1] \to \mathbb{R}_+$ is continuous.

In [4], p. 31, they proved that

$$B_N^{(M)}(|\cdot - x|)(x) \leq \frac{6}{\sqrt{N+1}}, \quad \forall\ x \in [0,1], \forall\ N \in \mathbb{N}. \tag{29}$$

And in [2] was proved that

$$B_N^{(M)}(|\cdot - x|^m)(x) \leq \frac{6}{\sqrt{N+1}}, \quad \forall\ x \in [0,1], \forall\ m, N \in \mathbb{N}. \tag{30}$$

We will also use

Corollary 8 *(to Theorem 5, case of $n = 1$) Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$ and let $x \in Q$. Let $f \in C^1(Q, \mathbb{R}_+)$, $h > 0$. We assume that $\frac{\partial f(x)}{\partial x_i} = 0$, for $i = 1, ..., k$. Let $\{L_N\}_{N \in \mathbb{N}}$ be positive sublinear operators from $C_+(Q)$ into $C_+(Q) : L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. Then*

$$|L_N(f)(x) - f(x)| \leq \left(\max_{i=1,...,k} \omega_1\left(\frac{\partial f}{\partial x_i}, h\right)\right) \cdot$$

$$\left[\frac{k}{2h}\left(\sum_{i=1}^{k} L_N\left((t_i - x_i)^2\right)(x)\right) + \frac{1}{2}\left(\sum_{i=1}^{k} L_N(|t_i - x_i|)(x)\right) + \frac{h}{8}\right], \tag{31}$$

$\forall\ N \in \mathbb{N}$.

In this article we study quantitatively the approximation properties of multivariate Max-product operators to the unit. These are special cases of positive sublinear operators. We give also general results regarding the convergence to the unit of positive sublinear operators. Special emphasis is given in our study about approximation under differentiability. Our work is motivated by [4].

6

2 Main Results

From now on $Q = [0,1]^k$, $k \in \mathbb{N} - \{1\}$, except otherwise specified.

We mention

Definition 9 *Let $f \in C_+\left([0,1]^k\right)$, and $\overrightarrow{N} = (N_1, ..., N_k) \in \mathbb{N}^k$. We define the multivariate Max-product Bernstein operators as follows:*

$$B_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=0}^{N_1} \bigvee_{i_2=0}^{N_2} ... \bigvee_{i_k=0}^{N_k} p_{N_1,i_1}(x_1)\, p_{N_2,i_2}(x_2) ... p_{N_k,i_k}(x_k)\, f\left(\frac{i_1}{N_1}, ..., \frac{i_k}{N_k}\right)}{\bigvee_{i_1=0}^{N_1} \bigvee_{i_2=0}^{N_2} ... \bigvee_{i_k=0}^{N_k} p_{N_1,i_1}(x_1)\, p_{N_2,i_2}(x_2) ... p_{N_k,i_k}(x_k)}, \quad (32)$$

$\forall\ x = (x_1, ..., x_k) \in [0,1]^k$. Call $N_{\min} := \min\{N_1, ..., N_k\}$.

The operators $B_{\overrightarrow{N}}^{(M)}(f)(x)$ are positive sublinear and they map $C_+\left([0,1]^k\right)$ into itself, and $B_{\overrightarrow{N}}^{(M)}(1) = 1$.

See also [4], p. 123 the bivariate case. We also have

$$B_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=0}^{N_1} \bigvee_{i_2=0}^{N_2} ... \bigvee_{i_k=0}^{N_k} p_{N_1,i_1}(x_1)\, p_{N_2,i_2}(x_2) ... p_{N_k,i_k}(x_k)\, f\left(\frac{i_1}{N_1}, ..., \frac{i_k}{N_k}\right)}{\prod_{\lambda=1}^{k}\left(\bigvee_{i_\lambda=0}^{N_\lambda} p_{N_\lambda,i_\lambda}(x_\lambda)\right)}, \quad (33)$$

$\forall\ x \in [0,1]^k$, by the maximum multiplicative principle, see (27).

We make

Remark 10 *The coordinate Max-product Bernstein operators are defined as follows ($\lambda = 1, ..., k$):*

$$B_{N_\lambda}^{(M)}(g)(x_\lambda) := \frac{\bigvee_{i_\lambda=0}^{N_\lambda} p_{N_\lambda,i_\lambda}(x_\lambda)\, g\left(\frac{i_\lambda}{N_\lambda}\right)}{\bigvee_{i_\lambda=0}^{N_\lambda} p_{N_\lambda,i_\lambda}(x_\lambda)}, \quad (34)$$

$\forall\ N_\lambda \in \mathbb{N}$, and $\forall\ x_\lambda \in [0,1]$, $\forall\ g \in C_+([0,1]) := \{g : [0,1] \to \mathbb{R}_+ \text{ continuous}\}$.

Here we have

$$p_{N_\lambda,i_\lambda}(x_\lambda) = \binom{N_\lambda}{i_\lambda} x_\lambda^{i_\lambda} (1 - x_\lambda)^{N_\lambda - i_\lambda}, \quad \text{for all } \lambda = 1, ..., k;\ x_\lambda \in [0,1]. \quad (35)$$

In case of $f \in C_+\left([0,1]^k\right)$ is such that $f(x) := g(x_\lambda)$, $\forall\ x \in [0,1]^k$, where $x = (x_1, ..., x_\lambda, ..., x_k)$ and $g \in C_+([0,1])$, we get that

$$B_{\overrightarrow{N}}^{(M)}(f)(x) = B_{N_\lambda}^{(M)}(g)(x_\lambda), \quad (36)$$

by the maximum multiplicative principle (27) and simplification of (33).

Clearly it holds that

$$B_{\overrightarrow{N}}^{(M)}(f)(x) = f(x), \ \forall \ x = (x_1, ..., x_k) \in [0,1]^k : x_\lambda \in \{0,1\}, \ \lambda = 1, ..., k. \tag{37}$$

We present

Theorem 11 *Let $x \in [0,1]^k$, $k \in \mathbb{N} - \{1\}$, be fixed, and let $f \in C^n\left([0,1]^k, \mathbb{R}_+\right)$, $n \in \mathbb{N} - \{1\}$. We assume that $f_\alpha(x) = 0$, for all $\alpha : |\alpha| = 1, ..., n$. Then*

$$\left|B_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \leq 6\left(\max_{\alpha:|\alpha|=n}\left(\omega_1\left(f_\alpha, \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{n+1}}\right)\right)\right) \cdot \tag{38}$$

$$\left[\frac{k^{n+1}}{(n+1)!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n}{n+1}} + \frac{k^n}{2n!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right) + \frac{k^{n-1}}{8(n-1)!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n+2}{n+1}}\right],$$

$\forall \ \overrightarrow{N} \in \mathbb{N}^k$, where $N_{\min} := \min\{N_1, ..., N_k\}$.

We have that $\lim_{\overrightarrow{N} \to (\infty,...,\infty)} B_{\overrightarrow{N}}^{(M)}(f)(x) = f(x)$.

Proof. By (26) we get:

$$\left|B_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \overset{(36)}{\leq} \left(\max_{\alpha:|\alpha|=n} \omega_1(f_\alpha, h)\right) \cdot$$

$$\left[\frac{k^n}{(n+1)!h}\left(\sum_{i=1}^k B_{N_i}^{(M)}\left(|t_i - x_i|^{n+1}\right)(x_i)\right) + \frac{k^{n-1}}{2n!}\left(\sum_{i=1}^k B_{N_i}^{(M)}\left(|t_i - x_i|^n\right)(x_i)\right)\right. \tag{39}$$

$$\left.+\frac{hk^{n-2}}{8(n-1)!}\left(\sum_{i=1}^k B_{N_i}^{(M)}\left(|t_i - x_i|^{n-1}\right)(x_i)\right)\right] \overset{(30)}{\leq}$$

$$\left(\frac{6}{\sqrt{N_{\min}+1}}\right)\left(\max_{\alpha:|\alpha|=n} \omega_1(f_\alpha, h)\right)\left[\frac{k^{n+1}}{(n+1)!h} + \frac{k^n}{2n!} + \frac{hk^{n-1}}{8(n-1)!}\right] =: (\xi).$$

Above notice $\sum_{i=1}^k B_{N_i}^{(M)}\left(|t_i - x_i|^n\right)(x_i) \overset{(30)}{\leq} \sum_{i=1}^k \frac{6}{\sqrt{N_i+1}} \leq \frac{6k}{\sqrt{N_{\min}+1}}$, etc.

Next we choose $h := \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{n+1}}$, then $h^n = \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n}{n+1}}$ and $h^{n+1} = \frac{1}{\sqrt{N_{\min}+1}}$.

We have

$$(\xi) = 6\left(\max_{\alpha:|\alpha|=n}\left(\omega_1\left(f_\alpha, \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{n+1}}\right)\right)\right) \cdot \tag{40}$$

8

$$\left[\frac{k^{n+1}}{(n+1)!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n}{n+1}}+\frac{k^n}{2n!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)+\frac{k^{n-1}}{8(n-1)!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n+2}{n+1}}\right],$$

proving the claim. ■

We also give

Proposition 12 *Let $x \in [0,1]^k$, $k \in \mathbb{N}-\{1\}$, be fixed and let $f \in C^1\left([0,1]^k, \mathbb{R}_+\right)$. We assume that $\frac{\partial f(x)}{\partial x_i}=0$, for $i=1,...,k$. Then*

$$\left|B_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right| \leq \left(\max_{i=1,\ldots,k}\omega_1\left(\frac{\partial f}{\partial x_i}, \frac{1}{\sqrt[4]{N_{\min}+1}}\right)\right)\cdot \tag{41}$$

$$\left[\frac{3k^2}{\sqrt[4]{N_{\min}+1}}+\frac{3k}{\sqrt{N_{\min}+1}}+\frac{1}{8\left(\sqrt[4]{N_{\min}+1}\right)}\right],$$

$\forall\ \overrightarrow{N} \in \mathbb{N}^k$, where $N_{\min} := \min\{N_1,...,N_k\}$.

Also it holds $\lim_{\overrightarrow{N}\to(\infty,\ldots,\infty)} B_{\overrightarrow{N}}^{(M)}(f)(x) = f(x)$.

Proof. By (31) we get:

$$\left|B_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right| \overset{(36)}{\leq} \left(\max_{i=1,\ldots,k}\omega_1\left(\frac{\partial f}{\partial x_i}, h\right)\right)\cdot$$

$$\left[\frac{k}{2h}\left(\sum_{i=1}^{k} B_{N_i}^{(M)}\left((t_i-x_i)^2\right)(x_i)\right)+\frac{1}{2}\left(\sum_{i=1}^{k} B_{N_i}^{(M)}\left(|t_i-x_i|\right)(x_i)\right)+\frac{h}{8}\right] \tag{42}$$

(next we choose $h := \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{2}}$, then $h^2=\frac{1}{\sqrt{N_{\min}+1}}$)

$$\overset{(30)}{\leq} \left(\max_{i=1,\ldots,k}\omega_1\left(\frac{\partial f}{\partial x_i}, \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{2}}\right)\right)\cdot \tag{43}$$

$$\left[3k^2\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{2}}+3k\left(\frac{1}{\sqrt{N_{\min}+1}}\right)+\frac{1}{8}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{2}}\right],$$

proving the claim. ■

We need

Theorem 13 *Let Q with $\|\cdot\|$ the l_1-norm, be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N}-\{1\}$, and $f \in C_+(Q)$; $h>0$. We denote $\omega_1(f,h) := \sup_{\substack{x,y\in Q:\\ \|x-y\|\leq h}} |f(x)-f(y)|$, the modulus of continuity of f. Let $\{L_N\}_{N\in\mathbb{N}}$ be positive sublinear operators from $C_+(Q)$ into itself such that $L_N(1)=1$, $\forall\ N\in\mathbb{N}$. Then*

$$|L_N(f)(x)-f(x)| \leq \omega_1(f,h)\left(1+\frac{1}{h}L_N(\|t-x\|)(x)\right) \leq$$

9

$$\omega_1(f,h)\left(1+\frac{1}{h}\left(\sum_{i=1}^{k} L_N(|t_i-x_i|)(x)\right)\right), \tag{44}$$

$\forall\ N\in\mathbb{N}$, $\forall\ x\in Q$, *where* $x:=(x_1,...,x_k)$; $t=(t_1,...,t_k)\in Q$.

Proof. We have that ([1], pp. 208-209)

$$|f(t)-f(x)|\leq\omega_1(f,h)\left\lceil\frac{\|t-x\|}{h}\right\rceil\leq\omega_1(f,h)\left(1+\frac{\|t-x\|}{h}\right), \tag{45}$$

$\forall\ t,x\in Q$.

By (25) we get:

$$|L_N(f)(x)-f(x)|\leq L_N(|f(t)-f(x)|)(x)\leq \tag{46}$$

$$\omega_1(f,h)\left(1+\frac{1}{h}L_N(\|t-x\|)(x)\right),\ \ \forall\ N\in\mathbb{N},$$

proving the claim. ■

We give

Theorem 14 *Let* $f\in C_+\left([0,1]^k\right)$, $k\in\mathbb{N}-\{1\}$. *Then*

$$\left|B^{(M)}_{\overrightarrow{N}}(f)(x)-f(x)\right|\leq(6k+1)\,\omega_1\left(f,\frac{1}{\sqrt{N_{\min}+1}}\right), \tag{47}$$

$\forall\ x\in[0,1]^k$, $\forall\ \overrightarrow{N}\in\mathbb{N}^k$, *where* $N_{\min}:=\min\{N_1,...,N_k\}$.

That is

$$\left\|B^{(M)}_{\overrightarrow{N}}(f)-f\right\|_\infty\leq(6k+1)\,\omega_1\left(f,\frac{1}{\sqrt{N_{\min}+1}}\right). \tag{48}$$

It holds that $\lim_{\overrightarrow{N}\to(\infty,...,\infty)} B^{(M)}_{\overrightarrow{N}}(f)(x)=f(x)$, *uniformly.*

Proof. We get that (use of (44))

$$\left|B^{(M)}_{\overrightarrow{N}}(f)(x)-f(x)\right|\overset{(36)}{\leq}\omega_1(f,h)\left(1+\frac{1}{h}\left(\sum_{i=1}^{k}B^{(M)}_{N_i}(|t_i-x_i|)(x_i)\right)\right)$$

$$\overset{(29)}{\leq}\omega_1(f,h)\left(1+\frac{1}{h}\left(\frac{6k}{\sqrt{N_{\min}+1}}\right)\right) \tag{49}$$

(setting $h:=\frac{1}{\sqrt{N_{\min}+1}}$)

$$=\omega_1\left(f,\frac{1}{\sqrt{N_{\min}+1}}\right)(6k+1),\ \ \forall\ x\in[0,1]^k,\ \forall\ \overrightarrow{N}\in\mathbb{N}^k,$$

proving the claim. ■

We continue with

Definition 15 *([4], p. 123) We define the bivariate Max-product Bernstein type operators:*

$$A_N^{(M)}(f)(x,y) := \frac{\bigvee_{i=0}^{N} \bigvee_{j=0}^{N-i} \binom{N}{i} \binom{N-i}{j} x^i y^j (1-x-y)^{N-i-j} f\left(\frac{i}{N}, \frac{j}{N}\right)}{\bigvee_{i=0}^{N} \bigvee_{j=0}^{N-i} \binom{N}{i} \binom{N-i}{j} x^i y^j (1-x-y)^{N-i-j}}, \tag{50}$$

$\forall\ (x,y) \in \Delta := \{(x,y) : x \geq 0,\ y \geq 0,\ x+y \leq 1\}$, $\forall\ N \in \mathbb{N}$, *and* $\forall\ f \in C_+(\Delta)$.

Remark 16 *By [4], p. 137, Theorem 2.7.5 there, $A_N^{(M)}$ is a positive sublinear operator mapping $C_+(\Delta)$ into itself and $A_N^{(M)}(1) = 1$, furthermore it holds*

$$\left|A_N^{(M)}(f) - A_N^{(M)}(g)\right| \leq A_N^{(M)}(|f-g|), \quad \forall\ f,g \in C_+(\Delta),\ \forall\ N \in \mathbb{N}. \tag{51}$$

By [4], p. 125 we get that $A_N^{(M)}(f)(1,0) = f(1,0)$, $A_N^{(M)}(f)(0,1) = f(0,1)$, and $A_N^{(M)}(f)(0,0) = f(0,0)$.

By [4], p. 139, we have that ($(x,y) \in \Delta$):

$$A_N^{(M)}(|\cdot - x|)(x,y) = B_N^{(M)}(|\cdot - x|)(x), \tag{52}$$

and

$$A_N^{(M)}(|\cdot - y|)(x,y) = B_N^{(M)}(|\cdot - y|)(y). \tag{53}$$

Working exactly the same way as (52), (53) are proved we also derive ($m \in \mathbb{N}$, $(x,y) \in \Delta$):

$$A_N^{(M)}(|\cdot - x|^m)(x,y) = B_N^{(M)}(|\cdot - x|^m)(x), \tag{54}$$

and

$$A_N^{(M)}(|\cdot - y|^m)(x,y) = B_N^{(M)}(|\cdot - y|^m)(y). \tag{55}$$

We present

Theorem 17 *Let $x := (x_1, x_2) \in \Delta$ be fixed, and $f \in C^n(\Delta, \mathbb{R}_+)$, $n \in \mathbb{N} - \{1\}$. We assume that $f_\alpha(x) = 0$, for all $\alpha : |\alpha| = 1, ..., n$. Then*

$$\left|A_N^{(M)}(f)(x_1,x_2) - f(x_1,x_2)\right| \leq 6\left(\max_{\alpha:|\alpha|=n} \omega_1\left(f_\alpha, \left(\frac{1}{\sqrt{N+1}}\right)^{\frac{1}{n+1}}\right)\right) \cdot \tag{56}$$

$$\left[\frac{2^{n+1}}{(n+1)!}\left(\frac{1}{\sqrt{N+1}}\right)^{\frac{n}{n+1}} + \frac{2^{n-1}}{n!}\left(\frac{1}{\sqrt{N+1}}\right) + \frac{2^{n-4}}{(n-1)!}\left(\frac{1}{\sqrt{N+1}}\right)^{\frac{n+2}{n+1}}\right],$$

$\forall\ N \in \mathbb{N}$.

It holds $\lim_{N\to\infty} A_N^{(M)}(f)(x_1,x_2) = f(x_1,x_2)$.

Proof. By (26) we get (here $x := (x_1, x_2) \in \Delta$):

$$\left| A_N^{(M)}(f)(x_1, x_2) - f(x_1, x_2) \right| \leq \left(\max_{\alpha : |\alpha| = n} \omega_1(f_\alpha, h) \right) \cdot$$

$$\left[\frac{2^n}{(n+1)!h} \left(\sum_{i=1}^{2} A_N^{(M)} \left(|t_i - x_i|^{n+1} \right)(x) \right) + \frac{2^{n-2}}{n!} \left(\sum_{i=1}^{2} A_N^{(M)} \left(|t_i - x_i|^n \right)(x) \right) \right. \tag{57}$$

$$\left. + \frac{h 2^{n-5}}{(n-1)!} \left(\sum_{i=1}^{2} A_N^{(M)} \left(|t_i - x_i|^{n-1} \right)(x) \right) \right] \overset{\text{(by (54), (55))}}{=}$$

$$\left(\max_{\alpha : |\alpha| = n} \omega_1(f_\alpha, h) \right) \left[\frac{2^n}{(n+1)!h} \left(\sum_{i=1}^{2} B_N^{(M)} \left(|t_i - x_i|^{n+1} \right)(x_i) \right) + \right.$$

$$\left. \frac{2^{n-2}}{n!} \left(\sum_{i=1}^{2} B_N^{(M)} \left(|t_i - x_i|^n \right)(x_i) \right) + \frac{h 2^{n-5}}{(n-1)!} \left(\sum_{i=1}^{2} B_N^{(M)} \left(|t_i - x_i|^{n-1} \right)(x_i) \right) \right] \tag{58}$$

$$\overset{(30)}{\leq} \frac{6 \left(\max_{\alpha : |\alpha| = n} \omega_1(f_\alpha, h) \right)}{\sqrt{N+1}} \left[\frac{2^{n+1}}{(n+1)!h} + \frac{2^{n-1}}{n!} + \frac{h 2^{n-4}}{(n-1)!} \right] =: (\xi).$$

Next we choose $h := \left(\frac{1}{\sqrt{N+1}} \right)^{\frac{1}{n+1}}$, then $h^n = \left(\frac{1}{\sqrt{N+1}} \right)^{\frac{n}{n+1}}$ and $h^{n+1} = \frac{1}{\sqrt{N+1}}$.

We have

$$(\xi) = 6 \left(\max_{\alpha : |\alpha| = n} \omega_1 \left(f_\alpha, \left(\frac{1}{\sqrt{N+1}} \right)^{\frac{1}{n+1}} \right) \right) \cdot \tag{59}$$

$$\left[\frac{2^{n+1}}{(n+1)!} \left(\frac{1}{\sqrt{N+1}} \right)^{\frac{n}{n+1}} + \frac{2^{n-1}}{n!} \left(\frac{1}{\sqrt{N+1}} \right) + \frac{2^{n-4}}{(n-1)!} \left(\frac{1}{\sqrt{N+1}} \right)^{\frac{n+2}{n+1}} \right],$$

proving the claim. ■

We also give

Theorem 18 *Let $x := (x_1, x_2) \in \Delta$ be fixed, and $f \in C^1(\Delta, \mathbb{R}_+)$. We assume that $\frac{\partial f}{\partial x_i}(x) = 0$, for $i = 1, 2$. Then*

$$\left| A_N^{(M)}(f)(x_1, x_2) - f(x_1, x_2) \right| \leq \left(\max_{i=1,2} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{1}{\sqrt[4]{N+1}} \right) \right) \cdot \tag{60}$$

$$\left[\frac{12}{\sqrt[4]{N+1}} + \frac{6}{\sqrt{N+1}} + \frac{1}{8} \left(\frac{1}{\sqrt[4]{N+1}} \right) \right],$$

$\forall\ N \in \mathbb{N}$.

It holds $\lim_{N \to \infty} A_N^{(M)}(f)(x_1, x_2) = f(x_1, x_2)$.

Proof. By (31) we get (here $x := (x_1, x_2) \in \Delta$):

$$\left| A_N^{(M)}(f)(x_1,x_2) - f(x_1,x_2) \right| \leq \left(\max_{i=1,2} \omega_1 \left(\frac{\partial f}{\partial x_i}, h \right) \right) \cdot$$

$$\left[\frac{1}{h} \left(\sum_{i=1}^{2} A_N^{(M)} \left((t_i - x_i)^2 \right)(x) \right) + \frac{1}{2} \left(\sum_{i=1}^{2} A_N^{(M)} (|t_i - x_i|)(x) \right) + \frac{h}{8} \right] \tag{61}$$

$$\overset{\text{(by (54), (55))}}{=} \left(\max_{i=1,2} \omega_1 \left(\frac{\partial f}{\partial x_i}, h \right) \right) \left[\frac{1}{h} \left(\sum_{i=1}^{2} B_N^{(M)} \left((t_i - x_i)^2 \right)(x_i) \right) + \right.$$

$$\left. \frac{1}{2} \left(\sum_{i=1}^{2} B_N^{(M)} (|t_i - x_i|)(x_i) \right) + \frac{h}{8} \right]$$

(next we choose $h := \left(\frac{1}{\sqrt{N+1}} \right)^{\frac{1}{2}}$, then $h^2 = \frac{1}{\sqrt{N+1}}$)

$$\overset{(30)}{\leq} \left(\max_{i=1,2} \omega_1 \left(\frac{\partial f}{\partial x_i}, \left(\frac{1}{\sqrt{N+1}} \right)^{\frac{1}{2}} \right) \right) \cdot \tag{62}$$

$$\left[12 \left(\frac{1}{\sqrt{N+1}} \right)^{\frac{1}{2}} + \left(\frac{6}{\sqrt{N+1}} \right) + \frac{1}{8} \left(\frac{1}{\sqrt{N+1}} \right)^{\frac{1}{2}} \right],$$

proving the claim. ■

We further obtain

Theorem 19 *Let $f \in C_+(\Delta)$. Then*

$$\left| A_N^{(M)}(f)(x_1,x_2) - f(x_1,x_2) \right| \leq 13 \omega_1 \left(f, \frac{1}{\sqrt{N+1}} \right), \tag{63}$$

$\forall\ (x_1, x_2) \in \Delta$, $\forall\ N \in \mathbb{N}$.

That is

$$\left\| A_N^{(M)}(f) - f \right\|_{\infty,\Delta} \leq 13 \omega_1 \left(f, \frac{1}{\sqrt{N+1}} \right), \tag{64}$$

$\forall\ N \in \mathbb{N}$.

It holds that $\lim_{N \to \infty} A_N^{(M)}(f) = f$, *uniformly,* $\forall\ f \in C_+(\Delta)$.

Proof. Using (44) ($x := (x_1, x_2) \in \Delta$) we get:

$$\left| A_N^{(M)}(f)(x_1,x_2) - f(x_1,x_2) \right| \leq$$

$$\omega_1(f,h) \left(1 + \frac{1}{h} \left(\sum_{i=1}^{2} A_N^{(M)} (|t_i - x_i|)(x) \right) \right) \overset{\text{(by (52), (53))}}{=}$$

13

$$\omega_1(f,h)\left(1+\frac{1}{h}\left(\sum_{i=1}^{2} B_N^{(M)}(|t_i-x_i|)(x_i)\right)\right) \overset{(29)}{\leq}$$

$$\omega_1(f,h)\left(1+\frac{2}{h}\cdot\frac{6}{\sqrt{N+1}}\right) \tag{65}$$

(setting $h := \frac{1}{\sqrt{N+1}}$)

$$= 13\omega_1\left(f, \frac{1}{\sqrt{N+1}}\right), \ \ \forall\ (x_1,x_2)\in\Delta,\ \forall\ N\in\mathbb{N},$$

proving the claim. ■

We make

Remark 20 *The Max-product truncated Favard-Szász-Mirakjan operators*

$$T_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} s_{N,k}(x)\, f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} s_{N,k}(x)}, \ \ x\in[0,1],\ N\in\mathbb{N},\ f\in C_+([0,1]), \tag{66}$$

$s_{N,k}(x) = \frac{(Nx)^k}{k!}$, *see also [4], p. 11.*

By [4], p. 178-179, we get that

$$T_N^{(M)}(|\cdot - x|)(x) \leq \frac{3}{\sqrt{N}}, \ \ \forall\ x\in[0,1],\ \forall\ N\in\mathbb{N}. \tag{67}$$

And from [2] we have

$$T_N^{(M)}(|\cdot - x|^m)(x) \leq \frac{3}{\sqrt{N}}, \ \ \forall\ x\in[0,1],\ \forall\ N,m\in\mathbb{N}. \tag{68}$$

We make

Definition 21 *Let* $f \in C_+\left([0,1]^k\right)$, $k\in\mathbb{N}-\{1\}$, *and* $\overrightarrow{N} = (N_1,...,N_k)\in\mathbb{N}^k$. *We define the multivariate Max-product truncated Favard-Szász-Mirakjan operators as follows:*

$$T_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

$$\frac{\vee_{i_1=0}^{N_1}\vee_{i_2=0}^{N_2}...\vee_{i_k=0}^{N_k} s_{N_1,i_1}(x_1)\, s_{N_2,i_2}(x_2)...s_{N_k,i_k}(x_k)\, f\left(\frac{i_1}{N_1},...,\frac{i_k}{N_k}\right)}{\vee_{i_1=0}^{N_1}\vee_{i_2=0}^{N_2}...\vee_{i_k=0}^{N_k} s_{N_1,i_1}(x_1)\, s_{N_2,i_2}(x_2)...s_{N_k,i_k}(x_k)}, \tag{69}$$

$\forall\ x=(x_1,...,x_k)\in[0,1]^k$. *Call* $N_{\min} := \min\{N_1,...,N_k\}$.

The operators $T_{\overrightarrow{N}}^{(M)}(f)(x)$ *are positive sublinear mapping* $C_+\left([0,1]^k\right)$ *into itself, and* $T_{\overrightarrow{N}}^{(M)}(1) = 1$.

We also have

$$T_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

14

$$\frac{\bigvee_{i_1=0}^{N_1} \bigvee_{i_2=0}^{N_2} \dots \bigvee_{i_k=0}^{N_k} s_{N_1,i_1}(x_1) s_{N_2,i_2}(x_2) \dots s_{N_k,i_k}(x_k) f\left(\frac{i_1}{N_1}, \dots, \frac{i_k}{N_k}\right)}{\prod_{\lambda=1}^{k} \left(\bigvee_{i_\lambda=0}^{N_\lambda} s_{N_\lambda,i_\lambda}(x_\lambda)\right)}, \quad (70)$$

$\forall\ x \in [0,1]^k$, *by the maximum multiplicative principle, see (27).*

We make

Remark 22 *The coordinate Max-product truncated Favard-Szász-Mirakjan operators are defined as follows* ($\lambda = 1, \dots, k$)*:*

$$T_{N_\lambda}^{(M)}(g)(x_\lambda) := \frac{\bigvee_{i_\lambda=0}^{N_\lambda} s_{N_\lambda,i_\lambda}(x_\lambda)\, g\left(\frac{i_\lambda}{N_\lambda}\right)}{\bigvee_{i_\lambda=0}^{N_\lambda} s_{N_\lambda,i_\lambda}(x_\lambda)}, \quad (71)$$

$\forall\ N_\lambda \in \mathbb{N}$, *and* $\forall\ x_\lambda \in [0,1]$, $\forall\ g \in C_+([0,1])$.

Here we have

$$s_{N_\lambda,i_\lambda}(x_\lambda) = \frac{(N_\lambda x_\lambda)^{i_\lambda}}{i_\lambda!},\ \ \lambda = 1, \dots, k;\ \ x_\lambda \in [0,1]. \quad (72)$$

In case of $f \in C_+\left([0,1]^k\right)$ *such that* $f(x) := g(x_\lambda)$, $\forall\ x \in [0,1]^k$, *where* $x = (x_1, \dots, x_\lambda, \dots, x_k)$ *and* $g \in C_+([0,1])$, *we get that*

$$T_{\overrightarrow{N}}^{(M)}(f)(x) = T_{N_\lambda}^{(M)}(g)(x_\lambda), \quad (73)$$

by the maximum multiplicative principle (27) and simplification of (70).

We present

Theorem 23 *Let* $x \in [0,1]^k$, $k \in \mathbb{N} - \{1\}$, *be fixed, and let* $f \in C^n\left([0,1]^k, \mathbb{R}_+\right)$, $n \in \mathbb{N} - \{1\}$. *We assume that* $f_\alpha(x) = 0$, *for all* $\alpha : |\alpha| = 1, \dots, n$. *Then*

$$\left|T_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \leq 3\left(\max_{\alpha:|\alpha|=n}\left(\omega_1\left(f_\alpha, \left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{1}{n+1}}\right)\right)\right) \cdot$$

$$\left[\frac{k^{n+1}}{(n+1)!}\left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{n}{n+1}} + \frac{k^n}{2n!}\left(\frac{1}{\sqrt{N_{\min}}}\right) + \frac{k^{n-1}}{8(n-1)!}\left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{n+2}{n+1}}\right], \quad (74)$$

$\forall\ \overrightarrow{N} \in \mathbb{N}^k$, *where* $N_{\min} := \min\{N_1, \dots, N_k\}$.

We have that $\lim_{\overrightarrow{N} \to (\infty, \dots, \infty)} T_{\overrightarrow{N}}^{(M)}(f)(x) = f(x)$.

Proof. By (26) we get:

$$\left|T_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \overset{(73)}{\leq} \left(\max_{\alpha:|\alpha|=n} \omega_1(f_\alpha, h)\right) \cdot$$

15

$$\left[\frac{k^{n}}{(n+1)!h}\left(\sum_{i=1}^{k}T_{N_{i}}^{(M)}\left(\left|t_{i}-x_{i}\right|^{n+1}\right)(x_{i})\right)+\frac{k^{n-1}}{2n!}\left(\sum_{i=1}^{k}T_{N_{i}}^{(M)}\left(\left|t_{i}-x_{i}\right|^{n}\right)(x_{i})\right)\right. \tag{75}$$

$$\left.+\frac{hk^{n-2}}{8\,(n-1)!}\left(\sum_{i=1}^{k}T_{N_{i}}^{(M)}\left(\left|t_{i}-x_{i}\right|^{n-1}\right)(x_{i})\right)\right]\overset{(68)}{\leq}$$

$$\frac{3}{\sqrt{N_{\min}}}\left(\max_{\alpha:|\alpha|=n}\omega_{1}\left(f_{\alpha},h\right)\right)\left[\frac{k^{n+1}}{(n+1)!h}+\frac{k^{n}}{2n!}+\frac{hk^{n-1}}{8\,(n-1)!}\right]=:(\xi)\,.$$

Above notice that $\sum_{i=1}^{k}T_{N_{i}}^{(M)}\left(\left|t_{i}-x_{i}\right|^{n}\right)(x_{i})\overset{(68)}{\leq}\sum_{i=1}^{k}\frac{3}{\sqrt{N_{i}}}\leq\frac{3k}{\sqrt{N_{\min}}}$, etc.

Next we choose $h:=\left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{1}{n+1}}$, then $h^{n}=\left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{n}{n+1}}$ and $h^{n+1}=\frac{1}{\sqrt{N_{\min}}}$.

We have

$$(\xi)=3\left(\max_{\alpha:|\alpha|=n}\left(\omega_{1}\left(f_{\alpha},\left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{1}{n+1}}\right)\right)\right)\cdot$$

$$\left[\frac{k^{n+1}}{(n+1)!}\left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{n}{n+1}}+\frac{k^{n}}{2n!}\left(\frac{1}{\sqrt{N_{\min}}}\right)+\frac{k^{n-1}}{8\,(n-1)!}\left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{n+2}{n+1}}\right], \tag{76}$$

proving the claim. ■

We also give

Proposition 24 *Let $x\in[0,1]^{k}$, $k\in\mathbb{N}-\{1\}$, be fixed and let $f\in C^{1}\left([0,1]^{k},\mathbb{R}_{+}\right)$. We assume that $\frac{\partial f(x)}{\partial x_{i}}=0$, for $i=1,...,k$. Then*

$$\left|T_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right|\leq\left(\max_{i=1,\dots,k}\omega_{1}\left(\frac{\partial f}{\partial x_{i}},\frac{1}{\sqrt[4]{N_{\min}}}\right)\right)\cdot$$

$$\left[\frac{3k^{2}}{2}\left(\frac{1}{\sqrt[4]{N_{\min}}}\right)+\frac{3k}{2}\left(\frac{1}{\sqrt{N_{\min}}}\right)+\frac{1}{8}\left(\frac{1}{\sqrt[4]{N_{\min}}}\right)\right], \tag{77}$$

$\forall\ \overrightarrow{N}\in\mathbb{N}^{k}$, where $N_{\min}:=\min\{N_{1},...,N_{k}\}$.

Also it holds $\lim_{\overrightarrow{N}\to(\infty,\dots,\infty)}T_{\overrightarrow{N}}^{(M)}(f)(x)=f(x)$.

Proof. By (31) we get:

$$\left|T_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right|\overset{(73)}{\leq}\left(\max_{i=1,\dots,k}\omega_{1}\left(\frac{\partial f}{\partial x_{i}},h\right)\right)\cdot$$

$$\left[\frac{k}{2h}\left(\sum_{i=1}^{k}T_{N_{i}}^{(M)}\left((t_{i}-x_{i})^{2}\right)(x_{i})\right)+\frac{1}{2}\left(\sum_{i=1}^{k}T_{N_{i}}^{(M)}\left(\left|t_{i}-x_{i}\right|\right)(x_{i})\right)+\frac{h}{8}\right] \tag{78}$$

16

(next we choose $h := \left(\frac{1}{\sqrt{N_{\min}}}\right)^{\frac{1}{2}}$, then $h^2 = \frac{1}{\sqrt{N_{\min}}}$)

$$\overset{(68)}{\leq} \left(\max_{i=1,\dots,k} \omega_1\left(\frac{\partial f}{\partial x_i}, \frac{1}{\sqrt[4]{N_{\min}}}\right)\right) \cdot$$

$$\left[\frac{3k^2}{2}\left(\frac{1}{\sqrt[4]{N_{\min}}}\right) + \frac{3k}{2}\left(\frac{1}{\sqrt{N_{\min}}}\right) + \frac{1}{8}\left(\frac{1}{\sqrt[4]{N_{\min}}}\right)\right], \tag{79}$$

proving the claim. ■

It follows

Theorem 25 *Let $f \in C_+\left([0,1]^k\right)$, $k \in \mathbb{N} - \{1\}$. Then*

$$\left|T_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \leq (3k+1)\,\omega_1\left(f, \frac{1}{\sqrt{N_{\min}}}\right), \tag{80}$$

$\forall\, x \in [0,1]^k$, $\forall\, \overrightarrow{N} \in \mathbb{N}^k$, where $N_{\min} := \min\{N_1, ..., N_k\}$.

That is

$$\left\|T_{\overrightarrow{N}}^{(M)}(f) - f\right\|_{\infty} \leq (3k+1)\,\omega_1\left(f, \frac{1}{\sqrt{N_{\min}}}\right). \tag{81}$$

It holds that $\lim_{\overrightarrow{N} \to (\infty,\dots,\infty)} T_{\overrightarrow{N}}^{(M)}(f) = f$, uniformly.

Proof. We get that (use of (44))

$$\left|T_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \overset{(73)}{\leq} \omega_1(f,h)\left(1 + \frac{1}{h}\left(\sum_{i=1}^{k} T_{N_i}^{(M)}(|t_i - x_i|)(x)\right)\right)$$

$$\overset{(67)}{\leq} \omega_1(f,h)\left(1 + \frac{1}{h}\left(\frac{3k}{\sqrt{N_{\min}}}\right)\right) \tag{82}$$

(setting $h := \frac{1}{\sqrt{N_{\min}}}$)

$$= \omega_1\left(f, \frac{1}{\sqrt{N_{\min}}}\right)(3k+1), \quad \forall\, x \in [0,1]^k,\ \forall\, \overrightarrow{N} \in \mathbb{N}^k,$$

proving the claim. ■

We make

Remark 26 *We mention the truncated Max-product Baskakov operator (see [4], p. 11)*

$$U_N^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} b_{N,k}(x)\, f\left(\frac{k}{N}\right)}{\bigvee_{k=0}^{N} b_{N,k}(x)}, \quad x \in [0,1],\ f \in C_+([0,1]),\ \forall\, N \in \mathbb{N}, \tag{83}$$

17

where

$$b_{N,k}(x) = \binom{N+k-1}{k} \frac{x^k}{(1+x)^{N+k}}. \tag{84}$$

From [4], pp. 217-218, we get ($x \in [0,1]$)

$$\left(U_N^{(M)}(|\cdot - x|)\right)(x) \leq \frac{12}{\sqrt{N+1}}, \;\; N \geq 2, \; N \in \mathbb{N}. \tag{85}$$

And as in [2], we obtain ($m \in \mathbb{N}$)

$$\left(U_N^{(M)}(|\cdot - x|^m)\right)(x) \leq \frac{12}{\sqrt{N+1}}, \;\; N \geq 2, \; N \in \mathbb{N}, \; \forall \; x \in [0,1]. \tag{86}$$

Definition 27 *Let* $f \in C_+\left([0,1]^k\right)$, $k \in \mathbb{N} - \{1\}$, *and* $\overrightarrow{N} = (N_1, ..., N_k) \in \mathbb{N}^k$. *We define the multivariate Max-product truncated Baskakov operators as follows:*

$$U_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=0}^{N_1} \bigvee_{i_2=0}^{N_2} ... \bigvee_{i_k=0}^{N_k} b_{N_1,i_1}(x_1)\, b_{N_2,i_2}(x_2) ... b_{N_k,i_k}(x_k)\, f\left(\frac{i_1}{N_1}, ..., \frac{i_k}{N_k}\right)}{\bigvee_{i_1=0}^{N_1} \bigvee_{i_2=0}^{N_2} ... \bigvee_{i_k=0}^{N_k} b_{N_1,i_1}(x_1)\, b_{N_2,i_2}(x_2) ... b_{N_k,i_k}(x_k)}, \tag{87}$$

$\forall \; x = (x_1, ..., x_k) \in [0,1]^k$. *Call* $N_{\min} := \min\{N_1, ..., N_k\}$.

The operators $U_{\overrightarrow{N}}^{(M)}(f)(x)$ *are positive sublinear mapping* $C_+\left([0,1]^k\right)$ *into itself, and* $U_{\overrightarrow{N}}^{(M)}(1) = 1$.

We also have

$$U_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=0}^{N_1} \bigvee_{i_2=0}^{N_2} ... \bigvee_{i_k=0}^{N_k} b_{N_1,i_1}(x_1)\, b_{N_2,i_2}(x_2) ... b_{N_k,i_k}(x_k)\, f\left(\frac{i_1}{N_1}, ..., \frac{i_k}{N_k}\right)}{\prod_{\lambda=1}^{k}\left(\bigvee_{i_\lambda=0}^{N_\lambda} b_{N_\lambda,i_\lambda}(x_\lambda)\right)}, \tag{88}$$

$\forall \; x \in [0,1]^k$, *by the maximum multiplicative principle, see (27).*

We make

Remark 28 *The coordinate Max-product truncated Baskakov operators are defined as follows* ($\lambda = 1, ..., k$):

$$U_{N_\lambda}^{(M)}(g)(x_\lambda) := \frac{\bigvee_{i_\lambda=0}^{N_\lambda} b_{N_\lambda,i_\lambda}(x_\lambda)\, g\left(\frac{i_\lambda}{N_\lambda}\right)}{\bigvee_{i_\lambda=0}^{N_\lambda} b_{N_\lambda,i_\lambda}(x_\lambda)}, \tag{89}$$

$\forall \; N_\lambda \in \mathbb{N}$, *and* $\forall \; x_\lambda \in [0,1]$, $\forall \; g \in C_+([0,1])$.

Here we have

$$b_{N_\lambda,i_\lambda}(x_\lambda) = \binom{N_\lambda + i_\lambda - 1}{i_\lambda} \frac{x_\lambda^{i_\lambda}}{(1+x_\lambda)^{N+i_\lambda}}, \;\; \lambda = 1, ..., k; \;\; x_\lambda \in [0,1].$$

In case of $f \in C_+\left([0,1]^k\right)$ such that $f(x) := g(x_\lambda)$, $\forall\ x \in [0,1]^k$, where $x = (x_1, ..., x_\lambda, ..., x_k)$ and $g \in C_+([0,1])$, we get that

$$U_{\overrightarrow{N}}^{(M)}(f)(x) = U_{N_\lambda}^{(M)}(g)(x_\lambda), \tag{90}$$

by the maximum multiplicative principle (27) and simplification of (89).

We present

Theorem 29 *Let $x \in [0,1]^k$, $k \in \mathbb{N}-\{1\}$, be fixed, and let $f \in C^n\left([0,1]^k, \mathbb{R}_+\right)$, $n \in \mathbb{N}-\{1\}$. We assume that $f_\alpha(x) = 0$, for all $\alpha : |\alpha| = 1, ..., n$. Then*

$$\left|U_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \leq 12\left(\max_{\alpha:|\alpha|=n}\left(\omega_1\left(f_\alpha, \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{n+1}}\right)\right)\right)\cdot$$

$$\left[\frac{k^{n+1}}{(n+1)!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n}{n+1}} + \frac{k^n}{2n!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right) + \frac{k^{n-1}}{8(n-1)!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n+2}{n+1}}\right], \tag{91}$$

$\forall\ \overrightarrow{N} \in (\mathbb{N}-\{1\})^k$, where $N_{\min} := \min\{N_1, ..., N_k\}$.

We have that $\lim_{\overrightarrow{N}\to(\infty,...,\infty)} U_{\overrightarrow{N}}^{(M)}(f)(x) = f(x)$.

Proof. By (26) we get:

$$\left|U_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \overset{(90)}{\leq} \left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha, h)\right)\cdot$$

$$\left[\frac{k^n}{(n+1)!h}\left(\sum_{i=1}^k U_{N_i}^{(M)}\left(|t_i - x_i|^{n+1}\right)(x_i)\right) + \frac{k^{n-1}}{2n!}\left(\sum_{i=1}^k U_{N_i}^{(M)}\left(|t_i - x_i|^n\right)(x_i)\right)\right. \tag{92}$$

$$\left.+\frac{hk^{n-2}}{8(n-1)!}\left(\sum_{i=1}^k U_{N_i}^{(M)}\left(|t_i - x_i|^{n-1}\right)(x_i)\right)\right] \overset{(86)}{\leq}$$

$$\frac{12}{\sqrt{N_{\min}+1}}\left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha, h)\right)\left[\frac{k^{n+1}}{(n+1)!h} + \frac{k^n}{2n!} + \frac{hk^{n-1}}{8(n-1)!}\right] =: (\xi).$$

Above notice that $\sum_{i=1}^k U_{N_i}^{(M)}(|t_i - x_i|^n)(x_i) \overset{(86)}{\leq} \sum_{i=1}^k \frac{12}{\sqrt{N_i+1}} \leq \frac{12k}{\sqrt{N_{\min}+1}}$, etc.

Next we choose $h := \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{n+1}}$, then $h^n = \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n}{n+1}}$ and $h^{n+1} = \frac{1}{\sqrt{N_{\min}+1}}$.

We have

$$(\xi) = 12\left(\max_{\alpha:|\alpha|=n}\left(\omega_1\left(f_\alpha, \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{n+1}}\right)\right)\right)\cdot$$

$$\left[\frac{k^{n+1}}{(n+1)!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n}{n+1}}+\frac{k^{n}}{2n!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)+\frac{k^{n-1}}{8\,(n-1)!}\left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{n+2}{n+1}}\right], \tag{93}$$

proving the claim. ■

We also give

Proposition 30 *Let $x \in [0,1]^k$, $k \in \mathbb{N}-\{1\}$, be fixed and let $f \in C^1\left([0,1]^k, \mathbb{R}_+\right)$. We assume that $\frac{\partial f(x)}{\partial x_i}=0$, for $i=1,...,k$. Then*

$$\left|U_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right| \leq \left(\max_{i=1,\ldots,k}\omega_1\left(\frac{\partial f}{\partial x_i},\frac{1}{\sqrt[4]{N_{\min}+1}}\right)\right)\cdot \tag{94}$$

$$\left[\frac{6k^2}{\sqrt[4]{N_{\min}+1}}+\frac{6k}{\sqrt{N_{\min}+1}}+\frac{1}{8\left(\sqrt[4]{N_{\min}+1}\right)}\right],$$

$\forall \overrightarrow{N} \in (\mathbb{N}-\{1\})^k$, where $N_{\min} := \min\{N_1,...,N_k\}$.

Also it holds $\lim_{\overrightarrow{N}\to(\infty,\ldots,\infty)} U_{\overrightarrow{N}}^{(M)}(f)(x)=f(x)$.

Proof. By (31) we get:

$$\left|U_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right| \overset{(90)}{\leq} \left(\max_{i=1,\ldots,k}\omega_1\left(\frac{\partial f}{\partial x_i},h\right)\right)\cdot$$

$$\left[\frac{k}{2h}\left(\sum_{i=1}^{k}U_{N_i}^{(M)}\left((t_i-x_i)^2\right)(x_i)\right)+\frac{1}{2}\left(\sum_{i=1}^{k}U_{N_i}^{(M)}\left(|t_i-x_i|\right)(x_i)\right)+\frac{h}{8}\right] \tag{95}$$

(next we choose $h := \left(\frac{1}{\sqrt{N_{\min}+1}}\right)^{\frac{1}{2}}$, then $h^2=\frac{1}{\sqrt{N_{\min}+1}}$)

$$\overset{(85)}{\leq} \left(\max_{i=1,\ldots,k}\omega_1\left(\frac{\partial f}{\partial x_i},\frac{1}{\sqrt[4]{N_{\min}+1}}\right)\right)\cdot$$

$$\left[\frac{6k^2}{\sqrt[4]{N_{\min}+1}}+\frac{6k}{\sqrt{N_{\min}+1}}+\frac{1}{8\left(\sqrt[4]{N_{\min}+1}\right)}\right], \tag{96}$$

proving the claim. ■

It follows

Theorem 31 *Let $f \in C_+\left([0,1]^k\right)$, $k \in \mathbb{N}-\{1\}$. Then*

$$\left|U_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right| \leq (12k+1)\,\omega_1\left(f,\frac{1}{\sqrt{N_{\min}+1}}\right), \tag{97}$$

$\forall\, x \in [0,1]^k$, $\forall\, \overrightarrow{N} \in (\mathbb{N}-\{1\})^k$, where $N_{\min} := \min\{N_1,...,N_k\}$.

That is

$$\left\| U_{\overrightarrow{N}}^{(M)}(f) - f \right\|_{\infty} \leq (12k+1)\, \omega_1 \left(f, \frac{1}{\sqrt{N_{\min}+1}} \right). \tag{98}$$

It holds that $\lim_{\overrightarrow{N} \to (\infty,\ldots,\infty)} U_{\overrightarrow{N}}^{(M)}(f) = f$, *uniformly.*

Proof. We get that (use of (44))

$$\left| U_{\overrightarrow{N}}^{(M)}(f)(x) - f(x) \right| \overset{(90)}{\leq} \omega_1(f,h) \left(1 + \frac{1}{h} \left(\sum_{i=1}^{k} U_{N_i}^{(M)}(|t_i - x_i|)(x_i) \right) \right)$$

$$\overset{(85)}{\leq} \omega_1(f,h) \left(1 + \frac{1}{h} \left(\frac{12k}{\sqrt{N_{\min}+1}} \right) \right) \tag{99}$$

(setting $h := \frac{1}{\sqrt{N_{\min}+1}}$)

$$= \omega_1 \left(f, \frac{1}{\sqrt{N_{\min}+1}} \right) (12k+1), \quad \forall\, x \in [0,1]^k, \ \forall\, \overrightarrow{N} \in (\mathbb{N} - \{1\})^k,$$

proving the claim. ■

We make

Remark 32 *Here we mention the Max-product truncated sampling operators (see [4], p. 13) defined by*

$$W_N^{(M)}(f)(x) := \frac{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin(Nx-k\pi)}{Nx-k\pi}}, \quad x \in [0,\pi], \tag{100}$$

$f : [0,\pi] \to \mathbb{R}_+$, *continuous,*

and

$$K_N^{(M)}(f)(x) := \frac{\bigvee_{k=0}^{N} \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2} f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \frac{\sin^2(Nx-k\pi)}{(Nx-k\pi)^2}}, \quad x \in [0,\pi], \tag{101}$$

$f : [0,\pi] \to \mathbb{R}_+$, *continuous.*

By convention we talk $\frac{\sin(0)}{0} = 1$, *which implies for every* $x = \frac{k\pi}{N}$, $k \in \{0,1,\ldots,N\}$ *that we have* $\frac{\sin(Nx-k\pi)}{Nx-k\pi} = 1$.

We define the Max-product truncated combined sampling operators

$$M_N^{(M)}(f)(x) := \frac{\bigvee_{k=0}^{N} \rho_{N,k}(x) f\left(\frac{k\pi}{N}\right)}{\bigvee_{k=0}^{N} \rho_{N,k}(x)}, \quad x \in [0,\pi], \tag{102}$$

$f \in C_+([0,\pi])$, *where*

$$M_N^{(M)}(f)(x) := \begin{cases} W_N^{(M)}(f)(x), \ \text{if } \rho_{N,k}(x) := \frac{\sin(Nx-k\pi)}{Nx-k\pi}, \\ K_N^{(M)}(f)(x), \ \text{if } \rho_{N,k}(x) := \left(\frac{\sin(Nx-k\pi)}{Nx-k\pi}\right)^2. \end{cases} \tag{103}$$

21

By [4], p. 346 and p. 352 we get

$$\left(M_N^{(M)}\left(|\cdot - x|\right)\right)(x) \leq \frac{\pi}{2N}, \tag{104}$$

and by [3] ($m \in \mathbb{N}$) we have

$$\left(M_N^{(M)}\left(|\cdot - x|^m\right)\right)(x) \leq \frac{\pi^m}{2N}, \ \ \forall\ x \in [0,\pi],\ \forall\ N \in \mathbb{N}. \tag{105}$$

We give

Definition 33 *Let $f \in C_+\left([0,\pi]^k\right)$, $k \in \mathbb{N} - \{1\}$, and $\overrightarrow{N} = (N_1, ..., N_k) \in \mathbb{N}^k$. We define the multivariate Max-product truncated combined sampling operators as follows:*

$$M_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=0}^{N_1}\bigvee_{i_2=0}^{N_2}...\bigvee_{i_k=0}^{N_k}\rho_{N_1,i_1}(x_1)\,\rho_{N_2,i_2}(x_2)...\rho_{N_k,i_k}(x_k)\,f\left(\frac{i_1\pi}{N_1},\frac{i_2\pi}{N_2},...,\frac{i_k\pi}{N_k}\right)}{\bigvee_{i_1=0}^{N_1}\bigvee_{i_2=0}^{N_2}...\bigvee_{i_k=0}^{N_k}\rho_{N_1,i_1}(x_1)\,\rho_{N_2,i_2}(x_2)...\rho_{N_k,i_k}(x_k)}, \tag{106}$$

$\forall\ x = (x_1, ..., x_k) \in [0,\pi]^k$. *Call $N_{\min} := \min\{N_1, ..., N_k\}$.*

The operators $M_{\overrightarrow{N}}^{(M)}(f)(x)$ are positive sublinear mapping $C_+\left([0,\pi]^k\right)$ into itself, and $M_{\overrightarrow{N}}^{(M)}(1) = 1$.

We also have

$$M_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=0}^{N_1}\bigvee_{i_2=0}^{N_2}...\bigvee_{i_k=0}^{N_k}\rho_{N_1,i_1}(x_1)\,\rho_{N_2,i_2}(x_2)...\rho_{N_k,i_k}(x_k)\,f\left(\frac{i_1\pi}{N_1},\frac{i_2\pi}{N_2},...,\frac{i_k\pi}{N_k}\right)}{\prod_{\lambda=1}^{k}\left(\bigvee_{i_\lambda=0}^{N_\lambda}\rho_{N_\lambda,i_\lambda}(x_\lambda)\right)}, \tag{107}$$

$\forall\ x \in [0,\pi]^k$, *by the maximum multiplicative principle, see (27).*

We make

Remark 34 *The coordinate Max-product truncated combined sampling operators are defined as follows ($\lambda = 1, ..., k$):*

$$M_{N_\lambda}^{(M)}(g)(x_\lambda) := \frac{\bigvee_{i_\lambda=0}^{N_\lambda}\rho_{N_\lambda,i_\lambda}(x_\lambda)\,g\left(\frac{i_\lambda\pi}{N_\lambda}\right)}{\bigvee_{i_\lambda=0}^{N_\lambda}\rho_{N_\lambda,i_\lambda}(x_\lambda)}, \tag{108}$$

$\forall\ N_\lambda \in \mathbb{N}$, *and* $\forall\ x_\lambda \in [0,\pi]$, $\forall\ g \in C_+([0,\pi])$.

Here we have ($\lambda = 1, ..., k$; $x_\lambda \in [0,\pi]$)

$$\rho_{N_\lambda,i_\lambda}(x_\lambda) = \left\{\begin{array}{l} \frac{\sin(N_\lambda x_\lambda - i_\lambda\pi)}{N_\lambda x_\lambda - i_\lambda\pi}, \quad \text{if } M_{N_\lambda}^{(M)} = W_{N_\lambda}^{(M)}, \\ \left(\frac{\sin(N_\lambda x_\lambda - i_\lambda\pi)}{N_\lambda x_\lambda - i_\lambda\pi}\right)^2, \quad \text{if } M_{N_\lambda}^{(M)} = K_{N_\lambda}^{(M)}. \end{array}\right\}. \tag{109}$$

22

In case of $f \in C_{+}\left([0,\pi]^{k}\right)$ such that $f(x) := g(x_{\lambda})$, $\forall\ x \in [0,\pi]^{k}$, where $x = (x_1, ..., x_{\lambda}, ..., x_k)$ and $g \in C_{+}([0,\pi])$, we get that

$$M_{\overrightarrow{N}}^{(M)}(f)(x) = M_{N_{\lambda}}^{(M)}(g)(x_{\lambda}), \tag{110}$$

by the maximum multiplicative principle (27) and simplification of (107).

We present

Theorem 35 *Let $x \in [0,\pi]^{k}$, $k \in \mathbb{N}-\{1\}$, be fixed, and let $f \in C^{n}\left([0,\pi]^{k}, \mathbb{R}_{+}\right)$, $n \in \mathbb{N}-\{1\}$. We assume that $f_{\alpha}(x) = 0$, for all $\alpha : |\alpha| = 1, ..., n$. Then*

$$\left|M_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \leq \frac{(k\pi)^{n-1}}{2}\left(\max_{\alpha:|\alpha|=n}\omega_1\left(f_{\alpha}, \frac{1}{(N_{\min})^{\frac{1}{n+1}}}\right)\right)\cdot \tag{111}$$

$$\left[\frac{(k\pi)^2}{(n+1)!}\frac{1}{(N_{\min})^{\frac{n}{n+1}}} + \frac{k\pi}{2n!N_{\min}} + \frac{1}{8(n-1)!(N_{\min})^{\frac{n+2}{n+1}}}\right],$$

$\forall\ \overrightarrow{N} = (N_1, ..., N_k) \in \mathbb{N}^{k}$, where $N_{\min} := \min\{N_1, ..., N_k\}$.

We have that $\lim_{\overrightarrow{N}\to(\infty,...,\infty)} M_{\overrightarrow{N}}^{(M)}(f)(x) = f(x)$.

Proof. By (26) we get:

$$\left|M_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \overset{(110)}{\leq} \left(\max_{\alpha:|\alpha|=n}\omega_1(f_{\alpha}, h)\right)\cdot$$

$$\left[\frac{k^{n}}{(n+1)!h}\left(\sum_{i=1}^{k} M_{N_i}^{(M)}\left(|t_i - x_i|^{n+1}\right)(x_i)\right) + \frac{k^{n-1}}{2n!}\left(\sum_{i=1}^{k} M_{N_i}^{(M)}\left(|t_i - x_i|^{n}\right)(x_i)\right)\right. \tag{112}$$

$$\left.+\frac{hk^{n-2}}{8(n-1)!}\left(\sum_{i=1}^{k} M_{N_i}^{(M)}\left(|t_i - x_i|^{n-1}\right)(x_i)\right)\right] \overset{(105)}{\leq}$$

$$\frac{1}{2N_{\min}}\left(\max_{\alpha:|\alpha|=n}\omega_1(f_{\alpha}, h)\right)\left[\frac{k^{n+1}\pi^{n+1}}{(n+1)!h} + \frac{k^{n}\pi^{n}}{2n!} + \frac{hk^{n-1}\pi^{n-1}}{8(n-1)!}\right] =: (\xi).$$

Above notice that $\sum_{i=1}^{k} M_{N_i}^{(M)}\left(|t_i - x_i|^{n}\right)(x_i) \overset{(105)}{\leq} \sum_{i=1}^{k}\frac{\pi^{n}}{2N_i} \leq \frac{k\pi^{n}}{2N_{\min}}$, etc.

Next we choose $h := \left(\frac{1}{N_{\min}}\right)^{\frac{1}{n+1}}$, then $h^{n} = \left(\frac{1}{N_{\min}}\right)^{\frac{n}{n+1}}$ and $h^{n+1} = \frac{1}{N_{\min}}$.

We have

$$(\xi) = \frac{(k\pi)^{n-1}}{2}\left(\max_{\alpha:|\alpha|=n}\omega_1\left(f_{\alpha}, \frac{1}{(N_{\min})^{\frac{1}{n+1}}}\right)\right)\cdot \tag{113}$$

$$\left[\frac{(k\pi)^2}{(n+1)!}\frac{1}{(N_{\min})^{\frac{n}{n+1}}} + \frac{k\pi}{2n!N_{\min}} + \frac{1}{8(n-1)!(N_{\min})^{\frac{n+2}{n+1}}}\right],$$

proving the claim. ■

We also give

Proposition 36 *Let $x \in [0,\pi]^k$, $k \in \mathbb{N}-\{1\}$, be fixed and let $f \in C^1([0,\pi], \mathbb{R}_+)$. We assume that $\frac{\partial f(x)}{\partial x_i} = 0$, for $i = 1, ..., k$. Then*

$$\left| M_{\overrightarrow{N}}^{(M)}(f)(x) - f(x) \right| \leq \left(\max_{i=1,...,k} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{1}{\sqrt{N_{\min}}} \right) \right) \cdot$$

$$\left[\frac{(k\pi)^2}{4\sqrt{N_{\min}}} + \frac{k\pi}{4N_{\min}} + \frac{1}{8\left(\sqrt{N_{\min}}\right)} \right], \tag{114}$$

$\forall\ \overrightarrow{N} \in \mathbb{N}^k$, *where* $N_{\min} := \min\{N_1, ..., N_k\}$.

Also it holds $\lim_{\overrightarrow{N} \to (\infty,...,\infty)} M_{\overrightarrow{N}}^{(M)}(f)(x) = f(x)$.

Proof. By (31) we get:

$$\left| M_{\overrightarrow{N}}^{(M)}(f)(x) - f(x) \right| \overset{(110)}{\leq} \left(\max_{i=1,...,k} \omega_1 \left(\frac{\partial f}{\partial x_i}, h \right) \right) \cdot$$

$$\left[\frac{k}{2h} \left(\sum_{i=1}^{k} M_{N_i}^{(M)} \left((t_i - x_i)^2 \right)(x_i) \right) + \frac{1}{2} \left(\sum_{i=1}^{k} M_{N_i}^{(M)} (|t_i - x_i|)(x_i) \right) + \frac{h}{8} \right] \tag{115}$$

(next we choose $h := \left(\frac{1}{N_{\min}} \right)^{\frac{1}{2}}$, then $h^2 = \frac{1}{N_{\min}}$)

$$\overset{(105)}{\leq} \left(\max_{i=1,...,k} \omega_1 \left(\frac{\partial f}{\partial x_i}, \frac{1}{\sqrt{N_{\min}}} \right) \right) \cdot$$

$$\left[\frac{(k\pi)^2}{4\sqrt{N_{\min}}} + \frac{k\pi}{4N_{\min}} + \frac{1}{8\left(\sqrt{N_{\min}}\right)} \right], \tag{116}$$

proving the claim. ■

It follows

Theorem 37 *Let $f \in C_+\left([0,\pi]^k\right)$, $k \in \mathbb{N} - \{1\}$. Then*

$$\left| M_{\overrightarrow{N}}^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{k\pi}{2} + 1 \right) \omega_1 \left(f, \frac{1}{N_{\min}} \right), \tag{117}$$

$\forall\ x \in [0,\pi]^k$, $\forall\ \overrightarrow{N} \in \mathbb{N}^k$, *where* $N_{\min} := \min\{N_1, ..., N_k\}$.

That is

$$\left\| M_{\overrightarrow{N}}^{(M)}(f) - f \right\|_\infty \leq \left(\frac{k\pi}{2} + 1 \right) \omega_1 \left(f, \frac{1}{N_{\min}} \right). \tag{118}$$

It holds $\lim_{\overrightarrow{N} \to (\infty,...,\infty)} M_{\overrightarrow{N}}^{(M)}(f) = f$, *uniformly.*

24

Proof. We get that (use of (44))

$$\left|M_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right| \overset{(110)}{\leq} \omega_1(f,h)\left(1+\frac{1}{h}\left(\sum_{i=1}^{k} M_{N_i}^{(M)}(|t_i-x_i|)(x_i)\right)\right)$$

$$\overset{(104)}{\leq} \omega_1(f,h)\left(1+\frac{1}{h}\left(\frac{k\pi}{2N_{\min}}\right)\right) \tag{119}$$

(setting $h:=\frac{1}{N_{\min}}$)

$$=\omega_1\left(f,\frac{1}{N_{\min}}\right)\left(\frac{k\pi}{2}+1\right), \quad \forall\, x\in[0,\pi]^k,\ \forall\, \overrightarrow{N}\in\mathbb{N}^k,$$

proving the claim. ■

We make

Remark 38 *Let* $f\in C_+([-1,1])$. *Let the Chebyshev knots of second kind* $x_{N,k}=\cos\left(\left(\frac{N-k}{N-1}\right)\pi\right)\in[-1,1]$, $k=1,...,N$, $N\in\mathbb{N}-\{1\}$, *which are the roots of* $\omega_N(x)=\sin(N-1)t\sin t$, $x=\cos t\in[-1,1]$. *Notice that* $x_{N,1}=-1$ *and* $x_{N,N}=1$.

Define

$$l_{N,k}(x):=\frac{(-1)^{k-1}\omega_N(x)}{(1+\delta_{k,1}+\delta_{k,N})(N-1)(x-x_{N,k})}, \tag{120}$$

$N\geq 2$, $k=1,...,N$, *and* $\omega_N(x)=\prod_{k=1}^{N}(x-x_{N,k})$ *and* $\delta_{i,j}$ *denotes the Kronecher's symbol, that is* $\delta_{i,j}=1$, *if* $i=j$, *and* $\delta_{i,j}=0$, *if* $i\neq j$.

The Max-product Lagrange interpolation operators on Chebyshev knots of second kind, plus the endpoints ± 1, *are defined by ([4], p. 12)*

$$L_N^{(M)}(f)(x)=\frac{\bigvee_{k=1}^{N} l_{N,k}(x)f(x_{N,k})}{\bigvee_{k=1}^{N} l_{N,k}(x)}, \quad x\in[-1,1]. \tag{121}$$

By [4], pp. 297-298 and [3], we get that

$$L_N^{(M)}(|\cdot-x|^m)(x)\leq\frac{2^{m+1}\pi^2}{3(N-1)}, \tag{122}$$

$\forall\, x\in(-1,1)$ *and* $\forall\, m\in\mathbb{N}$; $\forall\, N\in\mathbb{N}$, $N\geq 4$.

We see that $L_N^{(M)}(f)(x)\geq 0$ *is well defined and continuous for any* $x\in[-1,1]$. *Following [4], p. 289, because* $\sum_{k=1}^{N} l_{N,k}(x)=1$, $\forall\, x\in[-1,1]$, *for any* x *there exists* $k\in\{1,...,N\}: l_{N,k}(x)>0$, *hence* $\bigvee_{k=1}^{N} l_{N,k}(x)>0$. *We have that* $l_{N,k}(x_{N,k})=1$, *and* $l_{N,k}(x_{N,j})=0$, *if* $k\neq j$. *Furthermore it holds* $L_N^{(M)}(f)(x_{N,j})=f(x_{N,j})$, *all* $j\in\{1,...,N\}$, *and* $L_N^{(M)}(1)=1$.

By [4], pp. 289-290, $L_N^{(M)}$ *are positive sublinear operators.*

We give

Definition 39 *Let $f \in C_+\left([-1,1]^k\right)$, $k \in \mathbb{N}-\{1\}$, and $\overrightarrow{N}=(N_1,...,N_k) \in (\mathbb{N}-\{1\})^k$. We define the multivariate Max-product Lagrange interpolation operators on Chebyshev knots of second kind, plus the endpoints ± 1, as follows:*

$$L_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=1}^{N_1}\bigvee_{i_2=1}^{N_2}...\bigvee_{i_k=1}^{N_k} l_{N_1,i_1}(x_1)\, l_{N_2,i_2}(x_2)\,...l_{N_k,i_k}(x_k)\, f\left(x_{N_1,i_1},x_{N_2,i_2},...,x_{N_k,i_k}\right)}{\bigvee_{i_1=1}^{N_1}\bigvee_{i_2=1}^{N_2}...\bigvee_{i_k=1}^{N_k} l_{N_1,i_1}(x_1)\, l_{N_2,i_2}(x_2)\,...l_{N_k,i_k}(x_k)}, \tag{123}$$

$\forall\ x=(x_1,...,x_k)\in[-1,1]^k$. Call $N_{\min}:=\min\{N_1,...,N_k\}$.

The operators $L_{\overrightarrow{N}}^{(M)}(f)(x)$ are positive sublinear mapping $C_+\left([-1,1]^k\right)$ into itself, and $L_{\overrightarrow{N}}^{(M)}(1)=1$.

We also have

$$L_{\overrightarrow{N}}^{(M)}(f)(x) :=$$

$$\frac{\bigvee_{i_1=1}^{N_1}\bigvee_{i_2=1}^{N_2}...\bigvee_{i_k=1}^{N_k} l_{N_1,i_1}(x_1)\, l_{N_2,i_2}(x_2)\,...l_{N_k,i_k}(x_k)\, f\left(x_{N_1,i_1},x_{N_2,i_2},...,x_{N_k,i_k}\right)}{\prod_{\lambda=1}^{k}\left(\bigvee_{i_\lambda=1}^{N_\lambda} l_{N_\lambda,i_\lambda}(x_\lambda)\right)}, \tag{124}$$

$\forall\ x=(x_1,...,x_\lambda,...,x_k)\in[-1,1]^k$, by the maximum multiplicative principle, see (27). Notice that $L_{\overrightarrow{N}}^{(M)}(f)\left(x_{N_1,i_1},...,x_{N_k,i_k}\right)=f\left(x_{N_1,i_1},...,x_{N_k,i_k}\right)$. The last is also true if $x_{N_1,i_1},...,x_{N_k,i_k}\in\{-1,1\}$.

We make

Remark 40 *The coordinate Max-product Lagrange interpolation operators on Chebyshev knots of second kind, plus the endpoints ± 1, are defined as follows ($\lambda=1,...,k$):*

$$L_{N_\lambda}^{(M)}(g)(x_\lambda) := \frac{\bigvee_{i_\lambda=1}^{N_\lambda} l_{N_\lambda,i_\lambda}(x_\lambda)\, g\left(x_{N_\lambda,i_\lambda}\right)}{\bigvee_{i_\lambda=1}^{N_\lambda} l_{N_\lambda,i_\lambda}(x_\lambda)}, \tag{125}$$

$\forall\ N_\lambda\in\mathbb{N}$, $N_\lambda\geq 2$, and $\forall\ x_\lambda\in[-1,1]$, $\forall\ g\in C_+([-1,1])$.

Here we have ($\lambda=1,...,k$; $x_\lambda\in[-1,1]$)

$$l_{N_\lambda,i_\lambda}(x_\lambda) = \frac{(-1)^{i_\lambda-1}\,\omega_{N_\lambda}(x_\lambda)}{\left(1+\delta_{i_\lambda,1}+\delta_{i_\lambda,N_\lambda}\right)(N_\lambda-1)\left(x_\lambda-x_{N_\lambda,i_\lambda}\right)}, \tag{126}$$

$N_\lambda\geq 2$, $i_\lambda=1,...,N_\lambda$ and $\omega_{N_\lambda}(x_\lambda)=\prod_{i_\lambda=1}^{N_\lambda}\left(x_\lambda-x_{N_\lambda,i_\lambda}\right)$; where $x_{N_\lambda,i_\lambda}=\cos\left(\left(\frac{N_\lambda-i_\lambda}{N_\lambda-1}\right)\pi\right)\in[-1,1]$, $i_\lambda=1,...,N_\lambda$ ($N_\lambda\geq 2$) are roots of $\omega_{N_\lambda}(x_\lambda)=\sin(N_\lambda-1)\,t_\lambda\sin t_\lambda$, $x_\lambda=\cos t_\lambda$. Notice that $x_{N_\lambda,1}=-1$, $x_{N_\lambda,N_\lambda}=1$.

In case of $f\in C_+\left([-1,1]^k\right)$ such that $f(x):=g(x_\lambda)$, $\forall\ x\in[-1,1]^k$, where $x=(x_1,...,x_\lambda,...,x_k)$ and $g\in C_+([-1,1])$, we get that

$$L_{\overrightarrow{N}}^{(M)}(f)(x) = L_{N_\lambda}^{(M)}(g)(x_\lambda), \tag{127}$$

by the maximum multiplicative principle (27) and simplification of (124).

We present

Theorem 41 *Let* $x \in (-1,1)^k$, $k \in \mathbb{N}-\{1\}$, *be fixed, and let* $f \in C^n\left([-1,1]^k, \mathbb{R}_+\right)$, $n \in \mathbb{N}-\{1\}$. *We assume that* $f_\alpha(x) = 0$, *for all* $\alpha : |\alpha| = 1, ..., n$. *Then*

$$\left|L_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \leq \frac{(2k)^{n-1}\pi^2}{3}\left(\max_{\alpha:|\alpha|=n}\omega_1\left(f_\alpha, \frac{1}{\sqrt[n+1]{N_{\min}-1}}\right)\right)\cdot \tag{128}$$

$$\left[\frac{8k^2}{(n+1)!\,(N_{\min}-1)^{\frac{n}{n+1}}} + \frac{2k}{n!\,(N_{\min}-1)} + \frac{1}{4(n-1)!\,(N_{\min}-1)^{\frac{n+2}{n+1}}}\right],$$

$\forall\ \overrightarrow{N} = (N_1, ..., N_k) \in \mathbb{N}^k$; $N_i \geq 4$, $i = 1, ..., k$, *and* $N_{\min} := \min\{N_1, ..., N_k\}$.

We have that $\lim_{\overrightarrow{N}\to(\infty,...,\infty)} L_{\overrightarrow{N}}^{(M)}(f)(x) = f(x)$.

Proof. By (26) we get:

$$\left|L_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \overset{(127)}{\leq} \left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha, h)\right)\cdot$$

$$\left[\frac{k^n}{(n+1)!h}\left(\sum_{i=1}^k L_{N_i}^{(M)}\left(|t_i - x_i|^{n+1}\right)(x_i)\right) + \frac{k^{n-1}}{2n!}\left(\sum_{i=1}^k L_{N_i}^{(M)}\left(|t_i - x_i|^n\right)(x_i)\right)\right. \tag{129}$$

$$\left. + \frac{hk^{n-2}}{8(n-1)!}\left(\sum_{i=1}^k L_{N_i}^{(M)}\left(|t_i - x_i|^{n-1}\right)(x_i)\right)\right] \overset{(122)}{\leq}$$

$$\frac{\pi^2}{3(N_{\min}-1)}\left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha, h)\right)\left[\frac{k^{n+1}2^{n+2}}{(n+1)!h} + \frac{k^n 2^{n+1}}{2n!} + \frac{hk^{n-1}2^n}{8(n-1)!}\right] =: (\xi).$$

Above we notice that $\sum_{i=1}^k L_{N_i}^{(M)}(|t_i - x_i|^n)(x_i) \overset{(122)}{\leq} \sum_{i=1}^k \frac{2^{n+1}\pi^2}{3(N_i-1)} \leq \frac{2^{n+1}\pi^2 k}{3(N_{\min}-1)}$, etc.

Next we choose $h := \left(\frac{1}{N_{\min}-1}\right)^{\frac{1}{n+1}}$, then $h^n = \left(\frac{1}{N_{\min}-1}\right)^{\frac{n}{n+1}}$ and $h^{n+1} = \frac{1}{N_{\min}-1}$.

We have

$$(\xi) = \frac{\pi^2}{3}\left(\max_{\alpha:|\alpha|=n}\omega_1\left(f_\alpha, \frac{1}{\sqrt[n+1]{N_{\min}-1}}\right)\right)\cdot \tag{130}$$

$$\left[\frac{k^{n+1}2^{n+2}}{(n+1)!}\frac{1}{(N_{\min}-1)^{\frac{n}{n+1}}} + \frac{k^n 2^n}{n!\,(N_{\min}-1)} + \frac{k^{n-1}2^{n-1}}{4(n-1)!}\frac{1}{(N_{\min}-1)^{\frac{n+2}{n+1}}}\right],$$

proving the claim. ■

We also give

Proposition 42 *Let $x \in (-1,1)^k$, $k \in \mathbb{N}-\{1\}$, be fixed, and let $f \in C^1\left([-1,1]^k, \mathbb{R}_+\right)$. We assume that $\frac{\partial f(x)}{\partial x_i} = 0$, for $i = 1, ..., k$. Then*

$$\left|L_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \leq \left(\max_{i=1,...,k} \omega_1\left(\frac{\partial f}{\partial x_i}, \frac{1}{\sqrt{N_{\min}-1}}\right)\right) \cdot \tag{131}$$

$$\left[\frac{(4/3)(k\pi)^2}{\sqrt{N_{\min}-1}} + \frac{(2/3)k\pi^2}{(N_{\min}-1)} + \frac{1}{8\left(\sqrt{N_{\min}-1}\right)}\right],$$

$\forall\ \overrightarrow{N} = (N_1, ..., N_k) \in \mathbb{N}^k$; $N_i \geq 4$, $i = 1, ..., k$, *and* $N_{\min} := \min\{N_1, ..., N_k\}$.

We have that $\lim_{\overrightarrow{N} \to (\infty,...,\infty)} L_{\overrightarrow{N}}^{(M)}(f)(x) = f(x)$.

Proof. By (31) we get:

$$\left|L_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \overset{(127)}{\leq} \left(\max_{i=1,...,k} \omega_1\left(\frac{\partial f}{\partial x_i}, h\right)\right) \cdot$$

$$\left[\frac{k}{2h}\left(\sum_{i=1}^{k} L_{N_i}^{(M)}\left((t_i - x_i)^2\right)(x_i)\right) + \frac{1}{2}\left(\sum_{i=1}^{k} L_{N_i}^{(M)}(|t_i - x_i|)(x_i)\right) + \frac{h}{8}\right] \tag{132}$$

(next we choose $h := \left(\frac{1}{N_{\min}-1}\right)^{\frac{1}{2}}$, then $h^2 = \frac{1}{N_{\min}-1}$)

$$\overset{(122)}{\leq} \left(\max_{i=1,...,k} \omega_1\left(\frac{\partial f}{\partial x_i}, \frac{1}{\sqrt{N_{\min}-1}}\right)\right) \cdot$$

$$\left[\frac{(4/3)(k\pi)^2}{\sqrt{N_{\min}-1}} + \frac{(2/3)k\pi^2}{(N_{\min}-1)} + \frac{1}{8\left(\sqrt{N_{\min}-1}\right)}\right], \tag{133}$$

proving the claim. ■

It follows

Theorem 43 *Let any $x \in [-1,1]^k$, $k \in \mathbb{N}-\{1\}$, and let $f \in C_+\left([-1,1]^k\right)$. Then*

$$\left|L_{\overrightarrow{N}}^{(M)}(f)(x) - f(x)\right| \leq \left(1 + \frac{4\pi^2 k}{3}\right)\omega_1\left(f, \frac{1}{(N_{\min}-1)}\right), \tag{134}$$

and

$$\left\|L_{\overrightarrow{N}}^{(M)}(f) - f\right\|_\infty \leq \left(1 + \frac{4\pi^2 k}{3}\right)\omega_1\left(f, \frac{1}{(N_{\min}-1)}\right), \tag{135}$$

$\forall\ \overrightarrow{N} = (N_1, ..., N_k) \in \mathbb{N}^k$; $N_i \geq 4$, $i = 1, ..., k$, *and* $N_{\min} := \min\{N_1, ..., N_k\}$.

We have that $\lim_{\overrightarrow{N} \to (\infty,...,\infty)} L_{\overrightarrow{N}}^{(M)}(f)(x) = f(x)$, $\forall\ x := (x_1, ..., x_k) \in [-1,1]^k$, *uniformly.*

Proof. We get that (use of (44))

$$\left|L_{\overrightarrow{N}}^{(M)}(f)(x)-f(x)\right| \overset{(127)}{\leq} \omega_1(f,h)\left(1+\frac{1}{h}\left(\sum_{i=1}^{k} L_{N_i}^{(M)}(|t_i-x_i|)(x)\right)\right)$$

$$\overset{(122)}{\leq} \omega_1(f,h)\left(1+\frac{1}{h}\left(\sum_{i=1}^{k}\frac{2^2\pi^2}{3(N_i-1)}\right)\right) \leq \omega_1(f,h)\left(1+\frac{1}{h}\left(\frac{4\pi^2 k}{3(N_{\min}-1)}\right)\right) \tag{137}$$

(setting $h := \frac{1}{N_{\min}-1}$)

$$= \omega_1\left(f,\frac{1}{(N_{\min}-1)}\right)\left(1+\frac{4\pi^2 k}{3}\right), \quad \forall\, x \in (-1,1)^k,$$

proving the claim. ■

We make

Remark 44 *The Chebyshev knots of first kind* $x_{N,k} := \cos\left(\frac{(2(N-k)+1)}{2(N+1)}\pi\right) \in (-1,1)$, $k \in \{0,1,...,N\}$, $-1 < x_{N,0} < x_{N,1} < ... < x_{N,N} < 1$, *are the roots of the first kind Chebyshev polynomial* $T_{N+1}(x) := \cos((N+1)\arccos x)$, $x \in [-1,1]$.

Define ($x \in [-1,1]$)

$$h_{N,k}(x) := (1 - x \cdot x_{N,k})\left(\frac{T_{N+1}(x)}{(N+1)(x-x_{N,k})}\right)^2, \tag{138}$$

the fundamental interpolation polynomials.

The Max-product interpolation Hermite-Fejér operators on Chebyshev knots of the first kind (seep. 12 of [4]) are defined by

$$H_{2N+1}^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{N} h_{N,k}(x) f(x_{N,k})}{\bigvee_{k=0}^{N} h_{N,k}(x)}, \quad \forall\, N \in \mathbb{N}, \tag{139}$$

for $f \in C_+([-1,1])$, $\forall\, x \in [-1,1]$.

By [4], p. 287, we have

$$H_{2N+1}^{(M)}(|\cdot - x|)(x) \leq \frac{2\pi}{N+1}, \quad \forall\, x \in [-1,1], \forall\, N \in \mathbb{N}. \tag{140}$$

And by [3], we get that

$$H_{2N+1}^{(M)}(|\cdot - x|^m)(x) \leq \frac{2^m\pi}{N+1}, \quad \forall\, x \in [-1,1], \forall\, m, N \in \mathbb{N}. \tag{141}$$

Notice $H_{2N+1}^{(M)}(1) = 1$, *and* $H_{2N+1}^{(M)}$ *maps* $C_+([-1,1])$ *into itself, and it is a positive sublinear operator. Furthermore it holds* $\bigvee_{k=0}^{N} h_{N,k}(x) > 0$, $\forall\, x \in [-1,1]$. *We also have* $h_{N,k}(x_{N,k}) = 1$, *and* $h_{N,k}(x_{N,j}) = 0$, *if* $k \neq j$, *and* $H_{2N+1}^{(M)}(f)(x_{N,j}) = f(x_{N,j})$, *for all* $j \in \{0,1,...,N\}$, *see [4], p. 282.*

We need

Definition 45 *Let $f \in C_+\left([-1,1]^k\right)$, $k \in \mathbb{N}-\{1\}$, and $\overrightarrow{N} = (N_1, ..., N_k) \in \mathbb{N}^k$. We define the multivariate Max-product interpolation Hermite-Fejér operators on Chebyshev knots of the first kind, as follows:*

$$H^{(M)}_{2\overrightarrow{N}+1}(f)(x) :=$$

$$\frac{\bigvee_{i_1=0}^{N_1} \bigvee_{i_2=0}^{N_2} ... \bigvee_{i_k=0}^{N_k} h_{N_1,i_1}(x_1) h_{N_2,i_2}(x_2) ... h_{N_k,i_k}(x_k) f(x_{N_1,i_1}, x_{N_2,i_2}, ..., x_{N_k,i_k})}{\bigvee_{i_1=0}^{N_1} \bigvee_{i_2=0}^{N_2} ... \bigvee_{i_k=0}^{N_k} h_{N_1,i_1}(x_1) h_{N_2,i_2}(x_2) ... h_{N_k,i_k}(x_k)}, \tag{142}$$

$\forall\ x = (x_1, ..., x_k) \in [-1,1]^k$. Call $N_{\min} := \min\{N_1, ..., N_k\}$.

The operators $H^{(M)}_{2\overrightarrow{N}+1}(f)(x)$ are positive sublinear mapping $C_+\left([-1,1]^k\right)$ into itself, and $H^{(M)}_{2\overrightarrow{N}+1}(1) = 1$.

We also have

$$H^{(M)}_{2\overrightarrow{N}+1}(f)(x) :=$$

$$\frac{\bigvee_{i_1=0}^{N_1} \bigvee_{i_2=0}^{N_2} ... \bigvee_{i_k=0}^{N_k} h_{N_1,i_1}(x_1) h_{N_2,i_2}(x_2) ... h_{N_k,i_k}(x_k) f(x_{N_1,i_1}, x_{N_2,i_2}, ..., x_{N_k,i_k})}{\prod_{\lambda=1}^{k}\left(\bigvee_{i_\lambda=0}^{N_\lambda} h_{N_\lambda,i_\lambda}(x_\lambda)\right)}, \tag{143}$$

$\forall\ x = (x_1, ..., x_\lambda, ..., x_k) \in [-1,1]^k$, by the maximum multiplicative principle, see (27). Notice that $H^{(M)}_{2\overrightarrow{N}+1}(f)(x_{N_1,i_1}, ..., x_{N_k,i_k}) = f(x_{N_1,i_1}, ..., x_{N_k,i_k})$.

We make

Remark 46 *The coordinate Max-product interpolation Hermite-Fejér operators on Chebyshev knots of the first kind, are defined as follows ($\lambda = 1, ..., k$):*

$$H^{(M)}_{2N_\lambda+1}(g)(x_\lambda) := \frac{\bigvee_{i_\lambda=0}^{N_\lambda} h_{N_\lambda,i_\lambda}(x_\lambda) g(x_{N_\lambda,i_\lambda})}{\bigvee_{i_\lambda=0}^{N_\lambda} h_{N_\lambda,i_\lambda}(x_\lambda)}, \tag{144}$$

$\forall\ N_\lambda \in \mathbb{N}$, and $\forall\ x_\lambda \in [-1,1]$, $\forall\ g \in C_+([-1,1])$.

Here we have ($\lambda = 1, ..., k$; $x_\lambda \in [-1,1]$)

$$h_{N_\lambda,i_\lambda}(x_\lambda) = (1 - x_\lambda \cdot x_{N_\lambda,i_\lambda})\left(\frac{T_{N_\lambda+1}(x_\lambda)}{(N_\lambda+1)(x_\lambda - x_{N_\lambda,i_\lambda})}\right)^2, \tag{145}$$

where the Chebyshev knots $x_{N_\lambda,i_\lambda} = \cos\left(\frac{(2(N_\lambda - i_\lambda)+1)}{2(N_\lambda+1)}\pi\right) \in (-1,1)$, $i_\lambda \in \{0,1,...,N_\lambda\}$, $-1 < x_{N_\lambda,0} < x_{N_\lambda,1} < ... < x_{N_\lambda,N_\lambda} < 1$ are the roots of the first kind Chebyshev polynomial $T_{N_\lambda+1}(x_\lambda) = \cos((N_\lambda+1)\arccos x_\lambda)$, $x_\lambda \in [-1,1]$.

In case of $f \in C_+\left([-1,1]^k\right)$ such that $f(x) := g(x_\lambda)$, $\forall\ x \in [-1,1]^k$ and $g \in C_+([-1,1])$, we get that

$$H^{(M)}_{2\overrightarrow{N}+1}(f)(x) = H^{(M)}_{2N_\lambda+1}(g)(x_\lambda), \tag{146}$$

by the maximum multiplicative principle (27) and simplification of (143).

30

We present

Theorem 47 *Let $x \in [-1,1]^k$, $k \in \mathbb{N}-\{1\}$, be fixed, and let $f \in C^n\left([-1,1]^k, \mathbb{R}_+\right)$, $n \in \mathbb{N}-\{1\}$. We assume that $f_\alpha(x) = 0$, for all $\alpha : |\alpha| = 1, ..., n$. Then*

$$\left|H^{(M)}_{2\overrightarrow{N}+1}(f)(x) - f(x)\right| \leq 2^{n-2}k^{n-1}\pi\left(\max_{\alpha:|\alpha|=n}\omega_1\left(f_\alpha, \frac{1}{\sqrt[n+1]{N_{\min}+1}}\right)\right) \cdot \tag{147}$$

$$\left[\frac{8k^2}{(n+1)!(N_{\min}+1)^{\frac{n}{n+1}}} + \frac{2k}{n!(N_{\min}+1)} + \frac{1}{4(n-1)!(N_{\min}+1)^{\frac{n+2}{n+1}}}\right],$$

$\forall\ \overrightarrow{N} = (N_1, ..., N_k) \in \mathbb{N}^k$, and $N_{\min} := \min\{N_1, ..., N_k\}$.

We have that $\lim_{\overrightarrow{N}\to(\infty,...,\infty)} H^{(M)}_{2\overrightarrow{N}+1}(f)(x) = f(x)$.

Proof. By (26) we get:

$$\left|H^{(M)}_{2\overrightarrow{N}+1}(f)(x) - f(x)\right| \overset{(146)}{\leq} \left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha, h)\right) \cdot$$

$$\left[\frac{k^n}{(n+1)!h}\left(\sum_{i=1}^k H^{(M)}_{2N_i+1}\left(|t_i - x_i|^{n+1}\right)(x_i)\right) + \frac{k^{n-1}}{2n!}\left(H^{(M)}_{2N_i+1}\left(|t_i - x_i|^n\right)(x_i)\right)\right. \tag{148}$$

$$\left. + \frac{hk^{n-2}}{8(n-1)!}\left(\sum_{i=1}^k H^{(M)}_{2N_i+1}\left(|t_i - x_i|^{n-1}\right)(x_i)\right)\right] \overset{(141)}{\leq}$$

$$\left(\frac{\pi}{N_{\min}+1}\right)\left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha, h)\right)\left[\frac{k^{n+1}2^{n+1}}{(n+1)!h} + \frac{k^n 2^n}{2n!} + \frac{hk^{n-1}2^{n-1}}{8(n-1)!}\right] =: (\xi).$$

Next we choose $h := \left(\frac{1}{N_{\min}+1}\right)^{\frac{1}{n+1}}$, then $h^n = \left(\frac{1}{N_{\min}+1}\right)^{\frac{n}{n+1}}$ and $h^{n+1} = \frac{1}{N_{\min}+1}$.

We have

$$(\xi) = \pi\left(\max_{\alpha:|\alpha|=n}\omega_1\left(f_\alpha, \frac{1}{\sqrt[n+1]{N_{\min}+1}}\right)\right) \cdot \tag{149}$$

$$\left[\frac{(2k)^{n+1}}{(n+1)!(N_{\min}+1)^{\frac{n}{n+1}}} + \frac{2^{n-1}k^n}{n!(N_{\min}+1)} + \frac{2^{n-2}k^{n-1}}{4(n-1)!(N_{\min}+1)^{\frac{n+2}{n+1}}}\right],$$

proving the claim. ■

We also give

Proposition 48 *Let $x \in [-1,1]^k$, $k \in \mathbb{N}-\{1\}$, be fixed, and let $f \in C^1\left([-1,1]^k, \mathbb{R}_+\right)$. We assume that $\frac{\partial f(x)}{\partial x_i} = 0$, for $i = 1, ..., k$. Then*

$$\left|H^{(M)}_{2\overrightarrow{N}+1}(f)(x) - f(x)\right| \leq \left(\max_{i=1,...,k}\omega_1\left(\frac{\partial f}{\partial x_i}, \frac{1}{\sqrt{N_{\min}+1}}\right)\right) \cdot \tag{150}$$

$$\left[\frac{2k^2\pi}{\sqrt{N_{\min}+1}}+\frac{k\pi}{(N_{\min}+1)}+\frac{1}{8\left(\sqrt{N_{\min}+1}\right)}\right],$$

$\forall\ \overrightarrow{N}=(N_1,...,N_k)\in\mathbb{N}^k,\ N_{\min}:=\min\{N_1,...,N_k\}$.

We have that $\lim_{\overrightarrow{N}\to(\infty,\ldots,\infty)} H^{(M)}_{2\overrightarrow{N}+1}(f)(x)=f(x)$.

Proof. By (31) we get

$$\left|H^{(M)}_{2\overrightarrow{N}+1}(f)(x)-f(x)\right|\overset{(146)}{\leq}\left(\max_{i=1,\ldots,k}\omega_1\left(\frac{\partial f}{\partial x_i},h\right)\right)\cdot$$

$$\left[\frac{k}{2h}\left(\sum_{i=1}^{k}H^{(M)}_{2N_i+1}\left((t_i-x_i)^2\right)(x_i)\right)+\frac{1}{2}\left(\sum_{i=1}^{k}H^{(M)}_{2N_i+1}\left(|t_i-x_i|\right)(x_i)\right)+\frac{h}{8}\right]\tag{151}$$

(next we choose $h:=\frac{1}{\sqrt{N_{\min}+1}}$, then $h^2=\frac{1}{N_{\min}+1}$)

$$\overset{(141)}{\leq}\left(\max_{i=1,\ldots,k}\omega_1\left(\frac{\partial f}{\partial x_i},\frac{1}{\sqrt{N_{\min}+1}}\right)\right)\cdot$$

$$\left[\frac{2k^2\pi}{\sqrt{N_{\min}+1}}+\frac{k\pi}{(N_{\min}+1)}+\frac{1}{8\left(\sqrt{N_{\min}+1}\right)}\right],\tag{152}$$

proving the claim. ■

It follows

Theorem 49 *Let* $f\in C_+\left([-1,1]^k\right)$, $k\in\mathbb{N}-\{1\}$. *Then*

$$\left|H^{(M)}_{2\overrightarrow{N}+1}(f)(x)-f(x)\right|\leq(2k\pi+1)\,\omega_1\left(f,\frac{1}{N_{\min}+1}\right),\tag{153}$$

$\forall\ x\in[-1,1]^k$, *and* $\forall\ \overrightarrow{N}=(N_1,...,N_k)\in\mathbb{N}^k$, *where* $N_{\min}:=\min\{N_1,...,N_k\}$.

That is

$$\left\|H^{(M)}_{2\overrightarrow{N}+1}(f)-f\right\|_{\infty}\leq(2k\pi+1)\,\omega_1\left(f,\frac{1}{N_{\min}+1}\right),\tag{154}$$

We get that

$$\lim_{\overrightarrow{N}\to(\infty,\ldots,\infty)}H^{(M)}_{2\overrightarrow{N}+1}(f)=f,\tag{155}$$

uniformly.

32

Proof. We get that (use of (44))

$$\left| H_{2\overrightarrow{N}+1}^{(M)}(f)(x) - f(x) \right| \overset{(146)}{\leq} \omega_1(f,h)\left(1 + \frac{1}{h}\left(\sum_{i=1}^{k} H_{2N_i+1}^{(M)}(|t_i - x_i|)(x_i)\right)\right)$$

$$\overset{(140)}{\leq} \omega_1(f,h)\left(1 + \frac{k}{h}\left(\frac{2\pi}{(N_{\min}+1)}\right)\right) \tag{156}$$

(setting $h := \frac{1}{N_{\min}+1}$)

$$= \omega_1\left(f, \frac{1}{N_{\min}+1}\right)(1 + 2k\pi), \quad \forall\, x \in [-1,1]^k,$$

proving the claim. ■

We make

Remark 50 *Let $\theta_{\overrightarrow{N}}^{(M)}$ denote any of the Max-product multivariate operators studied in this article: $B_{\overrightarrow{N}}^{(M)}$, $T_{N}^{(M)}$, $U_{\overrightarrow{N}}^{(M)}$, $T_{\overrightarrow{N}}^{(M)}$, $M_{\overrightarrow{N}}^{(M)}$, $L_{\overrightarrow{N}}^{(M)}$ and $H_{2\overrightarrow{N}+1}^{(M)}$. We observe that an important contraction property holds:*

$$\left\| \theta_{\overrightarrow{N}}^{(M)}(f) \right\|_{\infty} \leq \|f\|_{\infty}, \tag{157}$$

and

$$\left\| \theta_{\overrightarrow{N}}^{(M)}\left(\theta_{\overrightarrow{N}}^{(M)}(f)\right) \right\|_{\infty} \leq \left\| \theta_{\overrightarrow{N}}^{(M)}(f) \right\|_{\infty} \leq \|f\|_{\infty}, \tag{158}$$

i.e.

$$\left\| \left(\theta_{\overrightarrow{N}}^{(M)}\right)^2(f) \right\|_{\infty} \leq \|f\|_{\infty}, \tag{159}$$

and in general holds

$$\left\| \left(\theta_{\overrightarrow{N}}^{(M)}\right)^n(f) \right\|_{\infty} \leq \left\| \left(\theta_{\overrightarrow{N}}^{(M)}\right)^{n-1}(f) \right\|_{\infty} \leq \ldots \leq \|f\|_{\infty}, \quad \forall\, n \in \mathbb{N}. \tag{160}$$

We need the following Holder's type inequality:

Theorem 51 *Let Q, with the l_1-norm $\|\cdot\|$, be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$ and $L : C_+(Q) \to C_+(Q)$, be a positive sublinear operator and $f, g \in C_+(Q)$, furthermore let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume that $L\left((f(\cdot))^p\right)(s_*)$, $L\left((g(\cdot))^q\right)(s_*) > 0$ for some $s_* \in Q$. Then*

$$L(f(\cdot)\, g(\cdot))(s_*) \leq \left(L\left((f(\cdot))^p\right)(s_*)\right)^{\frac{1}{p}} \left(L\left((g(\cdot))^q\right)(s_*)\right)^{\frac{1}{q}}. \tag{161}$$

Proof. Let $a, b \geq 0$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. The Young's inequality says

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{162}$$

Then

$$\frac{f(s)}{\left(L\left((f(\cdot))^p\right)(s_*)\right)^{\frac{1}{p}}} \cdot \frac{g(s)}{\left(L\left((g(\cdot))^q\right)(s_*)\right)^{\frac{1}{q}}} \leq$$

$$\frac{(f(s))^p}{p\left(L\left((f(\cdot))^p\right)(s_*)\right)} + \frac{(g(s))^q}{q\left(L\left((g(\cdot))^q\right)(s_*)\right)}, \quad \forall\ s \in Q. \tag{163}$$

Hence it holds

$$\frac{L(f(\cdot)\,g(\cdot))(s_*)}{\left(L\left((f(\cdot))^p\right)(s_*)\right)^{\frac{1}{p}}\left(L\left((g(\cdot))^q\right)(s_*)\right)^{\frac{1}{q}}} \leq \tag{164}$$

$$\frac{\left(L\left((f(\cdot))^p\right)\right)(s_*)}{p\left(L\left((f(\cdot))^p\right)(s_*)\right)} + \frac{\left(L\left((g(\cdot))^q\right)\right)(s_*)}{q\left(L\left((g(\cdot))^q\right)(s_*)\right)} = \frac{1}{p} + \frac{1}{q} = 1, \quad \text{for } s_* \in Q,$$

proving the claim. ■

By (161), under the assumption $L_N\left(\|\cdot - x\|^{n+1}\right)(x) > 0$, and $L_N(1) = 1$, we obtain

$$L_N\left(\|\cdot - x\|^n\right)(x) \leq \left(L_N\left(\|\cdot - x\|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}, \tag{165}$$

in case of $n = 1$ we derive

$$L_N\left(\|\cdot - x\|\right)(x) \leq \sqrt{\left(L_N\left(\|\cdot - x\|^2\right)(x)\right)}. \tag{166}$$

We give

Theorem 52 *Let Q with $\|\cdot\|$ the l_1-norm, be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$, and $f \in C_+(Q)$. Let $\{L_N\}_{N\in\mathbb{N}}$ be positive sublinear operators from $C_+(Q)$ into itself, such that $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. We assume further that $L_N(\|t - x\|)(x) > 0$, $\forall\ N \in \mathbb{N}$. Then*

$$|L_N(f)(x) - f(x)| \leq 2\omega_1\left(f, L_N(\|t - x\|)(x)\right), \tag{167}$$

$\forall\ N \in \mathbb{N}$, $x = (x_1, ..., x_k) \in Q$; $t = (t_1, ..., t_k) \in Q$, where

$$\omega_1(f, h) := \sup_{\substack{x,y\in Q:\\ \|x-y\|\leq h}} |f(x) - f(y)|. \tag{168}$$

If $L_N(\|t - x\|)(x) \to 0$, then $L_N(f)(x) \to f(x)$, as $N \to +\infty$.

Proof. By Theorem 13. ■

We need

Theorem 53 *Let $(Q, \|\cdot\|)$, where $\|\cdot\|$ is the l_1-norm, be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$, and let $x \in Q$ ($x = (x_1, ..., x_k)$) be fixed. Let $f \in C^n(Q)$, $n \in \mathbb{N}$, $h > 0$. We assume that $f_\alpha(x) = 0$, for all $\alpha : |\alpha| = 1, ..., n$.*

Let $\{L_N\}_{N\in\mathbb{N}}$ be positive sublinear operators from $C_+(Q)$ into $C_+(Q)$, such that $L_N(1)=1$, $\forall\ N\in\mathbb{N}$. Then

$$|L_N(f)(x)-f(x)| \leq \left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha,h)\right)\cdot$$

$$\left[\frac{L_N\left(\|\cdot-x\|^{n+1}\right)(x)}{(n+1)!h}+\frac{L_N\left(\|\cdot-x\|^{n}\right)(x)}{2n!}+\frac{h}{8(n-1)!}L_N\left(\|\cdot-x\|^{n-1}\right)(x)\right], \tag{169}$$

$\forall\ N\in\mathbb{N}$.

Proof. By (19) and (25). ■

It follows

Theorem 54 *All as in Theorem 53. Additionally assume that $L_N\left(\|\cdot-x\|^{n+1}\right)(x) > 0$, $\forall\ N\in\mathbb{N}$. Then*

$$|L_N(f)(x)-f(x)| \leq \frac{1}{2n!}\left(3+\frac{n}{4(n+1)}\right)\cdot$$

$$\left(\max_{\alpha:|\alpha|=n}\omega_1\left(f_\alpha,\frac{1}{(n+1)}\left(L_N\left(\|\cdot-x\|^{n+1}\right)(x)\right)^{\frac{1}{n+1}}\right)\right)\left(L_N\left(\|\cdot-x\|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}, \tag{170}$$

$\forall\ N\in\mathbb{N}$, $x=(x_1,...,x_k)\in Q$, ω_1 as in (168) for f_α.

If $L_N\left(\|\cdot-x\|^{n+1}\right)(x)\to 0$, then $L_N(f)(x)\to f(x)$, as $N\to+\infty$.

Proof. By Theorem 51 notice also that

$$L_N\left(\|\cdot-x\|^{n-1}\right)(x) \leq \left(L_N\left(\|\cdot-x\|^{n+1}\right)(x)\right)^{\frac{n-1}{n+1}}. \tag{171}$$

We choose

$$h := \frac{1}{(n+1)}\left(L_N\left(\|\cdot-x\|^{n+1}\right)(x)\right)^{\frac{1}{n+1}} > 0. \tag{172}$$

That is

$$(h(n+1))^{n+1} = L_N\left(\|\cdot-x\|^{n+1}\right)(x). \tag{173}$$

We apply (169) to have (see also (165) and (171)).

$$|L_N(f)(x)-f(x)| \leq \left(\max_{\alpha:|\alpha|=n}\omega_1(f_\alpha,h)\right)\cdot$$

$$\left[\frac{L_N\left(\|\cdot-x\|^{n+1}\right)(x)}{(n+1)!h}+\frac{\left(L_N\left(\|\cdot-x\|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}}{2n!}+\right. \tag{174}$$

35

$$\frac{h}{8(n-1)!} L_N\left(\left(\|\cdot - x\|^{n+1}\right)(x)\right)^{\frac{n-1}{n+1}}\Bigg] =$$

$$\left(\max_{\alpha:|\alpha|=n} \omega_1\left(f_\alpha, \frac{1}{(n+1)}\left(L_N\left(\|\cdot - x\|^{n+1}\right)(x)\right)^{\frac{1}{n+1}}\right)\right) \cdot$$

$$\left[\frac{h^n (n+1)^{n+1}}{(n+1)!} + \frac{h^n (n+1)^n}{2n!} + \frac{h^n (n+1)^{n-1}}{8(n-1)!}\right] =$$

$$\left(\max_{\alpha:|\alpha|=n} \omega_1\left(f_\alpha, \frac{1}{(n+1)}\left(L_N\left(\|\cdot - x\|^{n+1}\right)(x)\right)^{\frac{1}{n+1}}\right)\right) \cdot$$

$$\left[\frac{(n+1)^{n+1}}{(n+1)!} + \frac{(n+1)^n}{2n!} + \frac{(n+1)^{n-1}}{8(n-1)!}\right] \frac{1}{(n+1)^n}\left(L_N\left(\|\cdot - x\|^{n+1}\right)(x)\right)^{\frac{n}{n+1}} =$$

$$\left[\frac{3}{2n!} + \frac{n}{8(n+1)!}\right]\left(\max_{\alpha:|\alpha|=n} \omega_1\left(f_\alpha, \frac{1}{(n+1)}\left(L_N\left(\|\cdot - x\|^{n+1}\right)(x)\right)^{\frac{1}{n+1}}\right)\right) \cdot$$

$$\left(L_N\left(\|\cdot - x\|^{n+1}\right)(x)\right)^{\frac{n}{n+1}}, \tag{175}$$

proving the claim. ■

Final application for $n = 1$ follows:

Corollary 55 *Let $(Q, \|\cdot\|)$, where $\|\cdot\|$ is the l_1-norm, be a compact and convex subset of $\mathbb{R}^k$, $k \in \mathbb{N} - \{1\}$, and let $x \in Q$ $(x = (x_1, ..., x_k))$ be fixed. Let $f \in C^1(Q)$. We assume that $\frac{\partial f}{\partial x_i}(x) = 0$, $i = 1, ..., k$. Let $\{L_N\}_{N\in\mathbb{N}}$ be positive sublinear operators from $C_+(Q)$ into $C_+(Q)$, such that $L_N(1) = 1$, $\forall\ N \in \mathbb{N}$. Assume that $L_N\left(\|\cdot - x\|^2\right)(x) > 0$, $\forall\ N \in \mathbb{N}$. Then*

$$|L_N(f)(x) - f(x)| \leq \frac{25}{16}\left(\max_{i=1,...,k} \omega_1\left(\frac{\partial f}{\partial x_i}, \frac{1}{2}\left(L_N\left(\|\cdot - x\|^2\right)(x)\right)^{\frac{1}{2}}\right)\right) \cdot$$

$$\left(L_N\left(\|\cdot - x\|^2\right)(x)\right)^{\frac{1}{2}}, \tag{176}$$

$\forall\ N \in \mathbb{N}$.

If $L_N\left(\|\cdot - x\|^2\right)(x) \to 0$, then $L_N(f)(x) \to f(x)$, as $N \to +\infty$.

References

[1] G. Anastassiou, *Moments in probability and approximation theory*, Pitman Research Notes in Mathematics Series, Longman Group UK, New York, NY, 1993.

[2] G. Anastassiou, *Approximation by Sublinear Operators*, submitted, 2017.

[3] G. Anastassiou, *Approximation by Max-Product Operators*, submitted, 2017.

[4] B. Bede, L. Coroianu, S. Gal, *Approximation by Max-Product type Operators*, Springer, Heidelberg, New York, 2016.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

NEW DYNAMIC INEQUALITIES ON TIME SCALES BY USING THE SNEAK-OUT PRINCIPLE

S. H. SAKER[1], M. M. OSMAN[1] AND I. ABOHELA[2]

Abstract. In this paper, we extend and improve some dynamic inequalities by using the sneak-out principle with different exponents on time scales. The main results can be used to formulate the corresponding discrete inequalities of Bennett and G-Erdmann type.

2010 Mathematics Subject Classification: 34A40, 34N05, 26D10, 26D15, 39A13.

Key words and phrases. Hardy's inequality, sneak-out principle, dynamic inequlities, time scales.

1. Introduction

In 1967 Littlewood [9] formulated some problems concerning elementary inequalities for infinite series in connection with some work on general theory of orthogonal series. One of the simplest (non-trivial) examples is the following inequality

$$\sum_{n=1}^{\infty} a_n^3 \left(\sum_{k=1}^{n} a_k^2 A_k \right) \leq K \sum_{n=1}^{\infty} a_n^4 A_n^2, \tag{1.1}$$

where a_n is a non-negative sequence and $A_n = \sum_{k=1}^{n} a_k$. One of such problems that has been proposed by Littlewood is to seek to know whether a constant K exists such that the inequality (1.1) holds. In other words, is it possible to get the term A_k out from the inner sum in (1.1) and if this happened what is the smallest value of K which preserves on the direction of the inequality? Bennett [4] proved this for the special case when the sequence a_n is decreasing, and he showed that the inequality (1.1) holds with $K = 2$. His proof based on the fact that $a_n \leq nA_n$ (noting that a_n is decreasing) and the application of Cauchy's inequality and the classical discrete Hardy's inequality. The generalization of the Littlewood inequality (1.1) which has not been considered before is given by

$$\sum_{n=1}^{\infty} a_n^{p(p-1)+1} A_k^{p-2} \left(\sum_{k=1}^{n} a_k^p A_k \right) \leq K \sum_{n=1}^{\infty} \left[a_n^p A_n \right]^p, \quad p > 1, \tag{1.2}$$

where K is a positive constant. Motivated by the work of Littlewood [9] Bennett and G-Erdmann [5] considered the inequality

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=n}^{\infty} A_k^{\alpha} g_k \right)^p \leq K(\alpha, p) \sum_{n=1}^{\infty} a_n A_n^{\alpha p} \left(\sum_{k=n}^{\infty} g_k \right)^p, \tag{1.3}$$

and determined the value of K for different values of p and α. In particular, Bennett and G-Erdmann [5, Theorem 8] proved that if $\alpha \geq 1$ and $p \geq 1$, then

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=n}^{\infty} A_k^{\alpha} g_k \right)^p \leq (1+\alpha p)^p \sum_{n=1}^{\infty} a_n A_n^{\alpha p} \left(\sum_{k=n}^{\infty} g_k \right)^p, \tag{1.4}$$

where g_n is a non-negative sequence and $A_n = \sum_{k=1}^{n} a_k$, for any $n \in \mathbb{N}$. In [5, Theorem 9] the authors proved that if $p \geq 1$ and $0 \leq \alpha \leq 1$, then

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=n}^{\infty} A_k^{\alpha} g_k \right)^p \leq (1+p)^p \sum_{n=1}^{\infty} a_n A_n^{\alpha p} \left(\sum_{k=n}^{\infty} g_k \right)^p. \tag{1.5}$$

Also in [5, Theorem 10] they proved that if $p \geq 1$ and $-1/p < \alpha \leq 0$, then

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=n}^{\infty} A_k^{\alpha} g_k \right)^p \geq \left(\frac{1+\alpha p}{1+p+\alpha p} \right)^p \sum_{n=1}^{\infty} a_n A_n^{\alpha p} \left(\sum_{k=n}^{\infty} g_k \right)^p. \tag{1.6}$$

Motivated by the above work, we believe that the study of dynamic inequalities will help in proving several results for classical integral inequalities and inequalities involving discrete sequences. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. We assume that the reader has a good background in time scale calculus. For dynamic inequalities on time scales, we refer the reader to the books [2, 3] and the papers [1, 7, 10, 11, 12, 13]. For instance, we recall some related results.

Saker, O'Regan and Agarwal [13] proved a new inequality of Hardy type of the form

$$\int_a^{\infty} \frac{(A^{\sigma}(t))^p}{(\sigma(t)-a)^{\gamma}} \Delta t \leq \left(\frac{p}{\gamma - 1} \right)^p \int_a^{\infty} \frac{(\sigma(t)-a)^{\gamma(p-1)}}{(t-a)^{(\gamma-1)p}} g^p(t) \Delta t, \quad p, \ \gamma > 1, \tag{1.7}$$

where $A(t) := \int_a^t g(s)\Delta s$, for $t \in [a, \infty)_{\mathbb{T}}$ and employed it in the proof of the extension of (1.2) on time scales. In particular they proved that if $p, \gamma > 1$ and g is a nonnegative rd-continuous and decreasing function, then

$$\int_a^{\infty} \frac{(a(t))^{p(p-1)+1}}{(A^{\sigma}(t))^{2-p}} \left(\int_a^{\sigma(t)} a^p(s) A^{\sigma}(s) \Delta s \right) \Delta t \leq \frac{p\gamma^p}{(p-1)} \int_a^{\infty} \left[a^p(t) A^{\sigma}(t) \right]^p \Delta t, \tag{1.8}$$

where $A(t) = \int_a^t a(s)\Delta s$, for $t \in [a, \infty)_{\mathbb{T}}$. Bohner and Saker in [7] employed the Minkowski inequality [6, Theorem 6.16] on time scales

$$\left(\int_a^b |h(t)| \, |u(t)+\upsilon(t)|^p \, \Delta t \right)^{1/p} \leq \left[\int_a^b |h(t)| \, |u(t)|^p \, \Delta t \right]^{\frac{1}{p}} + \left[\int_a^b |h(t)| \, |\upsilon(t)|^p \, \Delta t \right]^{\frac{1}{p}}, \tag{1.9}$$

where $a, \ b \in \mathbb{T}$, $u, \upsilon \in C_{rd}([a, \ b]_{\mathbb{T}}, \ \mathbb{R})$, $p > 1$ and established the time scale versions of the inequalities (1.4), (1.5) and (1.6). In more precisely, they proved

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

that if $a(t)$, $g(t)$ are nonnegative rd-continuous functions on $[t_0, \infty)_{\mathbb{T}}$, then for $\alpha \geq 1$ and $p \geq 1$

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \leq (1+\alpha p)^p \int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha p}\left(\int_t^{\infty} g(s)\Delta s\right)^p \Delta t, \tag{1.10}$$

where

$$\Psi(t) = \int_t^{\infty} \left(A^\sigma(s)\right)^{\alpha} g(s)\Delta s \quad \text{and} \quad A(t) = \int_{t_0}^{t} a(s)\Delta s,$$

and if $0 \leq \alpha \leq 1$, $p \geq 1$, then

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \leq (1+p)^p \int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha p}\left(\int_t^{\infty} g(s)\Delta s\right)^p \Delta t. \tag{1.11}$$

Also in [7] they proved that if $-1/p < \alpha \leq 0$ and $p \geq 1$, then

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \geq \left(\frac{1+\alpha p}{1+p+\alpha p}\right)^p \int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha p}\left(\int_t^{\infty} g(s)\Delta s\right)^p \Delta t. \tag{1.12}$$

Our aim in this paper is to apply the sneak-out principle which is given in the inequalities (1.10) and (1.11) to prove some new inequalities with different exponents for the given values of α. Also we prove a new dynamic inequality which as special case improves the inequality (1.12).

2. Main Results

Before we prove our main results, we briefly introduce some basic definitions and results concerning the delta calculus on time scales that will be used in the sequel; for more details we refer the reader to the book [6]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided f is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$. We define the time scale interval $[a, b]_{\mathbb{T}}$ by $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$. Recall the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$, here $g^\sigma = g \circ \sigma$) of two differentiable function f and g

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \text{ and } \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}. \tag{2.1}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

The chain rule formula on time scales [6] is given by (here $x : \mathbb{T} \to (0,\infty)$ is assumed to be differentiable)

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_0^1 [hx^{\sigma} + (1-h)x]^{\gamma-1}\, dh x^{\Delta}(t), \quad \gamma \in \mathbb{R}. \tag{2.2}$$

In this paper we will use the (delta) integral which we can define as follows. If $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(s)\Delta s := G(t) - G(a)$. The integration by parts formula on time scales reads

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{2.3}$$

Hölder's inequality [6, Theorem 6.13] states that any two rd-continuous functions $u, v : \mathbb{T} \to \mathbb{R}$ satisfy

$$\int_a^b |u(t)v(t)|\, \Delta t \le \left[\int_a^b |u(t)|^q \,\Delta t\right]^{\frac{1}{q}} \left[\int_a^b |v(t)|^p \,\Delta t\right]^{\frac{1}{p}}, \tag{2.4}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $a, b \in \mathbb{T}$. Throughout this paper, we will assume that the functions in the statements of the theorems are nonnegative and rd-continuous functions and the integrals considered are assumed to exist.

The following dynamic inequality of Copson's type on time scales [3], will be used later to prove the main results.

Theorem 2.1. *Assume that $a : \mathbb{T} \to \mathbb{R}$ is rd-continuous function and define $A(t) = \int_{t_0}^t a(s)\Delta s$, $t \in \mathbb{T}$. Let $\varphi : \mathbb{T} \to \mathbb{R}^+$ and define*

$$\bar{\Phi}(t) := \int_t^{\infty} a(s)\varphi(s)\Delta s, \quad t \in \mathbb{T}. \tag{2.5}$$

If $k > 1$ and $0 \le c < 1$, then

$$\int_{t_0}^{\infty} \frac{a(t)}{(A^{\sigma}(t))^c} \left(\bar{\Phi}(t)\right)^k \Delta t \le \left(\frac{k}{1-c}\right)^k \int_{t_0}^{\infty} a(t) \left(A^{\sigma}(t)\right)^{k-c} \varphi^k(t)\, \Delta t. \tag{2.6}$$

Our main results are given in the following. For simplicity, we define

$$\Omega(t) := \int_t^{\infty} g(s)\Delta s, \quad \text{and} \quad \Psi(t) := \int_t^{\infty} (A^{\sigma}(s))^{\alpha} g(s)\Delta s, \ t \in \mathbb{T}. \tag{2.7}$$

Theorem 2.2. *Let $t_0 \in \mathbb{T}$, $\alpha \ge 1$, $p \ge 1$ and q, $r > 1$ such that $r > q$ and $(r-q)/(p-q) > 1$. Then*

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \le K_1(\alpha,p,q,r) \left(\int_{t_0}^{\infty} \left((A^{\sigma}(t))^{\alpha}\, \Omega(t)\right)^{2r-q} \Delta t\right)^{\frac{p}{2r-q}}, \tag{2.8}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

where

$$\begin{aligned} K_1(\alpha,p,q,r) \quad : \quad &= \left[\frac{(1+\alpha r)^{r(p-q)}}{(1+\alpha q)^{q(p-r)}}\right]^{\frac{1}{r-q}} \\ &\times \left(\int_{t_0}^{\infty} a^{\frac{2r-q}{r-q}}(t)\,\Delta t\right)^{\frac{p-q}{2r-q}} \left(\int_{t_0}^{\infty} a^{\frac{2r-q}{2(r-q)}}(t)\,\Delta t\right)^{\frac{2(r-p)}{2r-q}}. \end{aligned}$$

Proof. We first observe that

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t = \int_{t_0}^{\infty} \left(a^{\frac{p-q}{r-q}}(t)\Psi^{\frac{r(p-q)}{r-q}}(t)\right)\left(a^{\frac{r-p}{r-q}}(t)\Psi^{\frac{q(r-p)}{r-q}}(t)\right)\Delta t.$$

Applying Hölder's inequality (2.4) with indices $(r-q)/(p-q)$ and $(r-q)/(r-p)$, we obtain

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \le \left(\int_{t_0}^{\infty} a(t)\Psi^r(t)\Delta t\right)^{\frac{p-q}{r-q}}\left(\int_{t_0}^{\infty} a(t)\Psi^q(t)\Delta t\right)^{\frac{r-p}{r-q}}.$$

By using (1.10) to the two integrals on the right-hand side with $p=r$ and also with $p=q$, we get that

$$\begin{aligned} \int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \quad &\le \quad (1+\alpha r)^{\frac{r(p-q)}{r-q}}\left(\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha r}\left(\int_t^\infty g(s)\Delta s\right)^r \Delta t\right)^{\frac{p-q}{r-q}} \\ &\times (1+\alpha q)^{\frac{q(r-p)}{r-q}}\left(\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha q}\left(\int_t^\infty g(s)\Delta s\right)^q \Delta t\right)^{\frac{r-p}{r-q}}. \end{aligned}$$

Applying Hölder's inequality (2.4) with indices $(2r-q)/r$ and $(2r-q)/(r-q)$ to the integral

$$\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha r}(\Omega(t))^r\,\Delta t,$$

also applying it again on the integral

$$\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha q}(\Omega(t))^q\,\Delta t,$$

with indices $(2r-q)/q$ and $(2r-q)/2(r-q)$ and combining the result, we get that

$$\int_{t_0}^{\infty} a(t)\left(\Psi(t)\right)^p\Delta t \le K_1(\alpha,p,q,r)\left(\int_{t_0}^{\infty}\left(\left(A^\sigma(t)\right)^\alpha\Omega(t)\right)^{2r-q}\Delta t\right)^{\frac{p}{2r-q}},$$

which is the desired inequality (2.8). The proof is complete. □

Proceeding as in the proof of Theorem 2.2 and using inequality (1.11) instead of (1.10), we can obtain the following result.

Theorem 2.3. *Let* $t_0\in\mathbb{T}$, $0\le\alpha\le 1$, $p\ge 1$ *and* q, $r>1$ *such that* $r>q$ *and* $(r-q)/(p-q)>1$. *Then*

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \le K_2(p,q,r)\left(\int_{t_0}^{\infty}\left(\left(A^\sigma(t)\right)^\alpha\Omega(t)\right)^{2r-q}\Delta t\right)^{\frac{p}{2r-q}}, \tag{2.9}$$

where

$$K_2(p,q,r) \quad : \quad = \left[\frac{(1+r)^{r(p-q)}}{(1+q)^{q(p-r)}}\right]^{\frac{1}{r-q}} \times \left(\int_{t_0}^{\infty} a^{\frac{2r-q}{r-q}}(t)\,\Delta t\right)^{\frac{p-q}{2r-q}} \left(\int_{t_0}^{\infty} a^{\frac{2r-q}{2(r-q)}}(t)\,\Delta t\right)^{\frac{2(r-p)}{2r-q}}.$$

The next result follows from Theorem 2.2 by choosing $r = p$ and $q = p - 1$.

Corollary 2.1. *Let $p \geq 1$ and $\alpha \geq 1$. Then*

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)\Delta t \leq K_1(\alpha,p)\left(\int_{t_0}^{\infty} \left((A^{\sigma}(t))^{\alpha}\,\Omega(t)\right)^{p+1}\Delta t\right)^{\frac{p}{p+1}}, \tag{2.10}$$

where

$$K_1(\alpha,p) = (1+\alpha p)^p \left(\int_{t_0}^{\infty} a^{p+1}(t)\,\Delta t\right)^{\frac{1}{p+1}}.$$

Remark 2.1. *In Theorem 2.2 when $\mathbb{T} = \mathbb{R}$, we have that*

$$\Psi(t) = \int_t^{\infty} A^{\alpha}(s)g(s)ds, \quad A(t) = \int_{t_0}^{t} a(s)ds \quad \text{and} \quad \Omega(t) = \int_t^{\infty} g(s)ds, \quad t \in \mathbb{R},$$

and then from (2.8) we obtain the following new integral inequality

$$\int_{t_0}^{\infty} a(t)\Psi^p(t)dt \leq K_1(\alpha,p,q,r)\left(\int_{t_0}^{\infty} A^{\alpha(2r-q)}(t)\left(\Omega(t)\right)^{2r-q} dt\right)^{\frac{p}{2r-q}}, \tag{2.11}$$

where

$$K_1(\alpha,p,q,r) \quad : \quad = \left[\frac{(1+\alpha r)^{r(p-q)}}{(1+\alpha q)^{q(p-r)}}\right]^{\frac{1}{r-q}} \times \left(\int_{t_0}^{\infty} a^{\frac{2r-q}{r-q}}(t)\,dt\right)^{\frac{p-q}{2r-q}} \left(\int_{t_0}^{\infty} a^{\frac{2r-q}{2(r-q)}}(t)\,dt\right)^{\frac{2(r-p)}{2r-q}}.$$

Remark 2.2. *In Theorem 2.2 when $\mathbb{T} = \mathbb{N}$ and $n_0 = 1$, we have that*

$$\Psi(n) = \sum_{k=n}^{\infty} A^{\alpha}(k)g(k), \; A(n) = \sum_{k=1}^{n} a(k), \quad n \in \mathbb{N},$$

and then from (2.8), we get the following discrete inequality of Bennett and G-Erdmann [5] *type*

$$\sum_{n=1}^{\infty} a(n)\Psi^p(n) \leq K_1(\alpha,p,q,r)\left(\sum_{n=1}^{\infty} A^{\alpha(2r-q)}(n)\left(\sum_{k=n}^{\infty} g(k)\right)^{2r-q}\right)^{\frac{p}{2r-q}}, \tag{2.12}$$

where

$$K_1(\alpha,p,q,r) \;:=\; \left[\frac{(1+\alpha r)^{r(p-q)}}{(1+\alpha q)^{q(p-r)}}\right]^{\frac{1}{r-q}} \times \left(\sum_{n=1}^{\infty} a^{\frac{2r-q}{r-q}}(n)\right)^{\frac{p-q}{2r-q}} \left(\sum_{n=1}^{\infty} a^{\frac{2r-q}{2(r-q)}}(n)\right)^{\frac{2(r-p)}{2r-q}}.$$

Remark 2.3. *Setting $r=p$ and $q=p-1$ in (2.12) yields the following inequality*

$$\sum_{n=1}^{\infty} a(n)\Psi^p(n) \le K_1(\alpha,p)\left(\sum_{n=1}^{\infty} A^{\alpha(p+1)}(n)\left(\sum_{k=n}^{\infty} g(k)\right)^{p+1}\right)^{\frac{p}{p+1}}, \tag{2.13}$$

where

$$K_1(\alpha,p) = (1+\alpha p)^p \left(\sum_{n=1}^{\infty} a^{p+1}(n)\right)^{\frac{1}{p+1}}.$$

An improvement of the dynamic inequality (1.12) is obtained in the following Theorem.

Theorem 2.4. *Let $t_0 \in \mathbb{T}$, $-1/p < \alpha \le 0$, $p \ge 1$ and $q,\ r > 1$ such that $r > q$ and $(r-q)/(p-q) > 1$. Then*

$$\begin{aligned} &\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha p}(\Omega(t))^p\,\Delta t \\ \le\;& K_3(\alpha,p,q,r)\left[\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha(p-r)}(\Psi(t))^r\,\Delta t\right]^{\frac{p-q}{r-q}} \\ &\times \left[\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha(p-q)}(\Psi(t))^q\,\Delta t\right]^{\frac{r-p}{r-q}}, \end{aligned} \tag{2.14}$$

where

$$K_3(\alpha,p,q,r) := \left(\frac{1+r+\alpha p}{1+\alpha p}\right)^{\frac{r(p-q)}{r-q}} \left(\frac{1+q+\alpha p}{1+\alpha p}\right)^{\frac{q(r-p)}{r-q}}.$$

Proof. In this proof for brivity, we set

$$b(t) := \left(A^\sigma(t)\right)^{\alpha} g(t).$$

Then the left hand side of (2.14) can be written in the form

$$\int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha p}\Omega^p(t)\Delta t = \int_{t_0}^{\infty} a(t)\left(A^\sigma(t)\right)^{\alpha p}\left(\int_t^{\infty}\frac{b(s)}{\left(A^\sigma(s)\right)^{\alpha}}\Delta s\right)^p\Delta t. \tag{2.15}$$

Integrating the term $\int_t^\infty \left(A^\sigma(s)\right)^{-\alpha} b(s)\Delta s$ by parts, with $u^\Delta(s) = b(s)$ and $v^\sigma(s) = \left(A^\sigma(s)\right)^{-\alpha}$, we have

$$\int_t^{\infty}\left(A^\sigma(s)\right)^{-\alpha} b(s)\Delta s = u(s)\left(A(s)\right)^{-\alpha}\big|_t^{\infty} - \int_t^{\infty} u(s)\left(\left(A(s)\right)^{-\alpha}\right)^{\Delta}\Delta s,$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

where $u(t) = -\int_t^\infty b(s)\Delta s = -\Psi(t)$, and so (note that $A(t) \leq A^\sigma(t)$ and $-\alpha > 0$)

$$\begin{aligned}\int_t^\infty (A^\sigma(s))^{-\alpha} b(s)\Delta s &= \Psi(t)(A(t))^{-\alpha} + \int_t^\infty \Psi(s)\left((A(s))^{-\alpha}\right)^\Delta \Delta s \\ &\leq \Psi(t)(A^\sigma(t))^{-\alpha} + \int_t^\infty \Psi(s)\left((A(s))^{-\alpha}\right)^\Delta \Delta s.\end{aligned}$$

Using the following inequality (see [7, Lemma 2.2])

$$(f^\gamma(t))^\Delta \leq f^\Delta(t)(f^\sigma(t))^{\gamma-1}, \quad \text{if } \; 0 \leq \gamma \leq 1,\; f^\Delta > 0, \tag{2.16}$$

with $f = A$ and $\gamma = -\alpha$, we observe that

$$\left((A(s))^{-\alpha}\right)^\Delta \leq \frac{a(s)}{(A^\sigma(s))^{\alpha+1}}, \quad (\text{note that } 0 \leq -\alpha \leq 1).$$

This gives us

$$\int_t^\infty (A^\sigma(s))^{-\alpha} b(s)\Delta s \leq \Psi(t)(A^\sigma(t))^{-\alpha} + \int_t^\infty \frac{a(s)\Psi(s)}{(A^\sigma(s))^{\alpha+1}}\Delta s. \tag{2.17}$$

Substitute (2.17) into (2.15) and using the Minkowski inequality [8, Theorem 2.1]

$$\begin{aligned}&\int_a^b |h(t)|\,|u(t)+v(t)|^p\,\Delta t \\ \leq\; &\left[\left(\int_a^b |h(t)|\,|u(t)|^r\,\Delta t\right)^{\frac{1}{r}} + \left(\int_a^b |h(t)|\,|v(t)|^r\,\Delta t\right)^{\frac{1}{r}}\right]^{\frac{r(p-q)}{r-q}} \\ &\times\left[\left(\int_a^b |h(t)|\,|u(t)|^q\,\Delta t\right)^{\frac{1}{q}} + \left(\int_a^b |h(t)|\,|v(t)|^q\,\Delta t\right)^{\frac{1}{q}}\right]^{\frac{q(r-p)}{r-q}}.\end{aligned} \tag{2.18}$$

for $r > q$ such that $r, q > 1$ and $(r-q)/(p-q) > 1$, we obtain

$$\begin{aligned}&\int_{t_0}^\infty a(t)(A^\sigma(t))^{\alpha p}\left(\int_t^\infty \frac{b(s)}{(A^\sigma(s))^\alpha}\Delta s\right)^p \Delta t \\ \leq\; &\int_{t_0}^\infty a(t)(A^\sigma(t))^{\alpha p}\left(\Psi(t)(A^\sigma(t))^{-\alpha} + \int_t^\infty \frac{a(s)\Psi(s)}{(A^\sigma(s))^{\alpha+1}}\Delta s\right)^p \Delta t \\ \leq\; &\left[\left(\int_{t_0}^\infty a(t)(A^\sigma(t))^{\alpha(p-r)}(\Psi(t))^r\,\Delta t\right)^{\frac{1}{r}} + \left(\int_{t_0}^\infty a(t)(A^\sigma(t))^{\alpha p}\left(\check{\Phi}(t)\right)^r \Delta t\right)^{\frac{1}{r}}\right]^{\frac{r(p-q)}{r-q}} \\ &\times\left[\left(\int_{t_0}^\infty a(t)(A^\sigma(t))^{\alpha(p-q)}(\Psi(t))^q\,\Delta t\right)^{\frac{1}{q}} + \left(\int_{t_0}^\infty a(t)(A^\sigma(t))^{\alpha p}\left(\check{\Phi}(t)\right)^q \Delta t\right)^{\frac{1}{q}}\right]^{\frac{q(r-p)}{r-q}},\end{aligned}$$

where

$$\check{\Phi}(t) := \int_t^\infty \frac{a(s)\Psi(s)}{(A^\sigma(s))^{\alpha+1}}\Delta s.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Applying Theorem 2.1 with $0 < c = -\alpha p < 1$, and $\varphi(t) = \Psi(t)/\left(A^{\sigma}(t)\right)^{\alpha+1}$, we have

$$\begin{aligned} &\int_{t_0}^{\infty} a(t)\left(A^{\sigma}(t)\right)^{\alpha p}\left(\check{\Phi}(t)\right)^{r}\Delta t \\ \leq\ & \left(\frac{r}{1+\alpha p}\right)^{r}\int_{t_0}^{\infty} a(t)\left(A^{\sigma}(t)\right)^{r+\alpha p}\left(\frac{\Psi(t)}{\left(A^{\sigma}(t)\right)^{\alpha+1}}\right)^{r}\Delta t \\ =\ & \left(\frac{r}{1+\alpha p}\right)^{r}\int_{t_0}^{\infty} a(t)\left(A^{\sigma}(t)\right)^{\alpha(p-r)}\left(\Psi(t)\right)^{r}\Delta t, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &\int_{t_0}^{\infty} a(t)\left(A^{\sigma}(t)\right)^{\alpha p}\left(\check{\Phi}(t)\right)^{q}\Delta t \\ \leq\ & \left(\frac{q}{1+\alpha p}\right)^{q}\int_{t_0}^{\infty} a(t)\left(A^{\sigma}(t)\right)^{\alpha(p-q)}\left(\Psi(t)\right)^{q}\Delta t. \end{aligned} \tag{2.20}$$

From (2.19) and (2.20), we get that

$$\begin{aligned} &\int_{t_0}^{\infty} a(t)\left(A^{\sigma}(t)\right)^{\alpha p}\left(\int_{t}^{\infty}\frac{b(s)}{\left(A^{\sigma}(s)\right)^{\alpha}}\Delta s\right)^{p}\Delta t \\ \leq\ & \left[\left(\frac{1+r+\alpha p}{1+\alpha p}\right)\left(\int_{t_0}^{\infty} a(t)\left(A^{\sigma}(t)\right)^{\alpha(p-r)}\left(\Psi(t)\right)^{r}\Delta t\right)^{\frac{1}{r}}\right]^{\frac{r(p-q)}{r-q}} \\ & \times\left[\left(\frac{1+q+\alpha p}{1+\alpha p}\right)\left(\int_{t_0}^{\infty} a(t)\left(A^{\sigma}(t)\right)^{\alpha(p-q)}\left(\Psi(t)\right)^{r}\Delta t\right)^{\frac{1}{q}}\right]^{\frac{q(r-p)}{r-q}} \\ =\ & \left(\frac{1+r+\alpha p}{1+\alpha p}\right)^{\frac{r(p-q)}{r-q}}\left[\int_{t_0}^{\infty} a(t)\left(A^{\sigma}(t)\right)^{\alpha(p-r)}\left(\Psi(t)\right)^{r}\Delta t\right]^{\frac{p-q}{r-q}} \\ & \times\left(\frac{1+q+\alpha p}{1+\alpha p}\right)^{\frac{q(r-p)}{r-q}}\left[\int_{t_0}^{\infty} a(t)\left(A^{\sigma}(t)\right)^{\alpha(p-q)}\left(\Psi(t)\right)^{q}\Delta t\right]^{\frac{r-p}{r-q}}, \end{aligned}$$

which is the desired inequality (2.14). The proof is complete. □

Remark 2.4. *As a special case of (2.14) when $r = p$, we get the inequality (1.12) which has been proved by Bohner and Saker.*

Remark 2.5. *In Theorem 2.4 if $\mathbb{T} = \mathbb{N}$ and $r = p$, then inequality (2.14) reduces to the discrete inequality (1.6) due to Bennett and G-Erdmann.*

References

[1] R. P. Agarwal, M. Bohner and S. H. Saker, Dynamic Littlewood-type inequalities, Proc.Amer. Math. Soc. 143 (2015), 667–677.

[2] R. P. Agarwal, D. O'Regan and S. H. Saker, *Dynamic Inequalities on Time Scales*, Springer Heidlelberg New York Drodrechet London, (2014).

[3] R. P. Agarwal, D. O'Regan and S. H. Saker, *Hardy Type Inequalities on Time Scales*, Springer International Publishing, Cham Heidlelberg New York Drodrechet London, (2016).

[4] G. Bennett, Some elementary inequalities, Quart. J. Math. 2 (1987), 401–425.
[5] G. Bennet and K.-G. Grosse-Erdmann, On series of positive terms, Houston Journal of Mathematics 31 (2005), 541–586.
[6] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*: An Introduction with Applications, Birkhäuser, Boston, (2001).
[7] M. Bohner and S. H. Saker, Sneak-out principle on time scales, J. Math. Ineq. 10 (2) (2016), 393–403.
[8] G. S. Chen, Some improvements of Minkowsk's integral inequality on time scales, J. Ineq. Appl. 2013, 2013:318, 1–6.
[9] J. E. Littlewood, Some new inequalities and unsolved problems, Inequalities (Editor O. Shisha), Academic Press, New York, 151–162 (1967).
[10] S. H. Saker, R. R. Mahmoud, M. M. Osman and R. P. Agarwal, Some new generalized forms of Hardy's type inequality on time scales, Mathematical Inequalities & Applications 20 (2017), 459–481.
[11] S. H. Saker, M. M. Osman, D. O'Regan and R. P. Agarwal, Inequalities of Hardy-type and generalizations on time scales, Analysis: International mathematical journal of analysis and its applications 38 (1) (2018), 47–62.
[12] S. H. Saker, M. M. Osman, D. O'Regan and R. P. Agarwal, Levinson type inequalities and their extensions via convexity on time scales, RACSAM 113 (1) (2019), 299–314.
[13] S. H. Saker, D. O'Regan and R. P. Agarwal, Littlewood and Bennett inequalities on time scales, Mediterranean Journal of Mathematics 8 (2014), 1–15.

[1]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt, e-mails:shsaker@mans.edu.eg, mmosman@mans.edu.eg, [2]College of Engineering, Applied Science University, Kingdom of Bahrain, e-mail: islam.abohela@asu.edu.bh

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

ADDITIVE-QUADRATIC FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES AND ITS STABILITY

CHANG IL KIM AND GILJUN HAN*

Abstract. In this paper, we investigate the functional inequality

$$N(f(2x+y)+f(2x-y)-6f(x)-2f(-x)-f(y)-f(-y),t)$$
$$\geq N(f(x+y)+f(x-y)-2f(x)-f(y)-f(-y),kt)$$

for some fixed real number k and prove the generalized Hyers-Ulam stability for it in fuzzy Banach spaces.

1. Introduction

In 1940, Ulam proposed the following stability problem (cf. [28]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists an unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [13] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki ([1]) for additive mappings and by Rassias [22] for linear mappings to consider the stability problem with unbounded Cauchy differences. During the last decades, the stability problem of functional equations have been extensively investigated by a number of mathematicians (see [3], [4], [5], [10], and [18]).

In 2008, for the first time, Mirmostafaee and Moslehian [15], [16] used the definition of a fuzzy norm in [2] to obtain a fuzzy version of the stability for the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

and the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.2}$$

In [11], Glányi showed that if a mapping $f : X \longrightarrow Y$ satisfies the following functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\|, \tag{1.3}$$

then f satisfies the Jordan-Von Neumann functional equation

$$2f(x) + 2f(y) - f(xy^{-1}) = f(xy).$$

2010 *Mathematics Subject Classification.* 39B62, 39B72, 54A40, 47H10.

Key words and phrases. Hyers-Ulam stability, additive-quadratic functional equation, fuzzy normed space, fixed point theorem.

* Corresponding author.

Glányi [12] and Fechner [9] proved the Hyers-Ulam stability of (1.3). Park, Cho, and Han [21] proved the Hyers-Ulam stability of the following functional inequality:

$$\|f(x)+f(y)+f(z)\| \le \|f(x+y+z)\|. \tag{1.4}$$

Further, Park [20] proved the generalized Hyers-Ulam stability of the Cauchy additive functional inequality (1.4) in fuzzy Banach spaces using the fixed point method if f is an odd mapping.

In this paper, we investigate the following functional inequality

$$\begin{aligned} &N(f(2x+y)+f(2x-y)-6f(x)-2f(-x)-f(y)-f(-y),t)\\ \ge\ &N(f(x+y)+f(x-y)-2f(x)-f(y)-f(-y),kt) \end{aligned} \tag{1.5}$$

for some fixed nonzero real number k and prove the generalized Hyers-Ulam stability for (1.5) in fuzzy Banach spaces by fixed point methods.

2. PRELIMINARIES

In this paper, we use the definition of fuzzy normed spaces given in [2], [16], and [17].

Definition 2.1. Let X be a real vector space. A function $N : X\times\mathbb{R} \longrightarrow [0,1]$ is called *a fuzzy norm on* X if for any $x, y \in X$ and any $s, t\in\mathbb{R}$,

(N1) $N(x,t) = 0$ for $t \le 0$;

(N2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;

(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(N4) $N(x+y, s+t) \ge \min\{N(x,s), N(y,t)\}$;

(N5) $N(x,\cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;

(N6) for any $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X,N) is called *a fuzzy normed space.*

Let (X,N) be a fuzzy normed space and $\{x_n\}$ a sequence in X. Then (i) $\{x_n\}$ is said to be *Cauchy in* (X,N) if for any $\varepsilon > 0$, there exists an $m\in\mathbb{N}$ such that $N(x_{n+p}-x_n,t) > 1-\varepsilon$ for all $n \ge m$, all positive integer p, and all $t > 0$ and (ii) $\{x_n\}$ is said to be *convergent in* (X,N) if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* X and one denotes it by $N-\lim_{n\to\infty} x_n = x$.

Sequences of fuzzy numbers using the fuzzy metric or the fuzzy norm was studied by Das [6], [7], Tripathy et al. [23], Tripathy and Borgohain [24], [25], Tripathy and Dutta [26], Tripathy and Debnath [27] and others.

Example 2.2. For example, it is well known that for any normed space $(X, ||\cdot||)$ and any nonnegative real number ε, the mapping $N_X : X\times\mathbb{R} \longrightarrow [0,1]$, defined by

$$N_X(x,t) = \begin{cases} 0, & \text{if } t \le 0\\ \frac{t}{t+\varepsilon||x||}, & \text{if } t > 0\ , \end{cases}$$

is a fuzzy norm on X([16], [17], and [18]).

It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space is said to be *complete* if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called *a fuzzy Banach space*.

In 1996, Isac and Rassias [14] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

Theorem 2.3. [8] *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integer n or there exists a positive integer n_0 such that*
(1) $d(J^n x, J^{n+1}x) < \infty$ *for all* $n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J ;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq \dfrac{1}{1-L}\, d(y, Jy)$ *for all* $y \in Y$.

Throughout this paper, we assume that X is a linear space, (Y, N) is a fuzzy Banach space, and (Z, N') is a fuzzy normed space.

3. Solutions of (1.5)

In this section, we investigate the solution of (1.5) in fuzzy spaces. For any mapping $f : X \longrightarrow Y$, let

$$A_f(x, y) = f(2x + y) + f(2x - y) - 6f(x) - 2f(-x) - f(y) - f(-y),$$
$$B_f(x, y) = f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y),$$
$$C_f(x, y) = f(x + y) - f(x) - f(y),\ D_f(x, y) = f(x - y) - f(x) + f(y),$$

and

$$f_o(x) = \frac{f(x) - f(-x)}{2},\ f_e(x) = \frac{f(x) + f(-x)}{2}.$$

Then f_o is an odd mapping and f_e is an even mapping. By (N5), we can easily prove the following lemma.

Lemma 3.1. *Let $\alpha_i : [0, \infty) \longrightarrow [0, \infty)(i = 1, 2, \cdots, n)$ be mappings and r a real number with $r > 1$ and $y, z, z_1, z_2, \cdot, \cdot, \cdot, z_n \in Y$. Then we have the following :*
(1) *If* $N(y, t) \geq \min\{N(z, r^k t), N(z_1, \alpha_1(t)), N(z_2, \alpha_2(t)), \cdots, N(z_n, \alpha_n(t))\}$ *for all* $t > 0$ *and all* $k \in \mathbb{N}$, *then*

$$N(y, t) \geq \min\{N(z_1, \alpha_1(t)), N(z_2, \alpha_2(t)), \cdots, N(z_n, \alpha_n(t))\}$$

for all $t > 0$.
(2) *If* $N(y, t) \geq \min\{N(y, rt), N(z_1, \alpha_1(t)), N(z_2, \alpha_2(t)), \cdots, N(z_n, \alpha_n(t))\}$ *for all* $t > 0$ *and* $\alpha_i(i = 1, 2, \cdots, n)$ *is non-decreasing, then*

$$N(y, t) \geq \min\{N(z_1, \alpha_1(t)), N(z_2, \alpha_2(t)), \cdots, N(z_n, \alpha_n(t))\}$$

for all $t > 0$.
(3) *If* $N(y, t) \geq N(y, rt)$ *for all* $t > 0$, *then* $y = 0$.

We establish the following theorem using Lemma 3.1 :

Theorem 3.2. *Let $f : X \longrightarrow Y$ be an odd mapping. Suppose that a and b are real numbers with $a > 4$ and $b > 2$. Then f is an additive mapping if and only if f satisfies the following inequality*

$$N(A_f(x,y),t) \geq \min\{N(B_f(x,y),at), N(B_f(y,2x),bt)\} \tag{3.1}$$

for all $x, y \in X$ and all $t > 0$.

Proof. Suppose that f is a solution of (3.1). Letting $x = 0$ and $y = 0$ in (3.1), we get $f(0) = 0$. Letting $y = 0$ in (3.1), by (N2), we get

$$f(2x) = 2f(x) \tag{3.2}$$

for all $x \in X$. Letting $y = 2y$ in (3.1), by (3.2), we have

$$N(B_f(x,y),t) \geq \min\{N(B_f(x,2y),2at), N(B_f(y,x),bt)\} \tag{3.3}$$

for all $x, y \in X$ and all $t > 0$. Putting $x = 2x + y$ and $y = x$ in (3.3), we get

$$\begin{aligned} &N(f(3x+y)+f(x+y)-2f(2x+y),t) \\ &\geq \min\{N(f(4x+y)+f(y)-2f(2x+y),2at), N(f(3x+y)-f(x+y)-2f(x),bt)\} \\ &\geq \min\Big\{N(f(4x+y)+f(y)-2f(2x+y),2at), N\Big(f(2x+y)-f(x+y)-f(x),\frac{b}{4}t\Big), \\ &N\Big(f(3x+y)+f(x+y)-2f(2x+y),\frac{b}{2}t\Big)\Big\} \end{aligned} \tag{3.4}$$

for all $x, y \in X$ and all $t > 0$. Since $b > 2$, by (3.4) and Lemma 3.1, we have

$$\begin{aligned} &N(f(3x+y)+f(x+y)-2f(2x+y),t) \\ &\geq \min\Big\{N(f(4x+y)+f(y)-2f(2x+y),2at), N\Big(f(2x+y)-f(x+y)-f(x),\frac{b}{4}t\Big)\Big\} \end{aligned} \tag{3.5}$$

for all $x, y \in X$ and all $t > 0$. Letting $x = x + y$ and $y = x$ in (3.3), by (3.5), we get

$$\begin{aligned} &N(f(2x+y)+f(y)-2f(x+y),t) \\ &\geq \min\{N(f(3x+y)-f(x-y)-2f(x+y),2at), N(f(2x+y)-f(y)-2f(x),bt)\} \\ &\geq \min\Big\{N\Big(f(3x+y)+f(x+y)-2f(2x+y),\frac{a}{2}t\Big), N\Big(f(2x+y)+f(y)-2f(x+y),\frac{a}{2}t\Big), \\ &N\Big(f(x+y)-f(x-y)-2f(y),\frac{a}{2}t\Big), N(f(2x+y)-f(y)-2f(x),bt)\Big\} \\ &\geq \min\Big\{N\Big(f(4x+y)+f(y)-2f(2x+y),a^2t\Big), N\Big(f(2x+y)-f(x+y)-f(x),\frac{ab}{8}t\Big), \\ &N\Big(f(2x+y)+f(y)-2f(x+y),\frac{a}{2}t\Big), N\Big(f(x+y)-f(x-y)-2f(y),\frac{a}{2}t\Big), \\ &N(f(2x+y)-f(y)-2f(x),bt)\Big\} \end{aligned} \tag{3.6}$$

for all $x, y \in X$ and all $t > 0$. Since $a > 4$, by (3.6) and Lemma 3.1, we have

$$\begin{aligned} &N(f(2x+y)+f(y)-2f(x+y),t) \geq \min\Big\{N\Big(f(4x+y)+f(y)-2f(2x+y),a^2t\Big), \\ &N\Big(f(2x+y)-f(x+y)-f(x),\frac{ab}{8}t\Big), N\Big(f(x+y)-f(x-y)-2f(y),\frac{a}{2}t\Big), \\ &N(f(2x+y)-f(y)-2f(x),bt)\Big\} \end{aligned} \tag{3.7}$$

for all $x, y \in X$ and all $t > 0$. Letting $y = 2y$ in (3.7), by (3.2), we have

$$\begin{aligned}
&N(f(x+y)+f(y)-f(x+2y),t)\\
&\geq \min\Big\{N(f(2x+y)+f(y)-2f(x+y),a^2t), N\Big(2f(x+y)-f(x+2y)-f(x),\frac{ab}{4}t\Big),\\
&N(f(x+2y)-f(x-2y)-4f(y),at), N(C_f(x,y),bt)\Big\}\\
&\geq \min\Big\{N(f(2x+y)+f(y)-2f(x+y),a^2t), N\Big(f(y)+f(x+y)-f(x+2y),\frac{ab}{8}t\Big),\\
&N(f(x+2y)-f(x-2y)-4f(y),at), N\Big(C_f(x,y),\min\Big\{\frac{ab}{8},b\Big\}t\Big)\Big\}\\
&\geq \min\Big\{N(f(2x+y)+f(y)-2f(x+y),a^2t), N\Big(f(y)+f(x+y)-f(x+2y),\frac{a}{4}t\Big),\\
&N\Big(f(x-2y)-f(x-y)+f(y),\frac{a}{4}t\Big), N\Big(D_f(x,y),\frac{a}{4}t\Big), N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}
\end{aligned} \tag{3.8}$$

for all $x, y \in X$ and all $t > 0$, because $b > 2$. Since $a > 4$, by (3.8), we have

$$\begin{aligned}
&N(f(x+y)+f(y)-f(x+2y),t) \geq \min\Big\{N(f(2x+y)+f(y)-2f(x+y),a^2t),\\
&N\Big(f(x-2y)-f(x-y)+f(y),\frac{a}{4}t\Big), N\Big(D_f(x,y),\frac{a}{4}t\Big), N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}
\end{aligned} \tag{3.9}$$

for all $x, y \in X$ and all $t > 0$. Interchanging x and y in (3.9), we have

$$\begin{aligned}
&N(f(2x+y)-f(x+y)-f(x),t) \geq \min\Big\{N(f(x+2y)+f(x)-2f(x+y),a^2t),\\
&N\Big(f(2x-y)-f(x-y)-f(x),\frac{a}{4}t\Big), N\Big(D_f(x,y),\frac{a}{4}t\Big), N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}\\
&\geq \min\Big\{N\Big(f(x+2y)-f(x+y)-f(y),\frac{a^2}{2}t\Big), N\Big(f(2x-y)-f(x-y)-f(x),\frac{a}{4}t\Big),\\
&N\Big(D_f(x,y),\frac{a}{4}t\Big), N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}\\
&\geq \min\Big\{N\Big(f(2x+y)-f(x+y)-f(x),\frac{a^4}{2}t\Big), N\Big(f(x-2y)-f(x-y)+f(y),\frac{a^3}{8}t\Big),\\
&N\Big(f(2x-y)-f(x-y)-f(x),\frac{a}{4}t\Big), N\Big(D_f(x,y),\frac{a}{4}t\Big), N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}
\end{aligned} \tag{3.10}$$

for all $x, y \in X$ and all $t > 0$. Hence by Lemma 3.1 and (3.10), we have

$$\begin{aligned}
&N(f(2x+y)-f(x+y)-f(x),t) \geq \min\Big\{N\Big(f(2x-y)-f(x-y)-f(x),\frac{a}{4}t\Big),\\
&N\Big(f(x-2y)-f(x-y)+f(y),\frac{a^3}{8}t\Big), N\Big(D_f(x,y),\frac{a}{4}t\Big), N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}
\end{aligned} \tag{3.11}$$

for all $x, y \in X$ and all $t > 0$, because $a > 4$. By (3.11), we have

$$\begin{aligned}
&N(f(2x+y)-f(x+y)-f(x),t) \geq \min\Big\{N\Big(f(2x+y)-f(x+y)-f(x),\frac{a^2}{2^4}t\Big),\\
&N\Big(f(x+2y)-f(x+y)-f(y),\frac{a^4}{2^5}t\Big), N\Big(f(x-2y)-f(x-y)+f(y),\frac{a^3}{8}t\Big),\\
&N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big), N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\}
\end{aligned} \tag{3.12}$$

for all $x, y \in X$ and all $t > 0$. Thus by Lemma 3.1 and (3.12), we have

$$\begin{aligned} &N(f(2x+y)-f(x+y)-f(x),t) \\ &\geq \min\Big\{N\Big(f(x+2y)-f(x+y)-f(y),\frac{a^4}{2^5}t\Big),N\Big(f(x-2y)-f(x-y)+f(y),\frac{a^3}{2^3}t\Big), \\ &N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big),N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\} \end{aligned} \tag{3.13}$$

for all $x, y \in X$ and all $t > 0$. Interchanging x and y in (3.13), we have

$$\begin{aligned} &N(f(x+2y)-f(x+y)-f(y),t) \\ &\geq \min\Big\{N\Big(f(2x+y)-f(x+y)-f(x),\frac{a^4}{2^5}t\Big),N\Big(f(2x-y)-f(x-y)-f(x),\frac{a^3}{2^3}t\Big), \\ &N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big),N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\} \\ &\geq \min\Big\{N\Big(f(x+2y)-f(x+y)-f(y),\frac{a^8}{2^{10}}t\Big),N\Big(f(x-2y)-f(x-y)+f(y),\frac{a^3}{2^3}t\Big), \\ &N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big),N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\} \end{aligned} \tag{3.14}$$

for all $x, y \in X$ and all $t > 0$. By Lemma 3.1 and (3.14), we get

$$\begin{aligned} &N(f(x+2y)-f(x+y)-f(y),t) \geq \min\Big\{N\Big(f(x-2y)-f(x-y)+f(y),\frac{a^3}{2^3}t\Big), \\ &N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big),N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\} \\ &\geq \min\Big\{N\Big(f(x+2y)-f(x+y)-f(y),\frac{a^6}{2^6}t\Big),N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big), \\ &N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\} \end{aligned} \tag{3.15}$$

for all $x, y \in X$ and all $t > 0$. By Lemma 3.1 and (3.15), we get

$$\begin{aligned} &N(f(x+2y)-f(x+y)-f(y),t) \geq \min\Big\{N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big), \\ &N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\} \end{aligned} \tag{3.16}$$

for all $x, y \in X$ and all $t > 0$. Interchanging x and y in (3.16), we have

$$\begin{aligned} &N(f(2x+y)-f(x+y)-f(x),t) \geq \min\Big\{N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big), \\ &N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\} \end{aligned} \tag{3.17}$$

for all $x, y \in X$ and all $t > 0$. Letting $y = y - x$ in (3.17), we get

$$\begin{aligned} &N(C_f(x,y),t) \geq \min\Big\{N\Big(f(2x-y)-f(x)-f(x-y),\min\Big\{\frac{a}{4},b\Big\}t\Big), \\ &N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\} \\ &\geq \min\Big\{N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big),N\Big(C_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big)\Big\} \end{aligned} \tag{3.18}$$

for all $x, y \in X$ and all $t > 0$. Since $\min\{\frac{a}{4}, b\} > 1$, by Lemma 3.1 and (3.18), we have

$$N(C_f(x,y),t) \geq N\Big(D_f(x,y),\min\Big\{\frac{a}{4},b\Big\}t\Big) \geq N\Big(C_f(x,y),\Big[\min\Big\{\frac{a}{4},b\Big\}\Big]^2t\Big)$$

for all $x, y \in X$ and all $t > 0$ and hence by Lemma 3.1, f is an additive mapping.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

The converse is trivial. □

Theorem 3.3. *Let $f : X \longrightarrow Y$ be an even mapping. Suppose that k is a real number with $k > 1$. Then f is a solution of the following functional equation*

$$N(A_f(x,y),t) \geq N(B_f(x,y),kt) \tag{3.19}$$

for all $x, y \in X$ if and only if f is a quadratic mapping.

Proof. Suppose that f is a solution of (3.19). Letting $x = 0$ and $y = 0$ in (1.5), we have

$$N(f(0),t) \geq N\Big(f(0),4kt\Big)$$

for all $t > 0$ and sicne $4k > 1$, by Lemma 3.1, we get $f(0) = 0$. Letting $y = 0$ in (3.19), by (N2), we get

$$f(2x) = 4f(x) \tag{3.20}$$

for all $x \in X$. Now, letting $x = 2x$ in (3.19), by (3.20), we have

$$\begin{aligned} N(f(4x+y)+f(4x-y)-32f(x)-2f(y),t) &\geq N(A_f(x,y),kt) \\ &\geq N(B_f(x,y),k^2t) \end{aligned} \tag{3.21}$$

for all $x, y \in X$. Letting $y = 2y$ in (3.21), by (3.19), we have

$$N(A_f(x,y),t) \geq N(B_f(2y,x),4k^2t) = N(A_f(y,x),4k^2t) \geq N(B_f(x,y),4k^3t) \tag{3.22}$$

for all $x, y \in X$. Letting $x = 2x$ in (3.22), by (3.19), we have

$$N(f(4x+y)+f(4x-y)-32f(x)-2f(y),t) \geq N(B_f(x,y),4k^4t)$$

for all $x, y \in X$. Hence by induction, we get

$$N(f(4x+y)+f(4x-y)-32f(x)-2f(y),t) \geq N(B_f(x,y),4^nk^{n+3}t)$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Since $k > 1$, by Lemma 3.1 and (N5), we have

$$f(4x+y)+f(4x-y)-32f(x)-2f(y) = 0$$

for all $x, y \in X$. Hence f is a quadratic mapping. □

4. The generalized Hyers-Ulam stability for (1.5)

Now, we will prove the generalized Hyers-Ulam stability for (1.5) in fuzzy normed spaces.

Theorem 4.1. *Assume that $\phi : X^3 \longrightarrow [0,\infty)$ is a function such that*

$$N'(\phi(2x,2y),t) \geq N'(4L\phi(x,y),t) \tag{4.1}$$

for all $x, y \in X$, $t > 0$ and some real number L with $0 < L < \frac{1}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(A_f(x,y),t) \geq \min\{N(B_f(x,y),kt), N'(\phi(x,y),t)\} \tag{4.2}$$

for all $x, y \in X$, $t > 0$ and some real number k with $k > 32$. Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N\Big(f(x) - F(x), \frac{1}{2(1-2L)}t\Big) \geq \min\{N'(\phi(x,0),t), N'(\phi(-x,0),t)\} \tag{4.3}$$

for all $x \in X$ and all $t > 0$.

Proof. By (4.2), we get

$$\begin{aligned} N(A_{f_o}(x,y),t) \geq \min\Big\{&N\Big(B_{f_o}(x,y), \frac{k}{2}t\Big), N\Big(B_{f_e}(x,y), \frac{k}{2}t\Big), \\ &N'(\phi(x,y),t), N'(\phi(-x,-y),t)\Big\} \end{aligned} \tag{4.4}$$

for all $x, y \in X$, $t > 0$ and

$$\begin{aligned} N(A_{f_e}(x,y),t) \geq \min\Big\{&N\Big(B_{f_o}(x,y), \frac{k}{2}t\Big), N\Big(B_{f_e}(x,y), \frac{k}{2}t\Big), \\ &N'(\phi(x,y),t), N'(\phi(-x,-y),t)\Big\} \end{aligned} \tag{4.5}$$

for all $x, y \in X$ and all $t > 0$. Letting $y = 0$ in (4.4) and (4.5), by (N2), we have

$$N(2f_o(2x) - 4f_o(x), t) \geq \min\{N'(\phi(x,0),t), N'(\phi(-x,0),t)\} \tag{4.6}$$

and

$$N(2f_e(2x) - 8f_e(x), t) \geq \min\{N'(\phi(x,0),t), N'(\phi(-x,0),t)\} \tag{4.7}$$

for all $y \in X$ and all $t > 0$. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$d(g,h) = \inf\{c \in [0,\infty) \mid N(g(x) - h(x), ct) \geq \phi_o(x,t), \forall x \in X, \forall t > 0\},$$

where $\phi_o(x,t) = \min\{N'(\phi(x,0),t), N'(\phi(-x,0),t)\}$. Then (S,d) is a complete metric space([19]). Define a mapping $J_o : S \longrightarrow S$ by $J_o g(x) = \frac{1}{2}g(2x)$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (4.1), we have

$$N(J_o g(x) - J_o h(x), 2cLt) = N\Big(g(2x) - h(2x), 4cLt\Big) \geq \phi_o(2x, 4Lt) \geq \phi_o(x,t)$$

for all $x \in X$ and all $t > 0$. Hence $d(J_o g, J_o h) \leq 2Ld(g,h)$ for any $g, h \in S$ and by (4.6), we have $d(J_o f_o, f_o) \leq \frac{1}{4} < \infty$. By Theorem 2.3, there exists a mapping $P : X \longrightarrow Y$ which is a fixed point of J_o such that

$$N\Big(f_o(x) - P(x), \frac{1}{4(1-2L)}t\Big) \geq \phi_o(x,t) \tag{4.8}$$

for all $x \in X$ and all $t > 0$. Moreover, $d(J_o^n f_o, A) \to 0$ as $n \to \infty$. That is,

$$P(x) = N - \lim_{n\to\infty} \frac{f_o(2^n x)}{2^n}$$

for all $x \in X$. Now, define a mapping $J_e : S \longrightarrow S$ by $J_e g(x) = \frac{1}{4}g(2x)$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (4.1), we have

$$N(J_e g(x) - J_e h(x), cLt) = N\Big(g(2x) - h(2x), 4cLt\Big) \geq \phi_o(2x, 4Lt) \geq \phi_o(x,t)$$

for all $x \in X$ and $t > 0$. Hence $d(J_e g, J_e h) \leq Ld(g,h)$ for any $g, h \in S$ and by (4.7), we have $d(J_e f_e, f_e) \leq \frac{1}{8} < \infty$. By Theorem 2.3, there exists a mapping $Q : X \longrightarrow Y$ which is a fixed point of J_e such that

$$N\Big(f_e(x) - Q(x), \frac{1}{8(1-L)}t\Big) \geq \phi_o(x,t) \tag{4.9}$$

for all $x \in X$ and all $t > 0$. Moreover, $d(J_e^n f_e, A) \to 0$ as $n \to \infty$. That is,

$$Q(x) = N - \lim_{n\to\infty} \frac{f_e(2^n x)}{2^{2n}} \tag{4.10}$$

for all $x \in X$. Replacing x, and y by $2^n x$ and $2^n y$ in (4.5), respectively, by (4.1), we have

$$\begin{aligned} &N\Big(\frac{1}{2^{2n}} A_{f_e}(2^n x, 2^n y), t\Big) \geq \min\Big\{N\Big(\frac{1}{2^n} B_{f_o}(2^n x, 2^n y), 2^{n-1}kt\Big), \\ &N\Big(\frac{1}{2^{2n}} B_{f_e}(2^n x, 2^n y), \frac{k}{2}t\Big), N'\Big(\phi(x,y), \frac{1}{L^n}t\Big), N'\Big(\phi(-x,-y), \frac{1}{L^n}t\Big)\Big\} \end{aligned} \tag{4.11}$$

for all $x, y \in X$, $t > 0$, and all $n \in \mathbb{N}$. By (N4) and (4.11), we have

$$\begin{aligned} &N(A_Q(x,y), t) \\ &\geq \min\Big\{N\Big(A_Q(x,y) - \frac{1}{2^{2n}} A_{f_e}(2^n x, 2^n y), \frac{t}{2}\Big), N\Big(\frac{1}{2^{2n}} A_{f_e}(2^n x, 2^n y), \frac{t}{2}\Big)\Big\} \\ &\geq \min\Big\{N\Big(A_Q(x,y) - \frac{1}{2^{2n}} A_{f_e}(2^n x, 2^n y), \frac{t}{2}\Big), N\Big(\frac{1}{2^n} B_{f_o}(2^n x, 2^n y), 2^{n-2}kt\Big), \\ &N\Big(\frac{1}{2^{2n}} B_{f_e}(2^n x, 2^n y), \frac{k}{4}t\Big), N'\Big(\phi(x,y), \frac{1}{2L^n}t\Big), N'\Big(\phi(-x,-y), \frac{1}{2L^n}t\Big)\Big\} \\ &\geq \min\Big\{N\Big(A_Q(x,y) - \frac{1}{2^{2n}} A_{f_e}(2^n x, 2^n y), \frac{t}{2}\Big), N\Big(\frac{1}{2^n} B_{f_o}(2^n x, 2^n y), 2^{n-2}kt\Big), \\ &N\Big(\frac{1}{2^{2n}} B_{f_e}(2^n x, 2^n y) - B_Q(x,y), \frac{k}{8}t\Big), N\Big(B_Q(x,y), \frac{k}{8}t\Big), \\ &N'\Big(\phi(x,y), \frac{1}{2L^n}t\Big), N'\Big(\phi(-x,-y), \frac{1}{2L^n}t\Big)\Big\} \end{aligned} \tag{4.12}$$

for all $x, y \in X$, $t > 0$, and all $n \in \mathbb{N}$. By (N4), we have

$$\begin{aligned} &N\Big(\frac{1}{2^n} B_{f_o}(2^n x, 2^n y), 2^n t\Big) \\ &\geq \min\Big\{N\Big(\frac{1}{2^n} B_{f_o}(2^n x, 2^n y) - B_P(x,y), 2^{n-1}t\Big), N\Big(B_P(x,y), 2^{n-1}t\Big)\Big\} \end{aligned} \tag{4.13}$$

for all $x, y \in X$, $t > 0$, and all $n \in \mathbb{N}$. Letting $n \to \infty$ in (4.13), by (N5), we have

$$\lim_{n\to\infty} N\Big(\frac{1}{2^{2n}} B_{f_o}(2^n x, 2^n y), t\Big) = 1 \tag{4.14}$$

for all $x, y \in X$, $t > 0$, and all $n \in \mathbb{N}$. Letting $n \to \infty$ in (4.12), by (4.10) and (4.14), we have

$$N(A_Q(x,y), t) \geq N\Big(B_Q(x,y), \frac{k}{8}t\Big) \tag{4.15}$$

for all $x, y \in X$ and all $t > 0$. Since f_e is even, by (4.10), Q is even and hence by (4.15) and Theorem 3.3, Q is a quadratic mapping. By (4.5) and (4.7), we have

$$\begin{aligned} &N(B_{f_e}(x,2y),t) \geq \min\Big\{N\Big(A_{f_e}(y,x),\frac{t}{2}\Big), N\Big(8f_e(y)-2f_e(2y),\frac{t}{2}\Big)\Big\} \\ &\geq \min\Big\{N\Big(B_{f_o}(y,x),\frac{k}{4}t\Big), N\Big(B_{f_e}(y,x),\frac{k}{4}t\Big), N'\Big(\phi(y,x),\frac{t}{2}\Big), \\ &N'\Big(\phi(-y,-x),\frac{t}{2}\Big), N'\Big(\phi(y,0),\frac{t}{2}\Big), N'\Big(\phi(-y,0),\frac{t}{2}\Big)\Big\} \end{aligned} \tag{4.16}$$

for all $x, y \in X$ and $t > 0$. By (4.7) and (4.16), we have

$$\begin{aligned} &N(B_{f_e}(x,y),t) = N(4B_{f_e}(x,y),4t) \\ &\geq \min\{N(B_{f_e}(2x,2y),2t), N(4B_{f_e}(x,y)-B_{f_e}(2x,2y),2t)\} \\ &\geq \min\Big\{N\Big(B_{f_o}(y,2x),\frac{k}{2}t\Big), N\Big(B_{f_e}(y,2x),\frac{k}{2}t\Big), \Phi_1(x,y,t)\Big\} \\ &\geq \min\Big\{N\Big(B_{f_o}(y,2x),\frac{k}{2}t\Big), N\Big(B_{f_o}(x,y),\frac{k^2}{8}t\Big), N\Big(B_{f_e}(x,y),\frac{k^2}{8}t\Big), \Phi_2(x,y,t)\Big\} \end{aligned} \tag{4.17}$$

for all $x, y \in X$ and all $t > 0$, where

$$\begin{aligned} &\Phi_1(x,y,t) = \min\Big\{N'(\phi(y,2x),t), N'(\phi(-y,-2x),t), N'(\phi(x+y,0),t), \\ &N'(\phi(-x-y,0),t), N'(\phi(x-y,0),t), N'(\phi(-x+y,0),t), \\ &N'\Big(\phi(x,0),\frac{t}{2}\Big), N'\Big(\phi(-x,0),\frac{t}{2}\Big), N'\Big(\phi(y,0),\frac{t}{2}\Big), N'\Big(\phi(-y,0),\frac{t}{2}\Big)\Big\} \end{aligned}$$

and

$$\Phi_2(x,y,t) = \min\Big\{\Phi_1(x,y,t), N'\Big(\phi(x,y),\frac{k}{4}t\Big), N'\Big(\phi(-x,-y),\frac{k}{4}t\Big)\Big\},$$

because $k > 32$. By Lemma 3.1 and (4.17), we have

$$N(B_{f_e}(x,y),t) \geq \min\Big\{N\Big(B_{f_o}(y,2x),\frac{k}{2}t\Big), N\Big(B_{f_o}(x,y),\frac{k^2}{8}t\Big), \Phi_2(x,y,t)\Big\} \tag{4.18}$$

for all $x, y \in X$ and all $t > 0$ and hence by (4.4) and (4.18), we have

$$\begin{aligned} N(A_{f_o}(x,y),t) \geq \min\Big\{&N\Big(B_{f_o}(x,y),\frac{k}{2}t\Big), N\Big(B_{f_o}(y,2x),\frac{k^2}{4}t\Big), \\ &\Phi_1\Big(x,y,\frac{k}{2}t\Big), N'(\phi(x,y),t), N'(\phi(-x,-y),t)\Big\} \end{aligned} \tag{4.19}$$

for all $x, y \in X$, $t > 0$ and replacing x and y by $2^n x$ and $2^n y$ in (4.19), respectively, by (4.1), we have

$$\begin{aligned} &N\Big(A_{f_o}(2^n x, 2^n y), 2^n t\Big) \\ &\geq \min\Big\{N(B_{f_o}(2^n x, 2^n y), 2^{n-1}kt), N(B_{f_o}(2^n y, 2^{n+1}x), 2^{n-2}k^2 t), \\ &\Phi_1\Big(x,y,\frac{k}{2(2L)^n}t\Big), N'\Big(\phi(x,y),\frac{1}{(2L)^n}t\Big), N'\Big(\phi(-x,-y),\frac{1}{(2L)^n}t\Big)\Big\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Similar to Q, we have

$$N(A_P(x,y),t) \geq \min\Big\{N\Big(B_P(x,y),\frac{k}{8}t\Big), N\Big(B_P(y,2x),\frac{k^2}{16}t\Big)\Big\} \tag{4.20}$$

for all $x, y \in X$ and all $t > 0$. Cleraly, P is an odd mapping and since $k > 32$, by Theorem 3.2, P is an additive mapping. Let $F = P + Q$. Then $F : X \longrightarrow Y$ is an additive-quadratic mapping. By (4.8) and (4.9), we have (4.3).

Now, we show the uniqueness of F. Let H be another additive-quadratic mapping with (4.3). Since F and H are addiitve-quadratic mappings, we have

$$F(x) = \frac{1+2^n}{2^{2n+1}}F(2^n x) + \frac{1-2^n}{2^{2n+1}}F(-2^n x),\ H(x) = \frac{1+2^n}{2^{2n+1}}H(2^n x) + \frac{1-2^n}{2^{2n+1}}H(-2^n x),$$

for all $x \in X$ and all positive integer n. Hence by (4.3), (N3) and (N4), we have

$$\begin{aligned}
&N(F(x) - H(x), t) \\
&\geq \min\Big\{N\Big(F(2^n x) - H(2^n x), \frac{2^{2n}}{1+2^n}t\Big), N\Big(F(-2^n x) - H(-2^n x), \frac{2^{2n}}{2^n-1}t\Big)\Big\} \\
&\geq \min\Big\{N\Big(F(2^n x) - f(2^n x), \frac{2^{2n-1}}{1+2^n}t\Big), N\Big(f(2^n x) - H(2^n x), \frac{2^{2n-1}}{1+2^n}t\Big), \\
&N\Big(F(-2^n x) - f(-2^n x), \frac{2^{2n-1}}{2^n-1}t\Big), N\Big(f(-2^n x) - H(-2^n x), \frac{2^{2n-1}}{2^n-1}t\Big)\Big\} \\
&\geq \min\Big\{\phi_o\Big(2^n x, \frac{2^{2n}(1-2L)}{1+2^n}t\Big), \phi_o\Big(2^n x, \frac{2^{2n}(1-2L)}{2^n-1}t\Big)\Big\} \\
&\geq \min\Big\{\phi_o\Big(x, \frac{1-2L}{(L)^n + (2L)^n}t\Big), \phi_o\Big(x, \frac{1-2L}{(2L)^n\Big(1-\frac{1}{2^n}\Big)}t\Big)\Big\}
\end{aligned}$$

for all $x \in X$, $t > 0$, and all $n \in \mathbb{N}$. Since $0 < L < \frac{1}{2}$, letting $n \to \infty$ in the above inequality, we have $F(x) = H(x)$ for all $x \in X$. □

By Theorem 4.1, we can show that the following corollaries:

Corollary 4.2. *Let ε and p be real numbers with $\varepsilon \geq 0$ and $0 < p < \frac{1}{2}$. Let $f : X \longrightarrow Y$ be a mapping such that*

$$N(A_f(x,y), t) \geq \min\Big\{N(B_f(x,y), kt),\ \frac{t}{t + \varepsilon(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)}\Big\} \tag{4.21}$$

for all $x, y \in X$, all $t > 0$ and some real number k with $k > 32$. Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x) - F(x), t) \geq \frac{(2 - 2^{2p})t}{(2 - 2^{2p})t + \varepsilon\|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$.

Corollary 4.3. *Assume that $\phi : X^3 \longrightarrow [0, \infty)$ is a function with (4.1). Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$N(rA_f(x,y) + B_f(x,y), t) \geq \min\{N(B_f(x,y), t), N'(\phi(x,y), t)\} \tag{4.22}$$

for all $x, y \in X$, all $t > 0$ and some real numbers r with $|r| > 64$. Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N\Big(f(x) - F(x), \frac{1}{2(1-2L)}t\Big) \geq \min\{N'(\phi(x,0), t), N'(\phi(-x,0), t)\}$$

for all $x \in X$ and all $t > 0$.

Proof. By (N5) and (4.22), we have

$$\begin{aligned}N(A_f(x,y),t) &\ge \min\Big\{N\Big(rA_f(x,y)+B_f(x,y),\frac{|r|}{2}t\Big),N\Big(B_f(x,y),\frac{|r|}{2}t\Big)\Big\}\\ &\ge \min\Big\{N\Big(B_f(x,y),\frac{|r|}{2}t\Big),N'\Big(\phi(x,y),\frac{|r|}{2}t\Big)\Big\}\\ &\ge \min\Big\{N\Big(B_f(x,y),\frac{|r|}{2}t\Big),N'\Big(\phi(x,y),t\Big)\Big\}\end{aligned}$$

for all $x,y\in X$ and all $t>0$. Hence we have the results. □

Corollary 4.4. *Let ε and p be real numbers with $\varepsilon\ge 0$ and $0<p<\frac{1}{2}$. Let $f:X\longrightarrow Y$ be a mapping such that*

$$N(rA_f(x,y)+B_f(x,y),t)\ge \min\Big\{N(B_f(x,y),t),\ \frac{t}{t+\varepsilon(\|x\|^{2p}+\|y\|^{2p}+\|x\|^p\|y\|^p)}\Big\}\tag{4.23}$$

for all $x,y\in X$, all $t>0$ and some real number r with $|r|>64$. Then there exists an unique additive-quadratic mapping $F:X\longrightarrow Y$ such that

$$N(f(x)-F(x),t)\ge \frac{(2-2^{2p})t}{(2-2^{2p})t+\varepsilon\|x\|^{2p}}$$

for all $x\in X$ and all $t>0$.

Related with Theorem 4.1, we can also have the following theorem. The proof is similar to that of Theorem 4.1.

Theorem 4.5. *Assume that $\phi:X^3\longrightarrow[0,\infty)$ is a function such that*

$$N'\Big(\phi\Big(\frac{x}{2},\frac{y}{2}\Big),t\Big)\ge N'\Big(\frac{L}{2}\phi(x,y),t\Big)\tag{4.24}$$

for all $x,y\in X$, $t>0$ and some real number L with $0<L<\frac{1}{2}$. Let $f:X\longrightarrow Y$ be a mapping such that $f(0)=0$ and (4.2). Then there exists an unique additive-quadratic mapping $F:X\longrightarrow Y$ such that

$$N\Big(f(x)-F(x),\frac{L}{2(1-L)}t\Big)\ge \min\{N'(\phi(x,0),t),N'(\phi(-x,0),t)\}$$

for all $x\in X$ and $t>0$.

Proof. Let $\phi_o(x,t)=\min\{N'(\phi(x,0),t),N'(\phi(-x,0),t)\}$. Letting $x=\frac{x}{2}$ in (4.6) and (4.7), by (4.24), we have

$$N\Big(2f_o(x)-4f_o\Big(\frac{x}{2}\Big),\frac{L}{2}t\Big)\ge\phi_o(x,t)\tag{4.25}$$

and

$$N\Big(2f_e(x)-8f_e\Big(\frac{x}{2}\Big),\frac{L}{2}t\Big)\ge\phi_o(x,t)\tag{4.26}$$

for all $y\in X$ and $t>0$. Consider the set $S=\{g\mid g:X\longrightarrow Y\}$ and the generalized metric d on S defined by

$$d(g,h)=\inf\{c\in[0,\infty)\mid N(g(x)-h(x),ct)\ge\phi_o(x,t),\forall x\in X,\forall t>0\}.$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Then (S, d) is a complete metric space([19]). Define a mapping $J_o : S \longrightarrow S$ by $J_o g(x) = 2g\Big(\frac{x}{2}\Big)$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g, h) \le c$ for some $c \in [0, \infty)$. Then by (4.1), we have

$$N\Big(J_o g(x) - J_o h(x), cLt\Big) = N\Big(g\Big(\frac{x}{2}\Big) - h\Big(\frac{x}{2}\Big), c\frac{L}{2}t\Big) \ge \phi_o\Big(\frac{x}{2}, \frac{L}{2}t\Big) \ge \phi_o(x, t)$$

for all $x \in X$ and $t > 0$. Hence $d(J_o g, J_o h) \le Ld(g, h)$ for any $g, h \in S$. By (4.25), we have $d(J_o f_o, f_o) \le \frac{L}{4} < \infty$ and by Theorem 2.3, there exists a mapping $P : X \longrightarrow Y$ which is a fixed point of J_o such that

$$N\Big(f_o(x) - P(x), \frac{L}{4(1-L)}t\Big) \ge \phi_o(x, t)$$

for all $x \in X$, all $t > 0$ and $d(J_o^n f_o, A) \to 0$ as $n \to \infty$.

Now, define a mapping $J_e : S \longrightarrow S$ by $J_e g(x) = 4g\Big(\frac{x}{2}\Big)$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g, h) \le c$ for some $c \in [0, \infty)$. Then by (4.1), we have

$$N(J_e g(x) - J_e h(x), 2cLt) = N\Big(g\Big(\frac{x}{2}\Big) - h\Big(\frac{x}{2}\Big), c\frac{L}{2}t\Big) \ge \phi_o\Big(\frac{x}{2}, \frac{L}{2}t\Big) \ge \phi_o(x, t)$$

for all $x \in X$ and $t > 0$. Hence $d(J_e g, J_e h) \le 2Ld(g, h)$ and by (4.26), we have $d(J_e f_e, f_e) \le \frac{L}{4} < \infty$. By Theorem 2.3, there exists a mapping $Q : X \longrightarrow Y$ which is a fixed point of J_e such that

$$N\Big(f_e(x) - Q(x), \frac{L}{4(1-2L)}t\Big) \ge \phi_o(x, t)$$

for all $x \in X$, all $t > 0$ and $d(J_e^n f_e, A) \to 0$ as $n \to \infty$.

The rest of the proof is similar to Theorem 4.1. □

By Theorem 4.5, we can show that the following corollaries:

Corollary 4.6. *Let ε and p be real numbers with $\varepsilon \ge 0$ and $p > 1$. Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ and (4.21). Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that*

$$N(f(x) - F(x), t) \ge \frac{(2^{2p} - 2)t}{(2^{2p} - 2)t + \varepsilon \|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$.

Corollary 4.7. *Assume that $\phi : X^3 \longrightarrow [0, \infty)$ is a function with (4.24). Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ and (4.22). Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that*

$$N(f(x) - F(x), \frac{L}{2(1-L)}t) \ge \min\{N'(\phi(x, 0), t), N'(\phi(-x, 0), t)\}$$

for all $x \in X$ and all $t > 0$.

Corollary 4.8. *Let ε and p be real numbers with $\varepsilon \ge 0$ and $p > 1$. Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ and (4.23). Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that*

$$N(f(x) - F(x), t) \ge \frac{(2^{2p} - 2)t}{(2^{2p} - 2)t + \varepsilon \|x\|^{2p}}$$

for all $x \in X$ and all $t > 0$.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2**(1950), 64-66.

[2] T. Bag and S. K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11**(2003), 687-705.

[3] P.W.Cholewa, *Remarkes on the stability of functional equations*, Aequationes Math. **27**(1984), 76-86.

[4] K Cieplinski, *Applications of fixed point theorems to the hyers-ulam stability of functional equation-A survey*, Ann. Funct. Anal. **3**(2012), no. 1, 151-164.

[5] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Bull. Abh. Math. Sem. Univ. Hamburg **62**(1992), 59-64.

[6] P. C. Das, *Fuzzy normed linear space valued sequence space* $\ell_p^{F(X)}$, Proyecciones **36(2)** (2017), 245-255.

[7] ———, *On fuzzy normed linear space valued statistically convergent sequences*, Proyecciones **36(3)** (2017), 511-527.

[8] J. B. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305-309.

[9] W. Fechner, *Stability of a functional inequalty associated with the Jordan-Von Neumann functional equation*, Aequationes Math. **71**(2006), 149-161.

[10] P. Găvruta, *A generalization of the Hyer-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184**(1994), 431-436.

[11] A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung*, Aequationes Mathematicae, **62**(2001), 303-309.

[12] A. Gilányi, *On a problem by K. Nikoden, Mathematical Inequalities and Applications*, **5**(2002), 701-710.

[13] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. **27**(1941), 222-224.

[14] G. Isac and Th. M. Rassias, *Stability of ψ-additive mappings, Appications to nonlinear analysis*, Internat. J. Math. and Math. Sci. **19**(1996), 219-228.

[15] A. K. Mirmostafaee and M. S. Moslehian, Fuzzy almost quadratic functions, Results Math. **52**(2008), 161-177.

[16] A. K. Mirmostafaee and M. S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem*, Fuzzy Sets Syst. **159**(2008), 720-729.

[17] A. K. Mirmostafaee, M. Mirzavaziri, and M. S. Moslehian, *Fuzzy stability of the Jensen functional equation*, Fuzzy Sets Syst. **159**(2008), 730-738.

[18] M. Mirzavaziri and M. S. Moslehian, *A fixed point approach to stability of a quadratic equation*, Bulletin of the Brazilian Mathematical Society **37**(2006), no. 3, 361-376

[19] M. S. Moslehian and T. H. Rassias, *Stability of functional equations in non-Archimedean spaces*, Applicable Anal. Discrete Math. **1**(2007), 325-334.

[20] C. Park, *Fuzzy Stability of Additive Functional Inequalities with the Fixed Point Alternative*, J. Inequal. Appl. **2009**(2009), 1-17.

[21] C. Park, Y. S. Cho, and M. H. Han, *Functional inequalities associated with Jordan-von Neumann type additive functional equations*, J. Inequal. Appl. **2007**(2007), 1-13.

[22] Th. M. Rassias, *On the stability of the linear mapping in Banach sapces*, Proc. Amer. Math. Sco. **72**(1978), 297-300.

[23] B. C. Tripathy, A. Baruah, M.Et and M. Gungor, *On almost statistical convergence of new type of generalized difference sequence of fuzzy numbers*, Iranian Journal of Science and Technology, Transacations A : Science, **36(2)**(2012), 147-155.

[24] B. C. Tripathy and S. Borgogain, *Some classes of difference sequence spaces of fuzzy real numbers defined by Orlicz function*, Advances in Fuzzy Systems, 2011, Article ID216414, 1-6.

[25] B. C. Tripathy and S. Borgogain, *On a class of n-normed sequences related to the ℓ_p-space*, Boletim da Sociedade Paranaense de Matemática, **31(1)**(2013), 167-173.

[26] B. C. Tripathy and S. Debnath, *On generalized difference sequence spaces of fuzzy numbers*, Acta Scientiarum Technology, **35(1)**(2013), 117-121.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

[27] B. C. Tripathy and A. J. Dutta, *On I-acceleration convergence of sequences of fuzzy real numbers*, Math. Modell. Analysis, **17(4)**(2012), 549-557.

[28] S. M. Ulam, Problems in modern mathematics, Science Editions John Wiley and Sons, Inc., New York, 1964.

Department of Mathematics Education, Dankook University, 152, Jukjeon-ro, Suji-gu, Yongin-si, Gyeonggi-do, 16890, Korea

E-mail address: kci206@hanmail.net

Department of Mathematics Education, Dankook University, 152, Jukjeon-ro, Suji-gu, Yongin-si, Gyeonggi-do, 16890, Korea

E-mail address: gilhan@dankook.ac.kr

NEW CHARACTERIZATIONS OF WEIGHTS IN HARDY'S TYPE INEQUALITIES VIA OPIAL'S DYNAMIC INEQUALITIES

S. H. SAKER, M. M. OSMAN AND I. ABOHELA

Abstract. In this paper, we prove some new characterizations of weights in some Hardy-type inequalities on time scales. The results as special cases contain the results due to Beesack and Heinig, Leindler and Bloom and Kerman. Some new integral and discrete inequalities related to Copson's, Flett's, Bliss's and Bennett's will be formulated. The main results will be proved by using new generalizations of Opial's type inequalities, Hölder's inequality, Minkowski's inequality and the chain rule on time scales.

Keywords: Hardy's inequality, Opial's inequality, time scales.

AMS Classif: 26A15, 26D10, 26D15, 39A13, 34A40.

1. Introduction

During the last decades the inequality

$$\left(\int_a^b r(t)\left(\int_a^t f(\tau)d\tau\right)^q dt\right)^{1/q} \le C\left(\int_a^b s(t)\, f^p(t)dt\right)^{1/p},\ 1<p\le q<\infty, \tag{1.1}$$

with two different positive weighted functions defined in $[a,b]\subset\mathbb{R}^+$ has been studied by several authors, we refer the reader to the papers [11, 23, 37, 38] and the books [20, 24]. The main idea is to give a relation between the functions r and s and to find the optimal value of the constant C such that the inequality (1.1) holds. A systematic investigation of this type of inequality of Hardy's type with two different weights started in the late fifties and early sixties by Beesack [7]. In particular Beesack proved that

$$\int_a^b r(t)\left(\int_0^t f(\tau)d\tau\right)^p dt \le \int_a^b s(t)\, f^p(t)dt, \tag{1.2}$$

where r and s satisfy the Euler-Lagrange differential equation

$$\frac{d}{dt}\left(s(t)\left(y'(t)\right)^{p-1}\right)+r(t)\,y^{p-1}(t)=0.$$

Also Beesack and Heinig [8] proved that if $0<p<1$ and $\int_0^\infty r(t)\left(\int_0^t f(\tau)d\tau\right)^p dt<\infty$, then

$$\int_0^\infty r(t)\left(\int_0^t f(\tau)d\tau\right)^p dt \ge p^p\int_0^\infty r^{1-p}(t)\left(\int_t^\infty r(\tau)\,d\tau\right)^p f^p(t)dt, \tag{1.3}$$

and if $\int_0^\infty r(t)\left(\int_t^\infty f(\tau)d\tau\right)^p dt<\infty$, then

$$\int_0^\infty r(t)\left(\int_t^\infty f(\tau)d\tau\right)^p dt \ge p^p\int_a^\infty r^{1-p}(t)\left(\int_0^t r(\tau)\,d\tau\right)^p f^p(t)dt. \tag{1.4}$$

Bloom and Kerman [10] proved that if $1<p<\infty$, $f\ge 0$ and $\int_0^\infty (s(t)\,f(t))^p\,dt<\infty$, then

$$\int_0^\infty\left(r(t)\int_0^t f(\tau)d\tau\right)^p dt \le C\int_0^\infty (s(t)\,f(t))^p\,dt, \tag{1.5}$$

holds if and only if

$$\int_t^\infty \left(s^{-1}(\tau)\int_\tau^\infty r^p(x)\,dx\right)^{p'} d\tau \le C\int_t^\infty r^p(\tau)\,d\tau.$$

By using a new approach depends on the application of Opial's type inequalities Agarwal et al. [4] proved that if r, s are nonnegative measurable functions on (a,b) and $p>0$, $k>1$, then

$$\int_a^b r(t)\left(\int_a^t f(\tau)d\tau\right)^{p+1} dt \le (p+1)\,K_1(p,1,k)\left[\int_a^b s(t)\,f^k(t)dt\right]^{\frac{p+1}{k}}, \tag{1.6}$$

where

$$K_1(p,1,k)=\left(\frac{1}{p+1}\right)^{\frac{1}{k}}\left(\int_a^b (R(t,b))^{\frac{k}{k-1}}(s(t))^{\frac{-1}{k-1}}\left(\int_a^t s^{\frac{-1}{k-1}}(\tau)\,d\tau\right)^p dt\right)^{\frac{k-1}{k}},$$

and $R(t,b)=\int_t^b r(\tau)\,d\tau$.

In the last decades the study of discrete results on l^p analogues for L^p-bounds has been proved by some authors. One of the reasons for this upsurge of interest in discrete cases is due to the fact that the discrete operators may even behave differently from their continuous counterparts. So it was natural to look on the discrete results on l^p analogues for the above L^p-results. We mention here that in some special cases it is possible to translate or adapt almost straightforward the objects and results from the continuous setting to the discrete setting or vice versa, however, in some other cases that is far from be trivial. But l^p-bounds for discrete analogues of more complicated operators are not implied by results in the continuous setting, and moreover the discrete analogues are resistant to conventional methods. The main challenge here is that there are no general methods to study these questions and the methods should to be developed starting from the basic definitions in the discrete space. For example, Leindler [22] established the discrete versions of (1.3) and (1.4), and proved that if $0<p\le 1$, $a_n\ge 0$ and $\lambda_n>0$, then

$$\sum_{n=1}^{\infty}\lambda_n\left(\sum_{k=1}^{n}a_k\right)^p \ge p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}\left(\sum_{k=n}^{\infty}\lambda_k\right)^p a_n^p, \tag{1.7}$$

and

$$\sum_{n=1}^{\infty}\lambda_n\left(\sum_{k=n}^{\infty}a_k\right)^p \ge p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}\left(\sum_{k=1}^{n}\lambda_k\right)^p a_n^p. \tag{1.8}$$

In recent years the study of dynamic equations and inequalities on time scales has received a lot of attention in the literature and has become a major field in pure and applied mathematics. The general idea is to prove a result for a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$, to avoid proving results twice, once for differential inequality and once again for difference inequality. This idea goes back to its founder Stefan Hilger [19] who started the study of dynamic equations on time scales. Since the integral and discrete inequalities are important in the analysis of qualitative properties of solutions of differential and difference equations, we also believe that the dynamic Hardy type inequalities with weights on time scales will play the same effective role in the analysis of qualitative properties of dynamic equations with boundary conditions like oscillation, nonoscillation and distribution of zeros of solutions. For related dynamic inequalities on time scales, we refer the reader to the papers [26, 27, 32, 33] and the books [2,3]. Our technique in this paper will overcame the lack of calculus in the discrete

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

space where there is no power rules and also there is no chain rule which are the main tools used in the proofs of the continuous case.

The aim of this paper is to prove some new dynamic inequalities by employing some Opial's type inequalities on an arbitrary time scale $\mathbb{T}$ which contain the integral and discrete inequalities (1.3)–(1.6) as special cases. For applications of the main results we get some well-known dynamic inequalities as special cases. The paper is divided into two sections. In Section 2, we introduce some preliminaries on time scales and establish some basic lemmas that will be needed in the proofs. In Section 3, we prove the main results and formulate some discrete results to show the application of the new results.

2. Preliminaries and Some Basic Lemmas

In this section, we present some basic definitions and results concerning the delta calculus on time scales; for more details we refer the reader to the book [14]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, and $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, where $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$, is said to be left–dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, is right-dense if $\sigma(t) = t$, is left–scattered if $\rho(t) < t$ and right–scattered if $\sigma(t) > t$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided f is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. Also, the set of functions that are differentiable and whose derivative is rd–continuous is denoted by $C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R})$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$. Without loss of generality, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[a, b]_\mathbb{T}$ by $[a, b]_\mathbb{T} := [a, b] \cap \mathbb{T}$. Recall of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$, here $g^\sigma = g \circ \sigma$) of two differentiable functions f and g

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \text{ and } \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}. \tag{2.1}$$

The first chain rule that we will use in this paper is

$$(f^\gamma(t))^\Delta = \gamma \int_0^1 [hf^\sigma + (1-h)f]^{\gamma-1} \, dh f^\Delta(t), \quad \gamma \in \mathbb{R}, \tag{2.2}$$

which is a simple consequence of Keller's chain rule [14, Theorem 1.90]. The second chain rule that we will use in this paper is given in the following. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable, then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and

$$f^\Delta (g(t)) = f'(g(d))\, g^\Delta(t), \quad \text{for} \quad d \in [t, \sigma(t)]. \tag{2.3}$$

In this paper we will refer to the (delta) integral which we can define as follows. If $F^\Delta(t) = f(t)$, then the Cauchy (delta) integral of f is defined by $\int_{t_0}^t f(s)\Delta s := F(t) - F(t_0)$. It can be shown (see [14]) that if $f \in C_{rd}(\mathbb{T})$, then the Cauchy integral $F(t) := \int_{t_0}^t f(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $F^\Delta(t) = f(t)$, $t \in \mathbb{T}$. An infinite integral is defined as $\int_a^\infty f(t)\Delta t = \lim_{b\to\infty} \int_a^b f(t)\Delta t$. Integration on discrete time scales is defined by

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)} \mu(t)f(t).$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

The integration by parts formula on time scales reads

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{2.4}$$

Hölder's inequality [5, Theorem 6.2] states that for $f,\ g \in C_{rd}([a,\ b]_{\mathbb{T}},\ \mathbb{R})$, we have

$$\int_a^b |f(t)g(t)|\,\Delta t \leq \left[\int_a^b |f(t)|^p\,\Delta t\right]^{1/p}\left[\int_a^b |g(t)|^q\,\Delta t\right]^{1/q}, \tag{2.5}$$

where $p > 1$, $1/p + 1/q = 1$ and $a,\ b \in \mathbb{T}$. This inequality is reversed if $0 < p < 1$ and $\int_a^b |g(t)|^q\,\Delta t > 0$, and it is also reversed if $p < 0$ and $\int_a^b |f(t)|^p\,\Delta t > 0$.

Throughout this paper, we will assume that $r(t)$, $s(t)$ and $f(t)$ are nonnegative rd-continuous functions and the integrals considered are assumed to exist. In order to prove our main results in Section 3, we need the following lemmas.

Lemma 2.1. *Assume $F : \mathbb{T} \to \mathbb{R}$ is differentiable and positive. If F^{Δ} is always positive, then*

$$\left(F^{\lambda}\right)^{\Delta} \geq F^{\Delta}\left(F^{\sigma}(t)\right)^{\lambda-1}, \quad \text{if} \quad \lambda \geq 1, \tag{2.6}$$

and

$$\left(F^{\lambda}\right)^{\Delta} \leq F^{\Delta}\left(F^{\sigma}(t)\right)^{\lambda-1}, \quad \text{if} \quad 0 \leq \lambda \leq 1, \tag{2.7}$$

Proof. If F is increasing and $\lambda \geq 1$, then $F^{\lambda-1}$ is increasing and thus $\left(F^{\lambda-1}\right)^{\Delta} > 0$ so that

$$\left(F^{\lambda}\right)^{\Delta} = \left(FF^{\lambda-1}\right)^{\Delta} = F^{\Delta}\left(F^{\sigma}(t)\right)^{\lambda-1} + F\left(F^{\lambda-1}\right)^{\Delta} \geq 0.$$

This shows (2.6), and (2.7) follows similarly. The proof is complete. □

Lemma 2.2. *Let $\mathbb{T}$ be a time scale with $a,\ b \in \mathbb{T}$. If $p > 0$, then*

$$\int_a^b r(t)\left(\int_a^{\sigma(t)} f(\tau)\,\Delta\tau\right)^{p+1}\Delta t \leq (p+1)\int_a^b R(t,b)\left(F^{\sigma}(t)\right)^p F^{\Delta}(t)\,\Delta t, \tag{2.8}$$

where

$$R(t,b) = \int_t^b r(\tau)\,\Delta\tau, \quad \text{and} \quad F(t) = \int_a^t f(\tau)\,\Delta\tau. \tag{2.9}$$

Proof. From (2.9) and applying integration by parts (2.4) with $u^{\Delta}(t) = R^{\Delta}(t,b)$ and $v^{\sigma}(t) = \left(F^{\sigma}(t)\right)^{p+1}$, we obtain

$$\begin{aligned}\int_a^b r(t)\left(\int_a^{\sigma(t)} f(\tau)\,\Delta\tau\right)^{p+1}\Delta t &= \int_a^b \left(-R^{\Delta}(t,b)\right)\left(F^{\sigma}(t)\right)^{p+1}\Delta t \\ &= -R(t,b)\,F^{p+1}(t)\Big|_a^b + \int_a^b R(t,b)\left(F^{p+1}(t)\right)^{\Delta}\Delta t.\end{aligned}$$

Using the fact that $R(b,b) = 0$ and $F(a) = 0$, we have

$$\int_a^b r(t)\left(\int_a^{\sigma(t)} f(\tau)\,\Delta\tau\right)^{p+1}\Delta t = \int_a^b R(t,b)\left(F^{p+1}(t)\right)^{\Delta}\Delta t. \tag{2.10}$$

By the chain rule (2.2) and the fact that $F^{\Delta}(t) = f(t) \geq 0$ yields

$$\begin{aligned}\left(F^{p+1}(t)\right)^{\Delta} &= (p+1)\int_0^1 \left[hF^{\sigma}(t) + (1-h)F(t)\right]^p F^{\Delta}(t) \\ &\leq (p+1)\int_0^1 \left[hF^{\sigma}(t) + (1-h)F^{\sigma}(t)\right]^p F^{\Delta}(t) \\ &= (p+1)\left(F^{\sigma}(t)\right)^p F^{\Delta}(t).\end{aligned}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Substituting into (2.10), we get (2.8). The proof is complete. □

Lemma 2.3. *Let $\mathbb{T}$ be a time scale with a, $b \in \mathbb{T}$. If $p > 0$, then*

$$\int_a^b r(t)\left(\int_t^b f(\tau)\Delta\tau\right)^{p+1}\Delta t \le (p+1)\int_a^b R(a,\sigma(t))\bar{F}^p(t) f(t)\Delta t, \tag{2.11}$$

where

$$R(a,t) = \int_a^t r(\tau)\Delta\tau, \quad \text{and} \quad \bar{F}(t) = \int_t^b f(\tau)\Delta\tau. \tag{2.12}$$

Proof. From (2.12) and applying integration by parts (2.4) with $v^\Delta(t) = R^\Delta(a,t)$ and $u(t) = \bar{F}^{p+1}(t)$, we obtain

$$\begin{aligned}\int_a^b r(t)\left(\int_t^b f(\tau)\Delta\tau\right)^{p+1}\Delta t &= \int_a^b R^\Delta(a,t)\bar{F}^{p+1}(t)\Delta t \\ &= R(a,t)\,\bar{F}^{p+1}(t)\Big|_a^b - \int_a^b R(a,\sigma(t))\left(\bar{F}^{p+1}(t)\right)^\Delta\Delta t.\end{aligned}$$

Using the fact that $R(a,a) = 0$ and $\bar{F}(b) = 0$, we have

$$\int_a^b r(t)\left(\int_t^b f(\tau)\Delta\tau\right)^{p+1}\Delta t = -\int_a^b R(a,\sigma(t))\left(\bar{F}^{p+1}(t)\right)^\Delta\Delta t. \tag{2.13}$$

By the chain rule (2.3) and the fact that $\bar{F}^\Delta(t) = -f(t) \le 0$ and $t \le d$, we see that

$$\left(\bar{F}^{p+1}(t)\right)^\Delta = (p+1)\bar{F}^p(d)F^\Delta(t) \ge (p+1)\bar{F}^p(t)\bar{F}^\Delta(t).$$

Substituting into (2.13), we get (2.11). The proof is complete. □

3. Main Results

In this section, we prove the main results.

Theorem 3.1. *Let $\mathbb{T}$ be a time scale with $a \in [0,\infty)_{\mathbb{T}}$, $0 < p < 1$. If*

$$\int_a^\infty r(t)\left(\int_a^{\sigma(t)} f(\tau)\Delta\tau\right)^p\Delta t < \infty,$$

then

$$\int_a^\infty r(t)\left(\int_a^{\sigma(t)} f(\tau)\Delta\tau\right)^p\Delta t \ge p^p\int_a^\infty r^{1-p}(t)\left(\int_t^\infty r(\tau)\Delta\tau\right)^p f^p(t)\Delta t. \tag{3.1}$$

Proof. Define $F(t) = \int_a^t f(\tau)\Delta\tau$. Integrating the left hand side of (3.1) by parts (2.4) with $u^\Delta(t) = r(t)$ and $v^\sigma(t) = (F^\sigma(t))^p$, we obtain

$$\begin{aligned}\int_a^\infty r(t)(F^\sigma(t))^p\Delta t &= u(t)F^p(t)|_a^\infty - \int_a^\infty u(t)(F^p(t))^\Delta\Delta t \\ &= \int_a^\infty(-u(t))(F^p(t))^\Delta\Delta t,\end{aligned} \tag{3.2}$$

where $u(t) = -\int_t^\infty r(\tau)\Delta\tau$. From (2.3), we have (note that $F^\Delta(t) = f(t) \ge 0$ and $d \le \sigma(t)$)

$$(F^p(t))^\Delta = pF^{p-1}(d)F^\Delta(t) \ge p(F^\sigma(t))^{p-1}f(t). \tag{3.3}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

Substitute (3.3) into (3.2) and applying Hölder's inequality (2.5) to get

$$\begin{aligned}\int_a^\infty r(t)(F^\sigma(t))^p\Delta t &\geq p\int_a^\infty f(t)(F^\sigma(t))^{p-1}\left(\int_t^\infty r(\tau)\Delta\tau\right)\Delta t\\ &= p\int_a^\infty f(t)r^{-1/p'}(t)\left(\int_t^\infty r(\tau)\Delta\tau\right)r^{1/p'}(t)(F^\sigma(t))^{p-1}\Delta t\\ &\geq p\left\{\int_a^\infty r^{1-p}(t)\left(\int_t^\infty r(\tau)\Delta\tau\right)^p f^p(t)\Delta t\right\}^{1/p}\\ &\quad\times\left\{\int_a^\infty r(t)(F^\sigma(t))^p\Delta t\right\}^{1/p'},\end{aligned}$$

and consequently, we obtain

$$\left\{\int_a^\infty r(t)(F^\sigma(t))^p\Delta t\right\}^{1/p}\geq p\left\{\int_a^\infty r^{1-p}(t)\left(\int_t^\infty r(\tau)\Delta\tau\right)^p f^p(t)\Delta t\right\}^{1/p},$$

which is (3.1). The proof is complete. □

Remark 3.1. *If $\mathbb{T}=\mathbb{R}$, then inequality (3.1) reduces to the Beesack and Heinig integral inequality (1.3).*

Remark 3.2. *If $\mathbb{T}=\mathbb{N}$, then inequality (3.1) reduces to the Leindler discrete inequality (1.7).*

Here, we state the Minkowski inequality [29, Lemma 2.6] on time scales which is needed in the proof of our next main result.

Lemma 3.1. *Let $\mathbb{T}$ be a time scale with $a, b\in\mathbb{T}$ and let f, g be nonnegative rd-continuous functions on $[a,b]_\mathbb{T}$. If $\gamma\geq 1$, then*

$$\left(\int_a^b f(x)\left(\int_a^{\sigma(x)} g(t)\Delta t\right)^\gamma\Delta x\right)^{1/\gamma}\leq\int_a^b g(t)\left(\int_t^b f(x)\Delta x\right)^{1/\gamma}\Delta t. \tag{3.4}$$

Theorem 3.2. *Let $\mathbb{T}$ be a time scale with $a\in[0,\infty)_\mathbb{T}$, $0<p<1$. If*

$$\int_a^\infty r(t)\left(\int_t^\infty f(\tau)\Delta\tau\right)^p\Delta t<\infty,$$

then

$$\int_a^\infty r(t)\left(\int_t^\infty f(\tau)\Delta\tau\right)^p\Delta t\geq p^p\int_a^\infty r^{1-p}(t)\left(\int_a^{\sigma(t)} r(\tau)\Delta\tau\right)^p f^p(t)\Delta t. \tag{3.5}$$

Proof. Define $\bar{F}(t):=\int_t^\infty f(\tau)\Delta\tau$. Since

$$\bar{F}^p(t)=-\int_t^\infty\left(\bar{F}^p(\tau)\right)^\Delta\Delta\tau, \tag{3.6}$$

so, from (2.3), we have (note that $\bar{F}^\Delta(\tau)=-f(\tau)\leq 0$, and $d\geq\tau$)

$$\left(\bar{F}^p(\tau)\right)^\Delta=p\bar{F}^{p-1}(d)\bar{F}^\Delta(\tau)\leq -p\bar{F}^{p-1}(\tau)f(\tau). \tag{3.7}$$

Substitute (3.7) into (3.6) gives

$$\bar{F}^p(t)\geq p\int_t^\infty\bar{F}^{p-1}(\tau)f(\tau)\Delta\tau.$$

Applying Minkowski's inequality and Hölder's inequality to get

$$\begin{aligned}\int_a^\infty \bar{F}^p(t)\, r(t)\Delta t &\geq p\int_a^\infty r(t)\left(\int_t^\infty \bar{F}^{p-1}(\tau)f(\tau)\Delta\tau\right)\Delta t \\ &\geq p\int_a^\infty f(\tau)r^{-1/p'}(\tau)\left(\int_a^{\sigma(\tau)} r(t)\,\Delta t\right)\bar{F}^{p-1}(\tau)r^{1/p'}(\tau)\,\Delta\tau \\ &= p\left\{\int_a^\infty r^{1-p}(\tau)\left(\int_a^{\sigma(\tau)} r(t)\,\Delta t\right)^p f^p(\tau)\Delta\tau\right\}^{1/p} \\ &\quad\times\left\{\int_a^\infty \bar{F}^p(\tau)r(\tau)\,\Delta\tau\right\}^{1/p'},\end{aligned}$$

and consequently, we obtain

$$\left(\int_a^\infty \bar{F}^p(t)\, r(t)\Delta t\right)^{1/p} \leq p\left(\int_a^\infty r^{1-p}(\tau)\left(\int_a^{\sigma(\tau)} r(t)\,\Delta t\right)^p f^p(\tau)\Delta\tau\right)^{1/p},$$

which is the desired inequality (3.5). The proof is complete. □

Remark 3.3. *If $\mathbb{T}=\mathbb{R}$, then inequality (3.5) reduces to the Beesack and Heinig integral inequality (1.4).*

Remark 3.4. *If $\mathbb{T}=\mathbb{N}$, then inequality (3.5) reduces to the Leindler discrete inequality (1.8). (See also [28, Remark 3.5])*

Theorem 3.3. *Let $\mathbb{T}$ be a time scale with $a\in[0,\infty)_{\mathbb{T}}$, $1<p<\infty$. If*

$$\int_a^\infty (s(t)\, f(t))^p\,\Delta t<\infty,$$

and

$$\int_t^\infty \left(s^{-1}(\tau)\int_\tau^\infty r^p(x)\,\Delta x\right)^{p'}\Delta\tau \leq C\int_t^\infty r^p(\tau)\,\Delta\tau<\infty, \tag{3.8}$$

then

$$\int_a^\infty \left(r(t)\int_a^{\sigma(t)} f(\tau)\Delta\tau\right)^p\Delta t \leq C\int_a^\infty (s(t)\, f(t))^p\,\Delta t, \tag{3.9}$$

Proof. Assume first that (3.8) holds and define $F(t)=\int_a^t f(\tau)\Delta\tau$. Integrating the left hand side of (3.9) by parts (2.4) with $u^\Delta(t)=r^p(t)$ and $v^\sigma(t)=(F^\sigma(t))^p$, we obtain

$$\begin{aligned}\int_a^\infty r^p(t)\,(F^\sigma(t))^p\,\Delta t &= u(t)\,F^p(t)|_a^\infty-\int_a^\infty u(t)\,(F^p(t))^\Delta\,\Delta t \\ &= \int_a^\infty (-u(t))\,(F^p(t))^\Delta\,\Delta t,\end{aligned}$$

where $u(t)=-\int_t^\infty r^p(\tau)\,\Delta\tau$. From (2.3), we have

$$(F^p(t))^\Delta \leq p\,(F^\sigma(t))^{p-1} f(t),$$

and so

$$\begin{aligned}\int_a^\infty r^p(t)\,(F^\sigma(t))^p\,\Delta t &\leq p\int_a^\infty f(t)\,(F^\sigma(t))^{p-1}\left(\int_t^\infty r^p(\tau)\,\Delta\tau\right)\Delta t \\ &= p\int_a^\infty s(t)\, f(t)\,(F^\sigma(t))^{p-1}\left(s^{-1}(t)\int_t^\infty r^p(\tau)\,\Delta\tau\right)\Delta t.\end{aligned}$$

If we assume that $\int_a^\infty (s(t) f(t))^p \Delta t = 1$, then Hölder's inequality (2.5) gives

$$\begin{aligned}\int_a^\infty r^p(t)(F^\sigma(t))^p \Delta t &\le p\left\{\int_a^\infty (s(t) f(t))^p \Delta t\right\}^{1/p} \\ &\quad \times \left\{\int_a^\infty (F^\sigma(t))^p \left(s^{-1}(t)\int_t^\infty r^p(\tau)\Delta\tau\right)^{p'} \Delta t\right\}^{1/p'} \\ &= p\left\{\int_a^\infty (F^\sigma(t))^p \left(s^{-1}(t)\int_t^\infty r^p(\tau)\Delta\tau\right)^{p'} \Delta t\right\}^{1/p'}.\end{aligned}$$

Using integration by parts with $u^\Delta(t) = \left(s^{-1}(t)\int_t^\infty r^p(\tau)\Delta\tau\right)^{p'}$ and $v^\sigma(t) = (F^\sigma(t))^p$ to get

$$\begin{aligned}&\int_a^\infty r^p(t)(F^\sigma(t))^p \Delta t \\ &\le p\left\{\int_a^\infty \left[\int_t^\infty \left(s^{-1}(\tau)\int_\tau^\infty r^p(x)\Delta x\right)^{p'} \Delta\tau\right](F^p(t))^\Delta \Delta t\right\}^{1/p'}.\end{aligned}$$

Using (3.8) and integration by parts again with $u(t) = \int_t^\infty r^p(\tau)\Delta\tau$ and $v^\Delta(t) = (F^p(t))^\Delta$, we obtain

$$\int_a^\infty r^p(t)(F^\sigma(t))^p \Delta t \le C\left\{\int_a^\infty r^p(t)(F^\sigma(t))^p \Delta t\right\}^{1/p'} < \infty,$$

and so $\int_a^\infty r^p(t)(F^\sigma(t))^p \Delta t \le C$. The proof is complete. □

To prove the next results, need the following two theorems which are adapted from [35] and [1].

Theorem 3.4. *If $p(t)$, $q(t) \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ are positive functions such that $\int_a^t (p(\tau))^{-1/(k-1)}\Delta\tau < \infty$, and $y \in C^1_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ with $y(a) = 0$, then for $k > 1$, $\lambda > 0$ and $0 < \gamma < k$, we have*

$$\int_a^b q(t)\,|y(t)|^\lambda \left|y^\Delta(t)\right|^\gamma \Delta t \le K_1(\lambda,\gamma,k)\left[\int_a^b p(t)\left|y^\Delta(t)\right|^k \Delta t\right]^{(\lambda+\gamma)/k}, \tag{3.10}$$

where

$$K_1(\lambda,\gamma,k) := \left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/k}\left[\int_a^b \left(\frac{q^k(t)}{p^\gamma(t)}\right)^{\frac{1}{k-\gamma}}\left(\int_a^t p^{\frac{-1}{k-1}}(\tau)\Delta\tau\right)^{\frac{\lambda(k-1)}{(k-\gamma)}}\Delta t\right]^{\frac{k-\gamma}{k}}.$$

If $y \in C^1_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ with $y(b) = 0$, then for $k > 1$, $\lambda > 0$ and $0 < \gamma < k$, we have that

$$\int_a^b q(t)\,|y(t)|^\lambda \left|y^\Delta(t)\right|^\gamma \Delta t \le K_2(\lambda,\gamma,k)\left[\int_a^b p(t)\left|y^\Delta(t)\right|^k \Delta t\right]^{(\lambda+\gamma)/k}, \tag{3.11}$$

where

$$K_2(\lambda,\gamma,k) := \left(\frac{\gamma}{\lambda+\gamma}\right)^{\gamma/k}\left[\int_a^b \left(\frac{q^k(t)}{p^\gamma(t)}\right)^{\frac{1}{k-\gamma}}\left(\int_t^b p^{\frac{-1}{k-1}}(\tau)\Delta\tau\right)^{\frac{\lambda(k-1)}{(k-\gamma)}}\Delta t\right]^{\frac{k-\gamma}{k}}.$$

Theorem 3.5. *If $p(t), q(t) \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ are positive functions and $y \in C^1_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ such that $y(a) = 0$, then for $\lambda \geq 1$, $\gamma \geq 0$ and $k > \gamma + 1$, we have that*

$$\int_a^b q(t) \left|(y^\lambda)^\Delta(t)(y^\Delta(t))^\gamma\right| \Delta t \leq G_1(\lambda,\gamma,k) \left\{\int_a^b p(t) \left|y^\Delta(t)\right|^k \Delta t\right\}^{\frac{\lambda+\gamma}{k}}, \tag{3.12}$$

where

$$G_1(\lambda,\gamma,k) := c\left\{\int_a^b (q(t))^{\frac{k}{k-\gamma-1}} (p(t))^{\frac{-k\gamma}{(k-1)(k-\gamma-1)}} \left(R^{\frac{k\lambda-\lambda-\gamma}{k-\gamma-1}}\right)^\Delta (t)\Delta t\right\}^{\frac{k-\gamma-1}{k}},$$

with

$$c = \lambda\left(\frac{k-\gamma-1}{k\lambda-\lambda-\gamma}\right)^{\frac{k-\gamma-1}{k}} \left(\frac{\gamma+1}{\lambda+\gamma}\right)^{\frac{\gamma+1}{k}}, \quad \text{and} \quad R(t) = \int_a^t \frac{\Delta\tau}{(p(\tau))^{\frac{1}{k-1}}}.$$

From (2.6), inequality (3.12) becomes as follow: If $p(t), q(t) \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ are positive functions and $y \in C^1_{rd}([a,b]_{\mathbb{T}}, \mathbb{R})$ with $y^\Delta > 0$ satisfies $y(a) = 0$, then for $\lambda \geq 1$, $\gamma \geq 0$ and $k > \gamma + 1$

$$\int_a^b q(t) \left|y^\sigma(t)\right|^{\lambda-1} \left|y^\Delta(t)\right|^{\gamma+1} \Delta t \leq G_1(\lambda,\gamma,k) \left\{\int_a^b p(t) \left|y^\Delta(t)\right|^k \Delta t\right\}^{\frac{\lambda+\gamma}{k}}, \tag{3.13}$$

where $G_1(\lambda,\gamma,k)$ is defined as in (3.12).

Theorem 3.6. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$. If $p > 0$ and $k > 1$, then*

$$\int_a^b r(t) \left(\int_a^{\sigma(t)} f(\tau)\Delta\tau\right)^{p+1} \Delta t \leq (p+1) G_1(p+1,k) \left[\int_a^b s(t)(f(t))^k \Delta t\right]^{\frac{p+1}{k}}, \tag{3.14}$$

where

$$G_1(p+1,k) := \left[\int_a^b (R(t,b))^{\frac{k}{k-1}} \left(\left(\int_a^t s^{\frac{-1}{k-1}}(\tau)\Delta\tau\right)^{p+1}\right)^\Delta \Delta t\right]^{\frac{k-1}{k}},$$

and $R(t,b)$ is defined as in (2.9).

Proof. Applying Opial's inequality (3.13) with $y(t) = F(t)$, $q(t) = R(t,b)$, $p(t) = s(t)$, $\lambda = p+1$ and $\gamma = 0$, we obtain

$$\int_a^b R(t,b)(F^\sigma(t))^p F^\Delta(t)\Delta t \leq G_1(p+1,k)\left[\int_a^b s(t)(f(t))^k \Delta t\right]^{\frac{p+1}{k}}. \tag{3.15}$$

The result follows from (2.8) and (3.15). The proof is complete. □

Theorem 3.7. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$. If $p > 0$ and $k > 1$, then*

$$\int_a^b r(t) \left(\int_t^b f(\tau)\Delta\tau\right)^{p+1} \Delta t \leq (p+1) K_2(p,1,k) \left[\int_a^b s(t)(f(t))^k \Delta t\right]^{\frac{p+1}{k}}, \tag{3.16}$$

where

$$K_2(p,1,k) := \left(\frac{1}{p+1}\right)^{\frac{1}{k}} \left[\int_a^b (R(a,\sigma(t)))^{\frac{k}{k-1}} s^{\frac{-1}{k-1}}(t) \left(\int_t^b s^{\frac{-1}{k-1}}(\tau)\Delta\tau\right)^p \Delta t\right]^{\frac{k-1}{k}},$$

and $R(a,t)$ is defined as in (2.12).

Proof. Applying Opial's inequality (3.11) with $y(t) = F(t)$, $q(t) = R(a, \sigma(t))$, $p(t) = s(t)$, $\lambda = p$ and $\gamma = 1$, we obtain

$$\int_a^b R(a, \sigma(t)) \bar{F}^p(t) f(t) \Delta t \leq K_2(p, 1, k) \left[\int_a^b s(t) (f(t))^k \Delta t \right]^{\frac{p+1}{k}}. \tag{3.17}$$

The result follows from (2.11) and (3.17). The proof is complete. □

The next result follows from Theorems 3.6 and 3.7 by choosing $k = p + 1$.

Corollary 3.1. *Let $\mathbb{T}$ be a time scale with $a, b \in \mathbb{T}$. If $k > 1$, then*

$$\int_a^b r(t) \left(\int_a^{\sigma(t)} f(\tau) \Delta\tau \right)^k \Delta t \leq k G_1(k) \int_a^b s(t) (f(t))^k \Delta t, \tag{3.18}$$

where

$$G_1(k) := \left[\int_a^b (R(t, b))^{\frac{k}{k-1}} \left(\left(\int_a^t s^{\frac{-1}{k-1}}(\tau) \Delta\tau \right)^k \right)^{\Delta} \Delta t \right]^{\frac{k-1}{k}},$$

and

$$\int_a^b r(t) \left(\int_t^b f(\tau) \Delta\tau \right)^k \Delta t \leq k K_2(k) \int_a^b s(t) (f(t))^k \Delta t, \tag{3.19}$$

where

$$K_2(k) := \left(\frac{1}{k} \right)^{\frac{1}{k}} \left[\int_a^b (R(a, \sigma(t)))^{\frac{k}{k-1}} s^{\frac{-1}{k-1}}(t) \left(\int_t^b s^{\frac{-1}{k-1}}(\tau) \Delta\tau \right)^{k-1} \Delta t \right]^{\frac{k-1}{k}}.$$

Remark 3.5. *Note that Theorems 3.6 and 3.7 are consequences of the weighted Hardy-type inequality due to Saker et al. [29, 36] with $p + 1 = q$ and $k = p$.*

As special cases of Theorems 3.6 and 3.7 when $\mathbb{T} = \mathbb{N}$, we have the following new discrete results

Corollary 3.2. *Let $\{x_n\}$, $\{\lambda_n\}$ and $\{w_n\}$ be nonnegative sequences. If $p > 0$ and $k > 1$, then*

$$\sum_{n=1}^N r_n \left(\sum_{i=1}^n x_i \right)^{p+1} \leq (p+1) G_1(p+1, k) \left(\sum_{n=1}^N s_n x_n^k \right)^{\frac{p+1}{k}},$$

where

$$G_1(p+1, k) := \left[\sum_{n=1}^N (R(n, N))^{\frac{k}{k-1}} \Delta \left(\sum_{i=1}^n (s_i)^{\frac{-1}{k-1}} \right)^{p+1} \right]^{\frac{k-1}{k}},$$

with $R(n, N) = \sum_{i=n}^N r_i$.

Corollary 3.3. *Let $\{x_n\}$, $\{\lambda_n\}$ and $\{w_n\}$ be nonnegative sequences. If $p > 0$ and $k > 1$, then*

$$\sum_{n=1}^N r_n \left(\sum_{i=n}^N x_i \right)^{p+1} \leq (p+1) K_2(p, 1, k) \left(\sum_{n=1}^N s_n x_n^k \right)^{\frac{p+1}{k}},$$

where

$$K_2(p, 1, k) := \left(\frac{1}{p+1} \right)^{\frac{1}{k}} \left[\sum_{n=1}^N (R(1, n+1))^{\frac{k}{k-1}} (s_n)^{\frac{-1}{k-1}} \left(\sum_{i=n}^N (s_i)^{\frac{-1}{k-1}} \right)^p \right]^{\frac{k-1}{k}},$$

with $R(1, n+1) = \sum_{i=1}^n r_i$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

By making suitable substitutions for the two weighted functions $r(t)$ and $s(t)$, we get some extensions related to the dynamic inequalities due to Řehák [25] and Saker et al. [30, 31] respectively. Also when $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=\mathbb{R}$, we get consequences due to Bennett [6], Bliss [9] and Flett [15]. For illustrations, we will present these special cases in the following examples.

Example 3.1. *If $r(t)=(\sigma(t)-a)^{-k}$ and $s(t)=1$, then inequality (3.18) reduces to the following extension of the Hardy-type inequality due to Řehák [25, Theorem 2.1]*

$$\int_a^\infty \left(\frac{1}{\sigma(t)-a}\int_a^{\sigma(t)} f(\tau)\Delta\tau\right)^k \Delta t \leq kR_1 \int_a^\infty f^k(t)\Delta t,$$

where

$$R_1 := \left[\int_a^\infty (R(t,\infty))^{\frac{k}{k-1}} \left((t-a)^k\right)^\Delta \Delta t\right]^{\frac{k-1}{k}}.$$

Example 3.2. *If we choose $r(t)=1/t^\gamma$ and $s(t)=1/t^{\gamma-k}$, $\gamma>1$ in Corollary 3.1, we get the inequality*

$$\int_a^\infty \frac{1}{t^\gamma}\left(\int_a^{\sigma(t)} f(\tau)\Delta\tau\right)^k \Delta t \leq kR_2 \int_a^\infty \frac{1}{t^{\gamma-k}} f^k(t)\Delta t,$$

which is related to the inequality due to Saker and O'Regan [30, Theorem 2.2], where

$$R_2 := \left[\int_a^\infty (R(t,\infty))^{\frac{k}{k-1}} \left(\left(\int_a^t \left(\frac{1}{\tau^{\gamma-k}}\right)^{\frac{-1}{k-1}} \Delta\tau\right)^k\right)^\Delta \Delta t\right]^{\frac{k-1}{k}}.$$

Example 3.3. *If we choose $r(t)=1/\sigma^\gamma(t)$ and $s(t)=1/\sigma^{\gamma-k}(t)$ in Corollary 3.1, we have the inequality*

$$\int_a^\infty \frac{1}{\sigma^\gamma(t)}\left(\int_t^\infty f(\tau)\Delta\tau\right)^k \Delta t \leq kR_3 \int_a^\infty \frac{1}{\sigma^{\gamma-k}(t)} f^k(t)\Delta t,$$

which is related to the inequality due to Saker and O'Regan [30, Theorem 2.1], where

$$R_3 := \left(\frac{1}{k}\right)^{\frac{1}{k}} \left[\int_a^\infty (R(a,\sigma(t)))^{\frac{k}{k-1}} (\sigma(t))^{\frac{\gamma-k}{k-1}} \left(\int_t^\infty (\sigma(\tau))^{\frac{\gamma-k}{k-1}} \Delta\tau\right)^{k-1} \Delta t\right]^{\frac{k-1}{k}}.$$

Example 3.4. *If we take $f(t)=\lambda(t)g(t)$,*

$$r(t)=\frac{\lambda(t)}{(\Lambda^\sigma(t))^\gamma},\ s(t)=\lambda^{1-k}(t)(\Lambda^\sigma(t))^{k-\gamma},\ k\geq\gamma>1,$$

in Corollary 3.1, we have the inequality

$$\int_a^b \frac{\lambda(t)}{(\Lambda^\sigma(t))^\gamma}\left(\int_a^{\sigma(t)} \lambda(\tau)g(\tau)\Delta\tau\right)^k \Delta t \leq kR_4 \int_a^b \lambda(t)(\Lambda^\sigma(t))^{k-\gamma} g^k(t)\Delta t,$$

which is related to the inequality due to Saker et al. [31, Theorem 2.1], where $\Lambda(t)=\int_a^t \lambda(\tau)\Delta\tau$ and

$$R_4 := \left[\int_a^b (R(t,b))^{\frac{k}{k-1}} \left(\left(\int_a^t s^{\frac{-1}{k-1}}(\tau)\Delta\tau\right)^k\right)^\Delta \Delta t\right]^{\frac{k-1}{k}}.$$

Example 3.5. *If we choose* $r(t) = t^{-1-(p+1)\lambda}/t^{p+1}$ *and* $s(t) = t^{-1-k\lambda}$, $\lambda > -1$ *in Theorem 3.6, we obtain the inequality*

$$\int_a^b t^{-1-(p+1)\lambda}\left(\frac{\int_a^{\sigma(t)} f(\tau)\Delta\tau}{t}\right)^{p+1}\Delta t \leq (p+1)\,R_5\left[\int_a^b t^{-1-k\lambda}f^k(t)\Delta t\right]^{\frac{p+1}{k}},$$

which is related to the inequalities due to Flett [15] and Bliss [9], Hardy and Littlewood [18] (with $\lambda = -1/k$*), where*

$$R_5 := \left[\int_a^b (R(t,b))^{\frac{k}{k-1}}\left(\left(\int_a^t s^{\frac{-1}{k-1}}(\tau)\Delta\tau\right)^{p+1}\right)^{\Delta}\Delta t\right]^{\frac{k-1}{k}}.$$

Example 3.6. *If we take*

$$r_n = \frac{\lambda_n}{\Lambda_n^{1-\frac{(p+1)}{k}(1-c)}},\ s_n = \lambda_n^{1-k}\Lambda_n^{k-c},\ c > 1 \text{ and } x_n = \lambda_n y_n,$$

in Corollary 3.2, we get the inequality

$$\sum_{n=1}^N \lambda_n\Lambda_n^{\frac{(p+1)(1-c)}{k}-1}\left(\sum_{i=1}^n \lambda_i y_i\right)^{p+1} \leq (p+1)\,R_6\left(\sum_{n=1}^N \lambda_n\Lambda_n^{k-c}y_n^k\right)^{\frac{p+1}{k}},$$

which is related to Bennett's inequality [6, Corollary 7], where $\Lambda_n = \sum_{i=1}^n \lambda_i$ *and*

$$R_6 := \left[\sum_{n=1}^N (R(n,N))^{\frac{k}{k-1}}\,\Delta\left(\sum_{i=1}^n (s_i)^{\frac{-1}{k-1}}\right)^{p+1}\right]^{\frac{k-1}{k}},$$

with $R(n,N) = \sum_{i=n}^N \lambda_i\Lambda_i^{\frac{(p+1)(1-c)}{k}-1}$.

Remark 3.6. *As an application, we can apply Opial's inequalities together with a Hardy-type inequality (3.16) on time scales to establish some lower bounds of the distance between zeros of a solution and/or its derivatives for the fourth-order dynamic equation (see [13, Theorem 5.1])*

$$\left(r(t)y^{\Delta^3}(t)\right)^{\Delta} - \left(p(t)y^{\Delta}(t)\right)^{\Delta} + q(t)y^{\sigma}(t) = 0, \quad t \in [a,b]_{\mathbb{T}}. \tag{3.20}$$

Availability of supporting data: *The authors declare that all data and materials in the article are available and veritable.*

Competing interests: *The authors declare that they have no competing interests.*

Funding: *Not applicable*

Authors' contributions: *The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.*

Acknowledgements: *The authors sincerely thank the reviewers for their valuable suggestions and useful comments that have led to the present improved version of the original manuscript.*

Authors' information: **S. H. Saker**: Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt, `E-mails:shsaker@mans.edu.eg`

M. Osman: Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt, `E-mails: mmosman@mans.edu.eg`

Islam Abohela: College of Engineering, Applied Science University, Kingdom of Bahrain, `E-mail: islam.abohela@asu.edu.bh`

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

References

[1] R. Agarwal, M. Bohner, D. O'Regan, M. Osman, and S. Saker, A general dynamic inequality of Opial type, Appl. Math. & Infor. Sci. 10 (2016), 1–5.
[2] R. P. Agarwal, D. O'Regan and S. H. Saker, *Dynamic Inequalities on Time Scales*, Springer Cham Heidlelberg New York Drodrechet London, (2014).
[3] R. P. Agarwal, D. O'Regan and S. H. Saker, *Hardy Type Inequalities on Time Scales*, Springer International Publishing, Cham Heidlelberg New York Drodrechet London, (2016).
[4] R. P. Agarwal, D. O'Regan and S. H. Saker, Some Hardy type inequalities with weighted functions via Opial type inequalities, Advances in Dyn. Syst. Appl. 10 (2015), 1–9.
[5] M. Anwar, R. Bibi, M. Bohner and J. Pečarić, Integral inequalities on time scales via the theory of isotonic linear functionals, Abstr. Appl. Anal. Vol. 2011, Article ID 483595, 16 pages, 2011.
[6] G. Bennett, Some elementary inequalities, The Quart. J. Math. Oxford 38 (1987), 401–425.
[7] P. R. Bessack, Hardy's inequality and its extensions, Pacific J. Math. 11 (1961), 39–61.
[8] P. R. Bessack and H. P. Heinig, Hardy's inequalities with indices less than 1, Proc. Amer. Math. Soc. 83 (3) (1981), 532–536.
[9] G. A. Bliss, An integral inequality, J. Lond. Math. Soc. 5 (1930), 40–46.
[10] S. Bloom and R. Kerman, Weighted norm inequalities for operators of Hardy type, Proc. Amer. Math. Soc. 113 (1) (1991), 135–141.
[11] J. S. Bradly, Hardy inequalities with mixed norms, Cand. Math. Bull. 21 (4) (1978), 405–408.
[12] M. Bohner and B. Kaymakçalan, Opial inequalities on time scales, Annales Polo. Math. 77 (2001), 11–20.
[13] M. J. Bohner, M. M. Osman and S. H. Saker, General higher order dynamic Opial inequalities with applications, Dyn. Syst. Appl. 26 (2017), 65–80.
[14] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*: An Introduction with Applications, Birkhäuser, Boston, (2001).
[15] T. M. Flett, A note on some inequalities, Proc. Glasgow Math. Assoc. 4 (1958), 7–15.
[16] G. H. Hardy, Note on a theorem of Hilbert, Math. Zeitschrift 6 (1920), 314–317.
[17] G. H. Hardy, Notes on some points in the integral calculus (LX): An inequality between integrals, Messenger of Math. 54 (1925), 150–156.
[18] G. H. Hardy, J. E. Littlewood, Notes on the theory of series (XII): on certain inequalities connected with the calculus of variations, J. London. Math. Soc. 5 (1930), 34–39.
[19] S. Hilger, Analysis on measure chains —a unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18–56.
[20] A. Kufner and L.-E. Persson, *Weighted Inequalities of Hardy Type,* World Scientific Publishing Co., Singapore, (2003).
[21] L. Leindler, Generalization of inequalities of Hardy and Littlewood, Acta Scient. Math. (Szeged) 31 (1970), 279–285.
[22] L. Leindler, Further sharpening of inequalities of Hardy and Littlewood, Acta Scient. Math. (Szeged) 54 (1990), 285–289.
[23] B. Muckenhoupt, Hardy's inequality with weights, Studia Math. 44 (1) (1972), 31–38.
[24] B. Opic and A. Kufner, *Hardy-Type Inequalities,* Pitman Research Notes in Mathematics Vol. 219, Longman Scientific & Technical, Harlow (1990).
[25] P. Řehák, Hardy inequality on time scales and its application to half-linear dynamic equations, J. Ineq. Appl. 5 (2005), 495–507.
[26] S. H. Saker, Hardy-Leindler type inequalities on time scales, Appl. Math. & Infor. Sci. 8 (2014), 2975–2981.
[27] S. H. Saker, R. R. Mahmoud, M. M. Osman and R. P. Agarwal, Some new generalized forms of Hardy's type inequality on time scales, Math. Ineq. Appl. 20 (2017), 459–481.
[28] S. H. Saker, R. R. Mahmoud and A. Peterson, Some Bennett-Copson type inequalities on time scales, J. Math. Ineq. 10 (2016), 471–489.
[29] S. H. Saker, R. R. Mahmoud and A. Peterson, Weighted Hardy-type inequalities on time scales with Applications, Mediter. J. Math. 13 (2016), 585-606.
[30] S.H. Saker and D. O'Regan, Hardy and Littlewood inequalities on time scales, Bull. Malay. Math. Sci. Soc. 39 (2016), 527–543.
[31] S.H. Saker, D. O'Regan and R. P. Agarwal, Generalized Hardy, Copson, Leindler and Bennett inequalities on time scales, Math. Nachr. 287 (2014), 687–698.
[32] S. H. Saker, M. M. Osman, D. O'Regan and R. P. Agarwal, Inequalities of Hardy-type and generalizations on time scales, Analysis: Inter. Math. J. Anal. Appl. 38 (1) (2018), 47–62.
[33] S. H. Saker, M. M. Osman, D. O'Regan and R. P. Agarwal, Levinson type inequalities and their extensions via convexity on time scales, RACSAM 113 (1) (2019), 299–314.
[34] S. H. Saker, M. M. Osman, D. O'regan and R. P. Agarwal, Lyapunov inequalities for dynamic equations via new Opial type inequalities, Hacet. J. Math. Stat. 47 (6) (2018), 1544–1558.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO.6, 2020, COPYRIGHT 2020 EUDOXUS PRESS, LLC

[35] S. H. Saker, M. M. Osman, D. O'Regan and R. P. Agarwal, Some new Opial dynamic inequalities with weighted functions on time scales, Math. Ineq. Appl. 18 (2015), 1171–1187.
[36] S. H. Saker, M. M. Osman and A. Peterson, New characterizations of weights in Hardy's type dynamic inequalities with application, Analy. Math. Physics (Submitted)
[37] G. J. Sinnamon, Weighted Hardy and Opial-type inequalities, J. Math. Anal. Appl. 160 (1991), 434–445.
[38] G. Talenti, Osservazioni sopra une classe di disuguaglianze, Rend. Sem. Mat. Fis. Milano 39 (1969), 171–185.

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 28, NO. 6, 2020